Curvature estimates for stable marginally trapped surfaces

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Abstract. We derive integral and sup- estimates for the curvature of stably marginally outer trapped surfaces in a sliced space-time. The estimates bound the shear of a marginally outer trapped surface in terms of the intrinsic and extrinsic curvature of a slice containing the surface. These estimates are well adapted to situations of physical insterest, such as dynamical horizons.

1 Introduction

The celebrated regularity result for stable minimal surfaces, due to Schoen, Simon, and Yau [SSY75], gives a bound on the second fundamental form in terms of ambient curvature and area of the surface. The proof of the main result of [SSY75] makes use of the Simons formula [Sim68] for the Laplacian of the second fundamental form, together with the fact that the second variation of the area functional is an elliptic operator. In this paper we will prove a generalization of the regularity result of Schoen, Simon, and Yau to the natural analogue of stable minimal surfaces in the context of Lorentz geometry, stable marginally trapped surfaces.

Let Σ be a spacelike surface of codimension two in a 3+1 dimensional Lorentz manifold L and let $l^{\pm}$ be the two independent future directed null sections of the normal bundle of Σ, with corresponding mean curvatures, or null expansions, $\theta^{\pm}$. Σ is called trapped if the future directed null rays starting at Σ converge, *i.e.* $\theta^{\pm} < 0$. If L contains a trapped surface and satisfies certain causal conditions,

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then if in addition the null energy condition is satisfies, L is future causally incomplete [Pen65] Let l^+ be the outgoing null normal. If L is an asymptotically flat spacetime this notion is well defined, otherwise the outgoing direction can be fixed by convention. We call Σ a marginally outer trapped surface (MOTS) if the outgoing lightrays are marginally converging, *i.e.* if $\theta^+ = 0$. If Σ is contained in a time symmetric Cauchy surface, then $\theta^+ = 0$ if and only if Σ is minimal.

Marginally trapped surfaces are of central importance in general relativity, where they play the role of apparent horizons, or quasilocal black hole boundaries. The conjectured Penrose inequality, proved in the Riemannian case by Huisken and Ilmanen [HI01] and Bray [Bra01], may be formulated as an inequality relating the area of the outermost apparent horizon and the ADM mass. The technique of excising the interior of black holes using apparent horizons as excision boundaries plays a crucial role in current work in numerical relativity, where much of the focus is on modelling binary black hole collisions. In spite of the importance of marginally trapped surfaces in the geometry of spacetimes, the extent of our knowledge of the regularity and existence of these objects is rather limited compared to the situation for minimal surfaces.

A smooth marginally outer trapped surface is stationary with respect to variations of area within its outgoing null cone, in view of the formula

$$\delta_{fl^+}\mu_\Sigma = f\theta^+\mu_\Sigma$$

where f is a function on Σ. The second variation of area in the direction l^+ is

$$\delta_{fl^+}\theta^+ = -(|\chi^+|^2 + G(l^+, l^+))f$$

where G denotes the Einstein tensor of L, and χ^+ is the second fundamental form of Σ with respect to l^+. This shows that in contrast with minimal surfaces in a Riemannian manifold, or maximal hypersurfaces in a Lorentz manifold, where the second variation operator is an elliptic operator of second order, the second variation operator for area of a MOTS, with respect to variations in the null direction l^+, is an operator of order zero. Therefore, although MOTS can be characterized as stationary points of area, this point of view alone is not sufficient to yield a useful regularity result. In spite of this, as we will see below, there is a natural generalization of the stability condition for minimal surfaces, as well as the regularity result of Schoen, Simon, and Yau, to marginally outer trapped surfaces.

It is worth remarking at this point that if we consider variations of area of spacelike hypersurfaces in a Lorentz manifolds, the stationary points are maximal surfaces. Maximal surfaces satisfy a quasilinear non-uniformly elliptic equation closely related to the minimal surface equation. However, due to the fact that maximal hypersurfaces are spacelike, they are Lipschitz submanifolds. Moreover, in a spacetime satisfying the timelike convergence condition, every maximal surface is stable. Hence, the regularity theory for maximal surfaces is of a different flavor than the regularity theory for minimal surfaces, cf. [Bar84].

Assume that L is provided with a reference foliation consisting of spacelike hypersurfaces $\{M_t\}$, and that Σ is contained in one of the leaves M of this foliation. Let (g, K) be the induced metric and the second fundamental form of M with respect to the future directed timelike normal n. Further, let ν be the outward pointing normal of Σ in M and let A be the second fundamental form of Σ with respect to ν. After possibly changing normalization, $l^{\pm} = n \pm \nu$, we have

$$\theta^{\pm} = H \pm \operatorname{tr}_{\Sigma} K$$

where $H = \operatorname{tr} A$ is the mean curvature of Σ and $\operatorname{tr}_{\Sigma} K$ is the trace of the projection of K to Σ. Thus the condition for Σ to be a MOTS, $\theta^{+} = 0$, is a prescribed mean curvature equation.

The condition that plays the role of stability for MOTS is the stably locally outermost condition, see [AMS05, New87]. Suppose Σ is contained in a spatial hypersurface M. Then Σ is stably locally outermost in M if there is an outward deformation of Σ, within M which does not decrease θ^{+}. This condition, which is equivalent to the condition that Σ is stable in case M is time symmetric, turns out to be sufficient to apply the technique of [SSY75] to prove a bound on the second fundamental form A of Σ in M.

The techniques of [SSY75] were first applied in the context of general relativity by Schoen and Yau [SY81], where existence and regularity for Jang's equation were proved. Jang's equation is an equation for a graph in $N = M \times \Re$, and is of a form closely related to the equation $\theta^{+} = 0$. Let u be a function on M, and let $\bar{K}$ be the pull-back to N of K along the projection $N \to M$. Jang's equation is the equation

$$\bar{g}^{ij} \left(\frac{D_i D_j u}{\sqrt{1 + |Du|^2}} + \bar{K}_{ij} \right) = 0$$

where $\bar{g}^{ij} = g^{ij} - \frac{D_i u D_j u}{1+|Du|^2}$ is the induced metric on the graph $\bar{\Sigma}$ of u in N. is the induced metric on the graph of u. Thus Jang's equation can be written as $\bar{\theta} = 0$ with

$$\bar{\theta} = \bar{H} + \operatorname{tr}_{\bar{\Sigma}} \bar{K},$$

where $\bar{H}$ is the mean curvature of $\bar{\Sigma}$ in N. This shows that Jang's equation $\bar{\theta} = 0$ is a close analog to the equation $\theta^{+} = 0$ characterizing a MOTS. Solutions to Jang's equation satisfy a stability condition closely related to the stably locally outermost condition stated above, due to the fact that Jang's equation is translation invariant in the sense that if u solves Jang's equation, then also $u + c$ is a solution where c is a constant.

