Marshall's and Milnor's Conjectures for Preordered von Neumann Regular Rings

M. Dickmann F. Miraglia *

November, 2005

The aim of this paper is to prove that, if R is a commutative regular ring in which 2 is a unit, then the reduced theory of quadratic forms with invertible coefficients in R, modulo a proper preorder T, satisfies Marshall's signature conjecture and Milnor's Witt ring conjecture (for precise statements, see Section 1 below). For that purpose we use the theory of special groups (abbreviated SG), presented in [DM2] (see also Section 2 of [DM1]), and the K-theory of those structures, developed in [DM3] and [DM6].

To a pair $\langle R, T \rangle$ as above, we associate a reduced special group (RSG), $G_T(R) = R^\times/T^\times$ ($R^\times =$ units of R). A result from [DM5] (Thm. 3.16, pp. 17-18) shows that, under these conditions —in fact, even under considerably more general conditions— $G_T(R)$ faithfully reflects the reduced theory of quadratic forms modulo T, over free R-modules.

The technique used to prove the stated result can be summarized as follows:

1) Marshall's signature conjecture was proved in [DM1] for Pythagorean fields, and in [DM3] for formally real fields modulo an arbitrary (proper) preorder. For this kind of fields, modulo sums of squares, the problem was posed by Lam in 1976. Our proofs use the theory of SGs, but depend on results of Voevodsky (and of Orlov-Vishik-Voevodsky in the latter case), to conclude.

2) For fields of characteristic zero, Milnor's Witt ring conjecture is a celebrated result of Voevodsky's.

3) An analysis of our proofs shows that, in fact, we establish the validity of a powerful K-theoretic property —the [SMC] property— which implies Marshall's signature conjecture. This property was explicitly formulated in [DM3] (Definition 4.3, p. 168), but occurs without a name already in [DM1] (Corollary 6.5, p. 275). The [SMC] property asserts, in the abstract context of RSGs, the analog of injectivity of Milnor's "multiplication by $\ell(-1)$" map at each level of the graded mod 2 K-theory ring. It follows from results in [DM1] and [DM3] that the [SMC] property is equivalent, for arbitrary RSGs, to the conjunction of Marshall's signature conjecture and Milnor's Witt ring conjecture (see Lemma 1.2 below).

4) In view of the foregoing observations, our efforts are directed at proving the [SMC] property for the RGS $G_T(R)$ associated to a pair $\langle R, T \rangle$ as above. In order to achieve this we use the well-known representation (originally due to Pierce ([Pi])) of a von Neumann-regular ring (hereafter a vN-ring) R as the ring of global sections of a (pre-)sheaf of rings over the Boolean space Spec(R) whose stalks are fields (this representation is just the Grothendieck structure sheaf of R). The presence of a (proper) preorder T on R forces at least one of the stalks to be preordered by the corresponding image of T. By considering a suitable quotient of R the situation gets reduced to the case where all the stalks are (properly) preordered (Proposition 6.15). By Theorem 6.4ff of [DM3] the [SMC] property holds, then, at the RSG associated to each stalk of the sheaf representation of R. Having previously established (Theorem 6.14) that the RSG construction induces a (pre-)sheaf of RSGs on Spec(R) and that the K-theory functor of special groups is geometrical (Proposition 2.7(1)) we conclude (Theorems 7.1(c) and 7.2) that its SG of global sections —which is just $G_T(R)$— also has the [SMC] property.

Some ingredients of our proof are valid in more general contexts, and so it seemed appropriate to register them, with moderate extra effort, at that level of generality.

*First version : January, 2005

In sections 2 and 3 we review a number of results that, though most are generally considered folklore, either are not immediately accessible in the literature or our use of them is at variance with that (unwritten) folklore (for example, the notion of a geometrical formula in 2.1(d)).

In section 4 we are concerned with rings with many units, a class of rings previously considered in the literature, larger than that of vN-rings. Under mild additional assumptions —namely that 2 is a unit and that all residue fields have cardinality > 7— quadratic form theory via special groups faithfully reflects, for this class of rings, quadratic form theory over free modules ([DM5], Theorem 3.16). Furthermore, under these conditions, we adapt the K-theory in [Gu] to our setting, showing that the ensuing mod 2 K-theory is isomorphic to the K-theory of the associated SG (Theorem 4.12).

In section 5 we deal with the presheaf representation of vN-rings and study the elementary properties of preorders in rings of this type, especially in connection with that representation.

Besides results mentioned above, section 6 includes a proof that the functor assigning to each preordered ring a (suitable fragment of a) reduced special group is a geometrical functor (Proposition 6.11). The notion of a proto-SG singles out those axioms satisfied by the SG construction as applied to arbitrary preordered rings (cf. 6.3).

Finally, in section 7 we show that, under fairly general circumstances, the SG of sections of a presheaf of SGs whose stalks verify the property [SMC], is also [SMC] (Theorem 7.1). This yields Theorem 7.2, proving our main result.

1 Preliminaries : The [SMC] Property for Special Groups

1.1 Notation and Remarks. Let $G = \langle G, \equiv_G, -1 \rangle$ be a special group (SG) and write D_G for the representation relation in G.

(1) The K-theory of G, introduced in [DM3], is the graded $\mathbb{F}_2$-algebra, $k_*G = \langle \mathbb{F}_2, k_1G, \ldots, k_nG, \ldots \rangle$, constructed as follows :

$*$ k_1G is G written additively, that is, we fix an **isomorphism**

$$\lambda : G \longrightarrow k_1G, \text{ with } \lambda(ab) = \lambda(a) + \lambda(b).$$

In particular, $\lambda(1)$ is the zero of k_1G and k_1G has exponent 2, i.e., for $a \in G$, $\lambda(a) = -\lambda(a)$;

$*$ k_*G is the quotient of the graded tensor algebra $\langle \mathbb{F}_2, k_1G, \ldots, \underbrace{k_1G \otimes \cdots \otimes k_1G}_{n\ times}, \ldots \rangle$ over $\mathbb{F}_2$, by the ideal generated by $\{\lambda(a)\lambda(ab) : a \in D_G(1, b)\}$. Thus, for each $n \geq 2$, k_nG is the quotient of the n-fold tensor product $k_1G \otimes \cdots \otimes k_1G$ over $\mathbb{F}_2$, by the subgroup consisting of finite sums of elements of the type $\lambda(a_1) \cdots \lambda(a_n)$, where for some $1 \leq i \leq n-1$ and $b \in G$, we have $a_{i+1} = a_ib$ and $a_i \in D_G(1, b)$. An element of the type $\lambda(x_1) \cdots \lambda(x_n)$ is called a **generator** of k_nG;

$*$ There is a graded ring morphism of degree 1, $\lambda(-1)\ (\cdot) : k_nG \longrightarrow k_{n+1}G$, taking $\eta \in k_nG$ to $\lambda(-1)\eta \in k_{n+1}G$. A special group is **[SMC]** if for all $n \geq 1$, multiplication by $\lambda(-1)$ is an injection. Any [SMC] special group must be *reduced*;

$*$ A SG-morphism, $f : G \longrightarrow H$, induces a morphism of degree 0 of graded $\mathbb{F}_2$-algebras [1]

$$f_* : k_*G \longrightarrow k_*H,$$

$f_* = \{f_n : n \geq 0\}$, where $f_0 = Id_{\mathbb{F}_2}$ and for $n \geq 1$, $f_n : k_nG \longrightarrow k_nH$ is the unique group morphism whose value on generators is given by $f_n(\lambda(a_1) \cdots \lambda(a_n)) = \lambda(f(a_1)) \cdots \lambda(f(a_n))$.

(2) Let $W(G)$ be the Witt ring of G and let $I(G)$ be the fundamental ideal in $W(G)$. For $n \geq 0$, set

$$\overline{I^n}(G) = I^n(G)/I^{n+1}(G),$$

where $I^0(G) = W(G)$. The sequence, $W_g(G) = \langle \mathbb{F}_2, \ldots, \overline{I^n}(G), \ldots \rangle$ is, as usual, the *graded Witt ring of G*. In [DM3] we constructed a graded ring morphism

$$s_* = (s_n)_{n \geq 0} : k_*(G) \longrightarrow W_g(G),$$

[1] That is, for $\eta \in k_nG$, $\xi \in k_mG$, $f_{n+m}(\eta\xi) = f_n(\eta)f_m(\xi)$.

such that for each $n \geq 0$, the following diagram is commutative, where $\otimes\, 2$ indicates product by the Pfister form $2 = \langle\, 1, 1\,\rangle$:

$$\begin{tikzcd} k_n(G) \arrow[r, "\otimes\, \lambda(-1)"] \arrow[d, "s_n"'] & k_{n+1}(G) \arrow[d, "s_{n+1}"] \\ \overline{I^n}(G) \arrow[r, "\otimes\, 2"'] & \overline{I^{n+1}}(G) \end{tikzcd} \qquad \text{(D)}$$

The special group G is [MWRC], i.e., satisfies **Milnor's Witt Ring Conjecture**, if s_n is an isomorphism for all $n \geq 0$; it is shown in [DM3] that this holds for $n \leq 2$.

(3) G is [MC] if it satisfies **Marshall's signature conjecture**, that is, for all $n \geq 1$ and all forms, φ, over G, if the total signature of φ is congruent to zero mod 2^n, then $\varphi \in I^n(G)$; any such group must be *reduced*. For a detailed account of this property, see [DM1] and [DM4]. $\diamondsuit$

The relation between properties [SMC], [MC] and [MWRC] is described by

Lemma 1.2 *If G is a reduced special group, then*

$$G \textit{ is } [\text{SMC}] \quad \textit{iff} \quad G \textit{ is } [\text{MC}] \textit{ and } [\text{MWRC}].$$

Proof. By Theorem 5.1 in [DM1], G is [MC] iff the map "multiplication by $2 = \langle\, 1, 1\,\rangle$" from $\overline{I^n}(G)$ to $\overline{I^{n+1}}(G)$ is injective. Hence, if G is [MWRC] and [MC], the commutative diagram (D) above entails that multiplication by $\lambda(-1)$ is injective in all degrees, that is, G is [SMC]. Conversely, by Corollary 4.2 in [DM3], every [SMC]-group is [MWRC] and once again the commutativity of diagram (D) above entails that multiplication by $\langle\, 1, 1\,\rangle$ in the graded Witt ring of G is injective in all degrees. Another application of Theorem 5.1 in [DM1] guarantees that G is [MC]. $\diamondsuit$

2 Geometric Theories and Functors

We assume the reader is familiar with first-order languages, their structures and morphisms. Standard references are [CK] and [Ho]. For the convenience of the reader, we recall the following

Definition 2.1 *Let L be a first-order language with equality.*

Let A, B be L-structures, let $f : A \longrightarrow B$ be a map and let $\varphi(v_1, \ldots, v_n)$ be a formula of L, in the free variables $\overline{v} = \langle\, v_1, \ldots, v_n\,\rangle$. For $\overline{a} = \langle\, a_1, \ldots, a_n\,\rangle \in A^n$, write $f(\overline{a})$ for $\langle\, f(a_1), \ldots, f(a_n)\,\rangle \in B^n$.

a) (1) f **preserves φ** *if for all $\overline{a} \in A^n$,* $A \models \varphi[\overline{a}] \;\Rightarrow\; B \models \varphi[f(\overline{a})]$;

(2) f **reflects φ** *is for all $\overline{a} \in A^n$,* $B \models \varphi[f(\overline{a})] \;\Rightarrow\; A \models \varphi[\overline{a}]$.

Thus,

$*$ *f is a* ***L*-morphism** *if it preserves atomic formulas; by induction on complexity, it will also preserve positive existential formulas, i.e., those of the form $\exists\overline{v}\varphi(\overline{v})$, where φ is positive* [2] *and quantifier-free;*

$*$ *f is a* ***L*-embedding** *if it preserves and reflects all atomic formulas. By induction, an embedding will preserve and reflect all quantifier-free formulas* [3]*; and it will preserve formulas of the type $\exists\overline{v}\varphi(v)$, where φ is quantifier-free, called* **existential** *formulas;*

$*$ *f is an* **elementary embedding** *if it preserves and reflects* **all formulas**.

b) If f is a L-morphism, we say that ***A* is positively existentially closed in *B* along *f*** *if f reflects all positive existential L-formulas. Whenever A is a substructure of B, we say that* ***A* is existentially closed in *B***.

[2] Constructed from atomic formulas using only conjunction ($\wedge$) and disjunction ($\vee$).

[3] Constructed from atomic formulas using all propositional connectives, including implication ($\rightarrow$) and negation ($\neg$).

c) Let **L-mod** *be the category of L-structures and L-morphisms. If Σ is a set of sentences in L, write* **Σ-mod** *for the subcategory of* **L-mod** *whose objects are the models of Σ.*

d) A formula of L is **geometrical** *if it is the negation of an atomic formula* **or** *a formula of the form $\forall \overline{v}\ (\varphi_1(\overline{v}) \longrightarrow \exists \overline{w}\ \varphi_2(\overline{v}; \overline{w}))$, where φ_1, φ_2 are positive and quantifier-free. A* **geometrical theory in *L*** *is a theory possessing a set of geometrical axioms.*

e) A formula in L is **positive primitive (pp-formula)** *if it is of the form $\exists \overline{v}\, \varphi(\overline{v})$, where φ is a conjunction of atomic formulas.*

2.2 We assume familiarity with the notions of inductive systems of first-order structures over a right-directed partially ordered set [4] (hereafter called a rd-poset) and of colimits (a.k.a. inductive limits) of such a system. Our notation for these objects is standard and we write

$$M \;=\; \varinjlim \mathcal{M} \;=\; \varinjlim_{i \in I} \mathcal{M}_i$$

to indicate that M is an inductive limit of $\mathcal{M}$. If $\mathcal{M}, \mathcal{N} : \langle I, \leq \rangle \longrightarrow$ **L-mod** are inductive systems of first-order structures, $\mathcal{M} = \langle \mathcal{M}_i; \{\mu_{ij} : i \leq j \text{ in } I\} \rangle$, $\mathcal{N} = \langle \mathcal{N}_i; \{\nu_{ij} : i \leq j \text{ in } I\} \rangle$, then :

$*$ A **dual cone** over $\mathcal{M}$ is a system $\langle A, \{\alpha_i : i \in I\} \rangle$, where A is a L-structure and $\alpha_i : \mathcal{M}_i \longrightarrow A$ are L-morphisms, such that for all $i \leq j$ in I, $\alpha_j \circ \mu_{ij} = \alpha_i$;

$*$ A **morphism**, $\eta : \mathcal{M} \longrightarrow \mathcal{N}$, is a family of L-morphisms, $\eta = \{\mathcal{M}_i \xrightarrow{\eta_i} \mathcal{N}_i : i \in I\}$, such that for all $i \leq j$ in I, we have $\eta_j \circ \mu_{ij} \;=\; \nu_{ij} \circ \eta_i$. $\diamondsuit$

The following result is essentially folklore. Item (e).(2), is a (slight) generalization of Tarski's union of chains theorem.

Theorem 2.3 *Let $\mathcal{M} : \langle I, \leq \rangle \longrightarrow$ L-mod be an inductive system of L-structures.*

a) $\varinjlim \mathcal{M}$ exists in L-mod *and is unique up to isomorphism. Moreover, if $J \subseteq I$ is cofinal in I, then $\varinjlim \mathcal{M}_{|J}$ [5] is naturally isomorphic to $\varinjlim \mathcal{M}$.*

b) A dual cone over $\mathcal{M}$, $\langle M, \mu_i : i \in I\} \rangle$, is (isomorphic to) *$\varinjlim \mathcal{M}$ iff it verifies :*

(1) $M \;=\; \bigcup_{i \in I} \mu_i(\mathcal{M}_i)$;

(2) *If $\varphi(v_1, \ldots, v_n)$ is an atomic formula in L, $i \in I$ and $\langle s_1, \ldots, s_n \rangle \in \mathcal{M}_i^n$, then*

$$M \models \varphi[\mu_i(s_1), \ldots, \mu_i(s_n)] \quad \Leftrightarrow \quad \begin{cases} \exists\, k \in I \text{ such that } k \geq i \quad \text{and} \\ \mathcal{M}_k \models \varphi[\mu_{ik}(s_1), \ldots, \mu_{ik}(s_n)]. \end{cases}$$

Since the maps μ_{ij} and μ_i are L-morphisms, the significant implication above is ($\Rightarrow$).

c) Suppose that L is a language of algebras, *that is, structures with operations of arbitrary* finite arity, *but whose only relation is equality. Assume that each $\mathcal{M}_i$ has, besides additional structure, that of a group, written additively and that the morphisms μ_{ij} preserve the group structure. Then, a dual cone over $\mathcal{M}$, $\langle M, \mu_i : i \in I\} \rangle$, is* (isomorphic to) *$\varinjlim \mathcal{M}$ iff it verifies* (1) *in item (b) above and*

(2*) *For all $i \in I$ and all $x \in \mathcal{M}_i$, $\mu_i(x) = 0 \quad \Rightarrow \quad \exists\, j \geq i$ such that $\mu_{ij}(x) = 0$.*

d) Let $\psi(v_1, \ldots, v_n)$ be a disjunction of geometric formulas in L and let $M = \varinjlim \mathcal{M}$. For $i \in I$, let $\overline{s} \in \mathcal{M}_i^n$ and set $S_\psi \;=\; \{k \in I : k \geq i \text{ and } \mathcal{M}_k \models \psi[\mu_{ik}(\overline{s})]\}$. If S_ψ is cofinal in I, then $M \models \psi[\mu_i(\overline{s})]$.

e) Suppose that $\varinjlim \mathcal{M} \;=\; \langle M, \{\mu_i : i \in I\} \rangle$. Then,

(1) *If for all $i \leq j$, μ_{ij} is a L-embedding, then μ_i is an L-embedding.*

(2) (Tarski) *If for $i \leq j$, μ_{ij} is an* **elementary** *embedding, then μ_i is an elementary embedding.*

f) (Colimit of morphisms) *Let $\mathcal{N} = \langle \mathcal{N}_i; \{\nu_{ij} : i \leq j \text{ in } I\} \rangle$ be an inductive system of L-structures over I and let $\eta = \{\eta_i : i \in I\}$ be a morphism from $\mathcal{M}$ to $\mathcal{N}$. Then, there is a* unique *L-morphism, $\varinjlim \eta : \varinjlim \mathcal{M} \longrightarrow \varinjlim \mathcal{N}$, satisfying the following conditions :*

[4] I.e., $\forall\, i, j \in I$, $\exists\, k \in I$ such that $i, j \leq k$.

[5] $\mathcal{M}_{|J}$ is the functor obtained by restricting $\mathcal{M}$ to the poset J.

(1) *For all* $i \in I$, $(\varinjlim \eta) \circ \mu_i = \nu_i \circ \eta_i$*;*

(2) *If each* η_i *is a* L*-embedding, the same is true of* $\varinjlim \eta$.

Proof. We comment only on item (d). Since a finite union of subsets of I is cofinal in I iff at least one of them is cofinal in I, it is enough to verify the statement for a geometrical formula. For instance, if ψ is the negation of an atomic formula, $\varphi(\overline{v})$, assume that $S_{\neg\varphi}$ is cofinal in I, but that $M \models \varphi[\mu_i(\overline{s})]$; by 2.3.(b).(2), there is $k \geq i$ such that $\mathcal{M}_k \models \varphi[\mu_{ik}(\overline{s})]$. But then, there is $j \in S_{\neg\varphi}$ with $j \geq k$ and so $\mathcal{M}_j \models \neg\varphi[\mu_{ij}(\overline{s})]$. Since $\mu_{kj}(\mu_{ik}(\overline{s})) = \mu_{ij}(\overline{s})$ and μ_{kj} is an L-morphism, we also have $\mathcal{M}_k \models \varphi[\mu_{ij}(\overline{s})]$, a contradiction. The case in which ψ is of the form $\forall\overline{v}(\varphi_1(\overline{v}) \longrightarrow \exists\overline{w}\ \varphi_2(\overline{v};\overline{w})))$ is straightforward. $\diamondsuit$

Example 2.4 a) The theory of groups of exponent 2 is geometrical. Write **2-Grp** for the category of groups of exponent 2.

b) The theory of unitary commutative rings ($1 \neq 0$) is geometrical. Write **UCR** for the category of unitary commutative rings.

c) The theory of special groups and of reduced special groups are both geometrical theories. The axioms for special groups (see Definition 1.2, [DM2], for details) include, besides those of π-SGs (see 6.7, below), the sentences

[SG 4] : $\forall\ a, b, c, d\ ((\langle a, b\rangle \equiv_G \langle c, d\rangle) \longrightarrow (\langle a, -c\rangle \equiv_G \langle -b, d\rangle))$;

[SG 6] : The isometry of forms of dimension 3 is transitive,

all geometrical sentences. For reducibility, we add

$* \ 1 \neq -1$; [red] : $\forall a\ ((\langle a, a\rangle \equiv_G \langle 1, 1\rangle) \longrightarrow a = 1)$. $\diamondsuit$

Remark 2.5 a) By 2.3.(d), the colimit of models of a geometrical theory are also models of that theory. Moreover, it is easily established that a geometrical theory is preserved under the product of a non-empty family of its models.

b) The empty product in ***L*-mod** is its final object, that is, the structure $\{0\}$, wherein all n-ary predicates are interpreted as $\{0\}^n$, all n-ary function are interpreted as the only possible map from $\{0\}^n$ to $\{0\}$ and all constants are interpreted by 0. This structure is a model of any sentence of the form $\forall\overline{x}(\varphi_1 \rightarrow \exists\overline{y}\varphi_2)$, with φ_i positive and quantifier-free, but, in general, it will not model the *negation* of atomic sentences. $\diamondsuit$

Definition 2.6 *Let* L, $L^\sharp$ *be a first-order languages with equality. Let* Σ, $\Sigma^\sharp$ *be sets of sentences in* L *and* $L^\sharp$*, respectively. A covariant functor,* $F : \mathbf{\Sigma\text{-mod}} \longrightarrow \mathbf{\Sigma^\sharp\text{-mod}}$*, is* **geometrical** *if it preserves finite products and right-directed colimits* [6].

