On a class of 2-surface observables in general relativity

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The boundary conditions for canonical vacuum general relativity is investigated at the quasi-local level. It is shown that fixing the *area element on the 2-surface* $\mathcal{S}$ (rather than the induced 2-metric) is enough to have a well defined constraint algebra, and a well defined Poisson algebra of basic Hamiltonians parameterized by shifts that are tangent to *and divergence free on* $\mathcal{S}$. The evolution equations preserve these boundary conditions, and the value of the basic Hamiltonians gives 2+2–covariant, gauge-invariant 2-surface observables. The meaning of these observables is also discussed.

1 Introduction

As is well known, in a spacetime that is asymptotically flat at spatial infinity the ten classical conserved quantities, viz. the energy-momentum and relativistic angular momentum (i.e. including the centre-of-mass), can be introduced in several different ways. One possibility is to use a canonical/Hamiltonian approach [1-4]. However, to have a deeper understanding e.g. of the (geometrical or thermodynamical) properties of black holes, for example their entropy, the conserved quantities, or, more generally, the observables of the gravitational 'field' must be introduced at the *quasi-local* level. Such investigations lead to the so-called surface degrees of freedom [5-8], and to the large variety of proposals for the quasi-local energy-momentum and angular momentum [9]. A further motivation of searching for quasi-local observables is the remarkable result that all the global observables for the vacuum gravitational field in a closed universe, built as spatial integrals of local functions of the initial data and their derivatives, are necessarily vanishing [10,11]. Thus in closed universes we can associate non-trivial, locally constructible observables only to subsystems, bounded by some closed spacelike 2-surface.

The aim of the present note is to discuss certain quasi-local, 2-surface observables within the framework of canonical vacuum general relativity. Although in the literature there is a nice and quite general analysis using explicit background structures (see e.g. [12,13]), here we follow a more traditional (and perhaps more 'pedestrian') approach, and no such background structure will be used. In the subsequent analysis, in addition to the functional differentiability of various functions on the phase space (due to Regge and Teitelboim [1]), three new requirements, already appeared in the asymptotically flat context [2-4,14], will be expected to be satisfied at the quasi-local level: a. The evolution equations should preserve the boundary conditions (i.e. the boundary conditions should be compatible with the evolution equations); b. The Hamiltonians, and hence, in particular, the constraints, should close to a Poisson algebra; c. The value of the Hamiltonian on the constraint surface should be a 2+2–covariant, gauge invariant observable.

We show that the observables introduced in [5-8] are well defined even under much weaker boundary conditions. It will be shown that 1. fixing the *area element on the 2-surface* $\mathcal{S}$ rather than the induced 2-metric is enough to have i. a well defined constraint algebra $\mathcal{C}$, and ii. a well defined Poisson algebra $\mathcal{H}_0$ of basic Hamiltonians parameterized by shifts that are tangent to $\mathcal{S}$ and *divergence free with respect to the intrinsic Levi-Civita connection on* $\mathcal{S}$. 2. The evolution equations preserve these boundary conditions; and 3. the value of the basic Hamiltonians give 2+2–covariant, gauge-invariant 2-surface observables.

In the next section the basic variational formula of the constraints is recalled, and the variations of the 3-metric near the boundary $\mathcal{S}$ is decomposed. Then, in Section 3, the boundary condition above is introduced and the constraints are discussed. The fourth section is devoted to the investigation of the basic Hamiltonians and the 2-surface observables. In particular, we calculate its value in axi-symmetric spacetimes and the small and large sphere limits.

Our notations and conventions are essentially those that used in [3,4,9]. In particular, we use the abstract index formalism, and the curvature is defined by $-R^a{}_{bcd}X^b := (D_cD_d - D_dD_c)X^a$. Though primarily we are interested in the physical 3+1 dimensional case, the analysis will be done in $n+1$ dimensions, $n \geq 2$, and the signature of the spacetime metric is $1-n$ (and hence the spatial metric is *negative* definite). Although here we consider only the vacuum case (with cosmological constant λ), in our formulae we retain the gravitational 'coupling constant' $\kappa = 8\pi G$. The analysis is based on certain formulae given explicitly in [3].

2 Variation of the constraint function

Let Σ be any smooth n dimensional compact manifold with smooth $(n-1)$-boundary $\mathcal{S} := \partial\Sigma$. Then the constraint function in the ADM phase space of the $n+1$ dimensional vacuum general relativity with cosmological constant λ, smeared by the function N and vector field N^a on Σ, is

$$C\big[N, N^a\big] := -\int_\Sigma \Big\{ \frac{1}{2\kappa}\Big[R - 2\lambda + \frac{4\kappa^2}{|h|}\big(\frac{1}{(n-1)}\tilde{p}^2 - \tilde{p}_{ab}\tilde{p}^{ab}\big)\Big] N\sqrt{|h|} + 2N^c h_{ca} D_b \tilde{p}^{ab} \Big\} \mathrm{d}^n x. \tag{2.1}$$

Here the canonical variables are h_{ab} and $\tilde{p}^{ab}$, D_e is the Levi-Civita covariant derivative determined by h_{ab} and R is its curvature scalar. In spacetime this constraint function is just the integral $\int_\Sigma \xi^a(G_{ab} + \lambda g_{ab})t^b \mathrm{d}\Sigma$, where t^a is the future pointing unit timelike normal to Σ in the spacetime, $\xi^a := Nt^a + N^a$, and in the momentum phase space their vanishing for all N and N^a define the constraint surface Γ. The canonical momentum in terms of the Lagrange variables, i.e. the metric and the extrinsic curvature χ_{ab} of Σ in the spacetime, is known to be $\tilde{p}^{ab} = \frac{1}{2\kappa}\sqrt{|h|}(\chi^{ab} - \chi h^{ab})$. Here χ is the h_{ab}-trace of χ_{ab}, the velocity of h_{ab} is $\dot{h}_{ab} = 2N\chi_{ab} + \text{Ł}_{\mathbf{N}} h_{ab}$ and N and N^a play the role of the lapse and the shift, respectively, in the spacetime. $\text{Ł}_{\mathbf{N}}$ denotes Lie derivative along N^a.