Statement of Results

The stability condition for MOTS which replaces the stability condition for minimal surfaces and which allows one to apply the technique of [SSY75] is the following.

Definition 1.1. *Σ is stably outermost if there is a function $f \geq 0$ on Σ, $f \neq 0$ somewhere, such that $\delta_{f\nu}\theta^+ \geq 0$.*

This is analogous to the stability condition for a minimal surface $N \subset M$. The condition that there exist a function f on N, $f \geq 0$, $f \neq 0$ somewhere, such that $\delta_{f\nu} H \geq 0$ is equivalent to the condition that N is stable.

The main result of this paper is the following theorem, cf. theorem 6.10 and corollary 6.11.

Theorem 1.2. *Suppose Σ is a stable MOTS surface in (M, g, K). Then the second fundamental form A satisfies the inequality*

$$|A| \leq C(|\Sigma|, \|K\|_\infty, \|\nabla K\|_\infty, \|{}^M\mathrm{Rm}\|_\infty, \mathrm{inj}(M, g))\,.$$

As an application we prove the following compactness result for stable MOTS, cf. theorem 7.1.

Theorem 1.3. *Let L be a smooth part of a space-time sliced by smooth space-like surfaces M_t, $t \in [t_0, t_1]$. Assume that uniformly in t*

$$\|{}^M\mathrm{Rm}_t\|_\infty \leq C\,,$$
$$\|K_t\|_\infty + \|{}^M\nabla_t K_t\|_\infty \leq C \text{ and}$$
$$\mathrm{inj}(M_t, g_t) \geq C^{-1}\,.$$

Here K_t and ${}^M\mathrm{Rm}_t$ are the intrinsic and extrinsic curvatures of M_t and $\mathrm{inj}(M_t, g_t)$ is the injectivity radius of (M_t, g_t).

Let Σ_n be a sequence of stable marginally outer trapped surfaces such that $\Sigma_n \subset M_{t_n}$ for some t_n, the area of the Σ_n is bounded $|\Sigma_n| \leq C$, and all Σ_n are contained in a compact subset of L.

Then the sequence Σ_n accumulates on a smooth stable MOTS Σ. $\square$

2 Preliminaries and Notation

In this section we set up notation and recall some preliminaries from differential geometry. In the sequel we will consider two-dimensional spacelike submanifolds Σ of a four dimensional manifold L. As a space time manifold, L will be equipped with a metric h of signature $(-, +, +, +)$. The inner product induced by H will frequently be denoted by $\langle \cdot, \cdot, \rangle$. In addition, we will assume, that Σ is contained in a spacelike hypersurface M in L. The metric on M induced by h will be denoted by g, the metric on Σ by γ. We will denote the tangent bundles by TL, TM, and $T\Sigma$, and the space of smooth tangential vector fields along the respective manifolds by $\mathcal{X}(\Sigma)$, $\mathcal{X}(M)$, and $\mathcal{X}(L)$. Unless otherwise stated, we will assume that all manifolds and fields are smooth.

We denote by n the future directed unit timelike normal of M in L, which we will assume to be a well defined vector field along M. The normal of Σ in M will be denoted by ν, which again is assumed to be a well defined vector field along Σ.

The two directions n and ν span the normal bundle $\mathcal{N}\Sigma$ of Σ in L, and moreover, we can use them to define two canonical null directions, which also span this bundle, namely $l^\pm := n \pm \nu$.

In addition to the metrics, h and its Levi-Civita connection ${}^L\nabla$ induce the second fundamental form K of M in L. It is the normal part of ${}^L\nabla$, in the sense that for all vector fields $X, Y \in \mathcal{X}(M)$

$$ {}^L\nabla_X Y = {}^M\nabla_X Y + K(X,Y)n \,. \tag{2.1} $$

The second fundamental form of Σ in M will be denoted by A. For vector fields $X, Y \in \mathcal{X}(\Sigma)$ we have

$$ {}^M\nabla_X Y = {}^\Sigma\nabla_X Y - A(X,Y)\nu \,. \tag{2.2} $$

For vector fields $X, Y \in \mathcal{X}(\Sigma)$, the connection of L therefore splits according to

$$ {}^L\nabla_X Y = {}^\Sigma\nabla_X Y + K^\Sigma(X,Y)n - A(X,Y)\nu = {}^\Sigma\nabla_X Y - \mathrm{I\!I}(X,Y) \,, \tag{2.3} $$

where $\mathrm{I\!I}(X,Y) = A(X,Y)\nu - K^\Sigma(X,Y)n$ is the second fundamental form of Σ in L. Here K^Σ denotes the restriction of K to $T\Sigma$, the tangential space of Σ.

The trace of $\mathrm{I\!I}$ with respect to γ, which is a vector in the normal bundle of Σ, is called the mean curvature vector and is denoted by

$$ \mathcal{H} = \sum_i \mathrm{I\!I}(e_i, e_i) \,, \tag{2.4} $$

for an orthonormal basis e_1, e_2 of Σ. Since $\mathcal{H}$ is normal to Σ, it satisfies

$$ \mathcal{H} = H\nu - Pn \tag{2.5} $$

where $H = \gamma^{ij} A_{ij}$ is the trace of A and $P = \gamma^{ij} K^\Sigma_{ij}$ the trace of K^Σ, with respect to γ. For completeness, we note that the norms of $\mathrm{I\!I}$ and $\mathcal{H}$ are given by

$$ |\mathrm{I\!I}|^2 = |A|^2 - |K^\Sigma|^2 \qquad \text{and} \tag{2.6} $$

$$ |\mathcal{H}|^2 = H^2 - P^2 \,. \tag{2.7} $$

Recall that since $\mathcal{H}$ and $\mathrm{I\!I}$ have values normal to Σ, the norms are taken with respect to h and are therefore not necessarily nonnegative.