Here are some examples of geometrical functors. Others will arise in the sections that follow.

Proposition 2.7 *The following are geometrical functors :*

(1) *The* K*-theory functor of special groups, that is, for each* $n \geq 0$*, the functor from* **SG** *to* **2-Grp***, the category of groups of exponent* 2*, given by*

$$\begin{cases} G & \longmapsto & k_n G; \\ f : G \longrightarrow H & \longmapsto & f_n : k_n G \longrightarrow k_n H; \end{cases}$$

(2) *The Witt-ring functor,* $W : \mathbf{SG} \longrightarrow \mathbf{UCR}$*;*

(3) *The graded Witt-ring functor, i.e., for each* $n \geq 0$*, the functor from* **SG** *to* **2-Grp***, given by*

$$\begin{cases} G & \longmapsto & \overline{I^n} G; \\ f : G \longrightarrow H & \longmapsto & W_n(f) : \overline{I^n} G \longrightarrow \overline{I^n} H, \end{cases}$$

where, for $\varphi = \sum_{i=1}^m \bigotimes_{j=1}^n \langle 1, a_{ij}\rangle \in I^n(G)$, $W_n(f)(\varphi \mod I^{n+1}(G)) = \left(\sum_{i=1}^m \bigotimes_{j=1}^n \langle 1, f(a_{ij})\rangle\right) \mod I^{n+1}(H)$.

[6] That is, if these constructions exists in **Σ-mod**, then they exist in **Σ♯-mod** and F takes one to the other.

Proof. For (1), Theorems 4.5 and 5.1 in [DM6] guarantee the preservation of right-directed colimits and of finite products, respectively. The preservation of products and inductive limits in (2) and (3) follow from the results in [DM4], especially Proposition 3.1, Theorem 3.3 and Proposition 3.5. $\diamondsuit$

3 Presheaves of First-Order Structures

In this section we follow the lead set by [El]. General references on sheaf of algebraic structures are [Te], [Go], [GR] and [KS].

3.1 Notation. Let X be a topological space.

a) $\Omega(X)$ be the collection of opens of X, while $B(X)$ is the Boolean algebra (BA) of clopens in X.

b) A subset $\mathcal{B}$ of $\Omega(X)$ is a **basis** for X if it is closed under finite intersections and all opens in X are the union of elements of $\mathcal{B}$. Whenever convenient, we assume that $\emptyset$, $X \in \mathcal{B}$.

c) Any partially ordered set (poset), $\langle\, I, \leq \rangle$, can be seen as a category, whose set of objects is I and whose morphisms are given, for i, $j \in I$, by

$$\mathrm{Mor}_I(i,j) \quad = \quad \begin{cases} \emptyset & \text{if } i \not\leq j; \\ \{\emptyset\} & \text{if } i \leq j. \end{cases}$$

in particular, a basis for X, which is a poset under inclusion, can be treated as a category.

d) Write $U \subseteq_o V$ to mean that $V \in \Omega(X)$ and U is an *open* subset of V. $\diamondsuit$

Definition 3.2 (Essentially in [El]) *Let X be a topological space and let L be a first-order language with equality. Let $\mathcal{B}$ be a basis for X.*

a) A **presheaf basis of L-structures over $\mathcal{B}$**, *is a contravariant functor, $\mathfrak{A} : \mathcal{B} \longrightarrow$* **$L$-mod**,

$$U \ \longmapsto \ \mathfrak{A}(U) \quad \text{and} \quad U \subseteq_o V \ \longmapsto \ \alpha_{VU} : \mathfrak{A}(V) \longrightarrow \mathfrak{A}(U),$$

satisfying the following separation or extensionality *condition*

[ext] *If $\overline{s} \in \mathfrak{A}(U)^n$, R is a n-ary relation in L, $U \in \mathcal{B}$, and $\{U_i \subseteq_o U : i \in I\} \subseteq \mathcal{B}$ is a covering of U, then,* $\forall\, i \in I, \ \ \mathfrak{A}(U_i) \models R[\alpha_{UU_i}(s_1), \ldots, \alpha_{UU_i}(s_n)] \ \Rightarrow \ \mathfrak{A}(U) \models R[\overline{s}]$.

For $U \in \mathcal{B}$, the L-structure $\mathfrak{A}(U)$ is the **L-structure of sections of $\mathfrak{A}$ over U** *and the L-morphism $\alpha_{VU} : \mathfrak{A}(V) \longrightarrow \mathfrak{A}(U)$, $U \subseteq_o V$ in $\mathcal{B}$, is the* **restriction morphism**; *when no confusion is extant, this morphism, is written as $\cdot_{|U}$. In this notation, condition* [ext] *may be expressed as*

$$\forall\, i \in I, \ \ \mathfrak{A}(U_i) \models R[\overline{s}_{|U_i}] \quad \Rightarrow \quad \mathfrak{A}(U) \models R[\overline{s}] \qquad\qquad (\overline{s}_{|U_i} \ = \ \langle\, s_{1|U_i}, \ldots, s_{n|U_i} \,\rangle).$$

We shall assume that :

(1) *The L-structures $\mathfrak{A}(U)$ are* pairwise disjoint [7]*;*

(2) $\mathfrak{A}(\emptyset) \ = \ \{0\}$, *the final object in* **L-mod** *(as in 2.5.(b)).*

The set $|\mathfrak{A}| \ = \ \bigcup_{U \in \mathcal{B}} \mathfrak{A}(U)$ is the **domain of $\mathfrak{A}$** *and an element of $|\mathfrak{A}|$ is called a* **section of $\mathfrak{A}$**. *For each $s \in |\mathfrak{A}|$, let*

$$Es \ = \ \text{the unique } U \in \mathcal{B} \text{ such that } s \in \mathfrak{A}(U),$$

called the **extent of s**. *A section whose extent is X is a* **global section of $\mathfrak{A}$**. *We say that s, $t \in |\mathfrak{A}|$ are* **compatible**, *if $s_{|Es \cap Et} = t_{|Es \cap Et}$. In view of (2), sections with disjoint extents are compatible.*

b) If Σ is a set of sentences in L, a presheaf basis of **models of Σ** *over $\mathcal{B}$ is a presheaf basis of L-structures over $\mathcal{B}$, such that* **for all $U \neq \emptyset$** *in $\mathcal{B}$, $\mathfrak{A}(U)$ is a model of Σ.*

c) A **presheaf of L-structures over X** *is a presheaf basis such that $\mathcal{B} = \Omega(X)$.*

d) Let $\mathfrak{A}$ be a presheaf basis of L-structures over $\mathcal{B}$ and let $U \in \mathcal{B}$.

[7] There is no loss in generality; for $U \in \mathcal{B}$, substitute $\mathfrak{A}(U)$ by an isomorphic copy with domain $\{U\} \times \mathfrak{A}(U)$.

(1) $\mathfrak{A}$ *is* **finitely complete (fc)** *over* U *if for all* finite $S \subseteq |\mathfrak{A}|$ *such that* $U = \bigcup_{s \in S} Es$, *if the elements of* S *are pairwise compatible, then there is* $t \in \mathfrak{A}(U)$ *such that* $s = t_{|Es}$, *for all* $s \in S$; *because the extensionality condition* [ext] *applies to equality, this* t *is* unique *and is called the* **gluing of S** *in* $\mathfrak{A}$;

(2) $\mathfrak{A}$ *is* **complete** *over* U, *if the condition in* (1) *holds for* arbitrary *subsets* S *of* $|\mathfrak{A}|$, *satisfying* $U = \bigcup_{s \in S} Es$.

e) $\mathfrak{A}$ *is* **complete or finitely complete (fc) over $\mathcal{B}$** *if it is complete or fc over every* $U \in \mathcal{B}$, *respectively.*

f) A **sheaf of L-structures over X** *is a presheaf over* X *that is complete over all* $U \in \Omega(X)$.

g) If $\mathfrak{A}$, $\mathfrak{B}$ *are presheaf bases over* $\mathcal{B}$, *a* **morphism**, $f : \mathfrak{A} \longrightarrow \mathfrak{B}$, *is a natural transformation of contravariant functors, that is, a family of* L*-morphisms,* $f = \{\mathfrak{A}(U) \xrightarrow{f_U} \mathfrak{B}(U) : U \in \mathcal{B}\}$, *such that for* $U \subseteq_o V$ *in* $\mathcal{B}$ *and* $x \in \mathfrak{A}(V)$, $f_V(x)_{|U} = f_U(x_{|U})$.

Remark 3.3 a) Notation as in 3.2, if $U \in \mathcal{B}$ is **compact**, condition [ext] in 3.2.(a) is equivalent to

[extc] If $\overline{s} \in \mathfrak{A}(U)^n$, R is a n-ary relation in L and $\{U_1, \ldots, U_m\} \subseteq \mathcal{B}$ is a covering of U, then
$$\forall\, 1 \leq j \leq m,\ \ \mathfrak{A}(U_j) \models R[\overline{s}_{|U_j}] \quad \Rightarrow \quad \mathfrak{A}(U) \models R[\overline{s}].$$

If equality is the only relation symbol in L, that is, L is a language of algebras, then extensionality applies only to it. In particular, if $U \in \mathcal{B}$ is compact, then [extc] takes the form, for s, $t \in |\mathfrak{A}|$,

[ext=] If $\{U_1, \ldots, U_m\}$ is a covering of $Es = Et$ such that $s_{|U_j} = t_{|U_j}$, then $s = t$.

b) It is straightforward that the extensionality condition [ext] in Definition 3.2.(a) holds for a *conjunction of atomic formulas*, i.e., if $\varphi(v_1, \ldots, v_n)$ is a conjunction of atomic L-formulas, then for all $U \in \mathcal{B}$, all $\overline{s} = \langle\, s_1, \ldots, s_n \,\rangle \in \mathfrak{A}(U)^n$ and all coverings $\{U_i : i \in I\} \subseteq \mathcal{B}$ of U,

[ext] $$\forall\, i \in I,\ \ \mathfrak{A}(U_i) \models \varphi[s_{1|U_i}, \ldots, s_{n|U_i}] \quad \Rightarrow \quad \mathfrak{A}(U) \models \varphi[s_1, \ldots, s_n].$$

If $U \in \mathcal{B}$ is compact, item (a) above applies to show that it suffices to consider *finite* coverings of U. $\Diamond$

Presheaf bases are important because frequently the data for a sheaf are given only on a basis for the topology of X, as in the case of the affine scheme of a commutative ring, e.g., in section 5 below.

The following result gives, among other things, a useful criterion for a contravariant functor from the Boolean algebra of clopens of a Boolean space to $\boldsymbol{L}$-mod to be a fc and extensional presheaf basis.

Proposition 3.4 *Let* X *be a Boolean space and let* $\mathcal{B}$ *be the Boolean algebra of clopens in* X. *Let* $\mathfrak{A} : \mathcal{B} \longrightarrow$ **L-mod** *be a contravariant functor.*

a) The following are equivalent :

(1) $\mathfrak{A}$ *is an extensional, finitely complete presheaf basis over* $\mathcal{B}$;

(2) *For all* $U \in \mathcal{B}$, *if* $\overline{V} = \langle\, V_1, \ldots, V_n \,\rangle$ *is a clopen partition of* U, *then, the* L*-morphism* $\alpha_{\mathfrak{A}}(U; \overline{V}) : \mathfrak{A}(U) \longrightarrow \prod_{j=1}^{n} \mathfrak{A}(V_j)$, *given by* $s \longmapsto \langle\, s_{|V_1}, \ldots, s_{|V_n} \,\rangle$, *where* $\prod_{j=1}^{n} \mathfrak{A}(V_j)$ *has the product structure, is an isomorphism, making the following diagram commutative*

$$\begin{tikzcd}
\mathfrak{A}(U) \arrow[rr, "\alpha_{\mathfrak{A}}(U;\overline{V})"] \arrow[dr, "\cdot_{|V_j}"'] & & \mathfrak{A}(\overline{V}) \arrow[dl, "p_j"] \\
& \mathfrak{A}(V_j) &
\end{tikzcd} \qquad (1 \leq j \leq n)$$

where $p_j : \mathfrak{A}(\overline{V}) \longrightarrow \mathfrak{A}(V_j)$ *is the canonical coordinate projection.*

b) If $\mathfrak{A}$ *is a fc presheaf basis over* $\mathcal{B}$, *then :*

(1) *For all* $U \subseteq V$ *in* $\mathcal{B}$, *the restriction* L*-morphism from* $\mathfrak{A}(V)$ *to* $\mathfrak{A}(U)$ *is surjective;*

(2) *For all* $x \in X$ *and all* $U \in \mathcal{B}_x$, *the stalk* L*-morphism,* $s \in \mathfrak{A}(U) \longmapsto s_x \in \mathfrak{A}_x$, *is surjective* (*see Definition* 3.6 *below*).

Proof. a) Write $\mathfrak{A}(\overline{V})$ for the product structure $\prod_{j=1}^{n} \mathfrak{A}(V_j)$ and $\alpha(U;\overline{V})$ for $\alpha_{\mathfrak{A}}(U;\overline{V})$.

$\underline{(1) \Rightarrow (2)}$: Since $\mathfrak{A}$ is fc and extensional with respect to equality, the L-morphism $\alpha(U,\overline{V})$ is bijective; hence, to show it is an isomorphism it suffices to check that α reflects atomic formulas, that is, if $\varphi(v_1,\ldots,v_m)$ is an atomic formula in L and $\overline{s} = \langle s_1,\ldots,s_m \rangle \in \mathfrak{A}(U)^m$, then

$$\mathfrak{A}(\overline{V}) \models \varphi[\langle s_{1|V_1},\ldots,s_{m|V_1}\rangle, \ldots, \langle s_{1|V_n},\ldots,s_{m|V_n}\rangle] \quad \Rightarrow \quad \mathfrak{A}(U) \models \varphi[\overline{s}]. \tag{I}$$

The antecedent in (I) means $\mathfrak{A}(V_j) \models \varphi[s_{1|V_j},\ldots,s_{m|V_j}]$, $1 \leq j \leq n$, and hence Remark 3.3.(b) entails $\mathfrak{A}(U) \models \varphi[\overline{s}]$, as needed. It is clear that the displayed diagram in (2) is commutative for all $1 \leq j \leq n$.

$\underline{(2) \Rightarrow (1)}$: We fix $U \in \mathcal{B}$ and a clopen covering, $\mathcal{C}$, of U, whose elements are all contained in U. Let $\varphi(v_1,\ldots,v_m)$ be an atomic formula in L, let $\overline{s} = \langle s_1,\ldots,s_m \rangle \in \mathfrak{A}(U)^m$ and assume that for $O \in \mathcal{C}$, $\mathfrak{A}(O) \models \varphi[\overline{s}_{|O}]$, where $\overline{s}_{|O} = \langle s_{1|O},\ldots,s_{m|O}\rangle$; since U is compact, there is $\{U_1,\ldots,U_n\} \subseteq \mathcal{C}$ that is also a covering of U. Now consider

$$V_1 = U_1 \quad \text{and, for } 2 \leq j \leq n, \quad V_j \;=\; U_j \setminus \left(\textstyle\bigcup_{i<j} V_i\right).$$

Then, $\{V_1,\ldots,V_n\}$ is a pairwise disjoint clopen covering of U, subordinate to $\{U_1,\ldots,U_n\}$. Since $\mathfrak{A}(U_j) \models \varphi[\overline{s}_{|U_j}]$ and restriction is an L-morphism, we get that $\mathfrak{A}(V_j) \models \varphi[\overline{s}_{|V_j}]$, $1 \leq j \leq n$. Therefore, $\alpha(U;\overline{V}) : \mathfrak{A}(U) \longrightarrow \mathfrak{A}(\overline{V})$ being a L-isomorphism, we conclude that $\mathfrak{A}(U) \models \varphi[\overline{s}]$, establishing the extensionality of $\mathfrak{A}$.

For finite completeness, let $\{s_1,\ldots,s_n\}$ be a set of pairwise compatible sections in $|\mathfrak{A}|$, with $U = \bigcup_{j=1}^{n} Es_j$. The disjointing procedure used above yields a disjoint clopen covering of U, $\{W_1,\ldots,W_n\}$, subordinate to the covering $\{Es_1,\ldots,Es_n\}$. Since $\alpha(U;\overline{W})$ is an L-isomorphism, there is $t \in \mathfrak{A}(U)$ such that

$$\text{For all } 1 \leq j \leq n, \quad t_{|W_j} = s_{j|W_j}. \tag{II}$$

Fix j between 1 and n; then, if $A_{ji} = Es_j \cap W_i$, $\overline{A_j} = \{A_{j1},\ldots,A_{jn}\}$ is a disjoint clopen covering of Es_j; moreover, since $A_{ji} = Es_j \cap W_i \cap Es_i$ and the collection $\{s_1,\ldots,s_n\}$ is compatible, (II) yields, for $1 \leq i \leq n$,

$$s_{j|A_{ji}} \;=\; (s_{j|Es_i\cap Es_j})_{|W_i} \;=\; (s_{i|Es_i\cap Es_j})_{|W_i} \;=\; s_{i|A_{ji}} \;=\; (s_{i|W_i})_{|A_{ji}} \;=\; (t_{|W_i})_{|A_{ji}} \;=\; t_{|A_{ji}}.$$

Thus, since $U = \bigcup_{j,i} A_{ji}$, the extensionality of $\mathfrak{A}$ with respect to equality entails $t_{|Es_j} = s_j$, completing the proof of (a).

b) Item (1) follows from (a) because the map $s \in \mathfrak{A}(V) \longmapsto \langle s_{|U}, s_{|V\setminus U}\rangle$ is a L-isomorphism. For (2), fix $x \in U$ and let $\xi \in \mathfrak{A}_x$. Then, for some $W \in \mathcal{B}_x$, with $W \subseteq U$, there is $s \in \mathfrak{A}(W)$ such that $\xi = s_x$. By (1), there is $t \in \mathfrak{A}(U)$ such that $t_{|W} = s$, and so $t_x = s_x = \xi$, as needed. ♢

A presheaf basis on X can always be extended to a sheaf on X. Usually this construction involves taking projective limits (see [Te], Lemma 4.2.6, pp. 83-84), although there are better methods. It is important know when the extension process does not change the structure of sections originally given over a basis of X; this question is treated in item (4) of the next result, whose proof will be omitted.

Theorem 3.5 *Let $\mathcal{B}$ be a basis for the space X and let $\mathfrak{A}$, $\mathfrak{B}$ be presheaf bases of L-structures over $\mathcal{B}$.*

a) There is a unique *sheaf over X, $c\mathfrak{A}$, the* **completion or sheafification** *of $\mathfrak{A}$, together with an* **injective** *map, $c_A : |\mathfrak{A}| \longrightarrow |c\mathfrak{A}|$, satisfying the following conditions, for $s \in |\mathfrak{A}|$ and $U \in \mathcal{B}$:*

(1) $Ec_A(s) \;=\; Es$ *and* $c_A(s_{|U}) \;=\; c_A(s)_{|U}$.

(2) *If $t \in |c\mathfrak{A}|$, there is $S \subseteq |\mathfrak{A}|$ such that t is the gluing of $c_A(S) = \{c_A(s) \in |c\mathfrak{A}| : s \in S\}$;*

(3) *The restriction of c_A to $\mathfrak{A}(U)$ is a L-embedding of $\mathfrak{A}(U)$ into $c\mathfrak{A}(U)$, that is, if $\varphi(v_1,\ldots,v_n)$ is a positive, quantifier-free formula in L and $\overline{s} = \langle s_1,\ldots,s_n \rangle \in \mathfrak{A}(U)^n$, then*

$$\mathfrak{A}(U) \models \varphi[\overline{s}] \quad \Leftrightarrow \quad c\mathfrak{A}(U) \models \varphi[c_A(s_1),\ldots,c_A(s_n)].$$

(4) *The following are equivalent :*

(i) $\mathfrak{A}$ is complete over $U \in \mathcal{B}$; *(ii) $c_{A|\mathfrak{A}(U)} : \mathfrak{A}(U) \longrightarrow c\mathfrak{A}(U)$ is a L-isomorphism.*

If $U \in \mathcal{B}$ is compact [8], *(i) and (ii) are equivalent to $\mathfrak{A}$ being* finitely complete *over U.*

b) If $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a morphism of presheaf basis of L-structures over $\mathcal{B}$, then there is a unique *morphism of sheaves of L-structures, $cf : c\mathfrak{A} \longrightarrow c\mathfrak{B}$, such that $cf \circ c_A = c_B \circ f$.* ♢

By Theorem 3.5.(a).(3), a presheaf basis, $\mathfrak{A}$, is *embedded* in the sheaf $c\mathfrak{A}$ it generates. Since the spaces of interest to us here have a natural basis of compact opens, item (a).(4) in 3.5 will be particularly useful. **Whenever $\mathfrak{A}$ is clear from context, write c for the morphism c_A of 3.5.(a)**.