Let $N(u)$, $N^a(u)$, $h_{ab}(u)$ and $\tilde{p}^{ab}(u)$, $u \in (-\epsilon, \epsilon)$, be any smooth 1-parameter families of lapses, shifts, metrics and canonical momenta, respectively, and define the corresponding variation of any function of them, $F = F(N, N^a, h_{ab}, \tilde{p}^{ab})$, as $\delta F := (\mathrm{d}F(N(u), N^a(u), h_{ab}(u), \tilde{p}^{ab}(u))/\mathrm{d}u)|_{u=0}$. Then the corresponding variation of the constraint function $C[N, N^e]$, taken from [3], is

$$\begin{aligned}\delta C\big[N, N^e\big] =& C\big[\delta N, \delta N^e\big] + \int_\Sigma \Big(\frac{\delta C[N,N^e]}{\delta h_{ab}}\delta h_{ab} + \frac{\delta C[N,N^e]}{\delta \tilde{p}^{ab}}\delta\tilde{p}^{ab}\Big)\mathrm{d}^n x + \\ &+ \frac{1}{2\kappa}\oint_{\partial\Sigma}\Big\{ N\big(h^{ab}v^e(D_e\delta h_{ab}) - v^a(D^b\delta h_{ab})\big) + \big(v^aD^bN - h^{ab}v^eD_eN\big)\delta h_{ab} + \\ &+ \frac{2\kappa}{\sqrt{|h|}}\big(2N^av_e\tilde{p}^{eb} - N^ev_e\tilde{p}^{ab}\big)\delta h_{ab} + 4\kappa N_a v_b \frac{\delta\tilde{p}^{ab}}{\sqrt{|h|}}\Big\}\mathrm{d}\mathcal{S}.\end{aligned} \tag{2.2}$$

Here $\mathrm{d}\mathcal{S}$ is the induced volume $n-1$-form on $\mathcal{S}$, v^a is the *outward* pointing unit normal of $\mathcal{S}$ in Σ, and

$$\begin{aligned}\frac{\delta C[N,N^e]}{\delta h_{ab}} :=& \frac{1}{2\kappa}\sqrt{|h|}\Big\{ N\Big(R^{ab} - Rh^{ab} + 2\lambda h^{ab} + \frac{8\kappa^2}{|h|}\big(\tilde{p}^a{}_c\tilde{p}^{cb} - \frac{1}{(n-1)}h_{cd}\tilde{p}^{cd}\tilde{p}^{ab}\big)\Big) + \\ &+ D^aD^bN - h^{ab}D_cD^cN\Big\} - \text{Ł}_{\mathbf{N}}\tilde{p}^{ab} + \frac{1}{4\kappa}Nh^{ab}\sqrt{|h|}\Big(R - 2\lambda + \frac{4\kappa^2}{|h|}\big(\frac{1}{(n-1)}\tilde{p}^2 - \tilde{p}^{cd}\tilde{p}_{cd}\big)\Big),\end{aligned} \tag{2.3.a}$$

$$\frac{\delta C[N,N^e]}{\delta\tilde{p}^{ab}} := \frac{4\kappa}{\sqrt{|h|}}N\Big(\tilde{p}^{ab} - \frac{1}{(n-1)}\tilde{p}^{cd}h_{ca}h_{db}\Big) + \text{Ł}_{\mathbf{N}}h_{ab}. \tag{2.3.b}$$

Here R_{ab} is the Ricci tensor of D_e. Thus $C[N,N^e]$ is functionally differentiable (in the strict sense of [14,15]) with respect to the canonical variables only if the boundary integral in (2.2) is vanishing, whenever the functional derivatives themselves are given by (2.3). Then the vacuum evolution equations with cosmological constant are precisely the canonical equations

$$\dot{h}_{ab}=\frac{\delta C\big[N,N^e\big]}{\delta\tilde{p}^{ab}},\qquad \dot{\tilde{p}}^{ab}=-\frac{\delta C\big[N,N^e\big]}{\delta h_{ab}},\tag{2.4.a, b}$$

provided the constraint equations $C[N,N^e]=0$ are satisfied. Our ultimate aim is to find appropriate boundary conditions on the canonical variables $(h_{ab},\tilde{p}^{ab})$ and an appropriate class of fields N, N^a together with a boundary integral $B[N,N^e]$ such that $C[N,N^e]+B[N,N^e]$ be functionally differentiable, and the boundary conditions on the canonical variables be compatible with the evolution equations.

To find this boundary term and these conditions, it seems useful to split the variation of the metric h_{ab} at the points of $\mathcal{S}$ with respect to the boundary. Thus let $\Pi^a_b:=\delta^a_b+v^av_b$, the h_{ab}-orthogonal projection to $\mathcal{S}$, and define the induced metric $q_{ab}:=h_{cd}\Pi^c_a\Pi^d_b$, the corresponding Levi-Civita covariant derivative δ_e and another derivative operator simply by $\Delta_e:=\Pi^f_eD_f$. The extrinsic curvature of $\mathcal{S}$ in Σ will be defined by $\nu_{ab}:=\Pi^c_a\Pi^d_bD_cv_d$. At the points of $\mathcal{S}$ the splitting $h_{ab}=q_{ab}-v_av_b$ implies the variation $\delta h_{ab}=\delta q_{ab}-v_a\delta v_b-v_b\delta v_a$. Since v_a is a normal 1-form of the submanifold $\mathcal{S}$, for any X^a tangent to $\mathcal{S}$ one has $v_a(u)X^a=0$, implying that $\delta v_a\Pi^a_b=0$. Taking the u-derivative of $q_{ab}(u)v^a(u)=0$ we obtain that $\delta q_{ab}v^av^b=0$ and $\delta q_{ab}v^a\Pi^b_c=-\delta v^aq_{ac}$, and taking the derivative of $v^a(u)v_a(u)=1$ we obtain $\delta v^av_a=-\delta v_av^a$. Thus, the various projections of the variation δh_{ab} are

$$\delta h_{cd}\Pi^c_a\Pi^d_b=\delta q_{cd}\Pi^c_a\Pi^d_b,\qquad \delta h_{cd}v^c\Pi^d_b=-\delta v^aq_{ad},\qquad \delta h_{cd}v^cv^d=2v^a\delta v_a=-2v_a\delta v^a.\tag{2.5}$$

Therefore, the independent variations can be represented by $\delta q_{cd}\Pi^c_a\Pi^d_b$ and δv^a.

3 The quasi-local constraint algebra

In this section we determine the boundary conditions under which the constraint functions are functional differentiable with respect to the canonical variables. We will see that, as a bonus, this already ensures that they form a Poisson algebra too. (In the asymptotically flat case it has been demonstrated that in vacuum general relativity this differentiability implies the Poisson algebra structure [2]. Similar result has been proven in a more general classical field theory context in [14]: Functional differentiability of functions together with the requirement that the corresponding Hamiltonian vector fields preserve the boundary conditions also imply the Poisson algebra structure.) Thus first let us determine the condition of the functional differentiability of $C[N,N^a]$. To do this, we decompose the boundary integral in (2.2) with respect to $\mathcal{S}$. Clearly, $C[N,N^a]$ is functionally differentiable with respect to N and N^a, independently of the boundary conditions at $\mathcal{S}$. A tedious but straightforward calculation yields that the vanishing of the boundary integral in (2.2) is just the condition

$$\begin{aligned}0=\oint_{\mathcal{S}}\Big(&\frac{1}{2\kappa}N\big(v^a(D^b\delta h_{ab})-h^{ab}v^e(D_e\delta h_{ab})\big)-\frac{1}{2\kappa}v^a\delta h_{ab}q^{bc}\Delta_cN+\frac{1}{2\kappa}v^e(D_eN)q^{ab}\delta h_{ab}-\\&-2v_e\frac{\tilde{p}^{ef}}{\sqrt{|h|}}N^d\Pi^a_f\Pi^b_d\delta h_{ab}+v_eN^e\big(\frac{\tilde{p}^{ab}}{\sqrt{|h|}}\delta h_{ab}+2v_e\frac{\tilde{p}^{ea}}{\sqrt{|h|}}\Pi^b_a\delta h_{bc}v^c-2v_ev_f\frac{\tilde{p}^{ef}}{\sqrt{|h|}}v^av^b\delta h_{ab}\big)+\\&+2v_ev_f\frac{\tilde{p}^{ef}}{\sqrt{|h|}}v^a\delta h_{ab}\Pi^b_cN^c\Big)\mathrm{d}\mathcal{S}.\end{aligned}\tag{3.1}$$