We use the following convention to represent the Riemannian curvature tensor ${}^\Sigma\mathrm{Rm}$, the Ricci tensor ${}^\Sigma\mathrm{Rc}$, and the scalar curvature ${}^\Sigma\mathrm{Sc}$ of Σ. Here $X, Y, U, V \in \mathcal{X}(\Sigma)$ are vector fields.

$$ {}^\Sigma\mathrm{Rm}(X,Y,U,V) = \left\langle {}^\Sigma\nabla_X {}^\Sigma\nabla_Y U - {}^\Sigma\nabla_Y {}^\Sigma\nabla_X U - {}^\Sigma\nabla_{[X,Y]} U, V \right\rangle \,, $$

$$ {}^\Sigma\mathrm{Rc}(X,Y) = \sum_i {}^\Sigma\mathrm{Rm}(X, e_i, e_i, Y) \,, $$

$$ {}^\Sigma\mathrm{Sc} = \sum_i {}^\Sigma\mathrm{Rc}(e_i, e_i) \,. $$

Analogous definitions hold for ${}^M\mathrm{Rm}, {}^M\mathrm{Rc}, {}^M\mathrm{Sc}$ and ${}^L\mathrm{Rm}, {}^L\mathrm{Rc}, {}^L\mathrm{Sc}$, with the exception that for ${}^L\mathrm{Rc}$ and ${}^L\mathrm{Sc}$ we thake the trace with respect to the indefinite metric h.

We recall the Gauss and Codazzi equations of Σ in L, which relate the respective curvatures. The Riemannian curvature tensors ${}^\Sigma\mathrm{Rm}$ and ${}^L\mathrm{Rm}$ of Σ and L respectively, are related by the Gauss equation. For vector fields X, Y, U, V we have

$$\begin{aligned} &{}^\Sigma\mathrm{Rm}(X,Y,U,V) = \\ &\qquad {}^L\mathrm{Rm}(X,Y,U,V) + \big\langle \mathrm{I\!I}(X,V), \mathrm{I\!I}(Y,U)\big\rangle - \big\langle \mathrm{I\!I}(X,U), \mathrm{I\!I}(Y,V)\big\rangle \,. \end{aligned} \tag{2.8}$$

In two dimensions, all curvature information of Σ is contained in its scalar curvature, which we will denote by ${}^\Sigma\mathrm{Sc}$. The scalar curvature of L will be denoted by ${}^L\mathrm{Sc}$. The information of the Gauss equation above is fully contained in the following equation, which emerges from the above one by first taking the trace with respect to Y, U and then with respect to X, V

$${}^\Sigma\mathrm{Sc} = {}^L\mathrm{Sc} + 2\,{}^L\mathrm{Rc}(n,n) - 2\,{}^L\mathrm{Rc}(\nu,\nu) - 2\,{}^L\mathrm{Rm}(\nu,n,n,\nu) + |\mathcal{H}|^2 - |\mathrm{I\!I}|^2 \,. \tag{2.9}$$

The Codazzi equation, which relates ${}^L\mathrm{Rm}$ to $\mathrm{I\!I}$, has the following form

$$\big\langle\, {}^L\nabla_X \mathrm{I\!I}(Y,Z), S\big\rangle = \big\langle \nabla_Y \mathrm{I\!I}(X,Z), S\big\rangle + {}^L\mathrm{Rm}(X,Y,S,Z) \tag{2.10}$$

for vector fields $X, Y, Z \in \mathcal{X}(\Sigma)$ and $S \in \Gamma(\mathcal{N}\Sigma)$.

There is also a version of the Gauss and Codazzi equations for the embedding of M in L. They relate the curvature ${}^L\mathrm{Rm}$ of L to the curvature ${}^M\mathrm{Rm}$ of M. For vector fields $X, Y, U, V \in \mathcal{X}(M)$ we have

$$\begin{aligned} &{}^M\mathrm{Rm}(X,Y,U,V) \\ &\quad = {}^L\mathrm{Rm}(X,Y,U,V) - K(Y,U)K(X,V) + K(X,U)K(Y,V)\,, \end{aligned} \tag{2.11}$$

$${}^M\nabla_X K(Y,U) - {}^M\nabla_Y K(X,U) = {}^L\mathrm{Rm}(X,Y,n,U)\,. \tag{2.12}$$

These equations also have a traced form, namely

$${}^M\mathrm{Sc} = {}^L\mathrm{Sc} + 2\,{}^L\mathrm{Rc}(n,n) - (\mathrm{tr}\,K)^2 + |K|^2 \text{ and} \tag{2.13}$$

$${}^M\mathrm{div}K - {}^M\nabla\,\mathrm{tr}\,K = {}^L\mathrm{Rc}(\cdot, n)\,. \tag{2.14}$$

We now investigate the connection ${}^N\nabla$ on the normal bundle $\mathcal{N}\Sigma$ of Σ. Recall that for sections N of $\mathcal{N}\Sigma$ and $X \in \mathcal{X}(\Sigma)$, this connection is defined as follows

$${}^N\nabla_X N = \big({}^L\nabla_X N\big)^\perp \,,$$

where again $(\cdot)^\perp$ means taking the normal part. We have

$$0 = X(1) = X\big(\langle n,n\rangle\big) = 2\langle\, {}^N\nabla_X n, n\rangle\,,$$

and similarly $\langle {}^N\nabla_X \nu, \nu\rangle = 0$. Therefore the relevant component of ${}^N\nabla$ is

$$\big\langle {}^N\nabla_X \nu, n\big\rangle = \big\langle {}^L\nabla_X \nu, n\big\rangle = -K(X,\nu)\,.$$

Recall that X is tangential to Σ. This lead us to define the 1-form S along Σ by the restriction of $K(\cdot,\nu)$ to $T\Sigma$.

$$S(X) := K(X,\nu)\,. \tag{2.15}$$

Then, for an arbitrary section N of $\mathcal{N}\Sigma$ with $N = f\nu + gn$, we have

$${}^N\nabla_X N = X(f)\nu + X(g)n + S(X)\big(fn + g\nu\big)\,.$$

In particular

$${}^N\nabla_X l^\pm = \pm S(X) l^\pm\,. \tag{2.16}$$

We will later consider the decomposition of $\mathrm{I\!I}$ into its null components. For $X, Y \in \mathcal{X}(\Sigma)$ let

$$\chi^\pm(X,Y) := \big\langle \mathrm{I\!I}(X,Y), l^\pm\big\rangle = K(X,Y) \pm A(X,Y)\,. \tag{2.17}$$

The traces of $\chi^\pm$ respectively will be called $\theta^\pm$

$$\theta^\pm = \langle \mathcal{H}, l^\pm\rangle = P \pm H\,. \tag{2.18}$$

The Codazzi-equation (2.10) implies a Codazzi equation for $\chi^\pm$.