An important construct associated to presheaves is that of *stalk at a point.*

Let $\mathcal{B}$ be a basis for the topological space X. Let $\mathfrak{A}$ be a presheaf basis of L-structures over $\mathcal{B}$. Write ν_x for the filter of *open* neighborhoods of $x \in X$ and define $\mathcal{B}_x = \nu_x \cap \mathcal{B}$. Note that :

∗ Since $\mathcal{B}$ and ν_x are closed under finite intersections, both ν_x and $\mathcal{B}$, are rd-posets (cf. 2.2) under the opposite of the partial order of inclusion, $\subseteq^{op}$; whence, the same is true of $\mathcal{B}_x$;

∗ Because $\mathcal{B}$ is a basis for the topology of X, $\mathcal{B}_x$ is cofinal in $\langle \nu_x, \subseteq^{op} \rangle$;

∗ Since $\mathfrak{A}$ is a *contravariant* functor from $\langle \mathcal{B}, \subseteq \rangle$, it yields, by restriction to $\langle \mathcal{B}_x, \subseteq^{op} \rangle$, a *covariant* functor from this rd-poset to L-mod, that is, an inductive system of L-structures over $\langle \mathcal{B}_x, \subseteq^{op} \rangle$.

Definition 3.6 *With notation as above, for $x \in X$, the* **stalk of $\mathfrak{A}$ at x** *is defined as*

$$\mathfrak{A}_x = \varinjlim \mathfrak{A}_{|\mathcal{B}_x}.$$

For $U \in \mathcal{B}_x$, let $\alpha_{Ux} : \mathfrak{A}(U) \longrightarrow \mathfrak{A}_x$ be the L-morphism given by the inductive limit construction. If $U \subseteq_o V$ are in $\mathcal{B}_x$, then diagram (D) *below is commutative :*

$$\begin{tikzcd}
\mathfrak{A}(V) \arrow[rr, "\alpha_{VU}"] \arrow[dr, "\alpha_{Vx}"'] & \arrow[d, phantom, "\text{(D)}"] & \mathfrak{A}(U) \arrow[dl, "\alpha_{Ux}"] \\
& \mathfrak{A}_x &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{A}(V) \arrow[rr, "\alpha_{VW}"] \arrow[dr, "\alpha_{Vx}"'] & & \mathfrak{A}(W) \arrow[dl, "\alpha_{Wx}"] & \mathfrak{A}(U) \arrow[l, "\alpha_{UW}"'] \arrow[dll, "\alpha_{Ux}"] \\
& \mathfrak{A}_x & &
\end{tikzcd}$$

If $s \in |\mathfrak{A}|$, $x \in Es$ and $U \in \mathcal{B}_x$ is such that $U \subseteq_o Es$ [9], *we define* **the germ of s at x** *to be the value*

$$s_x = \alpha_{Ux}(s_{|U}). \qquad (*)$$

Remark 3.7 Given any other $V \in \mathcal{B}_x$ such that $V \subseteq Es$, let $W = U \cap V$. Commutativity of the diagram above right shows : $\alpha_{Ux}(s_{|U}) = \alpha_{Wx}(\alpha_{UW}(s_{|U})) = \alpha_{Wx}(s_{|W}) = \alpha_{Vx}(s_{|V})$, i.e., $(*)$ is independent of the choice of $U \in \mathcal{B}_x$ contained in Es. In this notation, the commutativity of diagram (D) is expressed as

[germ] For all $U \subseteq_o V$ in $\mathcal{B}_x$ and all $s \in \mathfrak{A}(V)$, $s_x = (s_{|U})_x$. ♢

Lemma 3.8 *Let $\mathcal{B}$ be a basis for a topological space X and let $\mathfrak{A}$ be a presheaf basis over $\mathcal{B}$.*

a) If $\langle s_1, \ldots, s_n \rangle \in |\mathfrak{A}|^n$, $\varphi(v_1, \ldots, v_n)$ is a positive quantifier-free formula in L, and $x \in \bigcap_{i=1}^n Es_i$, then

$$\mathfrak{A}_x \models \varphi[s_{1x}, \ldots, s_{nx}] \quad \Leftrightarrow \quad \begin{cases} \exists\, V \in \mathcal{B}_x \text{ such that } V \subseteq \bigcap_{i=1}^n Es_i \\ \text{and } \mathfrak{A}(V) \models \varphi[s_{1|V}, \ldots, s_{n|V}]. \end{cases}$$

In particular, this applies to equality, that is, if $s, t \in |\mathfrak{A}|$ and $x \in Es \cap Et$, then

$$s_x = t_x \quad \Leftrightarrow \quad \begin{cases} \exists\, U \in \mathcal{B}_x \text{ such that } U \subseteq Es \cap Et \\ \text{and } s_{|U} = t_{|U}. \end{cases}$$

[8] Not necessarily Hausdorff, usually called quasi-compact.

[9] Es might not be in $\mathcal{B}$.

b) Let $\mathfrak{A} \xrightarrow{c} c\mathfrak{A}$ *be the completion of* $\mathfrak{A}$*, as in Theorem* 3.5.(a)*. For each* $x \in X$*, the map* c *induces a* **L-isomorphism***,* $c_x : \mathfrak{A}_x \longrightarrow c\mathfrak{A}_x$*, given by* $s_x \in \mathfrak{A}_x \quad \longmapsto \quad c(s)_x \in c\mathfrak{A}_x$*.*

c) A morphism of presheaf bases over $\mathcal{B}$*,* $f : \mathfrak{A} \longrightarrow \mathfrak{B} = \langle\, \mathfrak{B}(U);\, \beta_{VU} : U \subseteq V \text{ in } \mathcal{B} \,\rangle$*, induces, for each* $x \in X$*, a* L*-morphism,* $f_x : \mathfrak{A}_x \longrightarrow \mathfrak{B}_x$*, such that for all* $U \in \mathcal{B}$*,* $\beta_{Ux} \circ f_U \;\; = \;\; f_x \circ \alpha_{Ux}$*.*

Proof. a) Since a positive quantifier-free formula is constructed from atomic formulas using the connectives $\wedge$, $\vee$, and $\mathcal{B}$ is closed under finite intersections, it is enough to verify the stated equivalence for *atomic formulas.* But this follows readily from Theorem 2.3.(b).(2).

b) Fix $x \in X$; if $s, t \in |\mathfrak{A}|$ are such that $x \in Es \cap Et$ and $s_x = t_x$, by (a) there is $U \in \mathcal{B}_x$ contained in $Es \cap Et$ such that $s_{|U} = t_{|U}$. Since c preserves extent and commutes with restriction, we have

$$U \;\; = \;\; Ec(s_{|U}) \;\; = \;\; Ec(t_{|U}), \;\; \text{and} \;\; c(s)_{|U} \;\; = \;\; c(s_{|U}) \;\; = \;\; c(t_{|U}) \;\; = \;\; c(t)_{|U},$$

and another application of (a) yields $c(s)_x = c(t)_x$, showing that the map c_x is well-defined. The equivalence in (a), together with item (3) in Theorem 3.5.(a), entail that c_x is a L-embedding. It remains to check that c_x is surjective. This follows from Theorem 3.5.(a).(2). Indeed, given $t_x \in c\mathfrak{A}_x$, there is $S \subseteq |\mathfrak{A}|$, such that t is the gluing of $\{c(s) : s \in S\}$, whence, $Et = \bigcup_{s \in S} Ecs = \bigcup_{s \in S} Es$. Thus, there is $s \in S$ such that $x \in Es$; since $c(s) = t_{|Es}$, it follows that for all $U \in \mathcal{B}_x$ such that $U \subseteq Es$, $c(s_{|U}) \;\; = \;\; c(s)_{|U} \;\; = \;\; t_{|U}$, which in turn implies, by (a), $c(s)_x = t_x$, as needed.

c) The morphism f induces a morphism of inductive systems,

$$f_{\mathcal{B}_x} = \{f_U : U \in \mathcal{B}_x\} : \langle\, \mathfrak{A}(U);\, \alpha_{VU} : U \subseteq V \text{ in } \mathcal{B} \,\rangle \longrightarrow \langle\, \mathfrak{B}(U);\, \beta_{VU} : U \subseteq V \text{ in } \mathcal{B} \,\rangle.$$

Then, with $f_x \;\; = \;\; \varinjlim f_{\mathcal{B}_x}$, all conclusions follow from Theorem 2.3.(f), ending the proof. $\Diamond$

Definition 3.9 *Let* $\mathcal{B}$ *be a basis for the topological space* X *and let* $\mathfrak{A}$ *be a presheaf basis of* L*-structures over* $\mathcal{B}$*. If* $\varphi(v_1, \ldots, v_n)$ *is a formula of* L *and* $\overline{s} = \langle\, s_1, \ldots, s_n \,\rangle \in |\mathfrak{A}|^n$*, define*

$$\mathfrak{v}_{\mathfrak{A}}(\varphi(\overline{s})) \;\; = \;\; \{x \in \textstyle\bigcap_{i=1}^n Es_i : \mathfrak{A}_x \models \varphi[s_{1x}, \ldots, s_{nx}]\},$$

called the **Feferman-Vaught value of φ at $\overline{s}$***. Whenever* $\mathfrak{A}$ *is clear from context, its mention will be omitted from the notation. In general,* $\mathfrak{v}(\varphi(\overline{s}))$ **is not** *an open set in* X*. Moreover, in view of* 3.8.(b)*, for all* $\overline{s} \in |\mathfrak{A}|^n$*,* $\mathfrak{v}_{\mathfrak{A}}(\varphi(\overline{s})) \;\; = \;\; \mathfrak{v}_{c\mathfrak{A}}(\varphi(c(\overline{s}))$*, where* $c\mathfrak{A}$ *is the completion of* $\mathfrak{A}$ *over* X *and* $c(\overline{s}) \;\; = \;\; \langle\, c(s_1), \ldots, c(s_n) \,\rangle$*.*

Proposition 3.10 *Let* $\mathcal{B}$ *be a basis for the space* X *and let* $\mathfrak{A}$ *be a presheaf basis of* L*-structures over* $\mathcal{B}$*. Let* $\varphi(v_1, \ldots, v_n)$ *be a* L*-formula and let* $\overline{s} = \langle\, s_1, \ldots, s_n \,\rangle \in |\mathfrak{A}|^n$*. Set* $E\overline{s} \;\; = \;\; \bigcap_{i=1}^n Es_i$*.*

a) If φ *is positive and quantifier free, then*

$$\mathfrak{v}(\varphi(\overline{s})) \;\; = \;\; \bigcup \{V \in \mathcal{B} : V \subseteq E\overline{s} \;\; \text{and} \;\; \mathfrak{A}(V) \models \varphi[s_{1|V}, \ldots, s_{n|V}]\}.$$

In particular, $\mathfrak{v}(\varphi(\overline{s}))$ *is an open set in* X *(not necessarily in* $\mathcal{B}$*).*

b) If φ *is a conjunction of atomic formulas, then for all* $U \in \mathcal{B}$*,*

$$U \subseteq \mathfrak{v}(\varphi(\overline{s})) \quad \Rightarrow \quad \mathfrak{A}(U) \models \varphi[s_{1|U}, \ldots, s_{n|U}].$$

c) For $U \in \mathcal{B}$*, define* $\Gamma(U) \;\; = \;\; \prod_{x \in U} \mathfrak{A}_x$ *and consider the map*

$$\gamma^U : \mathfrak{A}(U) \longrightarrow \Gamma(U), \text{ given by } \;\; \gamma^U(s) \;\; = \;\; \langle\, s_x \,\rangle_{x \in U}.$$

If $\Gamma(U)$ *is endowed with the product* L*-structure, then* γ^U *is a* L*-embedding, and hence preserves and reflects all quantifier-free* L*-formulas.*

d) Suppose X *is Hausdorff and that* $\mathcal{B}$ *is a Boolean algebra of clopens in* X*. If* $U \in \mathcal{B}$ *is* compact *and* $\mathfrak{A}$ *is* finitely complete *over* U*, then* γ^U *reflects geometric sentences with parameters in* $\mathfrak{A}(U)$ [10].

Proof. a) If $\psi_1(v_1, \ldots, v_n)$ and $\psi_2(v_1, \ldots, v_n)$ are L-formulas and $\overline{s} \in |\mathfrak{A}|^n$, it is clear that

$$\mathfrak{v}([\psi_1 \wedge \psi_2](\overline{s})) \;\; = \;\; \mathfrak{v}(\psi_1(\overline{s})) \cap \mathfrak{v}(\psi_2(\overline{s})) \;\; \text{and} \;\; \mathfrak{v}([\psi_1 \vee \psi_2](\overline{s})) \;\; = \;\; \mathfrak{v}(\psi_1(\overline{s})) \cup \mathfrak{v}(\psi_2(\overline{s})), \qquad (\mathfrak{v})$$

and so, it is enough to verify the statement for atomic formulas. Suppose $\varphi(v_1, \ldots, v_n)$ is an atomic L-formula and $\overline{s} \in |\mathfrak{A}|^n$. By 3.8.(a), if $\mathfrak{A}_x \models \varphi[s_{1x}, \ldots, s_{nx}]$, there is $V \in \mathcal{B}_x$ with $V \subseteq E\overline{s}$ and

[10] As in Definition 2.1.(c).

$\mathfrak{A}(V) \models \varphi[s_{1|V}, \ldots, s_{n|V}]$. For $y \in V$, the germ maps, $\alpha_{Vy} : \mathfrak{A}(V) \longrightarrow \mathfrak{A}_y$ are L-morphisms, and hence preserve atomic formulas. But this entails the displayed equality in (a), as needed.

b) By the first equality in ($\mathfrak{v}$) above, it suffices to verify the statement for an atomic L-formula, φ. Suppose $U \subseteq_o \mathfrak{v}(\varphi(\overline{s}))$, with $U \in \mathcal{B}$. Then, for each $x \in U$, $\mathfrak{A}_x \models \varphi[s_{1x}, \ldots, s_{nx}]$. By Lemma 3.8.(a), there is $V \in \mathcal{B}_x$, with $V \subseteq E\overline{s}$, such that $\mathfrak{A}(V) \models \varphi[s_{1|V}, \ldots, s_{n|V}]$. Let $V_x = V \cap U$; note that $V_x \in \mathcal{B}_x$. Moreover, since the restriction maps are L-morphisms, we also have

$$\mathfrak{A}(V_x) \models \varphi[s_{1|V_x}, \ldots, s_{n|V_x}]. \tag{I}$$

Thus, we get a covering of U in $\mathcal{B}$, $\{V_x : x \in U\}$, with the property in (I). It now follows from the extensionality condition [ext] in Definition 3.2.(a), that $\mathfrak{A}(U) \models \varphi[\overline{s}_{|U}]$, as desired.

c) This is a consequence of item (b), upon verifying that γ^U reflects and preserves atomic L-formulas. Indeed, if $\psi(v_1, \ldots, v_n)$ is an atomic L-formula and $\overline{t} = \langle t_1, \ldots, t_n \rangle \in \mathfrak{A}(U)^n$, then the fact that the maps α_{Ux}, $x \in U$, are L-morphisms, immediately entails, because $\Gamma(U)$ has the product L-structure, that $\Gamma(U) \models \varphi[\gamma^U(\overline{t})]$; conversely, if this relation holds, then $\mathfrak{v}(\varphi(\overline{t})) = U \in \mathcal{B}$, and item (b) then guarantees that $\mathfrak{A}(U) \models \varphi[t_{1|U}, \ldots, t_{n|U}]$.

d) We first show that γ^U reflects positive existential $L_{\mathfrak{A}(U)}$-sentences. It is well-known that positive existential formulas are logically equivalent to a disjunction of pp-formulas (as in 2.1.(e)). Hence, it suffices to verify the statement for pp-sentences in $L_{\mathfrak{A}(U)}$. To simplify exposition, we shall also assume that such a pp-sentence has only one existential quantifier, i.e., it is of the form $\exists v\psi(v;\, t_1, \ldots, t_n)$, where ψ is a conjunction of atomic formulas in $L_{\mathfrak{A}(U)}$, whose parameters from $\mathfrak{A}(U)$ are $t_1, \ldots, t_n$. The reader will readily realize that the method extends, straightforwardly, to the general case. Moreover, write γ for the L-embedding γ^U (see (c)) .

Suppose $\Gamma(U) \models \exists v\varphi(v)[\gamma(\overline{t})]$; because $\Gamma(U)$ has the product L-structure, for every $x \in U$, $\mathfrak{A}_x \models \exists v\varphi(v)[t_{1x}, \ldots, t_{nx}]$. Therefore, for each $x \in U$, there is $z_x \in \mathfrak{A}_x$ such that

$$\mathfrak{A}_x \models \varphi[z_x;\, t_{1x}, \ldots, t_{nx}].$$

By Lemma 3.8.(a), there is $V_x \in \mathcal{B}_x \subseteq U$ and $z(x) \in \mathfrak{A}(V_x)$ such that

$$\mathfrak{A}(V_x) \models \varphi[z(x);\, t_{1|V_x}, \ldots, t_{n|V_x}]. \tag{II}$$

Since U is compact, there is a finite collection, $\{x_1, \ldots, x_m\} \subseteq U$ such that $\{V_{x_j} : 1 \leq j \leq m\}$ cover U. By a standard disjointing argument [11], there are disjoint clopens, $V_j \in \mathcal{B}$, $1 \leq j \leq n$, such that

$$V_j \subseteq V_{x_j} \quad \text{and} \quad U = \textstyle\bigcup_{j=1}^m V_j. \tag{III}$$

Let $Z = \{z(x_j)_{|V_j} : 1 \leq j \leq m\}$; since their extents are disjoint, with union U, and $\mathfrak{A}(U)$ is finitely complete, there is $z \in \mathfrak{A}(U)$ such that $z_{|V_j} = z(x_j)_{|V_j}$, $1 \leq j \leq m$. Moreover, since $V_j \subseteq V_{x_j}$, (II) and the fact that φ is a conjunction of atomic formulas entail

$$\text{For all } 1 \leq j \leq m, \quad \mathfrak{A}(V_j) \models \varphi[z_{|V_j};\, t_{1|V_j}, \ldots, t_{n|V_j}]. \tag{IV}$$

Now (III), (IV) and Remark 3.3.(b) imply that $\mathfrak{A}(U) \models \varphi[z;\, t_1, \ldots, t_n]$, i.e., $\exists v\varphi(v;\, t_1, \ldots, t_n)$ holds in $\mathfrak{A}(U)$, as needed. To complete the proof, suppose that $\sigma(t_1, \ldots, t_n)$ is a geometric $L_{\mathfrak{A}(U)}$-sentence. If σ is the negation of an atomic sentence, reflection follows immediately from the fact that γ is an L-embedding. Let $\sigma(\overline{t})$ is a $L_{\mathfrak{A}(U)}$-sentence of the form $\forall\overline{v}(\varphi_1(\overline{v};\, \overline{t}) \rightarrow \exists\overline{y}\varphi_2(\overline{v},\, \overline{y};\, \overline{t}))$, with φ_1, φ_2 positive and quantifier-free. Let $\overline{s} = \langle s_1, \ldots, s_n \rangle \in \mathfrak{A}(U)^n$ and suppose $\mathfrak{A}(U) \models \varphi_1[\overline{s}; \overline{t}]$. Since γ is a L-embedding, we have $\Gamma(U) \models \varphi_1[\gamma(\overline{s});\, \gamma(\overline{t})]$; since $\sigma(\overline{t})$ holds in $\Gamma(U)$, it follows that $\Gamma(U) \models \exists\overline{y}\, \varphi_2[\gamma(\overline{s}), \overline{y};\, \gamma(\overline{t})]$ and so, the fact that $\mathfrak{A}(U)$ is positively existentially closed in $\Gamma(U)$ along γ guarantees that $\exists\overline{y}\, \varphi_2(\overline{s},\, \overline{y};\, \overline{t})$ holds in $\mathfrak{A}(U)$, as needed. $\diamondsuit$

We now have

Theorem 3.11 *Let L, $L^\sharp$ be first-order languages with equality and let Σ, $\Sigma^\sharp$ be theories in L and $L^\sharp$, respectively. Let X be a Boolean space and let $\mathcal{B}$ be the Boolean algebra of clopens in X. Let $\mathfrak{A} : \mathcal{B} \longrightarrow$ **Σ-mod** be a finitely complete presheaf basis of models of Σ, with restriction L-morphisms $\{\rho_{VU} : U \subseteq V \text{ in } \mathcal{B}\}$. If $F :$ **Σ-mod** $\longrightarrow$ **$\Sigma^\sharp$-mod** is a geometrical functor, then*

*a) $F \circ \mathfrak{A} : \mathcal{B} \longrightarrow$ **$\Sigma^\sharp$-mod** is a finitely complete presheaf basis of models of $\Sigma^\sharp$.*

[11] By induction, set $V_1 = V_{x_1}$ and $V_k = V_{x_k} \setminus (\bigcup_{i<k} V_{x_i})$; it is here that it is crucial that $\mathcal{B}$ be a BA.

b) For all $x \in X$, the stalk of $F \circ \mathfrak{A}$ at x is $(F \circ \mathfrak{A})_x = \langle F(\mathfrak{A}_x); \{F(\rho_{Ux}) : U \in \mathcal{B}_x\} \rangle$.

