DAMTP-2006–3

Einstein–Maxwell–Dilaton metrics from three–dimensional Einstein–Weyl structures.

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Abstract

A class of time dependent solutions to $(3+1)$ Einstein–Maxwell-dilaton theory with attractive electric force is found from Einstein–Weyl structures in (2+1) dimensions corresponding to dispersionless Kadomtsev–Petviashvili and $SU(\infty)$ Toda equations. These solutions are obtained from time–like Kaluza–Klein reductions of $(3+2)$ solitons.

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1 Introduction

Singularity theorems of Hawking and Penrose assert that mild energy conditions imposed on the energy momentum tensor result in a gravitational collapse to a singularity. While the final singular state of a collapsing star is inevitable, not much is known about the dynamical mechanisms leading to the formations of the singularities. The relevant time dependent exact solutions to Einstein equations are unknown, and the numerical considerations are made difficult by the Birkhoff theorem which says that any spherically symmetric vacuum solution is static, which implies that the metric is Schwarzchild. To make progress one would need to draw conclusions from numerical evolution of a non-spherically symmetric initial data which is considerably more difficult.

One way to overcome these difficulties is to introduce matter fields to Einstein equations which allows the study of time evolution, while maintaining the spherical symmetry. The simplest choice corresponds to the massless scalar field. Christodoulou has given a complete analysis of this situation [?]. Interestingly enough his analysis revealed that certain data on a future null cone centred at the origin can evolve into a solution with naked singularities (i.e. singularities not hidden inside an event horizon) in contrary with the Cosmic Censorship Hypothesis (CCH). This is a mild violation of the CCH, as the initial data is given on a high co-dimension surface inside the null cone, and is unstable in a sense that any data away from this surface does not evolve into naked singularities.

Another way to evade Birkhoff's theorem is to go to more than four space–time dimensions. This was recently done in [?] where the following ansatz was made for a (4+1) metric[1]

$$g_{(4,1)} = -Ce^{-2\delta}\mathrm{d}t^2 + C^{-1}\mathrm{d}r^2 + \frac{1}{4}r^2(e^{2B}(\sigma_1^2+\sigma_2^2) + e^{-4B}\sigma_3^2). \tag{1.1}$$

Here $\sigma_i, i = 1, 2, 3$ are the left invariant one forms on $SU(2)$ satisfying the standard Maurer–Cartan relations, and the functions B, C, δ depend on (r, t). The authors of [?] have numerically studied the PDEs for these functions resulting from the Ricci-flatness of (??), and have shown that a (4+1) dimensional Schwarzchild black hole

$$C = 1 - \frac{\text{const}}{\rho^2}, \qquad B = 0, \qquad \delta = 0$$

is formed for a large initial data. This lead to an explicit numerical profile of settling down to a singularity.

[1]In this paper $g_{(r,s)}$ denotes a real pseudo–Riemannian metric of signature (r, s).

Gibbons [?] has pointed out that the gravitational collapse of (**??**) is not as inevitable as it may seem from the numerical analysis. A spherically symmetric star in (4+1) dimensions can also settle into a soliton, i.e. a non-singular topologically stable solution of the field equations. The simples case is the Kaluza–Klein monopole of Gross–Perry and Sorkin [?, ?], where the five–dimensional metric takes the form

$$g_{(4,1)} = -\mathrm{d}t^2 + g_{TAUB-NUT}, \tag{1.2}$$

where $g_{TAUB-NUT}$ is the simplest asymptotically locally flat (ALF) four dimensional gravitational instanton. Any ALF gravitational instanton would do, so a mild, triaxial, generalisation of the ansatz (**??**) admits a static soliton of the form (**??**) with $g_{TAUB-NUT}$ replaced by the Atiyah–Hitchin gravitational instanton [?], where the complex structures are rotated by the $SU(2)$ action. Using the gravitational instantons can even lead to explicit time dependent solutions. As pointed out in [?] exploiting the scaling symmetry in the A_k ALF multi–instanton leads to the explicit solution of the 4+1 Einstein equations