Lemma 2.1. *For vector fields $X, Y, Z \in \mathcal{X}(\Sigma)$ the following relation holds*

$$\nabla_X \chi^\pm(Y,Z) = \nabla_Y \chi^\pm(X,Z) + Q^\pm(X,Y,Z) \mp \chi^\pm(X,Z)S(Y) \pm \chi^\pm(Y,Z)S(X)\,. \tag{2.19}$$

Here $Q^\pm(X,Y,Z) = {}^L\mathrm{Rm}(X,Y,l^\pm,Z)$.

3 A Simons identity for $\chi^\pm$

We use the Codazzi equation we derived in the previous section to compute an identity for the laplacian of $\chi^\pm$, which is very similar to the Simons identity for the second fundamental form of a hypersurface [Sim68, SSY75].

The Laplacian on the surface Σ is defined as the operator

$${}^\Sigma\Delta = \gamma^{ij}\,{}^\Sigma\nabla^2_{ij}\,.$$

In the sequel, we will drop the superscript on ${}^\Sigma\Delta$ and ${}^\Sigma\nabla$, since all tensors below will be defined only along Σ. We will switch to index notation, since this is convenient for the computations to follow. In this notation

$$T^{i_1\cdots i_p}_{j_1\cdots j_q}$$

denotes a (p,q)-tensor T as the collection of its compontents in an arbitrary basis $\{\partial_i\}_{i=1}^2$ for the tangent spaces. To make the subsequent computations easier, we will usally pick a basis of normal coordinate vectors. Also note that we use latin indices ranging from 1 to 2 to denote components tangential to the surface Σ.

Recall, that the commutator of the connection is given by the Riemann curvature tensor, such that for a $(0,2)$-tensor T_{ij}

$$\nabla_k\nabla_l T_{ij} - \nabla_l\nabla_k T_{ij} = {}^\Sigma\mathrm{Rm}_{klmi}T_{mj} + {}^\Sigma\mathrm{Rm}_{klmj}T_{im}\,. \tag{3.1}$$

Note that we use the shorthand ${}^\Sigma\mathrm{Rm}_{klmj}T_{im} = {}^\Sigma\mathrm{Rm}_{klpj}T_{iq}\gamma^{pq}$, when there is no ambiguity. That is, we assume that we are in normal coordinates where $\gamma_{ij} = \gamma^{ij} = \delta_{ij}$. Also note that this fixes the sign convention for ${}^\Sigma\mathrm{Rm}_{ijkl}$ such that ${}^\Sigma\mathrm{Rc}_{ij} = {}^\Sigma\mathrm{Rm}_{ikkj}$ is positive on the round sphere.

Lemma 3.1. *The Laplacian of $\chi = \chi^+$ satisfies the following identity*

$$\begin{aligned}\chi_{ij}\Delta\chi_{ij} &= \chi_{ij}\nabla_i\nabla_j\theta^+ + \chi_{ij}\big({}^L\mathrm{Rm}_{kilk}\chi_{lj} + {}^L\mathrm{Rm}_{kilj}\chi_{kl}\big)\\ &\quad + \chi_{ij}\nabla_k\big(Q_{kij} - \chi_{kj}S_i + \chi_{ij}S_k\big) + \chi_{ij}\nabla_i\big(Q_{kjk} - \theta^+S_j + \chi_{jk}S_k\big)\\ &\quad - |\mathrm{I\!I}|^2|\chi|^2 + \theta^+\chi^+_{ij}\chi^+_{jk}\chi^+_{ki} - \theta^+\chi^+_{ij}\chi^+_{jk}K^\Sigma_{ki} - P\chi^+_{ij}\chi^+_{jk}\chi^+_{ki}\end{aligned}$$

where $P = \gamma^{ij}K^\Sigma_{ij}$ is the trace of K^Σ.

Proof. Recall that in coordinates the Codazzi equation (2.19) for χ_{ij} reads

$$\nabla_i\chi_{jk} = \nabla_j\chi_{ik} + Q_{ijk} - \chi_{ik}S_j + \chi_{jk}S_i\,. \tag{3.2}$$

Then compute, using (3.2) in the first and third step, and the commutator relation (3.1) in the second, to obtain

$$\begin{aligned}\nabla_k\nabla_l\chi_{ij} &= \nabla_k\nabla_i\chi_{lj} + \nabla_k\big(Q_{lij} - \chi_{lj}S_i + \chi_{ij}S_l\big)\\ &= \nabla_i\nabla_k\chi_{lj} + {}^\Sigma\mathrm{Rm}_{kiml}\chi_{mj} + {}^\Sigma\mathrm{Rm}_{kimj}\chi_{lm}\\ &\quad + \nabla_k\big(Q_{lij} - \chi_{lj}S_i + \chi_{ij}S_l\big)\\ &= \nabla_i\nabla_j\chi_{kl} + {}^\Sigma\mathrm{Rm}_{kiml}\chi_{mj} + {}^\Sigma\mathrm{Rm}_{kimj}\chi_{lm}\\ &\quad + \nabla_k\big(Q_{lij} - \chi_{lj}S_i + \chi_{ij}S_l\big) + \nabla_i\big(Q_{kjl} - \chi_{kl}S_j + \chi_{jl}S_k\big)\,.\end{aligned} \tag{3.3}$$

We will use the Gauss equation (2.8) to replace the ${}^\Sigma\mathrm{Rm}$-terms by ${}^L\mathrm{Rm}$-terms. Observe, that

$$\mathrm{I\!I}_{ij} = -\tfrac{1}{2}\chi^+_{ij}l^- - \tfrac{1}{2}\chi^-_{ij}l^+\,.$$