Proof. Item (b) follows immediately from (a) and the fact that F preserves right-directed colimits. For (a), let $U \in \mathcal{B}$ and let $\overline{V} = \{V_1, \ldots, V_n\}$ be a disjoint clopen covering of U. Since $\mathfrak{A}$ is extensional and fc, Proposition 3.4.(a) guarantees that the diagram below left is commutative, with $\alpha(U; \overline{V})$ a L-isomorphism, $1 \leq j \leq n$:

$$\begin{tikzcd} \mathfrak{A}(U) \arrow[rr, "\alpha(U;\overline{V})"] \arrow[dr, "\cdot|_{V_j}"'] & & \prod_{j=1}^n \mathfrak{A}(V_j) \arrow[dl, "p_j"] \\ & \mathfrak{A}(V_j) & \end{tikzcd} \qquad \begin{tikzcd} F(\mathfrak{A}(U)) \arrow[rr, "F(\alpha(U;\overline{V}))"] \arrow[dr, "\cdot|_{F(V_j)}"'] & & \prod_{j=1}^n F(\mathfrak{A}(V_j)) \arrow[dl, "F(p_j)"] \\ & F(\mathfrak{A}(V_j)) & \end{tikzcd}$$

Since F preserves finite products, $F(p_j)$ is the j-th coordinate projection and the diagram above right is commutative, $1 \leq j \leq n$. Moreover, $F(\alpha(U; \overline{V}))$ is clearly a $L^\sharp$-isomorphism. By the equivalence in Proposition 3.4.(a), $F \circ \mathfrak{A}$ is an extensional, finitely complete presheaf basis of models of $\Sigma^\sharp$, as needed. $\diamondsuit$

4 Rings with Many Units

In this section we first give a model-theoretic criterion for a subring to inherit the property of having many units and then show that if A is a ring with many units, the mod 2 counterpart of Milnor's K-theory of rings, introduced in [Gu], is canonically isomorphic to the K-theory of a special group naturally associated to A in [DM5]. To begin, we recall

Definition 4.1 *Let R be a ring.*

a) A polynomial $f \in R[X_1, \ldots, X_n]$ has **local unit values relative to maximal ideals** *if for all maximal ideals $\mathfrak{m}$ in R, there is $\overline{u} \in R^n$ such that $f(\overline{u}) \notin \mathfrak{m}$. Similarly, one defines the notion f having local unit values relative to prime ideals in R.*

b) R is a **ring with many units** *if for all $f \in R[X_1, \ldots, X_n]$, if f has local unit values relative to maximal ideals, then there is $\overline{y} \in R^n$ such that $f(\overline{y})$ is a unit in R.*

Remark 4.2 Since every maximal ideal is prime and all (proper) prime ideals are contained in a maximal ideal, a ring R has many units iff for all $f(X_1, \ldots, X_n) \in R[X_1, \ldots, X_n]$,

f has local unit values relative to all prime ideals in R	$\Rightarrow$	$\exists\, \overline{z} = \langle z_1, \ldots, z_n \rangle \in R^n$ such that $f(\overline{z})$ is a unit in R.

Examples of rings with many units are semi-local rings, arbitrary products of rings with many units and more generally, the ring of global sections of a sheaf of rings over a partitionable space, whose stalks are rings with many units. In particular, the ring of global sections of a sheaf of rings over a Boolean space, whose stalks are rings with many units, is a ring with many units. The reader can find more information, as well as the proof of these results in [DM5], where it is also shown that, under mild assumptions, the RSGs associated to rings of this type faithfully represent the quadratic form theory over free modules (Theorems 3.15 and 3.16, [DM5]). $\diamondsuit$

Proposition 4.3 *Let R be a ring with many units. If S is a positively existentially closed subring of R, then S is also a ring with many units.*

Proof. We shall use the equivalence noted in Remark 4.2. Let $f(X_1, \ldots, X_n)$ be a polynomial with coefficients in S, that has local unit values relative to all prime ideals in S. If P is a prime ideal in R, then $Q = P \cap S$ is a prime ideal in S and so there is $\overline{x} \in S^n$ such that $f(\overline{x}) \notin Q$. Because f has coefficients in S, it is clear that $f(\overline{x}) \in S$. Hence, $f(\overline{x})$ cannot be in P. Thus, f has local unit values relative to all prime ideals in R. Since R has many units, there is $\overline{r} \in R^n$, such that $f(\overline{r})$ is a unit in R. Now consider the sentence φ given by

$$\exists x_1 \cdots x_n \ \exists u \ (u \cdot f(x_1, \ldots, x_n) = 1).$$

Because f has coefficients in S, φ is a pp-sentence of the language of rings with parameters in S. Since S is positively existentially closed in R and $R \models \varphi$, the same is true in S and so f has unit values in S, as needed. $\diamondsuit$

We now adapt to our purposes a condition introduced in [Gu] (page 29) :

Definition 4.4 *Let A be a ring and let $m \geq 1$ be an integer. We say that*

a) A satisfies [H1-m] ($A \models$ [H1-m]) *if for all $n \geq 2$ and all $1 \leq k \leq m$, if $\{f_1, \ldots, f_k\}$ is a family of surjective linear forms over the free A-module A^n, there is $v \in A^n$ such that $f_j(v) \in A^*$, $1 \leq j \leq k$.*

b) A satisfies [H1] *if $A \models$* [H1-m] *for all $m \geq 1$.*

It is mentioned in the Examples given on page 33 of [Gu] that all semilocal rings whose residue fields are infinite verify [H1]. In particular, all infinite fields satisfy [H1]. Generalizing this observation we have

Proposition 4.5 *Let $m \geq 2$ be an integer. If A is a ring with many units, whose residue fields all have cardinality $\geq m$, then $A \models$* [H1-m].

Proof. We start with the following

Fact 4.6 *If F is a field of cardinality $\geq m$, then $F \models$* [H1-m]. *In particular, infinite fields verify* [H1].

Proof. By induction on $m \geq 1$. Clearly, *any ring* verifies [H1-1]. Assume the result true for m, that F has at least $m+1$ elements and that $\{f_1, \ldots, f_k\}$, $1 \leq k \leq m+1$, are surjective linear forms from F^n to F. If $k \leq m$, the induction hypothesis immediately implies the desired result. So, assume $k = m+1$. The induction hypothesis yields $v \in F^n$ such that $f_j(v) \neq 0$ (i.e., $f_j\ (v) \in F^*$), $1 \leq j \leq m$. If $f_{m+1}(v) \neq 0$, we are done. Otherwise, select w such that $f_{m+1}(w) \neq 0$ and consider the set

$$A \ = \ \{f_j(w)/f_j(v) : 1 \leq j \leq m\}.$$

Since A has at most m elements and F has at least $m+1$ elements, there is $\lambda \in F \setminus A$. Now consider $x = w - \lambda v \in F^n$; then,

$$f_{m+1}(x) \ = \ f_{m+1}(w) \ \neq \ 0 \quad \text{and, for } 1 \leq j \leq m, \quad f_j(x) \ = \ f_j(w) - \lambda f_j(v) \ \neq \ 0,$$

because $\lambda \notin A$, establishing Fact 4.6.

Now let A be a ring with many units, whose residue fields all have more than m elements, and let $\{f_1, \ldots, f_k\}$ be surjective linear forms from A^n to A $(k \leq m)$. Let $\{e_1, \ldots, e_n\}$ be the canonical basis of A^n and set, for $1 \leq j \leq k$ and $1 \leq l \leq n$, $a_{jl} \ =_{def} \ f_j(e_l)$. Now, let

$$p(X_1, \ldots, X_n) \ = \ \textstyle\prod_{j=1}^{k} \sum_{l=1}^{n} a_{jl} X_l \ = \ \prod_{j=1}^{k} f_j. \tag{I}$$

If $\mathfrak{m}$ is a maximal ideal in A and $1 \leq j \leq k$, the form f_j naturally induces a *surjective* linear form, $f_j/\mathfrak{m}$, from $(A/\mathfrak{m})^n$ to $A/\mathfrak{m}$, given by

$$x/\mathfrak{m} \ = \ (x_1/\mathfrak{m}, \ \ldots, \ x_n/\mathfrak{m}) \ \longmapsto \ f_j(x)/\mathfrak{m}.$$

Indeed, if $x_l - y_l \in \mathfrak{m}$, $1 \leq l \leq n$, then, with notation as in (I),

$$f_j(x) - f_j(y) \ = \ f_j(x-y) \ = \ \textstyle\sum_{l=1}^{n} a_{jl}(x_l - y_l) \ \in \ \mathfrak{m},$$

and $f_j/\mathfrak{m}$ is well defined. It is clear that $f_j/\mathfrak{m}$ is surjective. By Fact 4.6, there is $v/\mathfrak{m} \in (A/\mathfrak{m})^n$ such that $[f_j/\mathfrak{m}](v/\mathfrak{m}) \neq 0$, that is, $f_j(v) \notin \mathfrak{m}$, for all $1 \leq j \leq k$. Since $\mathfrak{m}$ is a prime ideal, (I) entails

$$p(v) \ = \ \textstyle\prod_{j=1}^{k} f_j(v) \ \notin \ \mathfrak{m},$$

and thus $p(X_1, \ldots, X_n)$ has local unit values in A. Hence, there is $x \in A^n$ such that $p(x) \in A^*$. But this immediately implies that $f_j(x) \in A^*$, $1 \leq j \leq k$, ending the proof. $\diamondsuit$

We now wish to present a mod 2 K-theory of rings, patterned after the construction in section 3 of [Gu]. Let A be a ring. We set $K_0A = \mathbb{Z}$ and let K_1A be A^* written additively, that is, we fix an isomorphism

$$l : A^* \longrightarrow K_1A, \text{ such that } l(ab) = l(a) + l(b), \ \forall\, a, b \in A^*.$$

Then, Milnor's K-theory of A is the graded ring (Definition 3.2, p. 47, [Gu])

$$K_*A = \langle \mathbb{Z}, K_1A, \ldots, K_nA, \ldots \rangle,$$

obtained as the quotient of the graded tensor algebra over $\mathbb{Z}$,

$$\langle \mathbb{Z}, K_1A, \ldots, \underbrace{K_1A \otimes \ldots \otimes K_1A}_{n \text{ times}}, \ldots \rangle$$

by the ideal generated by $\{l(a) \otimes l(b) : a, b \in A^* \text{ and } a + b = 1 \text{ or } 0\}$. Hence, for each $n \geq 2$, K_nA is the quotient of the n-fold tensor product over $\mathbb{Z}$, $K_1A \otimes \ldots \otimes K_1A$, by the subgroup consisting of sums of generators $l(a_1) \otimes \ldots \otimes l(a_n)$, such that for some $1 \leq i \leq n-1$, $a_i + a_{i+1} = 1$ or 0. As usual, we shall write the generators in K_nA as $l(a_1) \cdots l(a_n)$, omitting the tensor operation. As a consequence of (the proof of) Proposition 3.2.3 in [Gu] (p. 48) and Proposition 4.5 we have

Lemma 4.7 *Let A be a ring with many units whose residue fields all have more than 7 elements. Then, K_*A is the graded ring obtained as the quotient of the graded tensor algebra over $\mathbb{Z}$,*

$$\langle \mathbb{Z}, K_1A, \ldots, \underbrace{K_1A \otimes \ldots \otimes K_1A}_{n \text{ times}}, \ldots \rangle$$

by the graded ideal generated by $\{l(a)l(b) : a, b \in A^* \text{ and } a + b = 1\}$.

Proof. By Prop. 3.2.3 in [Gu], the result holds for rings satisfying [H1] in 4.4.(b). However, an analysis of the proof shows that what is needed is [H1-6], and the desired conclusion follows from 4.5. $\diamondsuit$

Definition 4.8 *If A is a ring, we define the* **mod 2 K-theory of A**, *as the graded ring*

$$k_*A = \langle k_0A, k_1A, \ldots, k_nA, \ldots \rangle =_{def} K_*A/2K_*A,$$

that is, for each $n \geq 0$, k_nA is the quotient of K_nA by the subgroup $\{2\eta \in K_nA : \eta \in K_nA\}$.

We have $k_0A = \mathbb{F}_2$ and $k_1A \approx A^*/A^{*2}$, via an isomorphism still denoted by l. A generator in k_nA will be written $l(a_1) \cdots l(a_n)$. Clearly, k_nA is a group of exponent 2, i.e., $\eta + \eta = 0$, for all $\eta \in k_nA$.

Lemma 4.9 *If A is a ring verifying* [H1-6]*, then for all $b, a, a_1, \ldots, a_n \in A^*$ and all permutations σ of $\{1, \ldots, n\}$*

a) In k_2A, $l(a)l(-a) = 0$.

b) In k_2A, $l(a)l(-1) = l(a)^2$.

c) In k_2A, $l(a)l(b) = l(b)l(a)$.

d) In k_nA, $l(a_1) \cdots l(a_n) = l(a_{\sigma(1)} \cdots l(a_{\sigma(n)})$.

e) If $t_1, \ldots, t_n \in A^*$, then in k_nA, $l(t_1^2 a_1) \cdots l(t_n^2 a_n) = l(a_1) \cdots l(a_n)$.

Proof. a) The proof of Prop. 3.2.3 in [Gu] shows that if A verifies [H1-6], then $l(a)l(-a) = 0$ in K_2A and so the same is true in k_2A.

b) From (a) we get $0 = l(a)l(-a) = l(a)[l(-1) + l(a)] = l(a)l(-1) + l(a)^2$. Since k_2A is a group of exponent two, the conclusion follows.

c) From (a) and (b) we get

$$\begin{aligned} 0 &= l(-ab)l(ab) = [l(-a) + l(b)][l(a) + l(b)] = l(b)l(a) + l(-a)l(b) + l(b)^2 \\ &= l(b)l(a) + [l(-1) + l(a)]l(b) + l(b)^2 = l(b)l(a) + l(a)l(b), \end{aligned}$$

and so, since k_2 is a group of exponent two, we obtain $l(a)l(b) = l(b)l(a)$, as needed. Item (c) implies that the conclusion in (d) holds for all transpositions. Since the symmetric group is generated by transpositions, the full statement in (d) follows immediately. For item (e), note that for $t, a \in A^*$, $l(t^2a) = 2l(t) + l(a) = l(a)$, since $2l(t) = 0$ in k_1A. $\diamondsuit$

Our next order of business is to connect the mod 2 K-theory of a ring with many units satisfying certain conditions with the K-theory of a special group naturally associated to it. With this purpose we set down the following

4.10 Construction. Let A be a ring. For $a, b \in A^*$ define

$$D_A(a, b) = \{c \in A^* : \exists\, s, t \in A \text{ such that } c = s^2a + t^2b\},$$

called *the set of units represented modulo squares by* a, b. Now let

$$G(A) = A^*/A^{*2} = \{\overline{a} : a \in A^*\}$$

be the group of exponent two of the square classes of elements of A^*. For $u, v \in A^*$,

$$\overline{u} = \overline{v} \quad \text{iff} \quad uv \in A^{*2} \quad \text{iff} \quad \exists\, t \in A^* \text{ such that } u = t^2v. \tag{I}$$

It follows straightforwardly from (I) that for $u, v, w, z \in A^*$,

$$\overline{u} = \overline{w} \quad \text{and} \quad \overline{v} = \overline{z} \quad \Rightarrow \quad D_A(u, v) = D_A(w, z). \tag{II}$$

We abuse notation and write $1, -1$ both for the elements in A^* and for $\overline{1}, \overline{-1}$, respectively.

Define the relation of *binary isometry* in $G(A)$ by the following clause : for $u, v, x, y \in A^*$

$$\langle \overline{u}, \overline{v} \rangle \equiv \langle \overline{x}, \overline{y} \rangle \quad \text{iff} \quad \overline{u}\,\overline{v} = \overline{xy} \quad \text{and} \quad D_A(u, v) = D_A(x, y).$$

Relation (II) above shows that $\equiv$ is well-defined, i.e., is independent of representatives in the square classes of u, v, x and y. $\diamondsuit$

Lemma 4.11 *Let A be a ring with many units, whose residue fields all have more than 7 elements. Let $a, b, a_1, \ldots, a_n \in A^*$, with $a \in D_A(1, b)$. If $a_i = a$ and $a_j = ab$ for some $1 \leq i \neq j \leq n$, then $l(a_1) \cdots l(a_n) = 0$ in k_nA.*

Proof. Let A be a ring as in the statement. It is noted in the proof of Theorem 3.16 in [DM5] that A satisfies the following property (therein called [w2t], cf. 3.11, p. 16) :

$$\forall\, u, v, w \in A^*, \quad w \in D_A(u, v) \quad \Rightarrow \quad \exists\, p, q \in A^* \text{ such that } w = p^2u + q^2v. \tag{$\natural$}$$

Hence, since $a \in D_A(1, b)$, there are $p, q \in A^*$ such that $a = p^2 + q^2b$. Hence,

$$1 = (p^2/a^2)\, a + (q^2/a^2)\, ba = (p/a)^2a + (q/a)^2ab,$$

and so, the definition of k_*A and 4.9.(e) yield $l(a)l(ab) = 0$ in k_2A. The general statement follows immediately from 4.9.(d). $\diamondsuit$

Theorem 4.12 *Let A be a ring with many units such that $2 \in A^*$ and whose residue fields all have more than 7 elements. Then, $G(A) = \langle G(A), \equiv, -1 \rangle$ (as in 4.10) is a special group. Moreover, the rules $\alpha_0 = Id_{\mathbb{F}_2}$ and $\alpha_n : k_nA \longrightarrow k_nG(A)$, defined on generators by $\alpha_n(l(a_1) \cdots l(a_n)) = \lambda(\overline{a}_1) \cdots \lambda(\overline{a}_n)$, for $n \geq 1$, determine a graded ring isomorphism between the mod 2 K-theory of A and the K-theory of the special group $G(A)$.*

Proof. The fact that $G(A)$ is a special group is established in Theorem 3.16 of [DM5]. Now, the proof of Theorem 2.5 in [DM3], yielding an analogous result for fields of characteristic $\neq 2$, with Lemma 4.11 in the role of Lemma 2.4 of [DM3], applies, *ipsis litteris*, to show that $\alpha = \{\alpha_n : n \geq 0\}$ is a graded ring isomorphism between k_*A and $k_*G(A)$. $\diamondsuit$

5 Presheaf Representation and Preorders of vN-Rings

5.1 Notation and Remarks. Let R be a ring.

a) For $a_1, \ldots, a_n$ in R, $(a_1, \ldots, a_n)$ is the ideal generated by $a_1, \ldots, a_n$ in R. As usual, an ideal is **principal** if it is of the form $(a) = Ra$, for $a \in R$.

b) $\mathrm{Spec}(R) = \{P \subseteq R : P$ is a (proper) prime ideal in $R\}$ is the *Zariski spectrum* of R. For $a \in R$, set $Z(a) = \{P \in \mathrm{Spec}(R) : a \notin P\}$. The collection $\mathcal{Z} = \{Z(a) : a \in R\}$ has the following properties :

(1) $Z(0) = \emptyset$, $Z(1) = \mathrm{Spec}(R)$; (2) $Z(ab) = Z(a) \cap Z(b)$;

(3) $(a) \subseteq (b) \Rightarrow Z(a) \subseteq Z(b)$;

(4) $Z(a) \subseteq Z(b)$ iff $\exists\, n \geq 1$ such that $a^n \in (b)$;

(5) $Z(a + b - ab) \subseteq Z(a) \cup Z(b)$. If $b^2 = b$, then $Z(a + b - ab) = Z(a) \cup Z(b)$.

Items (1) and (2) above guarantee that $\mathcal{Z}$ is a basis for a topology on $\mathrm{Spec}(R)$, the *Zariski topology*, that is (well-) known to be spectral and in which $Z(a)$ is open and compact, for all $a \in R$.

c) For $a \in R$, let R_a be the **ring of fractions of R with respect to a**, that is, $R_a = RM_a^{-1}$, where $M_a = \{1\} \cup \{a^n : n \geq 1\}$. Note that if a is nilpotent, then $R_a = \{0\}$, the zero ring. $\diamondsuit$

5.2 The Boolean Algebra of Idempotents. Let R, S be rings.

a) Let $B(R) = \{e \in R : e^2 = e\}$ be the set of idempotents in R. With the operations

$$e \wedge f = ef \quad \text{and} \quad e \vee f = e + f - ef,$$

$\langle B(R), \wedge, \vee, 0, 1\rangle$ is a Boolean algebra (BA), where the complement of e is $1 - e$. Note that for $e, f \in B(R)$, $e \leq f \Leftrightarrow ef = e \Leftrightarrow e \vee f = f$.