Taking into account that the variation of the induced volume $(n-1)$-form on the boundary is $\delta\varepsilon_{a_1\dots a_{n-1}} = \frac{1}{2}q^{cd}\delta q_{cd}\varepsilon_{a_1\dots a_{n-1}} = \frac{1}{2}q^{cd}\delta h_{cd}\varepsilon_{a_1\dots a_{n-1}}$, the boundary conditions $N|_{\mathcal{S}} = 0$, $N^a|_{\mathcal{S}} = 0$ and $\varepsilon_{a_1\dots a_{n-1}} =$ fixed ensure the functional differentiability of the constraint functions $C[N, N^a]$ with respect to h_{ab} and $\tilde{p}^{ab}$. Since the last term of the integrand in (3.1) is proportional to $2\kappa v_a v_b \tilde{p}^{ab} = \sqrt{|h|}q^{ab}\chi_{ab}$, which is not zero in general, the boundary condition $N^a|_{\mathcal{S}} = 0$ cannot be weakened to $v_a N^a|_{\mathcal{S}} = 0$ even if the induced metric q_{ab} on $\mathcal{S}$ (rather than only the corresponding volume $(n-1)$-form) is kept fixed. On the other hand, because of the fourth term in (3.1), $N|_{\mathcal{S}} = 0$ and $N^a|_{\mathcal{S}} = 0$ in themselves are not enough to ensure the functional differentiability with respect to h_{ab}.

These boundary conditions are preserved by the evolution equations. Indeed, since the only condition that we imposed on the canonical variables is $\delta\varepsilon_{a_1\dots a_{n-1}} = 0$, we should consider only (2.4.a), the evolution equation for the metric h_{ab}. By $N|_{\mathcal{S}} = 0$ this yields on the boundary that $\dot{h}_{ab}|_{\mathcal{S}} = 2D_{(a}N_{b)}$, and hence, by (2.5), $q^{ab}\dot{q}_{ab} = q^{ab}\dot{h}_{ab} = 2q^{ab}D_a N_b = 2\Delta_a N^a = 0$, where in the last step we used $N^a|_{\mathcal{S}} = 0$. Therefore, *the evolution equations preserve the boundary conditions*. Geometrically $N|_{\mathcal{S}} = 0$, $N^a|_{\mathcal{S}} = 0$ correspond to an evolution vector field $\xi^a = t^a N + N^a$ in the spacetime that is vanishing on $\mathcal{S}$; i.e. the corresponding diffeomorphism leaves $\mathcal{S}$ fixed *pointwise*. The one parameter family of diffeomorphisms generated by such a ξ^a maps Σ into a family Σ_t of Cauchy surfaces for the *same* globally hyperbolic domain $D(\Sigma)$ with the same boundary $\partial\Sigma_t = \mathcal{S}$, i.e. such a ξ^a is precisely a vector field that we would intuitively consider to be the generator of a gauge motion in the *spacetime*.

By the functional differentiability of the constraint functions (with vanishing smearing fields N and N^a on $\mathcal{S}$) we can take the Poisson bracket of any two constraint functions $C[N, N^a]$ and $C[\bar{N}, \bar{N}^a]$. These brackets, keeping all the boundary terms, have already been calculated [3]. They are

$$\Big\{C\big[0,N^a\big],C\big[0,\bar{N}^a\big]\Big\} = -\,C\big[0,[N,\bar{N}]^a\big] + \\ + \int_\Sigma D_e\Big(N^e\tilde{p}^{ab}\mathrm{L}_{\bar{N}}h_{ab} - \bar{N}^e\tilde{p}^{ab}\mathrm{L}_N h_{ab} - 2\tilde{p}^{ef}h_{fa}[N,\bar{N}]^a\Big)\mathrm{d}^n x, \tag{3.2.a}$$

$$\Big\{C\big[0,N^a\big],C\big[\bar{N},0\big]\Big\} = -\,C\big[N^e D_e\bar{N},0\big] + \\ + \frac{1}{\kappa}\int_\Sigma D_e\Big(\bar{N}\big(R^e{}_f - \frac{1}{2}R\delta^e_f\big)N^f + \lambda\bar{N}N^e + \frac{2\kappa^2}{|h|}\bar{N}N^e\big(\tilde{p}^{ab}\tilde{p}_{ab} - \frac{1}{n-1}\tilde{p}^2\big) + \\ + \big(\Delta_f N^e\big)\big(\Delta^f\bar{N}\big) - \big(D^e\bar{N}\big)\big(\Delta_f N^f\big)\Big)\sqrt{|h|}\mathrm{d}^n x, \tag{3.2.b}$$

$$\Big\{C\big[N,0\big],C\big[\bar{N},0\big]\Big\} = C\big[0,ND^a\bar{N} - \bar{N}D^a N\big] + 2\int_\Sigma D_e\Big(N\tilde{p}^{ef}D_f\bar{N} - \bar{N}\tilde{p}^{ef}D_f N\Big)\mathrm{d}^n x. \tag{3.2.c}$$

However, by the vanishing of the smearing fields on $\mathcal{S}$ all the boundary terms in (3.2) are vanishing, and the Lie product can be summarized as

$$\Big\{C\big[N,N^a\big],C\big[\bar{N},\bar{N}^a\big]\Big\} = C\big[\bar{N}^e D_e N - N^e D_e\bar{N}, ND^a\bar{N} - \bar{N}D^a N - [N,\bar{N}]^a\big]. \tag{3.3}$$

Furthermore, the new smearing fields $\bar{N}^e D_e N - N^e D_e \bar{N}$ and $ND^a\bar{N} - \bar{N}D^a N - [N,\bar{N}]^a$ are also vanishing on the boundary $\mathcal{S}$. Therefore, *the constraint functions with vanishing smearing fields on $\mathcal{S}$ close to a Poisson algebra $\mathcal{C}$*, the so-called quasi-local constraint algebra, provided the induced volume $(n-1)$-form $\varepsilon_{a_1\dots a_{n-1}}$ is fixed on $\mathcal{S}$. Clearly, this Lie algebra is isomorphic to that appearing in the asymptotically flat case [2-4].

The boundary condition yields the split of the quasi-local phase space $T^*\mathcal{Q}(\Sigma) := \{(h_{ab}, \tilde{p}^{ab})\}$ into the disjoint union of sectors $T^*\mathcal{Q}(\Sigma, \varepsilon_{a_1\dots a_{n-1}})$, labeled by the volume $(n-1)$-from $\varepsilon_{a_1\dots a_{n-1}}$ on $\mathcal{S}$: The constraint functions are differentiable in the directions tangent to these sectors and form the familiar Poisson algebra, and the evolution equations with lapse and shift vanishing on $\mathcal{S}$ also preserve this sector–structure.