Plugging this into the Gauss equation (2.8) gives

$${}^\Sigma\mathrm{Rm}_{ijkl} = {}^L\mathrm{Rm}_{ijkl} + \tfrac{1}{2}\big(\chi^+_{ik}\chi^-_{jl} + \chi^-_{ik}\chi^+_{jl} - \chi^+_{il}\chi^-_{jk} - \chi^-_{il}\chi^+_{jk}\big)\,.$$

Combining with (3.3), we infer that

$$\begin{aligned}\nabla_k\nabla_l\chi_{ij} &= \nabla_i\nabla_j\chi_{kl} + {}^L\mathrm{Rm}_{kiml}\chi_{mj} + {}^L\mathrm{Rm}_{kimj}\chi_{lm} \\ &\quad + \tfrac{1}{2}\big(\chi^+_{il}\chi^-_{km} + \chi^-_{il}\chi^+_{km} - \chi^+_{kl}\chi^-_{im} - \chi^-_{kl}\chi^+_{im}\big)\chi^+_{mj} \\ &\quad + \tfrac{1}{2}\big(\chi^+_{km}\chi^-_{ij} + \chi^-_{km}\chi^+_{ij} - \chi^+_{kj}\chi^-_{im} - \chi^-_{kj}\chi^+_{im}\big)\chi^+_{lm} \\ &\quad + \nabla_k\big(Q_{lij} - \chi_{lj}S_i + \chi_{ij}S_l\big) + \nabla_i\big(Q_{kjl} - \chi_{kl}S_j + \chi_{jl}S_k\big)\,.\end{aligned}$$

Taking the trace with respect to k, l yields

$$\begin{aligned}\Delta\chi_{ij} &= \nabla_i\nabla_j\theta^+ + {}^L\mathrm{Rm}_{kilk}\chi_{lj} + {}^L\mathrm{Rm}_{kilj}\chi_{kl} \\ &\quad + \nabla_k\big(Q_{kij} - \chi_{kj}S_i + \chi_{ij}S_k\big) + \nabla_i\big(Q_{kjk} - \theta^+S_j + \chi_{jk}S_k\big) \\ &\quad + \frac{1}{2}\big(\chi^-_{ij}|\chi^+|^2 + \langle\chi^+,\chi^-\rangle\chi^+_{ij} - \theta^+\chi^+_{jk}\chi^-_{ki} - \theta^-\chi^+_{jk}\chi^+_{ki}\big) \\ &\quad + \frac{1}{2}\big(\chi^+_{jk}\chi^-_{kl}\chi^+_{li} - \chi^-_{jk}\chi^+_{kl}\chi^+_{li}\big)\end{aligned}$$

We contract this equation with χ^+_{ij} and obtain

$$\begin{aligned}\chi_{ij}\Delta\chi_{ij} &= \chi_{ij}\nabla_i\nabla_j\theta^+ + \chi_{ij}\big({}^L\mathrm{Rm}_{kilk}\chi_{lj} + {}^L\mathrm{Rm}_{kilj}\chi_{kl}\big) \\ &\quad + \chi_{ij}\nabla_k\big(Q_{kij} - \chi_{kj}S_i + \chi_{ij}S_k\big) + \chi_{ij}\nabla_i\big(Q_{kjk} - \theta^+S_j + \chi_{jk}S_k\big) \\ &\quad + \langle\chi^+,\chi^-\rangle|\chi|^2 - \tfrac{1}{2}\theta^+\chi^+_{ij}\chi^+_{jk}\chi^-_{ki} - \tfrac{1}{2}\theta^-\chi^+_{ij}\chi^+_{jk}\chi^+_{ki}\,.\end{aligned}$$

Now observe that $\chi^-_{ij} = 2K^\Sigma_{ij} - \chi^+_{ij}$ and $\theta^- = 2P - \theta^+$. Substituting this into the last two terms, together with $\langle\chi^+,\chi^-\rangle = -|\mathrm{II}|^2$, we arrive at the identity we claimed. □

4 The Linearization of θ^+

This section is concerned with the linearization of the operator θ^+, as defined in equation (2.18). We begin by considering an arbitrary, spacelike hypersurface $\Sigma \subset L$. Assume that the normal bundle is spanned by the globally defined null vector fields $l^\pm$, such that $\langle l^+, l^-\rangle = -2$. We call such a frame a *normalized null frame*. As before, let $\theta^\pm := \langle\mathcal{H}, l^\pm\rangle$.

A variation of Σ is a differentiable map

$$F : \Sigma \times (-\varepsilon, \varepsilon) \to L : (x,t) \mapsto F(x,t)\,,$$

such that $F(\cdot, 0) = \mathrm{id}_\Sigma$ is the identity on Σ. The vector field $\frac{\partial F}{\partial t}\big|_{t=0} = V$ is called variation vector field of F. We will only consider variations, with variation vector fields V of the form $V = \alpha l^+ + \beta l^-$.

Note that in this setting, as a normalized null frame is not uniquely defined by its properties, the notion of θ^+ depends on the frame chosen. The freedom we have here is the following. Assume $k^\pm$ is another normalized null frame for the normal bundle of Σ, that is $h(k^\pm, k^\pm) = 0$ and $h(k^+, k^-) = -2$. Since the null cone at each point is unique, the directions of $k^\pm$ can be aligned with $l^\pm$. But their magnitudes can be different, so $k^+ = e^\omega l^+$ and $k^- = e^{-\omega} l^-$ with a function $\omega \in C^\infty(\Sigma)$.

Therefore, if we want to compute the linearization of θ^+, it will not only depend on the deformation of Σ, as encoded in the deformation vector V. It will also depend on the change of the frame, that is on the change of the vector l^+, which is an additional degree of freedom.

To expose the nature of that freedom, observe that if $l^\pm(t)$ is a null frame on each $\Sigma_t := F(\Sigma, t)$, then $\frac{\partial l^\pm}{\partial t}\Big|_{t=0}$ is still normal to Σ. On the other hand

$$
\begin{aligned}
0 &= \tfrac{\partial}{\partial t}\Big|_{t=0} \langle l^+, l^+ \rangle = 2\Big\langle \tfrac{\partial l^+}{\partial t}\Big|_{t=0}, l^+ \Big\rangle \qquad \text{and} \\
0 &= \tfrac{\partial}{\partial t}\Big|_{t=0} \langle l^+, l^- \rangle = \Big\langle \tfrac{\partial l^+}{\partial t}\Big|_{t=0}, l^- \Big\rangle + \Big\langle \tfrac{\partial l^-}{\partial t}\Big|_{t=0}, l^+ \Big\rangle
\end{aligned}
$$

Therefore $\frac{\partial l^\pm}{\partial t}\Big|_{t=0} = w l^\pm$ for a function $w \in C^\infty(\Sigma)$. Thus the linearized change of the frame is described by the single function w, which we will call the *variation of the null frame*.