$$g_{(4,1)} = -\mathrm{d}t^2 + V\ h_{flat} + V^{-1}(\mathrm{d}\theta + A)^2, \tag{1.3}$$

where

$$V = t + \sum_{i=1}^{k} \frac{m}{|\mathbf{r} - \mathbf{r_k}|} \tag{1.4}$$

is a solution to the three–dimensional Laplace equation $\triangle V = 0$ depending on a parameter t and the one–form A satisfies the monopole equation $\mathrm{d}A = *_3\mathrm{d}V$, where $*_3$ is the duality operator of the flat Euclidean metric h_{flat} in three dimensions. Here m is a constant mass parameter, and $\mathbf{r}_1, ..., \mathbf{r}_k$ are positions of fixed points in $\mathbb{R}^3$. The metric appears singular at these points, but in fact it is not if θ is taken to be periodic, and the constant m equals to half of the period. In particular choosing $k = 1$ and $\mathbf{r}_1 = \mathbf{0}$ leads to a time–dependent generalisation of the Taub–NUT Kaluza–Klein monopole metric which is an explicit solution to the Einstein equations imposed on (**??**).

From the (3+1) dimensional perspective the solutions discussed so far give rise to solutions of Einstein–Maxwell theory with a dilaton. This is the standard Kaluza–Klein reduction where the fifths dimension compactifies to a circle of a small radius. This corresponds to θ in (**??**) being periodic. If the radius is sufficiently small then low energy experiments will average over the fifths dimension thus leading to an effective four–dimensional theory with the Maxwell potential given by A, and the dilaton given by $-(\sqrt{3}/4)\log V$. One even gets one time dependent solution (**??**), but this seems to be an isolated case.

The purpose of this paper is to point out (Proposition **??**) that large families of explicit time dependent solutions can be found in the $(3+1)$ dimensional theory. They will come from time–like Kaluza–Klein reductions of pure Einstein equations in $(3+2)$ dimensions. The five–dimensional metrics $g_{(3,2)}$ are given by four–dimensional Ricci–flat metrics $g_{(2,2)}$ of signature $(2,2)$

$$g_{(3,2)} = \mathrm{d}z^2 + g_{(2,2)}. \tag{1.5}$$

To make the Kaluza–Klein reduction possible the metric $g_{(2,2)}$ must admit a Killing vector, and so be of the form

$$g_{(2,2)} = Vh_{(2,1)} - V^{-1}(\mathrm{d}\theta + A)^2 \tag{1.6}$$

where $h_{(2,1)}$ is a metric of signature $(2,1)$ on the three–dimensional space of orbits of the Killing vector $\partial/\partial\theta$, and (V, A) is a function and a one–form on this space. The physical metric $G_{\mu\nu}$ of signature $(3,1)$ is then given from (**??**) by

$$g_{(3,2)} = \exp{(-2\Phi/\sqrt{3})} G_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu - \exp{(4\Phi/\sqrt{3})}(\mathrm{d}\theta + A)^2. \tag{1.7}$$

This metric has a space–like Killing vector $\partial/\partial z$ but is not stationary if $h_{(2,1)}$ does not admit a time–like Killing vector. In the next section we shall construct explicit examples taking $g_{(2,2)}$ to be a Ricci–flat anti–self–dual metric with symmetry. This will imply that $h_{(2,1)}$ is a part of the so called Einstein–Weyl structure [**?**, **?**], and can in principle be found explicitly by twistor methods.

If the reader objects to using the metric $g_{(3,2)}$ of signature $(3,2)$ as non-physical, he should regard it as a mathematical trick for producing interesting Lagrangians in four dimensions. A more serious objection comes from performing the K-K reduction along the time–like symmetry. This will result in a change of the relative sign between the Ricci scalar and the Maxwell term in the four dimensional effective Lagrangain. The charges in the resulting electro–vacuum solutions will therefore attract rather than repel, thus ruling out the extremality condition. The formalism is nevertheless well adopted to studying the Einstein anti–Maxwell theory [**?**].