If $f : R \longrightarrow S$ is a ring-morphism, then $B(f) =_{def} f_{|B(R)}$ is a BA-morphism from $B(R)$ to $B(S)$; it is clear that this correspondence preserves composition and identity. Hence, we have a covariant functor from **UCR** to **BA**, the category of BAs.

If $e \in B(R)$, the principal ideal $(e) = Re$ is a ring, whose unit is e.

b) For $e \in B(R)$, let $\varphi_{1e} : R \longrightarrow Re$, be the ring morphism given by $\varphi_{1e}(a) = ae$. If $f \leq e$, write φ_{ef} for $(\varphi_{1f})_{|Re} : Re \longrightarrow Rf$; since $ef = f$, we have $\varphi_{ef}(ae) = af$. Note that

(1) $\varphi_{ee} = Id_{Re}$; (2) For $h \leq f \leq e$, $\varphi_{eh} = \varphi_{fh} \circ \varphi_{ef}$.

c) $e, f \in B(R)$ are **disjoint or orthogonal** if $ef = 0$; thus, $ef = 0 \Leftrightarrow f \leq 1 - e \Leftrightarrow e \leq 1 - f$. Clearly, if e and f are disjoint, then $e \vee f = e + f$.

d) A family $\{f_1, \ldots, f_n\} \subseteq B(R)$ is

(1) A **covering** of $e \in B(R)$ if $e = \bigvee_{i=1}^n f_i$;

(2) An **orthogonal decomposition of e** if the f_j are pairwise disjoint and

$$e = \textstyle\sum_{j=1}^n f_j = \bigvee_{j=1}^n f_j.$$

e) An orthogonal decomposition of $e \in B(R)$, $\{h_j \in B(R) : 1 \leq j \leq n\}$ induces a decomposition into a direct sum of rings, $Re = \bigoplus_{j=1}^n Rf_j$, defined by $ae \longmapsto \sum_{j=1}^n af_j$. Hence, $R = Re \oplus R(1 - e)$ and the map $\alpha_e : R/(1 - e) \longrightarrow Re$ given by $\alpha_e(a/(1 - e)) = ae$ is an **isomorphism**, with $\varphi_{1e} = \alpha_e \circ q_{1-e}$, where $q_{1-e} : R \longrightarrow R/(1 - e)$ is the canonical quotient map and φ_{1e} is as in (b) above. $\diamondsuit$

Proposition 5.3 *If R is a ring and $e \in B(R)$, let $\{f_j \in B(R) : 1 \leq j \leq n\}$ be a covering of e.*

a) There is an orthogonal decomposition of e, $\{e_j \in B(R) : 1 \leq j \leq n\}$, so that $e_j \leq f_j$, $1 \leq j \leq n$. Such an orthogonal decomposition is said to be **subordinate to the covering** *$\{f_j : 1 \leq j \leq n\}$.*

b) If $a, b \in R$, then $ae = be \Leftrightarrow \forall\, 1 \leq j \leq n$, $af_i = bf_i$.

c) Let $a_1, \ldots, a_n \in R$ be such that for all $1 \leq j, k \leq n$, $a_k f_k f_j = a_j f_j f_k$. Then, there is $a \in R$ such that $af_j = a_j f_j$, for all $1 \leq j \leq n$.

d) If R_e is the ring of fractions of R with respect to e (as in 5.1.(c)), then

(1) The map $\lambda_e : Re \longrightarrow R_e$, given by $\lambda_e(re) = re/1$ is a ring isomorphism;

(2) *For $f \leq e$, the map $\gamma_{ef} : R_e \longrightarrow R_f$, given by $\gamma_{ef}(re/1) \;=\; rf/1$, is a ring morphism, such that the following diagram is commutative*

$$\begin{tikzcd} Re \arrow[r, "\lambda_e"] \arrow[d, "\varphi_{ef}"'] & R_e \arrow[d, "\gamma_{ef}"] \\ Rf \arrow[r, "\lambda_f"'] & R_f \end{tikzcd}$$

where $\varphi_{ef} : Re \longrightarrow Rf$ is as in 5.2.(b).

e) If R is a ring with many units, the same is true of Re.

Proof. a) Set $e_1 = f_1$ and for $2 \leq j \leq n$, define $e_j = f_j(1-f_1)(1-f_2)\cdots(1-f_{j-1})$; it is straightforward that the $e_j \leq f_j$ are pairwise orthogonal and that for all $1 \leq k \leq n$, $\sum_{j=1}^{k} e_j \;=\; \bigvee_{j=1}^{k} f_j$. In particular, $\{e_j : 1 \leq j \leq n\}$ is a orthogonal decomposition of e, subordinate to the covering $\{f_j : 1 \leq j \leq n\}$.

b) It suffices to verify ($\Leftarrow$). Let $\{e_j : 1 \leq j \leq n\}$ be an orthogonal decomposition of e, subordinate to $\{f_j : 1 \leq j \leq n\}$. For $1 \leq j \leq n$, $ae_j = ae_jf_j = bf_je_j = be_j$, and so $ae = a\sum_{j=1}^{n} e_j = b\sum_{j=1}^{n} e_j = be$, as needed.

c) Let $\{e_j : 1 \leq j \leq n\}$ be an orthogonal decomposition of e, subordinate to $\{f_j : 1 \leq j \leq n\}$ and set $a \;=\; \sum_{j=1}^{n} a_je_j$. Then, for $1 \leq k \leq n$, $af_k \;=\; \sum_{j=1}^{n} a_je_jf_k \;=\; \sum_{j=1}^{n} a_je_jf_jf_k \;=\; \sum_{j=1}^{n} a_ke_jf_jf_k$

$$= \; a_kf_k\sum_{j=1}^{n} e_j \;=\; a_kf_ke \;=\; a_kf_k,$$

as desired.

d) (1) Clearly, λ_e is an injective ring morphism. For $x \in R$, since $e(x - xe) = 0$, in R_e we have

$$x/e \;=\; x/1 \;=\; xe/1, \tag{$\flat$}$$

and λ_e is also surjective, whence an isomorphism. Item (2) is straightforward.

e) Let $\alpha(\overline{X}) \in Re[X_1, \ldots, X_n]$ be a polynomial. Observe that for $\overline{a} \in R^n$

$$e\alpha(\overline{a}) \;=\; \alpha(\overline{a}) \;=\; \alpha(a_1e, \ldots, a_ne), \tag{I}$$

since for a monomial $(c_\nu e)X_1^{\nu_1}\cdots X_n^{\nu_n}$ in α, $(c_\nu e)a_1^{\nu_1}\cdots a_n^{\nu_n} = (c_\nu e)(a_1e)^{\nu_1}\cdots(a_ne)^{\nu_n}$. Suppose α has local unit values with respect to all prime ideals in Re (cf. 4.2), and consider

$$\beta(\overline{X}) \;=\; \alpha(\overline{X}) + (1-e) \;\in\; R(X_1, \ldots, X_n).$$

Let Q be a (proper) prime ideal in R; since $e(1-e) = 0$, we have two possibilities :

(*i*) $e \in Q$: In view of (I), for all $\overline{b} \in R^n$, $\beta(\overline{b}) \;=\; e\alpha(\overline{b}) + (1-e) \;\notin\; Q$ (otherwise $1 - e \in Q$ and Q would not be a proper ideal);

(*ii*) $e \notin Q$: $P = Q \cap Re$ is a proper prime ideal in Re and so there is $\overline{a} \in R^n$ such that $\alpha(\overline{a}) = \alpha(a_1e, \ldots, a_ne) \notin P$. Because $1 - e \in Q$, we conclude that $\beta(\overline{a}) = \alpha(\overline{a}) + (1-e) \notin Q$, otherwise $\alpha(\overline{a})$ would belong to $Q \cap Re = P$.

We have just shown that β has local unit values with respect to all prime ideals in R. Since R has many units, there is $u \in R^n$ and $\overline{c} \in R^n$ such that

$$1 \;=\; u\beta(\overline{c}) \;=\; u(\alpha(\overline{c}) + (1-e)) \;=\; u\alpha(c_1e, \ldots c_ne) + u(1-e).$$

Multiplying this equation by e, we get $e = (ue)\,\alpha(c_1e, \ldots, c_ne)$, and $\alpha(\overline{c})$ is a unit in Re, as needed. $\Diamond$

Item (d) in Proposition 5.3 yields the following generalization of Proposition 4.3 :

Corollary 5.4 *Let R, T be rings and let $f : R \longrightarrow T$ be a map that preserves addition, multiplication and 0* [12]*. If T has many units and R is positively existentially closed in T along f, then R has many units.*

[12]So f is a morphism with respect to the language of rings *without* identity.

Proof. First note that $e = f(1) \in B(T)$ and that f identifies S with a positively existentially closed subring of Te. The conclusion follows immediately from 4.3 and 5.3.(e). $\diamondsuit$

Definition 5.5 *A ring R is* **von Neumann regular (vN-ring)** *if every principal ideal is generated by an idempotent. Thus, if $a \in R$, there is $e \in B(R)$ such that $(a) = (e)$. Equivalently,*

$$\forall\, a \in R\ \exists\, e \in B(R) \text{ and } \exists\, b \in R \quad \text{such that} \quad ae = a \quad \text{and} \quad ab = e. \qquad \text{[vN]}$$

We refer to e as **the idempotent associated** *to a (clearly,it is unique). Yet another formulation of von Neumann regularity of a ring R, is to require that every element of R be divisible by its square. A vN-ring is also called* **absolutely flat**, *being precisely the rings with the property that all modules are flat. Write* **vN** *for the category of vN-rings and ring morphisms.*

Lemma 5.6 *Let R be a vN-ring and let $e \in B(R)$.*

a) All prime ideals in R are maximal and the map $P \in \mathrm{Spec}(R) \overset{\mathfrak{r}}{\longmapsto} P \cap B(R)$ is a natural bijective correspondence between $\mathrm{Spec}(R)$ and the maximal ideals in the Boolean algebra $B(R)$.

b) Let P be a prime ideal in R and let R_P be the localization of R at P. If $P \in Z(e)$, let

(1) $\lambda_{eP} : R_e \longrightarrow R_P$, *be given by* $\lambda_{eP}(x/e) \;=\; x/e$;

(2) $\varphi_{eP} : Re \longrightarrow R/P$, *be given by* $\varphi_{eP}(re) \;=\; r/P$;

(3) $\lambda_P : R/P \longrightarrow R_P$, *be given by* $\lambda_P(x/P) \;=\; x/1$.

Then, λ_{eP}, φ_{eP} are surjective *ring morphisms, λ_P is an* isomorphism *and diagram* (I) *below is commutative,*

$$\begin{tikzcd}
Re \arrow[r, "\lambda_e"] \arrow[d, "\varphi_{eP}"'] \arrow[dr, phantom, "\mathrm{(I)}"] & R_e \arrow[d, "\lambda_{eP}"] \\
R/P \arrow[r, "\lambda_P"'] & R_P
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Re \arrow[rr, "\varphi_{ef}"] \arrow[dr, "\varphi_{eP}"'] & \mathrm{(II)} & Rf \arrow[dl, "\varphi_{fP}"] \\
& R/P &
\end{tikzcd}$$

where λ_e is the isomorphism in 5.3.*(d)*.(1). *Moreover, if $f \in B(R)$ is such that $P \in Z(f)$ and $f \leq e$, then diagram* (II) *above is commutative.*

c) With the Zariski topology, $\mathrm{Spec}(R)$ *is a Boolean space, with a basis of clopens,*

$$\mathcal{Z} \;=\; \{Z(e) \subseteq \mathrm{Spec}(R) : e \in B(R)\}$$

that is a Boolean algebra isomorphic to $B(R)$ by the map $e \in B(R) \;\longmapsto\; Z(e) \in \mathcal{Z}$. Moreover,

(1) *The map $\mathfrak{r}$ in (a) is a homeomorphism between* $\mathrm{Spec}(R)$ *and (maximal ideal version of) the Stone space of $B(R)$;*

(2) *For all $P \in \mathrm{Spec}(R)$, the filter $\mathcal{Z}_P \;=\; \{Z(e) \in \mathcal{Z} : P \in Z(e)\}$ of clopen neighborhoods of P is order-isomorphic to the ultrafilter $\{e \in B(R) : e \notin P\}$ in $B(R)$.*

d) If I is an ideal of R, then R/I is a vN-ring and $\mathrm{Spec}(R/I)$ *is naturally homeomorphic to the set $V(I) \;=_{def}\; \{P \in \mathrm{Spec}(R) : I \subseteq P\}$, with the topology induced by* $\mathrm{Spec}(R)$. *In particular, Re is a vN-ring, with* $\mathrm{Spec}(Re) = Z(e)$.

Proof. a) If $P \in \mathrm{Spec}(R)$ and $a \notin P$, then condition [vN] in 5.5 yields $e \in B(R)$ and $b \in R$ such that $ab = e$ and $ae = a$, whence, $e \notin P$. But then, $1 \;=\; e + (1 - e) \;=\; ab + (1 - e)$, and the ideal generated by P and a is not proper. Hence, P is maximal in R. Note that

$$\forall\, e \in B(R), \forall\, P \in \mathrm{Spec}(R), \quad e \notin P \quad \Leftrightarrow \quad e/P = 1/P \text{ in the quotient ring } R/P. \qquad (\sharp)$$

If $P \in \mathrm{Spec}(R)$, it is clear that $P \cap B(R)$ is a prime ideal in the *Boolean algebra* $B(R)$, hence a maximal ideal of $B(R)$. If $P \neq Q$ in $\mathrm{Spec}(R)$, then [vN] in 5.5 entails that there is $e \in B(R)$ such that, say, $e \in P$ and $e \notin Q$, and $\mathfrak{r}$ is injective. It is straightforward to check that if $\mathfrak{m}$ is a maximal ideal in $B(R)$ and M is the ideal generated by $\mathfrak{m}$ in R, then $M \cap B(R) = \mathfrak{m}$, establishing that $\mathfrak{r}$ is bijective.

b) Clearly, λ_{eP} and λ_P are ring morphisms, while ($\sharp$) above implies not only that φ_{eP} is a ring morphism, but also that it is the natural quotient projection, $Re \longrightarrow Re/Pe$ [13], being, therefore, surjective. That λ_{eP} is also surjective will follow from the commutativity of diagram (I), once λ_P is shown to be an isomorphism. For $x \in R$, since P is prime, we have

$$x/1 \;=\; 0/1 \quad \Leftrightarrow \quad \exists\, u \notin P \text{ such that } ux = 0 \quad \Rightarrow \quad x \in P \quad \Leftrightarrow \quad x/P = 0,$$

and λ_P is injective. To prove it is onto, note that for all $x \in R$ and $q \notin P$ there is $y \in R$ such that, in R_P, $x/q \;=\; xy/1$. Indeed, let $e \in B(R)$ and $y \in R$ be such that $yq = e$ and $qe = q$. Then, $y, e \notin P$ and, since $e(x - xyq) = ex(1 - e) = 0$, we have $x/q \;=\; xy/1$, as needed. The commutativity of the diagrams displayed in the statement is straightforward.

c) As remarked in 5.1.(b), $\mathrm{Spec}(R)$ with the Zariski topology is a T0 compact space, having $\mathcal{Z} = \{Z(a) : a \in R\}$ as a basis of compact opens. In view of 5.1.(b.3), we have

$$\mathcal{Z} \;=\; \{Z(e) : e \in B(R)\},$$

with $Z(e)$ clopen in $\mathrm{Spec}(R)$ (its complement is $Z(1-e)$). Since a T0 space with a basis of clopens is Hausdorff, $\mathrm{Spec}(R)$ is a Boolean space. For all $e, f \in B(R)$, we have, in view of the above and items (2) and (5) in 5.1, that

$$\begin{cases} Z(e)^c \;=\; Z(1-e), \;\; Z(e \wedge f) \;=\; Z(ef) \;=\; Z(e) \cap Z(f) \;\; \text{and} \\ Z(e \vee f) \;=\; Z(e + f - ef) \;=\; Z(e) \cup Z(f), \end{cases} \tag{$\mathcal{Z}$}$$

and so $\mathcal{Z}$ is a BA and the map $e \in B(R) \;\longmapsto\; Z(e) \in \mathcal{Z}$ is a BA-isomorphism. The remaining statements in (c) are now clear.

d) If I is an ideal in R, $a \in R$ and e is the idempotent associated to a (as in 5.5), then $(a/I) = (e/I)$, and R/I is a vN-ring. If $q_I : R \longrightarrow R/I$ is the canonical projection, then

$$q_{I*} : \mathrm{Spec}(R/I) \longrightarrow \mathrm{Spec}(R), \text{ given by } q_{I*}(Q) \;=\; q_I^{-1}(Q),$$

is a continuous bijection between $\mathrm{Spec}(R/I)$ and $V(I)$; it is straightforward that for $P \in V(I)$, $q_I(P) \in \mathrm{Spec}(R/I)$ and so the Fundamental Theorem of morphism of rings guarantees bijectivity, while continuity stems from the fact that for all $a \in R$, $(q_{I*})^{-1}(Z(a)) \;=\; Z(a/I)$. Since $\mathrm{Spec}(R/I)$ and $V(I)$ are Boolean spaces, any continuous bijection is a homeomorphism. The last assertion follows from the fact that $Re \approx R/(1-e)$, noted in 5.2.(e), ending the proof. $\Diamond$

With these preliminaries, we state

Proposition 5.7 *Let R be a vN-ring and let $\mathcal{Z}$ be the Boolean algebra of clopens in* $\mathrm{Spec}(R)$.

a) The assignments

$$(\mathfrak{R}) \qquad \begin{cases} Z(e) \in \mathcal{Z} & \longmapsto \quad Re \\ Z(f) \subseteq Z(e) & \longmapsto \quad \varphi_{ef} : Re \longrightarrow Rf, \end{cases}$$

constitute a presheaf basis of vN-rings over $\mathcal{Z}$, $\mathfrak{R}$, with the following properties :

(1) *$\mathfrak{R}$ is finitely complete over all $Z(e) \in \mathcal{Z}$;*

(2) *Notation as in* 5.2.(*b*) *and* 5.6.(*b*).(3)*, for each $P \in \mathrm{Spec}(R)$, let*

$$\mathcal{Z}_P \;=\; \{Z(e) \in \mathcal{Z} : P \in Z(e)\} \;\approx\; \{e \in B(R) : e \notin P\}.$$

Then, the colimit of the inductive system $\langle\, Z(e); \{\varphi_{ef} : f \leq e,\, f \notin P\}\,\rangle$, is $\langle\, R/P; \{\varphi_{eP} : e \notin P\}\,\rangle$. In other words, the stalk of $\mathfrak{R}$ at P is the field R/P, i.e.,

$$\mathfrak{R}_P \;=\; R/P \;=\; \varinjlim_{e \notin P} Re.$$

b) The completion of $\mathfrak{R}$, $c\mathfrak{R}$, is (naturally isomorphic to) the affine scheme of R. Moreover, for all $e \in B(R)$, we have $c\mathfrak{R}(Z(e)) = Re$; in particular, the ring of global sections of $c\mathfrak{R}$ is precisely R.

Proof. a) Item (d) in Lemma 5.6, together with relations (1) and (2) in 5.2.(b), show that $\mathfrak{R}$ is a contravariant functor from $\langle\, \mathcal{Z}, \subseteq^{op}\,\rangle$ to the category of vN-rings. Since each $Z(e)$ is compact clopen, the extensionality of $\mathfrak{R}$ and its finite completeness over $Z(e)$ follow immediately from items (b) and (c)

[13] Note that $Pe = P \cap Re$.

of Proposition 5.3, respectively. It remains to prove (2). By 2.3.(c), it must be shown, in view of the definition of the maps φ_{ef} and φ_{eP} in 5.6.(b), that :

$*$ $R/P \;=\; \bigcup_{e \notin P} \varphi_{eP}(Re)$, which is clear from from 5.6.(b);

$*$ For all $e \notin P$ and $xe \in Re$, $\;x/P \;=\; 0 \;\;\Rightarrow\;\; \exists\, f \leq e$ such that $f \notin P$ and $xf \;=\; 0$.

Let h be the idempotent associated to x; since $x \in P$, the same is true of h, whence, $1-h \notin P$. Set $f = e(1-h)$; then, $f \leq e$, $f \notin P$ and $xf \;=\; xe(1-h) \;=\; xhe(1-h) \;=\; 0$, as needed.

b) If R is a ring, the classical presheaf basis associated to R, whose completion is its affine scheme, is the contravariant functor from $\mathcal{Z} = \{Z(a) : a \in R\}$ to **UCR**, defined by

(i) $Z(a) \in \mathcal{Z} \;\longmapsto\; R_a$, the ring of fractions of R with respect to a;

(ii) If $Z(a) \subseteq Z(b)$, then (by 5.1.(b).(4)) $a^n = ub$, for some $n \geq 1$ and $u \in R$, whence, b is invertible in R_a, $1/b = u/a^n$. By the universal property of rings of fractions, there is a *unique* ring morphism, $\rho_{ba} : R_a \longrightarrow R_b$, given, for $r \in R$, by $\rho_{ab}(r/b^m) \;=\; ru^n/a^{nm}$, and this definition is independent of the parameters $n \geq 1$ and u. The presheaf basis so defined is complete over any $Z(a)$, $a \in R$. Now, if R is a vN-ring, then

(iii) For all $a \in R$, $Z(a) = Z(e)$, where e is the idempotent associated to a;

(iv) For $f \leq e$, the ring morphisms ρ_{ef} are precisely the γ_{ef} of Proposition 5.3.(d).(2). Indeed, in this case we have $ef = f$ and so, recalling equality ($\flat$) in the proof of (d).(1) of 5.3 (page 17), we obtain, for $r \in R$, $\;\rho_{ef}(r/e^m) \;=\; \rho_{ef}(re/1) \;=\; rf/e \;=\; ref/1 \;=\; rf/1 \;=\; \gamma_{ef}(re/1)$.