If we fix both of the quantities V and w, a straight forward (but lengthy) computation gives the linearization of θ^+.

Lemma 4.1. *Assume $F : \Sigma \times (-\varepsilon, \varepsilon) \to L$ is a variation of Σ with variation vector field $V = \alpha l^+ + \beta l^-$. Assume further that the variation of the null frame is w. Then the variation of θ^+ is given by*

$$
\begin{aligned}
\delta_{V,w}\theta^+ &= 2\Delta\beta - 4S(\nabla\beta) - \alpha\big(|\chi^+|^2 + {}^L\mathrm{Rc}(l^+, l^+)\big) + 2\theta^+ w \\
&\quad - \beta\big(2\operatorname{div} S - 2|S|^2 - |\mathrm{I\!I}|^2 + {}^L\mathrm{Rc}(l^+, l^-) - \tfrac{1}{2}{}^L\mathrm{Rm}(l^+, l^-, l^-, l^+)\big)\,.
\end{aligned}
$$

If we consider marginally trapped surfaces, then the term $\theta^+ w$ in the previous calculation vanishes, and we get expressions independent of the change in the frame. As a consequence, we state the following two corollaries, which also restrict the variations we take into account.

Corollary 4.2. *Assume Σ is a marginally trapped surface, that is, it satisfies the equation $\theta^+ = 0$. Then the linearization of θ^+ in direction of $-l^-$ is given by*

$$
\delta_{-\beta l^-, w}\theta^+ = 2L_-\beta\,,
$$

where the operator L_- is given by

$$
L_-\beta = -\Delta\beta + 2S(\nabla\beta) + \beta\big(\operatorname{div} S - \tfrac{1}{2}|\mathrm{I\!I}|^2 - |S|^2 - \Psi_-\big)\,,
$$

and $\Psi_- = \frac{1}{4}{}^L\mathrm{Rm}(l^+, l^-, l^-, l^+) - \frac{1}{2}{}^L\mathrm{Rc}(l^+, l^-)$.

If we assume that $\Sigma \subset M$, where M is a three dimensional spacelike surface, then Σ can be deformed in the direction of ν, the normal of Σ in M. The linearization of θ^+ then turns out to be the following.

Corollary 4.3. *Assume Σ is a marginally trapped surface, then the linearization of θ^+ in the spatial direction of $\nu := \frac{1}{2}(l^+ - l^-)$ is given by*

$$\delta_{f\nu,w} = L_M f \,,$$

where the operator L_M is given by

$$L_M f = -\Delta f + 2S(\nabla f) + f\big(\operatorname{div} S - |\chi|^2 + \langle K^\Sigma, \chi^+\rangle - |S|^2 - \Psi_M\big) \,,$$

and $\Psi_M = \frac{1}{4}{}^L\mathrm{Rm}(l^+, l^-, l^-, l^+) + {}^L\mathrm{Rc}(\nu, l^+)$.

Remark 4.4. (i) Using the Gauss equation (2.9), we can rewrite the expression for L_M as follows

$$L_M f = -\Delta f + 2S(\nabla f) + f\big(\operatorname{div} S - \tfrac{1}{2}|\chi|^2 - |S|^2 + \tfrac{1}{2}{}^\Sigma\mathrm{Sc} - \tilde\Psi_M\big) \,. \qquad (4.1)$$

Here $\tilde\Psi_M = G(n, l^+)$ where $G = {}^L\mathrm{Rm} - \frac{1}{2}{}^L\mathrm{Sc}h$ denotes the Einstein tensor of h. Note that in view of the Gauss and Codazzi equations of the embedding $M \hookrightarrow L$, equations (2.13) and (2.14), the term $\tilde\Psi_M$ can be rewritten as

$$\tilde\Psi_M = \tfrac{1}{2}\big({}^M\mathrm{Sc} + (\operatorname{tr} K)^2 - |K|^2\big) - \langle {}^M\mathrm{div}K - {}^M\nabla \operatorname{tr} K, \nu\rangle =: 8\pi\big(\mu - J(\nu)\big) \,, \quad (4.2)$$

where $8\pi J = {}^M\mathrm{div}K - {}^M\nabla \operatorname{tr} K$ is the projection of $G(n,\cdot)$ to M and $16\pi\mu = {}^M\mathrm{Sc} + (\operatorname{tr} K)^2 - |K|^2 = G(n,n)$. The dominant energy condition is equivalent to $|J| \le \mu$. Thus, if the dominant energy condition holds, $\tilde\Psi_M$ turns out to be non-negative.
(ii) The same procedure gives that we can write L_- as

$$L_- f = -\Delta f + 2S(\nabla f) + f\big(\operatorname{div} S - |S|^2 + \tfrac{1}{2}{}^\Sigma\mathrm{Sc} - \tilde\Psi_-\big) \,. \qquad (4.3)$$

with $\tilde\Psi_- = G(l^+, l^-)$. Note that $\tilde\Psi_-$ is non-negative if the dominant energy condition holds. However, this representation does not contain a term $|\chi|^2$, which does not allow us to get estimates. □

5 Stability of marginally outer trapped surfaces

As before, consider a four dimensional space time L^4, with a three dimensional spacelike slice M^3. As in the previous sections, the future directed unit normal to M in L will be denoted by n. In M consider a two dimensional surface Σ, such that there exists a global unit normal vector field ν of Σ in M. The vector fields n and ν span the normal bundle of Σ in L and give rise to two canonical null vectors $l^\pm = n \pm \nu$.

In this section we will introduce two notions of stability for a marginally trapped surface. These are related to variations of the surface in different directions. The first definition is related to definition 2 in [AMS05]. There a *stably outermost marginally outer trapped surface*, is defined as surface, on which the principal eigenvalue of L_M is positive. Here an *L_M-stable MOTS* is defined as follows.