4 The basic Hamiltonian

4.1 The boundary conditions

Starting with the naive quasi-local Lagrange phase space $T\mathcal{Q}(\Sigma):=\{(h_{ab},\dot h_{ab})\}$ and the traditional Lagrangian $L:T\mathcal{Q}(\Sigma)\to\mathbf{R}$, given explicitly by $L:=\frac{1}{2\kappa}\int_\Sigma N(R-2\lambda+\chi^{ab}\chi_{ab}-\chi^2)\sqrt{|h|}{\rm d}^nx$, the basic Hamiltonian $H_0[N,N^a]:=\int_\Sigma\tilde p^{ab}\dot h_{ab}{\rm d}^nx-L$ on $T^*\mathcal{Q}(\Sigma)$ takes the form

$$H_0\big[N,N^e\big]=C\big[N,N^e\big]+\int_\Sigma 2D_a\Big(\tilde p^{ab}h_{bc}N^c\Big){\rm d}^nx. \tag{4.1}$$

Its total variation is

$$\begin{aligned}\delta H_0\big[N,N^e\big]=&C\big[\delta N,\delta N^e\big]+\int_\Sigma\Big(\frac{\delta C[N,N^e]}{\delta h_{ab}}\delta h_{ab}+\frac{\delta C[N,N^e]}{\delta\tilde p^{ab}}\delta\tilde p^{ab}\Big){\rm d}^nx+\\ &+\frac{1}{2\kappa}\oint_{\partial\Sigma}\Big\{N\big(h^{ab}v^e(D_e\delta h_{ab})-v^a(D^b\delta h_{ab})\big)+\big(v^aD^bN-h^{ab}v^eD_eN\big)\delta h_{ab}-\\ &-\frac{2\kappa}{\sqrt{|h|}}\big(v_eN^e\tilde p^{ab}\delta h_{ab}+2v_e\tilde p^{ea}h_{ab}\delta N^b\big)\Big\}{\rm d}\mathcal{S}.\end{aligned} \tag{4.2}$$

Thus $H_0[N,N^e]$ is functionally differentiable with respect to N and the canonical momentum $\tilde p^{ab}$, independently of the boundary conditions at $\mathcal{S}$.

The condition of the functional differentiability of $H_0[N,N^e]$ with respect to h_{ab} is the vanishing of the boundary term in (4.2) involving δh_{ab}, provided the variations δh_{ab} and δN^a are independent. We decompose its integrand with respect to the boundary $(n-1)$-surface, using spacetime quantities as well. In particular, if t^a is the future pointing unit timelike normal to Σ in spacetime and $A_e:=v_a\Delta_et^a:=v_a\Pi^f_e\nabla_ft^a$, the connection 1-form on the normal bundle of $\mathcal{S}$, where now $\Pi^a_b:=\delta^a_b+v^av_b-t^at_b$ is the g_{ab}-orthogonal projection to $\mathcal{S}$ (for the details see [9] and references therein), then a lengthy but direct calculation gives that it is

$$\begin{aligned}0=\oint_{\mathcal{S}}\Big\{&-Nv^e\Big(D_e\delta h_{ab}\Big)q^{ab}+\delta h_{ab}v^av^b\Big(-N(\Delta_ev^e)+v_fN^f(\Delta_et^e)\Big)+\delta h_{ab}v^aq^{bc}\Big(-2\Delta_cN-2A_cv_eN^e\Big)+\\ &+\delta h_{ab}q^{ac}q^{bd}\Big(q_{cd}v^e(D_eN)-N(\Delta_cv_d)+v_eN^e(\Delta_ct_d)-q_{cd}v_eN^e(\Delta_ft^f)+q_{cd}v_eN^ev^f(D_ft_g)v^g\Big)\Big\}{\rm d}\mathcal{S}.\end{aligned} \tag{4.3}$$

The simplest way to make the first term vanishing is the condition that N be vanishing on $\mathcal{S}$, whenever $v_eN^e|_{\mathcal{S}}=0$ and $\varepsilon_{a_1\dots a_{n-1}}=$ fixed already ensure the functional differentiability of $H_0[N,N^e]$ with respect to h_{ab}. Note that this condition is weaker than that we had for the constraint functions, because we should require only that N^a be tangent to $\mathcal{S}$ rather than vanishing on $\mathcal{S}$. If we want $(n-1)+2$-covariant conditions for N and N^a at $\mathcal{S}$, then by $N|_{\mathcal{S}}=0$ we must impose $v_aN^a|_{\mathcal{S}}=0$ too. Indeed, if we do not want to prefer any timelike normal to $\mathcal{S}$, then N and v_aN^a must be treated on an equal footing, because they are the two components of $\xi^a=t^aN+N^a$ orthogonal to $\mathcal{S}$. On the other hand, in the absence of additional conditions we loose the functional differentiability with respect to N^a.

By $N|_{\mathcal{S}}=0$ the evolution equation for the metric gives $q^{ab}\dot q_{ab}=q^{ab}\dot h_{ab}=2q^{ab}D_aN_b=2\Delta_aN^a=\delta_aN^a$, where in the last step we used $v_aN^a|_{\mathcal{S}}=0$. Therefore, in addition, *we must require that N^a on $\mathcal{S}$ be divergence-free with respect to the intrinsic geometry of $\mathcal{S}$ as well*, otherwise the evolution equations do not preserve the boundary condition $\varepsilon_{a_1\dots a_{n-1}}=$ fixed. At first sight the requirement that N^a on $\mathcal{S}$ be δ_e-divergence-free yields that the variation of the metric on $\mathcal{S}$ produces a variation of N^a on $\mathcal{S}$, and hence these variations on $\mathcal{S}$ are not quite independent. However, by $\delta(\delta_aN^a)=N^e\delta_e(\frac{1}{2}q^{ab}\delta q_{ab})+\delta_a(\delta N^a)$ and the boundary condition $q^{ab}\delta q_{ab}=0$ the variation of the metric alone does not yield any variation of δ_aN^a. In other words, if N^a is any shift such that $v_aN^a|_{\mathcal{S}}=0$ and N^a is δ_e-divergence-free, then it will be divergence-free with respect to the connection coming from any 1-parameter family $q_{ab}(u)$ of metrics provided the volume $(n-1)$-form is kept fixed. Geometrically, $N|_{\mathcal{S}}=0$ and $v_aN^a|_{\mathcal{S}}=0$ correspond to an evolution vector field ξ^a in the spacetime which is tangent to $\mathcal{S}$, and hence, by $\delta_aN^a|_{\mathcal{S}}=0$, it generates a volume preserving diffeomorphism of $\mathcal{S}$ to itself.