(v) For all $e \leq f$ in $B(R)$, the maps λ_e of 5.3.(d).(1) are isomorphisms, making the diagram displayed in 5.3.(d).(2) commutative.

From (i) $-$ (v), we conclude that the presheaf basis constructed in part (a) above is **isomorphic** to the classical presheaf basis associated to the affine scheme of R, and so their completions must also be isomorphic. That $c\mathfrak{R}(Z(e)) = Re$ follows immediately from item (a).(4) of Theorem 3.5. $\diamondsuit$

Proposition 5.7 shows that every vN-ring is represented as the ring of global sections of a sheaf of vN-rings over a Boolean space, whose stalks are fields, in fact, the residue fields at its maximal ideals. The converse of this statement is also true : the ring of global sections of *any* sheaf of rings over a Boolean space, whose stalks are fields, is a vN-ring. This correspondence, originally due to Pierce, can be found in [Pi]. We shall now deal with preorders in vN-rings.

5.8 Definition and Notation. Let R be a ring and let S be a subset of R.

a) Write

$*$ S^* for the set of units in S. In particular, R^* is the (multiplicative) group of units in R;

$*$ R^2 for the set of squares in R; $\qquad$ $*$ ΣR^2 for the set of sums of squares in R.

b) As usual, a **preorder** in a ring R is a set $T \subseteq R$ closed under addition and multiplication and containing R^2. T is **proper** if $T \neq R$; if $2 \in \dot{R}$, this is equivalent to $-1 \notin T$. In fact, if $-1 \in T$ and $r \in R$, then

$$r \;=\; \left(\frac{r+1}{2}\right)^2 + (-1)\left(\frac{r-1}{2}\right)^2 \;\in\; T.$$

If $P \in \mathrm{Spec}(R)$, and T is a preorder of R, let $\;T/P \;=_{def}\; \{a/P \in R/P : a \in T\}$ be the **preorder induced by T** on the quotient R/P.

c) A ring is **real** if for all $a_1, \ldots, a_n \in R$, $\;\sum_{i=1}^n a_i^2 \;=\; 0 \;\;\Rightarrow\;\; a_i = 0,\;\; 1 \leq i \leq n$.

d) A vN-ring, R, is **strongly formally real** if for all $P \in \mathrm{Spec}(R)$, $\;R/P$ is a formally real field.

e) A preorder T of a vN-ring R is **strict** if for all $P \in \mathrm{Spec}(R)$, $\;T/P$ is a *proper* preorder of the residue field R/P. $\diamondsuit$

Lemma 5.9 *Let R be a vN-ring and let T be a preorder of R.*

a) 2 is a unit in R $\;\Leftrightarrow\;$ all residue fields of R have characteristic $\neq 2$.

b) If 2 is a unit in R, then for all $f \in B(R)$, the following are equivalent :

(1) *For all* $0 \neq e \leq f$, Te *is a* proper *preorder of* Re;

(2) *For all* $P \in Z(f)$, T/P *is a proper preorder of* R/P.

c) If 2 *is a unit in* R, $e \in B(R)$ *and* $a \in R$, *the following are equivalent :*

(1) *For all* $P \in Z(e)$, $a/P \in T/P$; (2) $ae \in T$.

Proof. a) It suffices to prove ($\Leftarrow$). Let $2 \in \mathfrak{R}(Z(1)) = R$; by assumption, for each $P \in \mathrm{Spec}(R)$, 2_P is a unit in the stalk $\mathfrak{R}_P = R/P$. Hence, for each $P \in \mathrm{Spec}(R)$, there is $a_P \in \mathfrak{R}_P$ such that $2_P a_P = 1_P$. By Theorem 2.3.(b).(1) and Lemma 3.8.(a), there are $f^P \in B(R)$ and $z^P \in Rf^P = \mathfrak{R}(Z(f^P))$, such that $(z^P)_P = a_P$ and $2_{|Z(f^P)} \cdot z^P = 1_{|Z(f^P)}$. Since $\{Z(f^P) : P \in \mathrm{Spec}(R)\}$ is a clopen covering of $\mathrm{Spec}(R)$, there are $P_1, \ldots, P_n$ in $\mathrm{Spec}(R)$, such that, with $f_i = f^{P_i}$ and $z_i = z^{P_i}$, $1 \leq i \leq n$, we have

$$\bigvee_{i=1}^n f_i \;=\; 1^{14} \quad \text{and} \quad 2_{|Z(f_i)} \cdot z_i \;=\; 1_{|Z(f_i)}, \;\; 1 \leq i \leq n.$$

Let $\{e_i : 1 \leq i \leq n\}$ be an orthogonal decomposition of 1, subordinate to $\{f_i : 1 \leq i \leq n\}$, as in 5.3.(a), and set $z \;=\; \sum_{i=1}^n z_i e_i$. For each $1 \leq i \leq n$, since $e_i \leq f_i$, we get

$$\begin{aligned} 2_{|Z(e_i)} \cdot z_{|Z(e_i)} &= (2e_i) \cdot (ze_i) = (2e_i) \cdot (z_i e_i) = 2z_i e_i = (2z_i f_i) e_i = (2_{|Z(f_i)} \cdot z_i)_{|Z(e_i)} \\ &= 1_{|Z(e_i)} = e_i, \end{aligned}$$

wherefrom it follows, summing over i, that $2z = 1$ in R, as desired.

b) (1) $\Rightarrow$ (2) : Assume that (2) is false, and let $P \in Z(f)$ be so that $-1 \in T/P$, i.e., there is $t \in T$, such that $(t+1)_P = 0$. By Lemma 3.8.(a), there is $e \leq f$ such that $e \notin P$ and

$$(t+1)_{|Z(e)} \;=\; (t+1)e \;=\; te + e \;=\; 0.$$

But this means that $-e \in Te$, and so Te is not proper in Re (e is the identity of Re).

(2) $\Rightarrow$ (1) : If for some $\emptyset \neq Z(e) \subseteq Z(f)$, Te is improper, then, since $2_{|Z(e)}$ is a unit in Re (by (a)), we have $-e \in Te$, or equivalently, $(t+1)e = 0$, for some $t \in T$. If $P \in Z(e)$, then, $t + 1 \in P$, that is, $-1 \in T/P$, violating (2).

c) One should keep in mind that $B(R) \subseteq T$, since every idempotent is a square.

(1) $\Rightarrow$ (2) : By (1), for each $P \in Z(e)$, there is $t^P \in T$ such that $(t^P)/P = a/P$ holds in R/P. Hence, just as in the proof of item (a) above, compactness will lead to the existence of $f_1, \ldots, f_n \leq e$ and $t_1, \ldots, t_n \in T$, such that

$$\bigvee_{i=1}^n f_i \;=\; 1 \quad \text{and} \quad t_i f_i = a f_i, \;\; 1 \leq i \leq n.$$

Let $\{e_i : 1 \leq i \leq n\}$ be a orthogonal decomposition of 1, subordinate to $\{f_i : 1 \leq i \leq n\}$ and set $x \;=\; \sum_{i=1}^n t_i e_i$. Since t_i, $e_i \in T$, it is clear that $x \in T$. Moreover, for $1 \leq i \leq n$, we have $t_i e_i = t_i f_i e_i = a f_i e_i = a e_i$, wherefrom it follows, summing over i, that $ae = x \in T$, as needed.

(2) $\Rightarrow$ (1) : Since $ae \in T$, ($\sharp$) in page 18 and the fact that $e \notin P$ entail $ae/P = a/P \in T/P$. $\diamondsuit$

Corollary 5.10 *If* R *is a vN-ring in which* 2 *is a unit, the following are equivalent :*

(1) *For all* $e \in B(R)$, Re *is a real ring;*

(2) *For every* $P \in \mathrm{Spec}(R)$, R/P *is a formally real field.*

Proof. Just apply Lemma 5.9.(b) to the preorder $T = \Sigma R^2$. $\diamondsuit$

Lemma 5.11 *Let* R *be a vN-ring in which* 2 *is a unit and let* T *be a proper preorder of* R. *With notation as in Proposition* 5.7, *the assignments*

$$(\mathfrak{T}) \qquad \begin{cases} Z(e) \in \mathcal{Z} & \longmapsto \quad Te \\ Z(f) \subseteq Z(e) & \longmapsto \quad (\varphi_{ef})_{|Te} : Te \longrightarrow Tf, \end{cases}$$

constitute a finitely complete presheaf basis of preorders over $\mathcal{Z}$, $\mathfrak{T}$, *such that for all* $P \in \mathrm{Spec}(R)$,

$$\underrightarrow{lim}\; \langle\, Z(e);\; \{(\varphi_{ef})_{|Te} : f \leq e,\, f \notin P\}\,\rangle \;=\; \langle\, T/P;\; \{(\varphi_{eP})_{|Te} : e \notin P\}\,\rangle, \qquad (\mathfrak{T}_P)$$

that is, $\mathfrak{T}_P$ *is the preorder* T/P *of field* R/P.

[14] By 5.6.(c), this is equivalent to $\bigcup_{i=1}^n Z(f_i) = \mathrm{Spec}(R) = Z(1)$; see also the equalities ($\mathcal{Z}$) on page 19.

Proof. Clearly, $\mathfrak{T}$ is a contravariant functor from $\mathcal{Z}$ to the category of sets. The extensionality of $\mathfrak{T}$ follows immediately from that of $\mathfrak{R}$, because for all $e \in B(R)$, $\mathfrak{T}(Z(e)) = Te \subseteq \mathfrak{R}(Z(e))$. To check finite completeness, let $\{f_1, \ldots, f_n\} \subseteq B(R)$ and let $\{a_j f_j \in Tf_j : 1 \leq j \leq n\}$ be a compatible set of sections in $\mathfrak{T}$. This means that

$$\text{For all } 1 \leq i,\, j \leq n, \quad a_j f_j f_i \;=\; a_i f_i f_j. \tag{I}$$

Let $\{e_1, \ldots, e_n\}$ be a orthogonal decomposition of $f = \bigvee_{i=1}^n f_j$, subordinate to $\{f_1, \ldots, f_n\}$, as in 5.3.(a), and consider

$$z \;=\; \textstyle\sum_{i=1}^n a_i e_i.$$

Then, $z = zf \in Tf$ and for all $1 \leq j \leq n$, (I) and the fact that $e_k f_k = e_k$, yield

$$zf_j \;=\; \textstyle\sum_{i=1}^n a_i e_i f_j \;=\; \sum_{i=1}^n a_i e_i f_i f_j \;=\; \sum_{i=1}^n a_j e_i f_i f_j \;=\; a_j f_j \sum_{i=1}^n e_i \;=\; a_j f_j f \;=\; a_j f_j,$$

and so z is the gluing of $\{a_j f_j \in Tf_j : 1 \leq j \leq n\}$ in $\mathfrak{T}$. To establish $(\mathfrak{T}_P)$, we have to show :

(A) $T/P \;=\; \bigcup_{e \notin P} \varphi_{eP}(Te)$;

(B) For $e \notin P$, $\varphi_{eP}(ae) = a/P \in T/P \quad\Rightarrow\quad \exists\, f \leq e$ such that $\varphi_{ef}(ae) = af \in Tf$.

(A) is clear, since $T/P = \varphi_{1P}(T)$ and, by ($\sharp$) (page 18), for $e \notin P$, $\varphi_{eP}(re) \;=\; r/P \;=\; \varphi_{1P}(r)$, for all $r \in R$. The argument for (B) is similar to that in the proof of 5.7.(a). Given $a \in R$ such that $ae = a$, assume that for some $t \in T$, $(a - t) \in P$. Let h be the idempotent associated to $(a - t)$; then $h \in P$ and so $(1 - h) \notin P$. Take $f = e(1 - h)$; then, $e \geq f \notin P$ and $(a - t)f \;=\; (a - t)he(1 - h) \;=\; 0$, showing that $af = tf \in Tf$, as needed. $\diamondsuit$

6 The Presheaf of Special Groups of a Preordered vN-ring

Before presenting the presheaf basis of the title we shall make some general observations, that will simplify the exposition and may apply to more general situations.

Definition 6.1 *a) A* **(proper) preordered ring (p-ring)** *is a pair $\langle A, T \rangle$ such that*

[pr 1] : *A is a ring, such that $2 \in A^*$;*

[pr 2] : *T is a proper preorder of A, i.e., $-1 \notin T$* (cf. 5.8).

To avoid having to discuss trivial cases, as well as because this can effectively happen in practice, the pair $\langle A, A \rangle$ will also be consider a p-ring, the **trivial p-ring**.

b) A **morphism of p-rings**, *$f : \langle A, T \rangle \longrightarrow \langle A', T' \rangle$, is a ring morphism, $f : A \longrightarrow A'$, such that $f(T) \subseteq T'$. Let* **p-Ring** *be the category of p-rings and their morphisms.*

Remark 6.2 The language of p-rings is $L = \langle =, +, \cdot, 0, 1, -1, T \rangle$, i.e., the first-order language of unitary rings, with an additional *unary* predicate, T, interpreted as a preorder. Besides atomic formulas of the type $\tau_1 = \tau_2$, where τ_i are terms ($i = 1, 2$), we also have $\tau_1 \in T$. $\diamondsuit$

6.3 A Construction. If $\langle A, T \rangle$ is a p-ring, $T^* = T \cap A^*$ is a subgroup of the multiplicative group A^*. Indeed, if $t \in T^*$, then $1/t \;=\; t \cdot (1/t)^2 \in T$ because T is closed under products and contains A^2.

Given a p-ring $\langle A, T \rangle$, let $G_T(A) = A^*/T^*$ and $q_T : A^* \longrightarrow G_T(A)$ be the quotient group and canonical projection, respectively; to ease notation, write a^T for $q_T(a)$. Thus, for $a, b \in A^*$,

$$a^T = b^T \quad\Leftrightarrow\quad ab \in T^* \quad\Leftrightarrow\quad \exists\, t \in T^* \text{ such that } b = at \tag{*}$$

and $G_T(A) \;=\; \{a^T : a \in A^*\}$. We also abuse notation, denoting by 1 and -1 both the elements of A^*, and 1^T, $(-1)^T$, respectively. Because $A^2 \subseteq T$, $G_T(A)$ is a group of exponent 2; moreover,

$$\begin{cases} G_T(A) = \{1\} & \Leftrightarrow \quad \langle A, T \rangle \text{ is the trivial p-ring;} \\ 1 \neq -1 \text{ in } G_T(A) & \Leftrightarrow \quad \langle A, T \rangle \text{ is a proper p-ring.} \end{cases} \tag{pp}$$

For $x, y \in A^*$, define

$$D_T(x, y) \;=\; \{z \in A^* : \exists\, t_1, t_2 \in T \text{ such that } z \;=\; t_1 x + t_2 y\}, \tag{D_T}$$

called the **set of elements represented by x and y** in A^*. Since $0,\ 1 \in T$, it is clear that $\{x,\ y\} \subseteq D_T(x,\ y)$. The basic properties of these sets are contained in the following Fact; the proofs of Lemma 1.30 and Proposition 1.31 of [DM2] (pp. 22-23), done for fields of characteristic $\neq 2$, transfer straightforwardly to the case of p-rings.

Fact 6.4 *With notation as above, let $x,\ y,\ u,\ v \in A^*$ and $t \in T^*$.*

a) $uD_T(x,y) \ = \ D_T(ux,uy)$ *and* $D_T(x,y) \ = \ D_T(tx,ty)$.

b) $u \in D_T(x,y)$ *and* $u^T = v^T \ \Rightarrow \ v \in D_T(x,y)$.

c) $x^T = u^T$ *and* $y^T = v^T \ \Rightarrow \ D_T(x,y) \ = \ D_T(u,v)$.

d) $D_T(1,x)$ *is a subgroup of* A^*.

e) $x \in D_T(1,y) \ \Rightarrow \ D_T(x,xy) \ = \ xD_T(1,y) \ = \ D_T(1,y)$.

f) $u \in D_T(x,y) \ \Leftrightarrow \ D_T(u,\ uxy) \ = \ D_T(x,y)$.

g) The following are equivalent :

(1) $(xy)^T = (uv)^T$ *and* $D_T(x,y) \ = \ D_T(u,v)$;

(2) $(xy)^T = (uv)^T$ *and* $D_T(x,y) \cap D_T(u,v) \ \neq \ \emptyset$. $\diamondsuit$

Since the representation sets, $D_T(x,\ y)$, are invariant (or saturated) with respect to the equivalence generated by the subgroup T^* of A^* (6.4.(b), (c)), they can be seen in $G_T(A)$, that is,

$$D_T(x^T,\ y^T) \ = \ D_T(x,\ y)/T^* \ = \ \{z^T \in G_T(A) : \exists\ t_1,\ t_2 \in T \text{ such that } z = t_1x + t_2y\},$$

with $q_T^{-1}(D_T(x^T,\ y^T)) \ = \ D_T(x,\ y)$. Hence, for $x,\ y,\ u,\ v \in A^*$

$$\begin{cases} u \in D_T(x,\ y) & \Leftrightarrow \quad u^T \in D_T(x^T,\ y^T); \\ D_T(u,\ v) = D_T(x,\ y) & \Leftrightarrow \quad D_T(u^T,\ v^T) \ = \ D_T(x^T,\ y^T). \end{cases} \tag{rep}$$

It is important to observe that $D_T(1,\ x^T)$ is a *subgroup* of $G_T(A)$.

Define a binary relation, $\equiv_T$, called **binary isometry mod $\boldsymbol{T}$** on $G_T(A) \times G_T(A)$, as follows : for $a,\ b,\ c,\ d \in A^*$

$$\langle\, a^T, b^T\,\rangle \equiv_T \langle\, c^T, d^T\,\rangle \quad \Leftrightarrow \quad a^Tb^T \ = \ c^Td^T \ \text{ and } \ D_T(a,\ b) \ = \ D_T(c,\ d). \tag{$\equiv_T$}$$

Fact 6.4 yields

Fact 6.5 (cf. [DM2], Definition 1.2, p.2) *a) The relation $\equiv_T$ satisfies the following properties, for all $a,\ b,\ c,\ d,\ x \in A^*$:*

[SG 0] : $\equiv_T$ *is an equivalence relation on* $G_T(A) \times G_T(A)$.

[SG 1] : $\langle\, a^T, b^T\,\rangle \equiv_T \langle\, b^T, a^T\,\rangle$; [SG 2] : $\langle\, a^T,\ -a^T\,\rangle \equiv_T \langle\, 1, -1\,\rangle$;

[SG 3] : $\langle\, a^T, b^T\,\rangle \equiv_T \langle\, c^T, d^T\,\rangle \ \Rightarrow \ a^Tb^T \ = \ c^Td^T$;

[SG 5] : $\langle\, a^T, b^T\,\rangle \equiv_T \langle\, c^T, d^T\,\rangle \ \Rightarrow \ \langle\, x^Ta^T, x^Tb^T\,\rangle \equiv_T \langle\, x^Tc^T, x^Td^T\,\rangle$.

b) (Reducibility) $\langle\, a^T, a^T\,\rangle \equiv_T \langle\, 1, 1\,\rangle \ \Leftrightarrow \ a^T \ = \ 1 \ \Leftrightarrow \ a \in T^*$.

Proof. We comment only on [SG 2] and (b). For [SG 2], since $2 \in A^*$, it was noted in 5.8.(b) that any element in A is a difference of two squares. Hence, if $a \in A^*$, we have $a \in D_T(a,\ -a) \cap D_T(1,\ -1)$. Since $a^T(-a)^T = \ (-1)^T$, 6.4.(g) guarantees that $\langle\, a^T,\ -a^T\,\rangle \equiv_T \langle\, 1, -1\,\rangle$.

b) Since $(a^T)^2 = 1$, the isometry in the antecedent is equivalent to $D_T(a,\ a) = D_T(1,\ 1)$; in particular, $a \in D_T(1,\ 1)$, which is clearly equivalent to $a \in T^*$. $\diamondsuit$

Remark 6.6 Under the very general conditions in 6.3, axiom [SG 4] in Definition 1.2 of [DM2] may fail. The point is that all known proofs of this axiom resort to an analogue, for preorders, of the transversality condition ($\natural$), stated at the beginning of the proof of Lemma 4.11 (p. 15) : if $\langle\, A, T\,\rangle$ is a p-ring and $u,\ v,\ w \in A^*$

[T] $\qquad w \in D_T(u,\ v) \ \Rightarrow \ \exists\ p,\ q \in T^*$ so that $w = up + vq$.