Definition 5.1. *A two dimensional surface $\Sigma \subset M \subset L$ is called a L_M-*stable marginally outer trapped surface *if*
(i) Σ is marginally trapped with respect to l^+ , that is $\theta^+ = 0$.
(ii) There exists a function $f \geq 0, f \not\equiv 0$ such that $L_M f \geq 0$. Here L_M is the operator from corollary 4.3.

Remark 5.2. (i) Although L_M is not formally self-adjoint, the eigenvalue of L_M with the smallest real part is real and non-negative (cf. [AMS05, Lemma 1]). This definition is equivalent to saying, that the principal eigenvalue of L_M is nonnegative. This is seen as follows:
Let λ be the principal eigenvalue L_M. Then, since λ is real, the L^2-adjoint L_M^* of L_M has the same principal eigenvalue and a corresponding eigenfunction $g > 0$. Pick $f \geq 0$ as in the definition of L_M-stability, ie. $L_M f \geq 0$. Then compute

$$\lambda \int_\Sigma fg \, \mathrm{d}\mu = \int_\Sigma f L_M^* g \, \mathrm{d}\mu = \int_\Sigma L_M f g \, \mathrm{d}\mu \, .$$

As $f \geq 0$, $f \not\equiv 0$, $g > 0$ and $L_M f \geq 0$, this implies $\lambda \geq 0$.
The eigenfunction ψ of L_M with respect to the principal eigenvalue does not change sign. Therefore it can be chosen positive, $\psi > 0$. Thus, the definition in fact is equivalent to the existence of $\psi > 0$ such that $L_M \psi = \lambda\psi \geq 0$. We will use this fact frequently in the subsequent sections. Note that L_M-stability is equivalent to the notion of a *stably outermost MOTS* in [AMS05, Definition 2].
(ii) The conditions from the above definition are satisfied in the following situation. Let $\Sigma = \partial\Omega$ be the boundary of the domain Ω and satisfy $\theta^+ = 0$. Furthermore assume that there is a neighborhood U of Σ such that the exterior part $U \setminus \Omega$ does not contain any trapped surface, ie. a surface with $\theta^+ < 0$. Then Σ is stable. Assume not. Then the principal eigenvalue would be negative and the corresponding eigenfunction ψ would satisfy $L_M \psi < 0$, $\psi > 0$. This would imply the existence of trapped surfaces outside of Σ, since the variation of Σ in direction $\psi\nu$ would decrease θ^+. □

Note that the condition $\theta^+ = 0$ does not depend on the choice of the particular frame. Therefore, to say that a surface is marginally trapped, we do not need any additional information. In contrast the notion of stability required here does depend on the frame, since clearly there is no distinct selection of ν when only Σ — and not M — is specified.

To address this issue, we introduce the second notion of stability of marginally outer trapped surfaces, namely with reference to the direction $-l^-$. This definition is more in spirit of Newman [New87] and recent interest in the so called dynamical horizons [AK03, AG05].

Definition 5.3. *A two dimensional surface $\Sigma \subset M \subset L$ is called a L_-*-stable marginal outer trapped surface (L^--stable MOTS) *if*
(i) Σ is marginally trapped with respect to l^+ , that is $\theta^+ = 0$.
(ii) There exists a function $f \geq 0, f \not\equiv 0$ such that $L_- f \geq 0$. Here L_- is the operator from corollary 4.2.

Remark 5.4. It turns out that this notion of stability does not depend on the choice of the null frame. This is due to the natural transformation law of the stability operator L_- when changing the frame according to $\tilde{l}^+ = fl^+$ and $\tilde{l}^- = f^{-1}l^-$. Then the operator $\tilde{L}_-$ with respect to this frame satisfies $f^{-1}\tilde{L}(f\beta) = L\beta$ for all functions $\beta \in C^\infty(\Sigma)$, as it is expected from the facts that $\tilde{\theta}^+ = f\theta^+$ and $-\beta l^- = -\beta f \tilde{l}^-$. □

Remark 5.5. (i) Remark 5.2 is also valid here, in particular the definition implies that there exists a function $\psi > 0$ with $L_-\psi \geq 0$.
(ii) Technically speaking, the equation for a marginally trapped surface prescribes the mean curvature H of Σ in M to equal minus the value of a function $P : TM \to \mathbf{R} : (p, v) \mapsto trK - K_{ij}\nu^i\nu^j$, namely $H(p) = -P(p, \nu)$ for all $p \in \Sigma$. This is a degenerate quasilinear elliptic equation for the position of the surface. These equations do not allow estimates for second derivatives without any additional information. This is where the two stability conditions come into play. They give the additional piece of information needed in the estimates as in the case for stable minimal surfaces. □

We conclude with the remark that L_M-stability implies L_--stability.

Lemma 5.6. *Let (L, h) satisfy the null energy condition, i.e. assume that for all null vectors k we have that ${}^L\mathrm{Rc}(k, k) \geq 0$. Then if Σ is an L_M-stable MOTS, then it is also L_--stable.*

Proof. We use the notation from section 4, where we introduced the linearization of θ^+. For any function f compute

$$L_M f - L_- f = \delta_{f\nu,w}\theta^+ - \tfrac{1}{2}\delta_{fl^-,w}\theta^+ = \tfrac{1}{2}\delta_{fl^+,w}\theta^+ = -\tfrac{1}{2}f\left(|\chi^+|^2 + {}^L\mathrm{Rc}(l^+, l^+)\right).$$

If $f > 0$, then by the null energy condition, the right hand side is non-positive. If in addition $L_M f \geq 0$, as in the definition of L_M-stability, then this implies that

$$L_- f \geq L_M f \geq 0\,.$$

Hence Σ is also L_M stable. □

6 A priori estimates

In this section we derive the actual estimates for stable outermost marginally trapped surfaces. We will use both definitions for stability from section 5, since both yield the estimates needed. Note that L_--stability can be defined independently of M, the spatial slice containing the surfaces Σ in question, but the estimates presented here do depend on the geometry of the surrounding slice. We first begin with the observation, that stability of MOTS gives an L^2-estimate for the shear tensor χ^+.