4.2 The algebra of the basic Hamiltonians and 2-surface observables

Since the formal variational derivatives of the constraint functions and of the basic Hamiltonians are the same, the Poisson bracket of two basic Hamiltonians, $H_0[N,N^a]$ and $H_0[\bar N,\bar N^a]$, can be calculated easily by (3.2). By the boundary conditions $v_aN^a|_{\mathcal{S}}=v_a\bar N^a|_{\mathcal{S}}=0$ the boundary term in the Poisson bracket $\{H_0[0,N^a],H_0[0,\bar N^a]\}$ is vanishing, and there is no boundary term at all in the Poisson bracket $\{H_0[N,0],H_0[\bar N,0]\}$. On the other hand, the boundary term in the Poisson bracket $\{H_0[0,N^a],H_0[\bar N,0]\}$ is vanishing only if we use $\delta_aN^a|_{\mathcal{S}}=0$ too. This gives an additional justification of the condition $\delta_aN^a|_{\mathcal{S}}=0$. Then the Lie product of the basic Hamiltonians can be summarized as

$$\Big\{H_0\big[N,N^a\big],H_0\big[\bar N,\bar N^a\big]\Big\}=H_0\big[\bar N^eD_eN-N^eD_e\bar N,ND^a\bar N-\bar ND^aN-[N,\bar N]^a\big]. \tag{4.4}$$

Furthermore, if N^a and $\bar N^a$ are any two shifts which are tangent to $\mathcal{S}$ and δ_a-divergence-free on $\mathcal{S}$, then their Lie bracket $[N,\bar N]^a$ is also tangent to $\mathcal{S}$ and δ_a-divergence-free on $\mathcal{S}$. Hence the new lapse $\bar N^eD_eN-N^eD_e\bar N$ and the new shift $ND^a\bar N-\bar ND^aN-[N,\bar N]^a$ also satisfy the boundary conditions. Therefore, *the basic Hamiltonians parameterized by lapses and shifts satisfying $N|_{\mathcal{S}}=0$, $v_aN^a|_{\mathcal{S}}=0$ and $\delta_aN^a|_{\mathcal{S}}=0$ form a Poisson algebra $\mathcal{H}_0$.*

The value of the basic Hamiltonian on the constraint surface is

$$O\big[N^a\big]:=H_0\big[N,N^a\big]|_\Gamma=-\frac{1}{\kappa}\oint_{\mathcal{S}}N^aA_a\mathrm{d}\mathcal{S}. \tag{4.5}$$

Though A_a is not a gauge invariant object (namely, as we already mentioned, this is a connection 1-form in the normal bundle of $\mathcal{S}$ in the spacetime, and under an $SO(1,1)$ boost gauge transformation of the two normals, $(t^a,v^a)\mapsto(t^a\cosh(w)+v^a\sinh(w),v^a\cosh(w)+t^a\sinh(w))$, it transforms as a vector potential), by $\delta_aN^a|_{\mathcal{S}}=0$ the integral $O[N^a]$ is indeed boost gauge invariant. This is the third justification of the condition $\delta_aN^a|_{\mathcal{S}}=0$. Clearly, the constraint functions form an ideal in the algebra of the basic Hamiltonians, $\mathcal{C}\subset\mathcal{H}_0$, and the quotient $\mathcal{H}_0/\mathcal{C}$ can be parameterized by the value $O[N^a]$. By (4.4) this $O[N^a]$ defines a Lie algebra anti-homomorphism of the Lie algebra of the divergence-free vector fields on $\mathcal{S}$ into $\mathcal{H}_0/\mathcal{C}$: In fact, let N^a, N'^a and $\bar N^a$, $\bar N'^a$ be shift vectors such that they are tangent to $\mathcal{S}$ and δ_e-divergence-free on $\mathcal{S}$, furthermore $N^a|_{\mathcal{S}}=N'^a|_{\mathcal{S}}$ and $\bar N^a|_{\mathcal{S}}=\bar N'^a|_{\mathcal{S}}$. Then $O[N^a]=O[N'^a]$ and $\{H_0[0,N^a],H_0[0,\bar N^a]\}|_\Gamma=\{H_0[0,N'^a],H_0[0,\bar N'^a]\}|_\Gamma$, i.e. both $O[N^a]$ and the Poisson bracket $\{H_0[0,N^a],H_0[0,\bar N^a]\}$ evaluated on the constraint surface depend only on the restriction of the shifts to $\mathcal{S}$, and independent of their part inside Σ. Hence the Poisson bracket $\{O[N^a],O[\bar N^a]\}:=\{H_0[0,N^a],H_0[0,\bar N^a]\}|_\Gamma$ of $O[N^a]$ and $O[\bar N^a]$ is well defined and, by (4.4), it is $\{O[N^a],O[\bar N^a]\}=-O[[N,\bar N]^a]$.

It might be worth noting that the δ_e-divergence free vector fields on $\mathcal{S}$ can be given explicitly by using the Hodge decomposition (see e.g. [16]): If N^a is divergence free, then it necessarily has the form $\delta_bN^{ab}+{*\omega^a}$, where $N^{ab}=N^{[ab]}$ is an arbitrary bi-vector and ${*\omega^a}:=\frac{1}{(n-2)!}\varepsilon^{aa_1\dots a_{n-2}}\omega_{a_1\dots a_{n-2}}$ denotes the Hodge dual of a harmonic $(n-2)$-form $\omega_{a_1\dots a_{n-2}}$. The latter is an arbitrary linear combination of finitely many linearly independent harmonic forms $\omega^\alpha_{a_1\dots a_{n-2}}$, $\alpha=1,\dots,b$, where $b:=\dim H^{n-2}(\mathcal{S})$, the $(n-2)$th Betti number of $\mathcal{S}$. In terms of these the observable (4.5) takes the form $-\frac{1}{\kappa}\oint_{\mathcal{S}}(N^{ab}\delta_aA_b+{*\omega^a}A_a)\mathrm{d}\mathcal{S}$. In the physically important special case $n=3$ the bi-vector can always be written as $\varepsilon^{ab}\nu$ with an arbitrary real function ν, and the Betti number is $b=2g$, twice the genus of $\mathcal{S}$.

Formally, $O[N^a]$ is just the observable $O_M[N^a]$ of Balachandran, Chandar and Momen [5,6] (see also [7,8]). However, the present boundary conditions for the canonical variables are definitely weaker than those of them: They kept fixed the whole 3-metric h_{ab} on $\mathcal{S}$. On the other hand, without the extra condition $\delta_a N^a|_{\mathcal{S}}=0$, the observable $O_M[N^a]$ is *not* boost-gauge invariant. In addition, this extra condition on N^a ensures that the evolution equations preserve the weaker boundary conditions. Without this the evolution equations would preserve neither the boundary conditions of [5,6,8] nor the present, weaker ones. Similarly, the 'natural' boundary condition that the induced 2-metric q_{ab} is fixed is preserved by the evolution equation (2.4.a) only if N^a is vanishing on $\mathcal{S}$ or if $(\mathcal{S},q_{ab})$ admits N^a as a Killing vector. It could be interesting to note that the quasi-local quantity $L(N^a)$ of Yoon [17], obtained by following an $(n-1)+2$ analysis of the vacuum Einstein equations, as well as the '(generalized) angular momentum' of Brown and York [18], of Liu and Yau [19], and of Ashtekar and Krishnan [20] are just the observable $O[N^a]$ provided N^a on $\mathcal{S}$ is restricted to be tangent to $\mathcal{S}$ and δ_e-divergence-free on $\mathcal{S}$. Another (and quite obvious) observable is the surface integral of any integrable 'test' function f on $\mathcal{S}$: $A[f]:=\oint_{\mathcal{S}} f\,\mathrm{d}\mathcal{S}$.