A large class of rings with many units satisfy [T], which follows, in fact, from a more general transversality principle (see, [Wa], Propositions 3.6.1, p. 25, and 4.1.8, pp. 32-33). $\diamondsuit$

The construction above suggests the following

Definition 6.7 *a) A* **proto special group (π-SG)***, is a triple,* $G = \langle\, G, \equiv_G, -1\,\rangle$, *consisting of*

$*$ *A group, G, of exponent two, written multiplicatively (and so its identity is 1);*

$*$ *A distinguished element,* -1 *, in G; (we write* $-x$ *for* $-1 \cdot x$, $\forall\, x \in G$*);*

$*$ *A binary relation* $\equiv_G$ *on* $G \times G$*, satisfying the axioms* [SG 0] $-$ [SG 3] *and* [SG 5] *in* 6.5.(a).

G is **reduced (π-RSG)** *if* $1 \neq -1$ *and it satisfies the first equivalence in* 6.5.(b).

For $a, b, c \in G$*, write* $c\langle\, a, b\,\rangle$ *for* $\langle\, ca, cb\,\rangle$*. The product ab is the* **discriminant** *of* $\langle\, a, b\,\rangle$.

If $G = \langle\, G, \equiv_G, -1\,\rangle$ *is a π-SG and* $x, y \in G$*, define*

$$D_G(x, y) \;=\; \{z \in G : \langle\, z,\, zxy\,\rangle \equiv_G \langle\, x, y\,\rangle\},$$

the set of elements **represented by x and y** *in G. Since G has exponent two* ($x^2 = 1$, $\forall\, x$)

(i) *By* [SG 3], $\langle\, z, u\,\rangle \equiv_G \langle\, x, y\,\rangle$ *entails* $u = zxy$*;*

(ii) [SG 0] *implies* $\langle\, x, y\,\rangle \equiv_G \langle\, x, y\,\rangle$ *and so* $\{x, y\} \subseteq D_G(x, y)$*;*

(iii) *For* $x \in G$, $\;D_G(1, x) = \{z \in G : z\langle\, 1, x\,\rangle \equiv_G \langle\, 1, x\,\rangle\}$.

b) If $G_i = \langle\, G_i, \equiv_{G_i}, -1\,\rangle$ *are π-SGs,* $i = 1, 2$*, a* **morphism of π-SGs***,* $h : G_1 \longrightarrow G_2$*, is a morphism of the underlying groups, such that* $h(-1) = -1$ *and*

$$\forall\, a, b, c, d \in G_1, \;\; \langle\, a, b\,\rangle \equiv_{G_1} \langle\, c, d\,\rangle \quad \Rightarrow \quad \langle\, h(a), h(b)\,\rangle \equiv_G \langle\, h(c), h(d)\,\rangle.$$

Write **π-SG** *and* **π-RSG** *for the categories of π-SGs and π-RSGs, respectively.*

Lemma 6.8 *If* $G = \langle\, G, \equiv_G, -1\,\rangle$ *is a π-SG and* $a, b, c, d \in G$*, then*

a) $D_G(1, a)$ *is a* subgroup *of G.*

b) $\langle\, a, b\,\rangle \equiv_G \langle\, c, d\,\rangle \;\;\Leftrightarrow\;\; ab = cd$ *and* $ac \in D_G(1, cd)$.

c) If $H = \langle\, H, \equiv_H, -1\,\rangle$ *is a π-SG and* $G \xrightarrow{h} H$ *is a group morphism, such that* $h(-1) = -1$*, then h is a π-SG morphism iff for all* $a, b \in G$, $\;a \in D_G(1, b) \;\Rightarrow\; f(a) \in D_H(1, h(b))$.

d) If $\langle\, A, T\,\rangle$ *is p-ring, then* $G_T(A)$ *is a π-SG, which is reduced iff* $\langle\, A, T\,\rangle$ *is a non-trivial p-ring. Moreover, for all* $a, b, c, d \in A^*$

$$\langle\, a^T, b^T\,\rangle \equiv_T \langle\, c^T, d^T\,\rangle \quad \Leftrightarrow \quad a^T b^T \;=\; c^T d^T \;\; \text{and} \;\; ac \in D_T(1, cd).$$

Proof. Item (a) is straightforward. The proof of Lemma 1.5.(a) of [DM2] (p. 3) uses only [SG 3] and [SG 5] and yields (b). Item (c) is an immediate consequence of (b) and the definition of morphism in 6.7.(b), while (d) follows from Fact 6.5.(b), item (b) and the relations [pp] (page 22) and [rep] in 6.3 (page 23). $\diamondsuit$

Definition 6.9 *If* $\langle\, A, T\,\rangle$ *is a p-ring,* $G_T(A) = \langle\, G_T(A), \equiv_T, -1\,\rangle$ *is the* **π-SG associated to $\langle\, A, T\,\rangle$***. Note that*

$*$ *If* $\langle\, A, T\,\rangle$ *is non-trivial, then* $G_T(A)$ *is a π-RSG;*

$*$ *If* $\langle\, A, T\,\rangle$ *is trivial, then* $G_T(A)$ *is the trivial* **special group**, $\{1\}$.

In the case that $T = \Sigma A^2$*, write* $G_{red}(A)$ *for* $G_T(A)$.

Lemma 6.10 *A p-ring morphism,* $h : \langle\, A_1, T_1\,\rangle \longrightarrow \langle\, A_2, T_2\,\rangle$*, induces a morphism of π-SGs,*

$(*)$ $\qquad h^\pi : G_{T_1}(A_1) \longrightarrow G_{T_2}(A_2)$*, given by* $\;h^\pi(a^{T_1}) \;=\; h(a)^{T_2}$.

Furthermore, $Id^\pi_{A_1} = Id_{G_{T_1}(A_1)}$ *and if* $g : \langle\, A_2, T_2\,\rangle \longrightarrow \langle\, A_3, T_3\,\rangle$ *is a morphism of p-rings, then* $(g \circ h)^\pi = g^\pi \circ h^\pi$.

Proof. Since h is a p-ring morphism, $h^* = h_{|A_1^*} : A_1^* \longrightarrow A_2^*$ is a group morphism, with $h^*(-1) = -1$. In particular, $h^*(T_1^*) \subseteq T_2^*$. Hence h^* induces a group morphism given by $(*)$, such that $h^\pi(-1) = -1$. By 6.8.(c), h^π will be π-SG morphism if for $a, b \in A_1^*$,

$$a^{T_1} \in D_{T_1}(1, b^{T_1}) \quad \Rightarrow \quad h^\pi(a^{T_1}) = h(a)^{T_2} \in D_{T_2}(1, \ h(b^{T_1})) = D_{T_2}(1, \ h(b)^{T_2}). \tag{I}$$

The antecedent in (I) means that there are $t_1, t_2 \in T_1$ such that $a = t_1 + t_2 b$; thus,

$$h(a) \ = \ h(t_1) + h(t_1)h(b). \tag{II}$$

Since $h(T_1) \subseteq T_2$, (II) implies $h(a) \in D_{T_2}(1, h(b))$, which by the relations [rep] in 6.3 (page 23) is equivalent to the consequent in (I). The preservation of identity and composition is clear. $\diamondsuit$

Proposition 6.11 *The π-SG functor from* **p-Ring** *to* **π-SG**, *given by,*

$$\begin{cases} \langle A, T \rangle & \longmapsto & G_T(A) \\ \langle A_1, T_1 \rangle \xrightarrow{h} \langle A_2, T_2 \rangle & \longmapsto & G_{T_1}(A_1) \xrightarrow{h^\pi} G_{T_2}(A_2) \end{cases}$$

is a geometrical functor.

Proof. Regarding products, it is enough to check that the π-SG functor preserves binary products. If $\langle A_i, T_i \rangle$, $i = 1, 2$ are p-rings, then their product is the p-ring $\langle A, T \rangle = \langle A_1 \times A_2, \ T_1 \times T_2 \rangle$; note that $\langle A, T \rangle$ is trivial iff both components are trivial. Clearly, $p_i : \langle A, T \rangle \longrightarrow \langle A_i, T_i \rangle$, the canonical coordinate projections, are p-ring morphisms. Moreover, we have $A^* = A_1^* \times A_2^*$, $T^* = T_1^* \times T_2^*$ and

$$\langle x, y \rangle \in D_T(\langle 1, 1 \rangle, \ \langle u, v \rangle) \quad \text{iff} \quad x \in D_{T_1}(1, u) \ \text{ and } \ y \in D_{T_2}(1, v).$$

It is then straightforward to check that $G_T(A) \ = \ G_{T_1}(A_1) \times G_{T_2}(A_2)$, as well as that the projections are precisely p_i^π, $i = 1, 2$, as needed. It remains to check that the π-SG functor preserves right-directed colimits. This is the content of the following

Fact 6.12 *Let $\langle I, \leq \rangle$ be a rd-poset and let $\mathcal{A} \ = \ \langle \langle A_i, T_i \rangle; \ h_{ij} : i \leq j \text{ in } I \} \rangle$ be an inductive system of p-rings and p-ring morphisms. Let $\mathcal{G} \ = \ \langle G_{T_i}(A_i); \ \{h_{ij}^\pi : i \leq j \text{ in } I\} \rangle$ be the associated inductive system of π-SGs.*

a) Let $\langle A; \ \{h_i : i \in I\} \rangle \ = \ \varinjlim A_i$ in the category of rings and set $T \ = \ \bigcup_{i \in I} h_i(T_i)$. Then, $\langle A, T \rangle$ is a p-ring, $h_i : \langle A_i, T_i \rangle \longrightarrow \langle A, T \rangle$ is a morphism of p-rings and

$$\langle \langle A, T \rangle; \ \{h_i : i \in I\} \rangle \ = \ \varinjlim \mathcal{A} \quad \textit{in the category of p-rings.}$$

Moreover, $\langle A, T \rangle$ is a trivial p-ring iff $\mathfrak{t} = \{i \in I : \langle A_i, T_i \rangle$ is a trivial p-ring$\}$ is cofinal in I
iff $\{i \in I : \langle A_i, T_i \rangle$ is a proper p-ring$\}$ is not cofinal in I.

b) $\langle G_T(A); \ \{h_i^\pi : i \in I\} \rangle \ = \ \varinjlim \mathcal{G}$.

Proof. Since $A = \varinjlim_{i \in I} A_i$ in the category of rings, by 2.3.(c) we know that

(1) $A \ = \ \bigcup_{i \in I} h_i(A_i)$;

(2*) $\forall \ i \in I$ and $x \in A_i$, $\ h_i(x) = 0 \quad \Rightarrow \quad \exists \ k \geq i$ such that $h_{ik}(x) = 0$.

We first verify that T is a preorder of A. If $x, y \in T$, there are $i, j \in I$, together with $u \in T_i$ and $v \in T_j$ such that $h_i(u) = x$ and $h_j(v) = y$. Select $q \geq i, j$, and consider $w_x = h_{iq}(u)$ and $w_y = h_{jq}(v)$, both in T_q (the h_{ij} are p-ring morphisms). Then, $h_q(w_x) = x$ and $h_q(w_y) = y$, $w_x \dotplus w_y \in T_q$ and $x \dotplus y \ = \ h_q(w_x \dotplus w_y) \ \in \ T$, showing that $T \dotplus T \subseteq T$. Similarly, one verifies that $A^2 \subseteq T$, and that $-1 \in T \ \Leftrightarrow \ \mathfrak{t} = \{i \in I : \langle A_i, T_i \rangle$ is the trivial p-ring$\}$ is cofinal in I. Hence, if $\mathfrak{t}$ is cofinal in I, Theorem 2.3.(a) guarantees that items (a) and (b) in the statement hold true [15]. If $\mathfrak{t}$ *is not cofinal* in I, then the fact that $\langle I, \leq \rangle$ is rd immediately implies that its complement is cofinal in I. Thus, by 2.3.(a), we may, from now on, assume that for all $i \in I$, $\langle A_i, T_i \rangle$ is a *proper* p-ring, which entails that $\langle A, T \rangle$ is also a *proper* p-ring. The very definition of T guarantees that h_i is a p-ring morphism and that (1) above holds for T. By 2.3.(b), to finish the proof that $\langle A, T \rangle \ = \ \varinjlim_{i \in I} \langle A_i, T_i \rangle$ it suffices to check that

[15] For all i, j, h_{ij}^π and h_i^π are the only possible map from $\{1\}$ to $\{1\}$.

$$\forall\, i \in I, \forall\, x \in A_i, \quad h_i(x) \in T \quad \Rightarrow \quad \exists\, k \geq i \text{ such that } h_{ik}(x) \in T_k, \tag{I}$$

corresponding to 2.3.(b).(2) for (the predicate) T. If $h_i(x) \in T$, then there is $j \in I$ and $y \in T_j$ such that $h_j(y) = h_i(x)$. Select $q \geq i, j$ and consider $w_x = h_{iq}(x) \in A_q$ and $w_y = h_{jq}(y) \in T_q$. Note that, $h_q(w_x) = h_q(h_{iq}(x)) = h_i(x) = h_j(y) = h_q(h_{jq}(y)) = h_q(w_y) \in T_q$ and so (2*) above guarantees that there is $k \geq q$ such that $h_{qk}(w_x) = h_{qk}(w_y) \in T_q$. But then

$$h_{ik}(x) = h_{qk}(h_{iq}(x)) = h_{qk}(w_x) = h_{qk}(w_y) \in T_k,$$

as needed to establish (I) and to complete the proof of (a).

b) Recall our working hypothesis that all $\langle A_i, T_i \rangle$ are proper p-rings. To ease notation write

$*$ $G_i = \langle G_i, \equiv_i, -1 \rangle$ for the π-RSGs $G_{T_i}(A_i) = \langle G_{T_i}(A_i), \equiv_{T_i}, -1 \rangle$;

$*$ $G = \langle G, \equiv_T, -1 \rangle$ for $G_T(A) = \langle G_T(A), \equiv_T, -1 \rangle$.

$*$ The elements of G_i and G will still be denoted by a^{T_i} and a^T, respectively.

By Lemma 6.10, $\mathcal{G} = \langle G_i; \{h^\pi_{ij} : i \leq j \text{ in } I\} \rangle$ is an inductive system of π-RSGs, $h^\pi_i : G_i \longrightarrow G$ is a π-SG morphism and the following diagram is commutative, for $i \leq j$:

$$\begin{tikzcd}
G_i \arrow[rr, "h^\pi_{ij}"] \arrow[dr, "h^\pi_i"'] & & G_j \arrow[dl, "h^\pi_j"] \\
& G &
\end{tikzcd}$$

that is, $\mathfrak{G} = \langle G; \{h^\pi_i : i \in I\} \rangle$ is a dual cone over $\mathcal{G}$. By items (b) and (c) in Theorem 2.3, to show that $\mathfrak{G} = \underrightarrow{lim}\ \mathcal{G}$ we must verify the following conditions :

(A) $G = \bigcup_{i \in I} h^\pi_i(G_i)$;

(B) For all $i \in I$ and $x, y, u, v \in A^*_i$,

(B1) $h^\pi_i(x^{T_i}) = 1 \quad \Rightarrow \quad \exists\, k \geq i$ such that $h^\pi_{ik}(x^{T_i}) = 1$;

and by Lemma 6.8.(d),

$$\text{(B2)} \quad \left\{ \begin{array}{c} h_i(xy)^T = h^\pi_i((xy)^{T_i}) = h^\pi_i((uv)^{T_i}) = h_i(uv)^T \quad \text{and} \quad h_i(xu) \in D_T(1, h_i(uv)) \\ \Downarrow \\ \exists\, k \geq i \text{ such that } h_{ik}(xyuv) \in T_k \quad \text{and} \quad h_{ik}(xu) \in D_{T_k}(1, h_{ik}(uv)). \end{array} \right.$$

To establish (A) it suffices to verify that $A^* = \bigcup_{i \in I} h_i(A^*_i)$; once this is shown, we get $T^* = T \cap A^* = \bigcup_{i \in I} h_i(T^*_i)$, and so, $G = A^*/T^* = \bigcup_{i \in I} h^\pi_i(A^*_i/T^*_i)$. Since any ring morphism preserves units, it is enough to check that $A^* \subseteq \bigcup_{i \in I} h_i(A^*_i)$. Suppose $x \in A^*$, i.e., there is $y \in A$ such that $xy = 1$. By (1) (at the beginning of the proof), there are $i, j \in I$ and $a \in A_i$, $b \in A_j$ such that $h_i(a) = x$ and $h_j(b) = y$. Select $q \geq i, j$ and set $c = h_{iq}(a)$, $d = h_{jq}(b)$. Then, $h_q(c) = x$, $h_q(d) = y$ and we have $h_q(cd) = xy = 1 = h_q(1)$. Item (2*) (at the beginning of the proof) applied to $cd - 1$ yields $k \geq q$ such that $h_{qk}(cd) = h_{qk}(1) = 1$, that is, $h_{qk}(c) \in A^*_k$. Since, $h_k(c) = x$, the needed inclusion is proven.

The implication (B1) is immediate from (I), because for all $a \in A^*_i$, $h^\pi_i(a^{T_i}) = 1$ iff $h_i(a) \in T$. Note that we have just shown that $G = \underrightarrow{lim}_{i \in I}\ G_i$ in the category of groups. It remains to verify (B2); its antecedent means

$$h_i(xyuv) \in T \quad \text{and} \quad \exists\, t_1, t_2 \in T \text{ such that } h_i(xu) = t_1 + t_2 h_i(uv).$$

Since $T = \bigcup_{j \in I} h_j(T_j)$ and I is right-directed, a standard argument yields $k \geq i$ and representatives b_l of t_l $(l = 1, 2)$ and a of $h_i(xyuv)$ **in $\boldsymbol{T_k}$** (i.e., $h_l(b_l) = t_l$) so that $h_{ik}(xu) = b_1 + b_2 h_{ik}(uv)$. Hence, $a = h_{ik}(xyuv) \in T_k$ and $h_{ik}(xu) \in D_{T_k}(1, h_{ik}(uv))$, as required. $\diamondsuit$

We now discuss presheaf bases of p-rings over Boolean spaces and the presheaf bases of π-SGs that arise from them. We begin with the following

Remark 6.13 Let $\mathcal{B}$ be a basis for the topological space X and let $\mathfrak{P} : \mathcal{B} \longrightarrow$ **p-Ring**,

$$\begin{cases} U \in \mathcal{B} & \longmapsto & \langle \mathfrak{P}(U),\ T(U) \rangle; \\ U \subseteq_o V & \longmapsto & p_{VU} : \mathfrak{P}(U) \longrightarrow \mathfrak{P}(V), \end{cases}$$

be presheaf basis of p-rings over $\mathcal{B}$. The extensionality condition (2) in Definition 3.2 applies also to the predicate T, that is interpreted as the preorder on each ring of sections. Since the restriction maps are p-ring morphisms, the assignments

$$\begin{cases} U \in \mathcal{B} & \longmapsto & T(U); \\ U \subseteq_o V & \longmapsto & (p_{VU})_{|T(V)} : T(V) \longrightarrow T(U), \end{cases}$$

constitute a presheaf basis of preorders, $\mathfrak{T}$. Hence :

(1) Every presheaf basis of p-rings, $\mathfrak{P}$, comes equipped with a presheaf basis of preorders, $\mathfrak{T}$;

(2) The language of presheaves applies to $\mathfrak{T}$. For instance, for $U \in \mathcal{B}$, we may require that $\mathfrak{T}$ be **finitely complete over $\boldsymbol{U}$**, defined in 3.2.(c).(1). Note that this *does not* imply that $\mathfrak{P}$ is finitely complete over U, since a finite set of compatible sections in $|\mathfrak{P}|$, outside $|\mathfrak{T}|$, may not have a gluing in $\mathfrak{P}$. $\diamondsuit$

Theorem 6.14 *Let $\mathcal{B}$ be the BA of clopens of the Boolean space X. With notation as in 6.13, let $\mathfrak{P} : \mathcal{B} \longrightarrow$ **p-Ring** be presheaf basis of p-rings over $\mathcal{B}$, with associated presheaf of preorders, $\mathfrak{T}$, both of which are assumed to be finitely complete over all $U \in \mathcal{B}$. Let $\mathfrak{G} : \mathcal{B} \longrightarrow$ **$\boldsymbol{\pi}$-SG** be the composition of $\mathfrak{P}$ with the π-SG functor, i.e.,*

$$\begin{cases} U \in \mathcal{B} & \longmapsto & \mathfrak{G}(U) & = & G_{T(U)}(\mathfrak{P}(U)); \\ U \subseteq_o V & \longmapsto & p^{\pi}_{VU} : \mathfrak{G}(V) \longrightarrow \mathfrak{G}(U). \end{cases}$$

Then,

a) $\mathfrak{G}$ is a finitely complete presheaf basis of π-SGs over $\mathcal{B}$. For $x \in X$, let $\mathcal{B}_x = \{U \in \mathcal{B} : x \in U\}$ be the filter of clopen neighborhoods of x in X. If $\mathfrak{P}_x = \langle \langle \mathfrak{P}_x, T_x \rangle;\ \{p_{Ux} : U \in \mathcal{B}_x\} \rangle$ is the stalk of $\mathfrak{P}$ at x, then $\mathfrak{G}_x = \langle G_{T_x}(\mathfrak{P}_x);\ \{p^{\pi}_{Ux} : U \in \mathcal{B}_x\} \rangle$ is the stalk of $\mathfrak{G}$ at x.

b) The set $\tau_{prop} = \{x \in X : T_x$ is a proper preorder in $\mathfrak{P}_x\}$ is closed in X. Moreover,

(1) For all $U \in \mathcal{B}$, $U \cap \tau_{prop} \neq \emptyset \Leftrightarrow \mathfrak{G}(U)$ is a proper *p-ring. In particular, if $T(X)$ is a proper preorder in $\mathfrak{P}(X)$, then $\tau_{prop} \neq \emptyset$;*

(2) For all $x \in \tau_{prop}$, $\mathfrak{G}_x$ is a π-RSG.