2 Anti–Self–Dual Ricci–flat $(2,2)$ metrics with symmetry

Consider a $(2,2)$ signature metric $g_{(2,2)}$ on a four–dimensional manifold M. The construction outlined in the introduction demands that it is Ricci flat, but to find explicit examples we shall also assume that its curvature (when viewed as a two–form) is anti–self–dual. In the $(2,2)$ signature the spin group $Spin(2,2)$ decomposes as a product of two independent copies of $SL(2,\mathbb{R})$, and the representation space of the spin group splits up into a direct sum of two real two dimensional vector spaces S_+ and S_-. In the ASD Ricci flat case the curvature of the spin connection on S_+ is zero, and the holonomy effectively reduces to $SL(2,\mathbb{R})$. From the mathematical point of view $g_{(2,2)}$ is a pseudo-Riemannian analog of a four dimensional hyper–Kahler structure. The endomorphisms of the tangent bundle associated to three Kahler structures satisfy the algebra of pseudo–quaternions.

Any ASD Ricci–flat $(2,2)$ metric with a non-null symmetry is of the form (**??**) and, by the Jones–Tod correspondence [**?**], the ASD vacuum equations reduce down to the Lorentzian Einstein–Weyl equations in three–dimensional space of orbits W of the Killing vector $K=\partial/\partial\theta$. This means that there exists a torsion–free connection D on W such that the null geodesics of a conformal structure $[h]$ defined by $h_{(2,1)}$ are also geodesic of this connection. This compatibility condition implies the existence of a one–form ω on W such that

$$Dh_{(2,1)} = \omega \otimes h_{(2,1)}.$$

If we change this representative by $h \longrightarrow \psi^2 h$, then $\omega \longrightarrow \omega + 2\mathrm{d}\ln\psi$, where ψ is a non-vanishing function on W.

The pair $(D,[h])$ satisfies the conformally invariant Einstein–Weyl equations which assert that the symmetrized Ricci tensor of D is proportional to $h_{(2,1)}$. One can regard $h_{(2,1)}$ and ω as the unknowns in these equations. Once they have been found, the covariant differentiation w.r.t D is given by

$$D\chi = \nabla\chi - \frac{1}{2}(\chi\otimes\omega + (1-m)\omega\otimes V - h_{(2,1)}(\omega,\chi)h_{(2,1)}),$$

where χ is a one–form of conformal weight m, and ∇ is the Levi–Civita connection of h.

Given the Einstein–Weyl structure which arises from and ASD vacuum structure, the metric $g_{(2,2)}$ is given by solutions to the generalised monopole equation [**?**]

$$*_h(\mathrm{d}V + \frac{1}{2}\omega V) = \mathrm{d}A, \tag{2.8}$$

where $*_h$ is the duality operator in three dimensions corresponding to $h_{(2,1)}$, and the unknowns are the function V and the one–form A on W. The arbitrary solution to the generalised monopole equation would lead to conformally ASD metric $g_{(2,2)}$ which is not necessarily vacuum. One must therefore select a special class of solutions. This problem has been extensively analysed, and is known to lead to three possibilities depending on $(\nabla K)_+$, the self dual derivative of the Killing vector:

- If $(\nabla K)_+$ is zero then K preserves the self–dual two forms on M, and the moment map coordinates can be chosen. In this case If $h_{(2,1)}$ is flat,

$$g_{(2,2)} = V h_{flat} - V^{-1}(\mathrm{d}\theta + A)^2 \tag{2.9}$$

  and (V, A) satisfy the 2+1 monopole equation $*\mathrm{d}V = \mathrm{d}A$.

- If $(\nabla K)_+$ is a simple two form (i.e.$(\nabla K)_+ = \mathrm{d}p \wedge \mathrm{d}q$ for some functions (p, q) on M) then there exist local coordinates such that [?]

$$h_{(2,1)} = \mathrm{d}y^2 - 4\mathrm{d}x\mathrm{d}t - 4u\mathrm{d}t^2, \tag{2.10}$$

  and

$$g_{(2,2)} = \frac{u_x}{2} h_{(2,1)} - \frac{2}{u_x}(\mathrm{d}\theta - \frac{u_x \mathrm{d}y}{2} - u_y \mathrm{d}t)^2, \tag{2.11}$$

  where the function $u = u(x, y, t)$ satisfies the dispersionless Kadomtsev–Petviashvili equation

$$(u_t - uu_x)_x = u_{yy}. \tag{2.12}$$

  Note that $V = u_x/2 \neq 0$ for (**??**) to be well defined. The flat metric $g_{(2,2)}$ corresponds to $u = -x/t$.