In [5,6] a further 'observable' $O_H[T]$ was introduced, where T is the (not necessarily vanishing) *constant* value of the lapse on $\mathcal{S}$, and this was interpreted as some (not renormalized) form of energy. However, it depends on the choice for a preferred timelike normal to $\mathcal{S}$ too; i.e. *not* boost gauge invariant.

4.3 The various limits of the 2-surface observable

To clarify the meaning of the observable $O[N^a]$ it seems natural to consider various special 3+1 dimensional spacetimes and limits, such as axi-symmetric spacetimes, and the small and large sphere limits.

• *Axi-symmetric spacetimes*

Let the spacetime be axi-symmetric with Killing vector K^a. Then the angular momentum is usually defined by the 2-surface integral of the Komar superpotential built from K^a, and the value of this integral is well known to be invariant with respect to the continuous deformations of the 2-surface through *vacuum* regions (see e.g. [15]). To be able to compare the Komar expression and the observable, let us fix the 2-surface $\mathcal{S}$ and a foliation Σ_t of an open neighbourhood of $\mathcal{S}$ by smooth spacelike hypersurfaces such that $\mathcal{S}$ is lying in one leaf, e.g. in Σ_0, and let v^a denote the outward pointing unit normal of $\mathcal{S}$ in Σ_0. (This foliation should not be confused with the foliation of the globally domain whose 'edge' is the 2-surface $\mathcal{S}$: The former foliates an open neighbourhood of $\mathcal{S}$, whilst the latter collapses just on $\mathcal{S}$.) Let t^a be the future pointing unit normal of the leaves of the foliation, $P^a_b:=\delta^a_b-t^at_b$ the orthogonal projection to the leaves and let M be the lapse function of the foliation. Let us choose a shift vector M^a as well, i.e. specify an 'evolution vector field' $\xi^a:=Mt^a+M^a$. Then let $K^a=:Nt^a+N^a$ define the 3+1 pieces of the Killing field K^a with respect to the foliation. Then the time–space projection of the Killing operator acting on K_a is [3,4]

$$2MP^c_at^d\nabla_{(c}K_{d)}=\bigl(\mathrm{L}_\xi N_b\bigr)P^b_a-\bigl(\mathrm{L}_M N_b\bigr)P^b_a+MD_aN-ND_aM-2M\chi_{ab}N^b, \tag{4.6}$$

where $h_{ab}=g_{ab}-t_at_b$ is the induced metric on and χ_{ab} is the extrinsic curvature of the leaves. Using this, Komar's expression (normalized to get the correct value for the angular momentum in Kerr spacetime, see [21]) can be written as

$$\begin{aligned}\mathtt{I}_{\mathcal{S}}\bigl[K^a\bigr]:&=\frac{1}{2\kappa}\oint_{\mathcal{S}}\nabla^{[a}K^{b]}\frac{1}{2}\varepsilon_{abcd}=\frac{1}{\kappa}\oint_{\mathcal{S}}v^aP^c_at^d\bigl(\nabla_{[c}K_{d]}\bigr)\mathrm{d}\mathcal{S}=\\&=\frac{1}{\kappa}\oint_{\mathcal{S}}v^a\Bigl(-\frac{1}{2M}\bigl(\mathrm{L}_\xi N_b\bigr)P^b_a+\frac{1}{2M}\bigl(\mathrm{L}_M N_b\bigr)P^b_a+\frac{1}{2M}D_a\bigl(NM\bigr)\Bigr)\mathrm{d}\mathcal{S}=\\&=\frac{1}{\kappa}\oint_{\mathcal{S}}\Bigl(v^aD_aN-v^a\chi_{ab}N^b-v^aP^c_at^d\nabla_{(c}K_{d)}\Bigr)\mathrm{d}\mathcal{S}.\end{aligned} \tag{4.7}$$

Thus if the 2-surface $\mathcal{S}$ is chosen to be axi-symmetric (i.e. if K^a is tangent to $\mathcal{S}$ on $\mathcal{S}$) and K^a is tangent to Σ_0, then by $K^a = N^a$ the first term of the integrand is vanishing, the second term is $-N^a A_a$, and the third term is also zero because K^a is a Killing vector. Hence, in the special boost gauge defined by the hypersurface Σ_0 containing the integral curves of K^a, the observable $O[N^a]$ coincides with the Komar integral. Since, however, $O[N^a]$ is boost gauge invariant, we obtained that *the observable $O[N^a]$ for the Killing vector of axi-symmetry N^a and for the axi-symmetric 2-surface $\mathcal{S}$ coincides with the Komar integral $\mathtt{I}_{\mathcal{S}}[K^a]$*. Since $\mathtt{I}_{\mathcal{S}}[K^a]$ is invariant with respect to continuous deformations of $\mathcal{S}$ through vacuum regions, and in the definition of the Komar integral K^a is not required to be tangent to $\mathcal{S}$, the 2-surface is not required to be axi-symmetric. On the other hand, the observable $O[N^a]$ is well defined only for vector fields N^a tangent to the 2-surface, and hence $\mathcal{S}$ should be required to be axi-symmetric. Thus for axi-symmetric surfaces the observable $O[N^a]$ reproduces Komar's angular momentum, but for non-axi-symmetric surfaces in an axi-symmetric spacetime, whenever Komar's expression can still be calculated, $O[N^a]$ is not even well defined.

• *The small sphere limit*

To calculate $O[N^a]$ for small spheres $\mathcal{S}_r$ of radius r about a point $p \in M$ defined by the future pointing unit timelike vector t^a at p (for the standard definitions of all these limits see e.g. [9] and references therein), it seems more convenient to use the expression of N^a obtained form the application of the Hodge decomposition. Since no non-trivial harmonic form exists on spheres, we can write $N^a = \varepsilon^{ab}\delta_b\nu$ and ν is an arbitrary real function on $\mathcal{S}_r$. Since the field strength $-\varepsilon^{ab}\delta_a A_b$ is half the imaginary part of the complex Gauss curvature of $\mathcal{S}_r$ given in the well known GHP formalism by $K = -\psi_2 - \rho\rho' + \sigma\sigma' + \phi_{11} + \Lambda$, the observable (4.3) takes the form

$$O\big[N^a\big] = \frac{\mathrm{i}}{\kappa}\oint_{\mathcal{S}_r}\nu\Big(\psi_2 - \bar\psi_{2'} - \sigma\sigma' + \bar\sigma\bar\sigma'\Big)\mathrm{d}\mathcal{S}_r. \tag{4.8}$$