Proof. Since the theories of p-rings and of π-SGs are geometrical and the π-SG functor from **p-Ring** to **$\boldsymbol{\pi}$-SG** is geometrical (Proposition 6.11), all statements in (a) are immediate consequences of Theorem 3.11. As for (b), with notation as in Definition 3.9, observe that 5.8.(b) implies that τ^c_{prop} is the Feferman-Vaught value of the atomic sentence $-1 \in T$ (Proposition 3.10.(a)) :

$$\tau^c_{prop} = \mathfrak{v}_{\mathfrak{P}}(-1 \in T) = \bigcup \{U \in \mathcal{B} : \mathfrak{P}(U) \models -1 \in T(U)\},$$

which guarantees that τ^c_{prop} is open and implies (1). If $x \in \tau_{prop}$, (1) entails that for all $U \in \mathcal{B}_x$, $T(U)$ is a *proper* preorder of the ring $\mathfrak{P}(U)$ and (2) follows from the equivalences in item (a) of Fact 6.12, completing the proof. $\diamondsuit$

Being the ring of global sections of a sheaf of rings whose stalks are fields, Theorem 2.10 in [DM5] guarantees that any vN-ring, R, is a ring with many units and so, by Proposition 5.3.(e), for all $0 \neq e \in B(R)$, the ring Re is also a ring with many units.

By Theorems 3.15 and 3.16 of [DM5], if A is a ring with many units where $2 \in A^*$ and all residue fields of A have more than 7 elements, then if T is a proper preorder of A, the π-SG associated to $\langle A, T \rangle$, $G_T(A)$, is, in fact, a *reduced special group*, that faithfully represents the reduced theory, modulo T, of quadratic forms over free A-modules, with coefficients in A^*. If R is a vN-ring in which 2 is a unit and T is a *strict* preorder of R, then for all $P \in \mathrm{Spec}(R)$, T/P is a proper preorder of the residue field R/P, and so all residue fields of R are formally real. Hence, the results in [DM5] apply, yielding, in particular, that, **$\boldsymbol{G_T(R)}$ is a reduced special group whenever $\boldsymbol{T}$ is a strict preorder of $\boldsymbol{R}$.** Proposition

6.15 below, one of main reduction steps in our argument, will show that, in fact, if T is any **proper** preorder of a vN-ring R in which 2 is a unit, then $G_T(R)$ is a reduced special group.

Henceforth in this section, **fix a proper preordered vN-ring, $\langle\, \boldsymbol{R}, \boldsymbol{T}\,\rangle$, where $\boldsymbol{2} \in \boldsymbol{R}^*$.** Note that item (1) in Theorem 6.14.(b), together with relation $(\mathfrak{T}_P)$ in Lemma 5.11, guarantee that

$$\tau_{prop} \;=\; \{P \in \mathrm{Spec}(R) : T/P \text{ is a proper preorder in } R/P\}$$

is a **non-empty closed set** in $\mathrm{Spec}(R)$. Define $I = \bigcap \tau_{prop}$; clearly, I is an ideal in R. Let $q_I : R \longrightarrow R/I$ be the canonical quotient morphism. Clearly, 2 is a unit in the vN-ring R/I (5.6.(d)). We now have

Proposition 6.15 *With notation as above,*

a) For all $P \in \mathrm{Spec}(R)$, $\; I \subseteq P \;\Leftrightarrow\; P \in \tau_{prop}$. Moreover, if τ_{prop} is endowed with the topology induced by $\mathrm{Spec}(R)$, then, $Q \in \mathrm{Spec}(R/I) \;\longmapsto q_I^{-1}(Q) \in \tau_{prop}$ is a homeomorphism.

b) T/I is a **strict** *preorder on R/I.*

c) For $a \in R$, the following are equivalent :

(1) $a/I \in T/I$; (2) *There is $x \in T$ such that for all $P \in \mathrm{Spec}(R)$, $a/P = x/P$;*

(3) $a \in T$.

d) $q_I^{\pi} : G_T(R) \longrightarrow G_{T/I}(R/I)$ is an **isomorphism** *of reduced special groups.*

Proof. a) For the first assertion, it suffices to verify $(\Rightarrow)$. Suppose $e \in B(R)$ is such that $e \notin P$; hence, $e \notin I$, and its definition yields $Q \in \tau_{prop}$ such that $e \notin Q$. Hence, every clopen neighborhood of P has non-empty intersection with τ_{prop}; since it is closed, we get $P \in \tau_{prop}$, as needed. The equivalence just proven shows, with notation as in 5.6.(d), that $V(I) = \tau_{prop}$; the remaining assertion follows from that same result.

b) Clearly, T/I is a preorder of R/I; since $\tau_{prop} \neq \emptyset$, T/I is a proper preorder of R/I [16]. By (a), we may identify $\mathrm{Spec}(R/I)$ with τ_{prop} ; if $P \in \tau_{prop}$, then

$$(R/I)/(P/I) \;=\; R/P \quad\text{and}\quad (T/I)/(P/I) \;=\; T/P.$$

Since T/P is a proper of preorder of R/P, the contention is established.

c) $\underline{(1) \Rightarrow (2)}$: If $a/I \in T/I$, there is $t \in T$ such that $a/I = t/I$ and so, $a - t \in P$, for all $P \in \tau_{prop}$. Let e be the idempotent associated to $a - t$. Then,

(i) From $e(a - t) = a - t$, it follows that $(a - t)(1 - e) = 0$, i.e., $a(1 - e) = t(1 - e)$.

(ii) For all $P \in \tau_{prop}$, $e \in P$, that is, $Z(e) \cap \tau_{prop} = \emptyset$. If $Q \in Z(e)$, then $T/Q = R/Q$, whence $a/Q \in T/Q$. Since this holds for all $Q \in Z(e)$, Lemma 5.9.(c) guarantees that $ae \in T$.

Set $x = t(1 - e) + ae$; because t, $(1 - e)$, $ae \in T$, we get $x \in T$. Now, for $P \in \mathrm{Spec}(R)$:

$*$ If $P \in Z(e)$, i.e., $e \notin P$, then $1 - e \in P$ and so, recalling $(\sharp)$ (page 18),

$$x/P \;=\; t(1-e)/P + (ae)/P \;=\; (ae)/P \;=\; a/P;$$

$*$ If $P \in Z(1 - e)$, then $1 - e \notin P$ and $e \in P$, whence, in view of (i) and $(\sharp)$,

$$x/P \;=\; t(1-e)/P + (ae)/P \;=\; t(1-e)/P \;=\; a(1-e)/P \;=\; a/P,$$

as required. For (2) $\Rightarrow$ (3), just observe that (2) implies that the Feferman-Vaught value of the atomic formula $(v_1 = v_2)$ at the pair $\langle a, x\rangle$ of global sections is $\mathrm{Spec}(R)$. By Proposition 3.10.(b), this implies $a = x \in T$. That (3) implies (1) is obvious.

d) Since $q_I : \langle R, T\rangle \longrightarrow \langle R/I, T/I\rangle$ is a morphism p-rings, Lemma 6.10 guarantees that q_I^{π} is a morphism of π-SGs; since it is clearly surjective, it will be an isomorphism iff it reflects representation, that is, for a, $b \in R^*$,

$$(a/I)^{T/I} \;\in\; D_I(1,\;(b/I)^{T/I}) \;\;\Rightarrow\;\; a^T \in D_T(1, b^T), \tag{I}$$

where D_I denotes representation in $G_{T/I}(R/I)$. Because the π-groups in question are *reduced* (6.5.(b)), (I) implies that q_I^{π} is injective. The antecedent means that $a/I = (x + yb)/I$, for some x, $y \in T$;

[16] If for $t \in T$, $t + 1 \in I \subseteq P \in \tau_{prop}$, then T/P is not proper in R/P.

consequently, $a - (x + yb) \in T/I$ and item (c) entails $a - (x + yb) \in T$. Setting $t = a - (x + yb)$, we have $a = (x + t) + yb$, with $(x + t), y \in T$, establishing (I). As observed in the paragraphs preceding the statement of this Proposition, since T/I is a *strict* preorder on R/I, $G_{T/I}(R/I)$ is, in fact, a reduced special group, and so the same must be true of $G_T(R)$, ending the proof. $\diamondsuit$

Summarizing, we can state

Corollary 6.16 *Let R be a vN-ring where 2 is a unit and let T be a proper preorder of R. With notation as in* 5.7, 5.11, 6.14 *and* 6.15, *let $\langle \mathfrak{R}, \mathfrak{T} \rangle$ be the presheaf basis of p-rings over $\mathcal{Z}$, associated to $\langle R, T \rangle$. Then,*

a) $\mathfrak{G} = \mathfrak{G}_{\mathfrak{T}}(\mathfrak{R})$ is a finitely complete presheaf basis of **special groups**, *such that*

(1) *For all $P \in \mathrm{Spec}(R)$, the stalk of $\mathfrak{G}$ at P, $\mathfrak{G}_P$, is the special group $G_{T/P}(R/P)$, associated to the preorder T/P of the field R/P;*

(2) *$\tau_{prop} = \{P \in \mathrm{Spec}(R) : G_{T/P}(R/P)$ is a non-trivial RSG$\}$ is closed and non-empty in* $\mathrm{Spec}(R)$.

b) If T is a **strict** *preorder of R, then for all $e \in B(R)$, $\mathfrak{G}(Z(e)) = G_{Te}(Re)$ is a reduced special group and for all $P \in \mathrm{Spec}(R)$, $\mathfrak{G}_P$ is the reduced special group $G_{T/P}(R/P)$.*

Proof. a) We comment only on the first assertion in (a), since the others follow directly from the preceding discussion. If $0 \neq e$ is an idempotent in R, we have two possibilities :

$*$ Te is a proper preorder of Re : In this case, since Re is a vN-ring in which 2 is a unit, it follows from Proposition 6.15 that $\mathfrak{G}(Z(e)) = G_{Te}(Re)$ is a reduced special group;

$*$ $Te = Re$: Here we get $G_{Te}(Re) = \{1\}$, the trivial *special group*.

In any case, $\mathfrak{G}$ is a presheaf of special groups, as stated. $\diamondsuit$

7 The [SMC] property for properly preordered vN-rings

In this section we apply the K-theory of special groups developed in [DM3] and [DM6] to associate to a presheaf basis of special groups, $\mathfrak{G}$, a graded ring of presheaf bases of groups of exponent two

$$k_*\mathfrak{G} = \langle k_0\mathfrak{G}, k_1\mathfrak{G}, \ldots, k_n\mathfrak{G}, \ldots \rangle$$

together with a sequence $\omega = \langle \omega_1, \ldots, \omega_n, \ldots \rangle$ of morphisms of presheaf bases of groups,

$$\omega_n : k_n\mathfrak{G} \longrightarrow k_{n+1}\mathfrak{G}, \qquad (n \geq 1)$$

corresponding to multiplication by $\lambda(-1)$. K-theoretic notation is as in 1.1.(1).

Theorem 7.1 *Let X be a Boolean space and let $\mathcal{B}$ be the Boolean algebra of clopens in X. Let $\mathfrak{G}$ be a finitely complete presheaf basis of special groups over $\mathcal{B}$, with restriction morphisms $\{\rho_{VU} : U \subseteq V$ in $\mathcal{B}\}$.*

a) For each $n \geq 0$, the assignments
$$\begin{cases} U \in \mathcal{B} \longmapsto k_n\mathfrak{G}(U); \\ U \subseteq_o V \longmapsto (\rho_{UV})_n : k_n\mathfrak{G}(V) \longrightarrow k_n\mathfrak{G}(U), \end{cases}$$
constitute a finitely complete presheaf basis of groups, $k_n\mathfrak{G}$, such that

(1) *For all $n, m \geq 0$ and $U \in \mathcal{B}$, $\eta \in k_n\mathfrak{G}(U)$ and $\xi \in k_m\mathfrak{G}(U) \Rightarrow \eta\xi \in k_{n+m}\mathfrak{G}(U)$;*

(2) *For all $x \in X$, the map defined on generators by*
$$(\lambda(a_1)\cdots\lambda(a_n))_x \in (k_n\mathfrak{G})_x \longmapsto \lambda(a_{1x})\cdots\lambda(a_{nx}) \in k_n\mathfrak{G}_x$$
extends to a (natural) **isomorphism** *from $(k_n\mathfrak{G})_x$ to $k_n\mathfrak{G}_x$, by which these groups will be identified.*

b) For $n \geq 1$, define $\omega_n = \{\omega_{nU} : U \in \mathcal{B}\} : k_n\mathfrak{G} \longrightarrow k_{n+1}\mathfrak{G}$ by
$$\text{For each } U \in \mathcal{B} \text{ and } \eta \in \mathfrak{G}(U), \quad \omega_{nU}(\eta) = \lambda(-1_{|U})\eta.$$
Then, ω_n is a morphism of presheaf bases of groups and for each $x \in X$, $\omega_{nx} : k_n\mathfrak{G}_x \longrightarrow k_{n+1}\mathfrak{G}_x$ is precisely multiplication by $\lambda(-1_x)$, where $-1_x \in G_x$.

c) For $U \in \mathcal{B}$, if $\mathfrak{G}_x$ is [SMC] *for all $x \in U$, then $\mathfrak{G}(U)$ is* [SMC]. *In particular, if every stalk of $\mathfrak{G}$ is* [SMC], *then $\mathfrak{G}(X)$, the SG of global sections of $\mathfrak{G}$, is* [SMC].

Proof. a) By item (1) in Proposition 2.7, the K-theory functor from **SG** to **2-Gr** is geometrical, connecting the geometrical theories of special groups and groups of exponent 2. Hence, Theorem 3.11 applies to yield the desired conclusions.

b) It is clear that for $U \in \mathcal{B}$, ω_{nU} is a group morphism and that, for $U \subseteq V$ in $\mathcal{B}$ and $\eta \in k_n\mathfrak{G}(V)$, $\omega_{nV}(\eta)_{|U} = \omega_{nU}(\eta_{|U})$; hence, ω_n is a morphism of presheaf bases, as in 3.2.(f). For $x \in X$, let $\omega_{nx} = \underset{\longrightarrow}{lim}_{U \in \mathcal{B}_x} \omega_{nU}$; by (a).(2), given $\xi \in k_n\mathfrak{G}_x$, there is $U \in \mathcal{B}_x$ and $\eta \in k_n\mathfrak{G}(U)$ such that $\eta_x = \xi$. Then, Theorem 2.3.(f).(1) and another application of (a).(2) yield

$$\omega_{nx}(\xi) = \omega_{nx}(\eta_x) = (\omega_n(\eta))_x = (\lambda(-1_{|U})\eta)_x = \lambda(-1)_x\eta_x = \lambda(-1_x)\xi,$$

showing that ω_{nx} is multiplication by $\lambda(-1_x)$, as claimed.

c) If $n \geq 1$, since $k_n\mathfrak{G}$ is a presheaf basis over $\mathcal{B}$, (a).(2) and Proposition 3.10.(c) imply that the map

$$\gamma_n^U : k_n\mathfrak{G}(U) \longrightarrow \Gamma_n(U) = \textstyle\prod_{x \in U} k_n\mathfrak{G}_x$$

is a group embedding, where $\Gamma_n(U)$ has the product structure, defined coordinatewise. By item (b), the following diagram commutes :

$$\begin{tikzcd}
k_n\mathfrak{G}(U) \arrow[r, "\gamma_n^U"] \arrow[d, "\omega_{nU}"'] & \Gamma_n(U) \arrow[d, "\Pi_{x \in U}\ \omega_{nx}"] \\
k_{n+1}\mathfrak{G}(U) \arrow[r, "\gamma_{n+1}^U"'] & \Gamma_{n+1}(U)
\end{tikzcd}$$

Now let $\eta \in k_n\mathfrak{G}(U)$ be such that $\omega_{nU}(\eta) = \lambda(-1_{|U})\eta = 0$ in $k_{n+1}\mathfrak{G}(U)$. By the commutativity of the diagram above, we get that for all $x \in U$, $\omega_{nx}(\eta_x) = \lambda(-1_x)\eta_x = 0$ in $k_{n+1}\mathfrak{G}_x$; since $\mathfrak{G}_x$ is [SMC], we conclude that $\eta_x = 0$ in $k_n\mathfrak{G}_x$, for all $x \in U$. But then, the extensionality of $k_n\mathfrak{G}$ entails $\eta = 0$ in $k_n\mathfrak{G}(U)$, as needed to verify that $\mathfrak{G}(U)$ is [SMC]. $\diamondsuit$

We now have

Theorem 7.2 *If R is a vN-ring in which 2 is a unit and T is a proper preorder of R, then $G_T(R)$ is* [SMC]. *In particular, if R is a formally real vN-ring, $G_{red}(R)$ is* [SMC].

Proof. By Proposition 6.15.(b) it suffices to show that the result holds for a **strict** preorder on R. Indeed, with notation as in 6.15, since k_* is a functor, the map $(q_I^\pi)_* : k_*G_T(R) \longrightarrow k_*G_{T/I}(R/I)$ is an isomorphism, and one of these groups will be [SMC] iff the same is true of the other.

Assume that T is a strict preorder on R. By Corollary 6.16.(b), the stalk at each $P \in \mathrm{Spec}(R)$ of the presheaf basis, $\mathfrak{G}$, of RSGs associated to $\langle R, T \rangle$, is the RSG corresponding to the proper preorder T/P on the field R/P, i.e., $G_{T/P}(R/P)$. Since R/P is a formally real field, it follows from Theorem 6.4 and (the proof of) Theorem 6.9 in [DM3] that $G_{T/P}(R/P)$ is [SMC]. Hence, for all $x \in X$, $\mathfrak{G}_x$ is [SMC] and the desired conclusion follows from item (c) of Theorem 7.1. $\diamondsuit$

References

[CK] C. C. Chang, H. J. Kiesler, **Model Theory**, North-Holland Publ. Co., Amsterdam, 1990.

[DM1] M. Dickmann, F. Miraglia, *On quadratic forms whose total signature is zero mod 2^n. Solution to a problem of M. Marshall*, Invent. Math. 133 (1998), 243–278.

[DM2] M. Dickmann, F. Miraglia, **Special Groups : Boolean-Theoretic Methods in the Theory of Quadratic Forms**, Memoirs Amer. Math. Soc. 689, Providence, R.I., 2000.

[DM3] M. Dickmann, F. Miraglia, *Lam's Conjectur*e, Algebra Colloquium 10 (2003), 149–176.

[DM4] M. Dickmann, F. Miraglia, *Bounds for the representation of quadratic forms*, J. Algebra 268 (2003), 209–251.

[DM5] M. Dickmann, F. Miraglia, *Rings with Many Units and Special Groups*, in Séminaire de Structures Algébriques Ordonnées, 2003-04, Paris VII-CNRS, Prépublications 77 (May 2005), 25 pp.

[DM6] M. Dickmann, F. Miraglia, *Algebraic K-theory of Special Groups*, Journal of Pure and Applied Algebra 204 (2006), 195-234.

[El] D. P. Ellerman, *Sheaves of Structures and Generalized Ultraproducts*, Annals of Pure and Applied Logic **7** (1974), 165-195.

[Gu] D. Guin, *Homologie du groupe linéaire et K-theorie de Milnor des anneaux*, J. Algebra, **123** (1989), 27-89.

[Go] R. Godement, **Topologie Algébrique et Théorie des Faisceaux**, Hermann, Paris, 1958.

[GR] H. Grauert and R. Remmert, **Coherent Analytic Sheaves**, Grund. der Math. Wissen. **265**, Springer-Verlag, Berlin, 1984.

[Ho] W. Hodges, **Model Theory**, Encyclopedia of Mathematics and Its Applications, **42**, Cambridge Univ. Press, Cambridge, 1993.

[KS] M. Kashiwara, P. Schapira, **Sheaves on Manifolds**, Grund. Math. Wissen. **292**, Springer-Verlag, Berlin, 1994.

[Pi] R. S. Pierce, **Modules over Commutative Regular Rings**, Memoirs Amer. Math. Soc. **70**, Providence, RI, 1967.

[Te] B. R. Tennison, **Sheaf Theory**, London Math. Soc. Lecture Notes **20**, Cambridge University Press, Cambridge, 1975.

[Wa] L. Walters, *Quadratic Forms, Orderings and Quaternion Algebras over Rings with Many Units*, Master's Thesis, University of Saskatchewan, 1988.

M. Dickmann
Équipe de Logique Mathématique
Université de Paris VII, et
Projet Topologie et Géométrie Algébriques,
Institut de Mathématiques de Jussieu,
75251 Paris Cedex 05, France
e-mail : dickmann@logique.jussieu.fr

F. Miraglia
Departamento de Matemática
Instituto de Matemática e Estatística
Universidade de São Paulo
C. P. 66.281 (Ag. Cidade de São Paulo)
05311-970 S.Paulo, S.P. - Brazil
e-mail : miraglia@ime.usp.br