- Finally if $(\nabla K)_+ \wedge (\nabla K)_+ \neq 0$ there exist local coordinates such that

$$h_{(2,1)}{}^{\pm} = e^u(\mathrm{d}x^2 \pm \mathrm{d}y^2) \mp \mathrm{d}t^2, \tag{2.13}$$

  and

$$g_{(2,2)} = \frac{u_t}{2} h_{(2,1)} - \frac{2}{u_t}(\mathrm{d}\theta + A)^2, \tag{2.14}$$

  where the one–form A on W is a solution to the linear equation (**??**) with $V = u_t/2$ and $\omega = 2u_t\mathrm{d}t$. The ASD vacuum equations reduce [**?**, **?**, **?**] to the $SU(\infty)$ Toda equation

$$u_{xx} \pm u_{yy} \mp (e^u)_{tt} = 0. \tag{2.15}$$

  The $\pm$ sign in (**??**) correspond to two different Lorentzian slices $h_{(2,1)}$.

The reader should note that while the case (**??**) and (**??**) are indefinite analogues of positive–definite metrics, the case (**??**) is genuinely pseudo–Reimannian, as there are no null two–forms in the Riemannian case.

3 Kaluza–Klein reduction

Let $h_{(2,1)}$ be an Einstein–Weyl given by one of (**??**, **??**) or h_{flat}, and let (V, A) be the corresponding solution to the generalised monopole equation giving rise to a vacuum metric (**??**). Consider the $(3+2)$ dimensional metric $g_{(3,2)}$ given by (**??**). It is obviously Ricci–flat, and it admits two commuting Killing vectors $\partial/\partial z$ and $\partial/\partial\theta$. We preform the Kaluza-Klein reduction with respect to the time like vector $\partial/\partial\theta$. The four dimensional theory is invariant under the general coordinate transformations independent of θ. The translation of the fiber coordinate $\theta \rightarrow \theta + \Lambda(x^\mu)$ induces the $U(1)$ transformations of the Maxwell one–form. The scaling of

$$\theta \longrightarrow \lambda\theta, \qquad [g_{(3,2)}]_{\mu\theta} \longrightarrow \lambda^{-2}[g_{(3,2)}]_{\mu\theta},$$

is spontaneously broken by the Kaluza–Klein vacuum, since θ is a coordinate on a circle with a fixed radius. The scalar field corresponding to this symmetry breaking is called the dilaton. It is the usual practise to conformally rescale the resulting $(3+1)$ dimensional metric, and the dilaton so that the multiple of the Ricci scalar of $G_{\mu\nu}$ in the reduced Lagrangian is equal to $\sqrt{|\det(G_{\mu\nu})|}$. The corresponding Maxwell field is $F = \mathrm{d}A$, and the physical metric $G_{\mu\nu}$ in $(3+1)$ signature is given by (**??**). We can summarise our findings in the following proposition

Proposition 3.1 *Let* $([h], D, W)$ *be an Einstein–Weyl structure in 2+1 dimensions, and let* V *be function on* W *of conformal weight* -1 *which is a solution to the generalised monopole equation* (**??**) *such that the corresponding* $(++--)$ *ASD metric* (**??**) *is Ricci–flat. Then for any* $h_{(2,1)} \in [h]$ *the triple*

$$G = \sqrt{V} h_{(2,1)} + \frac{1}{\sqrt{V}} \mathrm{d}z^2, \qquad \Phi = -\frac{\sqrt{3}}{4} \log V, \qquad F = *_h DV$$

satisfies the Einstein–Maxwell–Dilaton equations arising from the Lagrangian density

$$\frac{1}{16\pi^2}(R - 2(\nabla\Phi)^2 + \frac{1}{4} e^{-2\sqrt{3}\Phi} F_{\mu\nu}F^{\mu\nu}). \tag{3.16}$$

To understand the unusual sign between the Ricci scalar and the Maxwell term in this Lagrangian notice that making the replacements

$$\theta \longrightarrow i\theta, \qquad A \longrightarrow iA$$

would lead to a more usual space–like Kaluza–Klein reduction from (4+1) to (3+1) dimensions, where the relative sign between the Ricci scalar and the Maxwell term is negative.