Expanding the Weyl spinor component as $\psi_2 = \psi_2^{(0)} + r\psi_2^{(1)} + r^2\psi_2^{(2)} + ...$ and substituting the solution of the Ricci identities for σ and σ' and the expression of $\mathrm{d}\mathcal{S}_r$ from [22] to (4.8), we obtain $\frac{\mathrm{i}}{\kappa}\oint_{\mathcal{S}_1}\nu(r^2[\psi_2^{(0)} - \bar\psi_{2'}^{(0)}] + r^3[\psi_2^{(1)} - \bar\psi_{2'}^{(1)}] + r^4([\psi_2^{(2)} - \bar\psi_{2'}^{(2)}] - \frac{1}{3}\psi_{00}^{(0)}[\psi_2^{(0)} - \bar\psi_{2'}^{(0)}] + \frac{2}{9}\phi_{20}^{(0)}\psi_0^{(0)} - \frac{2}{9}\phi_{02}^{(0)}\bar\psi_{0'}^{(0)}) + O(r^5))\mathrm{d}\mathcal{S}_1$. ($\mathrm{d}\mathcal{S}_1$ is, of course, the unit sphere area element.) To have a definite expression, we must specify the function ν by hand. Since $O[N^a]$ is usually expected to be something similar to spatial angular momentum, let us suppose that N^a is a linear combination of the three independent approximate spatial rotation Killing vectors in a neighbourhood of p that vanish at p and tangent to $\mathcal{S}_r$:

$$N^a = \frac{2\sqrt{2}r}{1+\zeta\bar\zeta}\Big(\bar m^a\big(M_{00}\zeta^2 + 2M_{01}\zeta + M_{11}\big) + m^a\big(\bar M_{0'0'}\bar\zeta^2 + 2\bar M_{0'1'}\bar\zeta + \bar M_{1'1'}\big)\Big) + O\big(r^2\big). \tag{4.9}$$

Here $M_{\underline{A}\,\underline{B}} = M_{(\underline{A}\,\underline{B})} = (M_{00}, M_{01}, M_{11})$ are complex constants satisfying $\bar M_{1'1'} = M_{00}$ and M_{01} is purely imaginary. (In Minkowski spacetime the leading order part of N_e is precisely the $2(M_{\underline{A}\,\underline{B}}K_f^{\underline{A}\,\underline{B}} + \bar M_{\underline{A}'\underline{B}'}\bar K_f^{\underline{A}'\underline{B}'})\Pi_e^f$ combination of the anti-self-dual boost-rotation Killing 1-forms $K_e^{\underline{A}\,\underline{B}}$ that vanish at p. For the details see [22].) Then the corresponding function ν is $4\mathrm{i}r^2(1+\zeta\bar\zeta)^{-1}(M_{00}\zeta + 2M_{01} - M_{11}\bar\zeta) + O(r^3)$. Substituting this into the general r^4 accurate approximate formula above we obtain that $O[N^a]$ is vanishing in the r^4 order, and in non-vacuum the first non-vanishing order is r^5. In vacuum $O[N^a]$ is vanishing in all orders up to (and including) r^6. Since here we considered only approximate *rotation* (but not boost) Killing fields, this result is compatible with the expectations of [9,22]: Although in general non-vacuum spacetime the leading term in the small sphere expression of any reasonable angular momentum expression must be of order r^4 and in vacuum it must be of order r^6, but these correspond to the centre-of-mass part of the relativistic angular momentum. The rotation part is expected to be only of order r^5 and r^7, respectively.

• *Large spheres near the future null infinity*
If $\mathcal{S}_r$ is a large sphere of radius r near the future null infinity (see e.g. [23]), then we can write $O[N^a]$ into the form (4.8). Taking into account the asymptotic form of the Weyl spinor component and the shears given in [23], and writing the function ν as $\nu = r^2\nu^{(2)} + r\nu^{(1)} + \nu^{(0)} + O(r^{-1})$, (4.8) takes the form

$$O[N^a] = \frac{\mathrm{i}}{\kappa}\oint_{\mathcal{S}_1}\Big\{r\nu^{(2)}\big({}_0\eth'^2\sigma^0 - {}_0\eth^2\bar\sigma^0\big)+ \\ +\nu^{(2)}\Big({}_0\eth\big(\bar\psi^0_{1'} + \bar\sigma^0\,{}_0\eth'\sigma^0\big) - {}_0\eth'\big(\psi^0_1 + \sigma^0\,{}_0\eth\bar\sigma^0\big)\Big) + \nu^{(1)}\big({}_0\eth'^2\sigma^0 - {}_0\eth^2\bar\sigma^0\big)\Big\}\mathrm{d}\mathcal{S}_1 + O\big(r^{-1}\big), \tag{4.10}$$

where ${}_0\eth$ is the standard edth operator on the metric unit sphere. $O[N^a]$ has finite $r\to\infty$ limit precisely when $\nu^{(2)} \in \ker{}_0\eth^2 \cap \ker{}_0\eth'^2$, or, explicitly, if $\nu^{(2)} = T^{\underline a}t_{\underline a}$ where $T^{\underline a}$ are arbitrary real numbers, $\underline a = 0, ..., 3$, and $t_0 := 1$, $t_1 := -(\bar\zeta+\zeta)(1+\zeta\bar\zeta)^{-1}$, $t_2 := -\mathrm{i}(\bar\zeta-\zeta)(1+\zeta\bar\zeta)^{-1}$, and $t_3 := -(\zeta\bar\zeta-1)(1+\zeta\bar\zeta)^{-1}$. (These are precisely the components of the independent BMS translations [24].) Then we have ${}_0\eth\nu^{(2)} = -2^{-\frac12}(1+\zeta\bar\zeta)^{-1}T^{\mathbf i}\xi_{\mathbf i}$ and ${}_0\eth'\nu^{(2)} = -2^{-\frac12}(1+\zeta\bar\zeta)^{-1}T^{\mathbf i}\bar\xi_{\mathbf i}$, where $\mathbf i = 1,2,3$, and $\xi_1 := 1-\zeta^2$, $\xi_2 := \mathrm{i}(1+\zeta^2)$ and $\xi_3 := 2\zeta$. Furthermore, direct calculation gives that ${}_0\eth{}_0\eth'\nu^{(2)} = {}_0\eth'{}_0\eth\nu^{(2)} = -T^{\mathbf i}t_{\mathbf i}$ holds. However, it is precisely the functions $\xi_{\mathbf i}$ that appear in the BMS rotation vector fields. Indeed, in the standard Bondi-type coordinate system $(u, r, \zeta, \bar\zeta)$ the general form of the BMS vector fields is

$$k^a = \Big(H + \big(b^{\mathbf i} + \bar b^{\mathbf i}\big)t_{\mathbf i}\;u\Big)\big(\frac{\partial}{\partial u}\big)^a + b^{\mathbf i}\frac{\sqrt2\xi_{\mathbf i}}{1+\zeta\bar\zeta}\bar{\hat m}^a + \bar b^{\mathbf i}\frac{\sqrt2\bar\xi_{\mathbf i}}{1+\zeta\bar\zeta}\hat m^a + O\big(r^{-1}\big), \tag{4.11}$$