The negative energy of the Maxwell field has peculiar physical consequences. No static multi–black hole solutions analogous to the Majumdar–Papapetrou extremal black holes can exist, as both gravity and electromagnetism are now attractive forces and the cancellations can not take place. Thus the theory only allows non–extremal black holes which can effectively increase their masses by radiating photons out! We can also encounter 'tachyonic' solutions invariant under $\mathbb{R} \times SO(2,1)$.

Proposition (**??**) also applies to the ordinary EMD solutions with positive Maxwell energy if one takes $([h], D, W)$ to be positive definite Einstein Weyl structure and preforms the usual space-like Kaluza–Klein reduction of the product metric $g_{(4)} - \mathrm{d}t^2$, where $g_{(4)}$ is an ASD Ricci flat metric corresponding to $([h], D, W)$. This will rule out solutions coming from the dKP equation (**??**). There is however another possible construction which we now outline. Let

$$h_{(3)} = \mathrm{d}z^2 + e^u(\mathrm{d}x^2 + \mathrm{d}y^2), \qquad \omega = 2u_z \mathrm{d}z$$

be a positive–definite EW structure corresponding to a solution $u = u(x, y, z)$ to the elliptic $SU(\infty)$ Toda equation $u_{xx} + u_{yy} + (e^u)_{zz} = 0$. If (V, A) is an arbitrary solution to the monopole equation (**??**) the resulting four–dimensional Riemannian metric

$$g_{(4)} = V h_{(3)} + V^{-1}(\mathrm{d}\theta + A)^2$$

is scalar–flat and Kahler [**?**]. One special solution $V = u_z/2$ makes $g_{(4)}$ Ricci–flat, and the Lorentzian version of this solution must be used in Proposition (**??**). If one instead takes $V = V_\Lambda$, where

$$V_\Lambda = -\frac{1}{3}\Lambda(1 - z u_z)$$

then $g_{(4)}$ is conformal to an Einstein metric $\hat{g}_{(4)}$ with a cosmological constant Λ [**?**]:

$$\hat{g}_{(4)} = \frac{V_\Lambda}{z^2} h_{(3)} + \frac{1}{z^2 V_\Lambda}(\mathrm{d}\theta + A)^2. \tag{3.17}$$

Given $\hat{g}_{(4)}$ we construct a 'cosmological' vacuum metric in 4+1 dimensions given by

$$g_{(4,1)} = f(t)^2 \hat{g}_{(4)} - \mathrm{d}t^2.$$

There are several possibilities for f: Choosing $f(t) = L^{-1}\cosh(Lt)$ yields a regular metric, and $f(t) = L^{-1}\sin(Lt)$ gives a generalised AdS solution with Big Bang and Big Crunch singularities at $t=0$ and $t=\pi/L$ respectively. The K-K reduction of this metric (now we need to change the relative sign between the two terms in (**??**)) gives the following solution to the EMD equations

$$\begin{aligned} G_{(3,1)} &= \frac{f}{z}\Big(\sqrt{V_\Lambda}\Big(\frac{f}{z}\Big)^2 h_{(3)} - \frac{1}{\sqrt{V_\Lambda}}\mathrm{d}t^2\Big), \qquad \Phi = -\frac{\sqrt{3}}{2}\log\Big(\frac{f}{z\sqrt{V_\Lambda}}\Big), \\ F &= -(V_\Lambda)_x \mathrm{d}y\wedge\mathrm{d}z - (V_\Lambda)_y \mathrm{d}z\wedge\mathrm{d}x - (V_\Lambda e^u)_z \mathrm{d}x\wedge\mathrm{d}y. \end{aligned}$$

4 Example

The simplest illustration of Proposition (**??**) corresponds to $g_{(2,2)}$ being an analytic continuation of a gravitational instanton in Riemannian signature. We shall examine the (3+1) solution arising from a $(2,2)$ analog of the Taub–Nut gravitational instanton, and emphasise that this example hints the semi-classical instability of the Einstein–anti Maxwell theories.

Consider (**??**), and take

$$V = \varepsilon + \frac{m}{\sqrt{x^2+y^2-t^2}}, \qquad \varepsilon = 1. \tag{4.18}$$

The NUT singularity is absent in $(3+2)$ dimensions. It is a fixed point of the Killing vector $\partial/\partial\theta$ which is regular if θ is periodic. The resulting metric (3+1) metric (**??**) written in cylindrical polar coordinates takes the form

$$\begin{aligned} \mathrm{d}s^2 &= \Big(\varepsilon + \frac{m}{\sqrt{\rho^2-t^2}}\Big)^{-1/2}\mathrm{d}z^2 + \Big(\varepsilon + \frac{m}{\sqrt{\rho^2-t^2}}\Big)^{1/2}(\mathrm{d}\rho^2 + \rho^2\mathrm{d}\theta^2 - \mathrm{d}t^2), \qquad (4.19) \\ \Phi &= -\frac{\sqrt{3}}{4}\log(\varepsilon + \frac{m}{\sqrt{\rho^2-t^2}}), \qquad A = \Big(\varepsilon + \frac{m}{\sqrt{\rho^2-t^2}}\Big)^{-1}\mathrm{d}z, \end{aligned}$$

and we can take z to be periodic[2].

The initial data on the surface $t=0$ is regular everywhere. The rescalling of the three–metric

$$g_{(3)} = \Big(\varepsilon + \frac{m}{\rho}\Big)^{-1/2}\mathrm{d}z^2 + \Big(\varepsilon + \frac{m}{\rho}\Big)^{1/2}(\mathrm{d}\rho^2 + \rho^2\mathrm{d}\theta^2)$$

is regular at $\rho=0$ if θ is periodic with a period 4π. This can be seen by setting $\rho=\hat{\rho}^2/m$, so that around $\hat{\rho}=0$

$$g_{(3)} \sim \mathrm{d}z^2 + 4\Big(\mathrm{d}\hat{\rho}^2 + \hat{\rho}^2\,\mathrm{d}\Big(\frac{\theta}{2}\Big)^2\Big).$$

The cylindrical mass of $g_{(3)}$ can be defined as a deficit angle at infinity, which is proportional to the integral of the Gaussian curvature of the metric induced on the surfaces of constant z.

[2]Another possibility is to take V as in (**??**) with $\varepsilon = z$ which leads to a non-asymptotically flat metric.

This metric evolves to a space-time with naked singularities on the cone $\rho^2 = t^2$. Near this cone the metric behaves like

$$\sqrt{m}^{-1} e^{-U/2} \mathrm{d}z^2 + e^{3U/2}\sqrt{m}(-\mathrm{d}T^2 + \cosh^2(T)\mathrm{d}\theta^2 + \mathrm{d}U^2), \qquad U \longrightarrow -\infty,$$

where $\rho = e^U \cosh(T), t = e^U \sinh(T)$. This solution represents a charged particle moving with the along the z axis. It can be interpreted as a tachyon in a sense of [?], as it is unstable and invariant under $\mathbb{R} \times SO(2,1)$.

The properties of (??) signal the semi-classical instability of the vacuum in Einstein–Maxwell–dilaton with attractive electric force. The argument is analogous to Witten's bubble of nothing [?]. If the coordinate z is periodic the solution is asymptotic to the flat metric on $\mathbb{R}^3 \times S^1$. The decay of $\mathbb{R}^3 \times S^1$ vacuum is described by the instanton obtained by replacing $t \rightarrow i\tau$ in the metric (??). This instanton has vanishing action [?], and the probability of the decay is given by the exponential of the negative action.

Acknowledgements

This work resulted from a seminar [?] given at the programme Global Problems in Mathematical Relativity held in Newton Institute in the Fall 2005. I would like to thank Gary Gibbons and other participants of the programme for useful comments.

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