where $H = H(\zeta,\bar\zeta)$ is an arbitrary real function, and $\hat m^a := \frac{1}{\sqrt2}(1+\zeta\bar\zeta)(\partial/\partial\bar\zeta)^a$, the Newman–Penrose complex null vector on the unit sphere normalized (with respect to the unit sphere metric) such that $\hat m^a\bar{\hat m}_a = -1$ (see e.g. [24,25]). Comparing the vector field N^a determined by $\nu^{(2)}$ and the BMS vector field above we obtain that *the vector field N^a corresponding to the function $\nu^{(2)}$ is the pure rotation BMS vector field with parameters $b^{\mathbf i} = \frac12\mathrm{i}T^{\mathbf i}$*. Thus it seems promising to calculate the observable $O[N^a]$ explicitly. It is

$$O[N^a] = \frac{1}{\kappa}\oint_{\mathcal{S}_1}\Big(-k_a\hat m^a\big(\bar\psi^0_{1'} + \bar\sigma^0{}_0\eth'\sigma^0\big) - k_a\bar{\hat m}^a\big(\psi^0_1 + \sigma^0{}_0\eth\bar\sigma^0\big) + \mathrm{i}\sigma^0\big({}_0\eth'^2\nu^{(1)}\big) - \mathrm{i}\bar\sigma^0\big({}_0\eth^2\nu^{(1)}\big)\Big)\mathrm{d}\mathcal{S}_1 + O\big(r^{-1}\big). \tag{4.12}$$

Though the first two terms of the integrand have some resemblance to several angular momentum expressions at future null infinity (see e.g. [25,26] and references therein), without additional restrictions on $\nu^{(1)}$ the last two terms make the whole expression totally ambiguous.

On the other hand, if the spacetime is stationary then the asymptotic shear is purely electric: $\sigma^0 = -{}_0\eth^2 S$ for some real function S (see e.g. [24]). Bramson [27] showed that in this case $2\bar\sigma^0{}_0\eth'\sigma^0 + {}_0\eth'(\sigma^0\bar\sigma^0) = 2{}_0\eth'^3\bar A + 2{}_0\eth\bar B$ for some functions A and B built from S and its ${}_0\eth$ and ${}_0\eth'$-derivatives. Furthermore, also in the stationary case, elementary calculation gives that ${}_0\eth'^2\sigma^0 = {}_0\eth^2\bar\sigma^0$. Substituting these into (4.10) or (4.12), and using $k_a\hat m^a = \mathrm{i}{}_0\eth(T^{\mathbf i}t_{\mathbf i})$ and $T^{\mathbf i}t_{\mathbf i} \in \ker{}_0\eth^2\cap\ker{}_0\eth'^2$, by partial integration we obtain

$$\begin{aligned} O[N^a] &= \frac{1}{\kappa}\oint_{\mathcal{S}_1}\Big(-k_a\hat m^a\big(\bar\psi^0_{1'} - \frac12{}_0\eth'(\sigma^0\bar\sigma^0) + {}_0\eth'^3\bar A + {}_0\eth\bar B\big) - \\ &\qquad - k_a\bar{\hat m}^a\big(\psi^0_1 - \frac12{}_0\eth(\sigma^0\bar\sigma^0) + {}_0\eth^3 A + {}_0\eth' B\big)\Big)\mathrm{d}\mathcal{S}_1 + O\big(r^{-1}\big) = \\ &= \frac{1}{\kappa}\oint_{\mathcal{S}_1}\Big(-k_a\hat m^a\psi^0_1 - k_a\bar{\hat m}^a\bar\psi^0_{1'}\Big)\mathrm{d}\mathcal{S}_1 + O\big(r^{-1}\big), \end{aligned} \tag{4.13}$$

which is the standard spatial angular momentum expression at future null infinity [27,25]. Thus in stationary spacetimes the ambiguities, coming from the arbitrariness of $\nu^{(1)}$, are canceled.

• *Large spheres near the spatial infinity*
Finally suppose that $\mathcal{S}_r$ is a large sphere of radius r near spatial infinity in an asymptotically flat slice. A straightforward calculation gives that

$$O[N^a]=-\frac{1}{\kappa}\oint_{\mathcal{S}_r}N^a\Pi^b_a\Big(\chi_{bc}-\chi h_{bc}\Big)v^c{\rm d}\mathcal{S}_r=2\int_{\Sigma_r}D_a\Big(\tilde p^{ab}N_b\Big){\rm d}^3x=2\int_{\Sigma_r}\Big(\big(D_a\tilde p^{ab}\big)N_b+\tilde p^{ab}D_{(a}N_{b)}\Big){\rm d}^3x, \tag{4.14}$$

whose $r\to\infty$ limit is the standard expression of the spatial angular momentum for the asymptotic rotation Killing vectors N^a [1-3]. However, to have finite and functionally differentiable global Hamiltonian the only N^a which is not vanishing at infinity must be an asymptotic translation or rotation. Hence by the condition $v_aN^a|_{\mathcal{S}_r}=0$ it must be a linear combination of the three independent asymptotic rotations. Therefore, *at spatial infinity $O[N^a]$ reduces to the standard spatial angular momentum.*

Therefore, to summarize: The basic Hamiltonian $H_0[N,N^a]$ is functionally differentiable with respect to the canonical variables on each sector $T^*\mathcal{Q}(\Sigma,\varepsilon_{a_1\dots a_{n-1}})$ provided N is vanishing and N^a is tangent to $\mathcal{S}$ on $\mathcal{S}$. This condition is $(n-1)+2$-covariant. If, in addition, N^a is required to be δ_a-divergence-free on $\mathcal{S}$, then the boundary conditions on the canonical variables are preserved by the evolution equations, the basic Hamiltonians form a Poisson algebra in which the constraints form an ideal, and the value of the basic Hamiltonian on the constraint surface defines a boost gauge-invariant, $(n-1)+2$-covariant quasi-local observable associated with the closed spacelike $(n-1)$-surface $\mathcal{S}$. In axi-symmetric spacetimes for axi-symmetric surfaces this observable coincides with the Komar angular momentum, at spatial infinity it reduces to the spatial angular momentum, for small spheres (with the approximate rotation Killing fields specified by hand) it is compatible with the expected behaviour of a reasonable quasi-local angular momentum expression, and in stationary spacetimes it reproduces the standard ambiguity-free angular momentum at null infinity. However, without additional restrictions on N^a (or on the still freely specifiable function ν) it is ambiguous at future null infinity of a radiative spacetime. Likewise, for general ν the integral $O[N^a]$ is not vanishing in Minkowski spacetime: That reduces only to the smeared average $\frac{\rm i}{\kappa}\oint_{\mathcal{S}}\nu(\bar\sigma\bar\sigma'-\sigma\sigma'){\rm d}\mathcal{S}$ of the two shears of $\mathcal{S}$. Thus the question arises whether we can find conditions on the function ν for which the observable $O[N^a]$ defines ambiguity-free angular momentum at full infinity, and, at the quasi-local level, $O[N^a]$ is vanishing in flat spacetime. This is still an open question.

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