Lemma 6.1. *Suppose Σ is an L_M-stable MOTS. Then*

$$\int_\Sigma |\chi|^2 \,\mathrm{d}\mu \le \int_\Sigma |K^\Sigma|^2 - 2\Psi_M \,\mathrm{d}\mu\,.$$

Proof. Take f as in the definition of a stable MOTS. From remark 5.2 we can assume $f > 0$. Then $f^{-1}L_M f \ge 0$. Integrate this equation, and expand L_M as in corollary 4.3. This yields

$$0 \le \int_\Sigma -f^{-1}\Delta f + 2f^{-1}S(\nabla f) - |\chi|^2 + \langle K^\Sigma, \chi\rangle - |S|^2 + \operatorname{div} S - \Psi_M \,\mathrm{d}\mu\,.$$

By sorting terms, and partial integration of the Laplacian, we obtain

$$\int_\Sigma |S|^2 + |\chi|^2 \,\mathrm{d}\mu \le \quad \int_\Sigma -f^{-2}|\nabla f|^2 + 2f^{-1}|\nabla f|\,|S| + |K^\Sigma|\,|\chi| - \Psi_M \,\mathrm{d}\mu\,.$$

By the Schwarz inequality

$$2\int_\Sigma f^{-1}|\nabla f|\,|S| \,\mathrm{d}\mu \le \int_\Sigma |S|^2 + f^{-2}|\nabla f|^2\,,$$

and

$$\int_\Sigma |K^\Sigma|\,|\chi| \,\mathrm{d}\mu \le \frac{1}{2}\int_\Sigma |K^\Sigma|^2 \,\mathrm{d}\mu + \frac{1}{2}\int_\Sigma |\chi|^2\,.$$

Cancelling the terms $\int_\Sigma |S|^2 \,\mathrm{d}\mu$ and $\frac{1}{2}\int_\Sigma |\chi|^2 \,\mathrm{d}\mu$ on both sides, we arrive at the desired estimate. □

We can use the alternative representation of L_M in equation (4.1) to derive a similar estimate with a different kind of right hand side. Note that in case the dominant energy condition holds, the right hand side can be estimated by 8π only, as the remaining term is negative.

Lemma 6.2. *Suppose Σ is an L_M-stable MOTS. Then*

$$\int_\Sigma |\chi|^2 \,\mathrm{d}\mu \le 8\pi - 2\int_\Sigma \tilde\Psi_M \,\mathrm{d}\mu\,.$$

Proof. We proceed as in the proof of the previous lemma. In addition, we invoke the Gauss-Bonnet theorem to conclude that $\int_\Sigma \mathrm{Scal}\,\mathrm{d}\mu \leq 8\pi$. □

An estimate in the same spirit holds for L_--stable surfaces.

Lemma 6.3. *Suppose Σ is an L_M-stable MOTS. Then*

$$\int_\Sigma |\chi|^2 \,\mathrm{d}\mu \leq 4 \int_\Sigma |K^\Sigma|^2 - \Psi_- \,\mathrm{d}\mu\,.$$

Proof. As in the proof of the previous lemma we take the function f from the definition of L_--stability and multiply the equation $L_- f \geq 0$ by f^{-1} and integrate. Proceeding as before we arrive at the estimate

$$\int_\Sigma |\mathrm{I\!I}|^2 \,\mathrm{d}\mu \leq -2 \int_\Sigma \Psi_- \,\mathrm{d}\mu\,.$$

Then observe that $|\mathrm{I\!I}|^2 = -\langle \chi^+, \chi^- \rangle = |\chi|^2 - 2\langle \chi, K^\Sigma \rangle$ as $\chi^- = -\chi^+ + 2K^\Sigma$. Hence $|\chi|^2 \leq 2|\mathrm{I\!I}|^2 + 4|K^\Sigma|^2$. This yields the estimate. □

Here, and in the sequel, for a tensor T, we denote $\|T\|_\infty = \sup_\Sigma |T|$. That is, ∞-norms are taken on Σ only.

Proposition 6.4. *Let Σ be an L_M-stable MOTS. For any $\varepsilon > 0$, and any $p \geq 2$ we have the estimate*

$$\begin{aligned}\int_\Sigma |\chi|^{p+2} \,\mathrm{d}\mu \\ \leq \tfrac{p^2}{4}(1+\varepsilon) \int_\Sigma |\chi|^{p-2} \big|\nabla|\chi|\big|^2 \,\mathrm{d}\mu + C(\varepsilon^{-1}, \|\Psi_M\|_\infty, \|K^\Sigma\|_\infty, \|S\|_\infty) \int_\Sigma |\chi|^p \,\mathrm{d}\mu.\end{aligned}$$

If Σ is an L_--stable MOTS, then the same estimate holds, with $\|\Psi_-\|_\infty$ replacing $\|\Psi_M\|_\infty$ in the constant on the right hand side.

Proof. We will restrict to the proof of the first statement, since the second is proved similarly.

As in the proof of the previous lemmas, take the function f from the definition of stability, multiply the equation $Lf \geq 0$ with $|\chi|^p f^{-1}$, and integrate to obtain

$$\begin{aligned}\int_\Sigma |\chi|^{p+2} + |\chi|^p |S|^2 \\ \leq \int_\Sigma -|\chi|^p f^{-1} \Delta f + 2|\chi|^p f^{-1} S(\nabla f) + |\chi|^p \langle K^\Sigma, \chi \rangle + |\chi|^p \operatorname{div} S + |\chi|^p |\Psi_M| \,\mathrm{d}\mu\,.\end{aligned}$$

For an arbitrary $\varepsilon > 0$, we can estimate the terms on the right hand side in the following manner

$$\begin{aligned}
&\int_\Sigma -|\chi|^p f^{-1}\Delta f \,\mathrm{d}\mu \\
&\quad = \int_\Sigma -|\chi|^p f^{-2}|\nabla f|^2 + p|\chi|^{p-1}f^{-1}\langle\nabla|\chi|,\nabla f\rangle \,\mathrm{d}\mu \\
&\quad \leq \int_\Sigma -|\chi|^p f^{-2}|\nabla f|^2 + (1-\varepsilon)|\chi|^p f^{-2}|\nabla f|^2 + \tfrac{p^2}{4}(1-\varepsilon)^{-1}|\chi|^{p-2}\big|\nabla|\chi|\big|^2 \,\mathrm{d}\mu\,, \\
&\int_\Sigma 2|\chi|^p f^{-1}S(\nabla f)\,\mathrm{d}\mu \leq \int_\Sigma \varepsilon|\chi|^p f^{-2}|\nabla f|^2 + \varepsilon^{-1}|\chi|^p|S|^2\,\mathrm{d}\mu\,, \\
&\int_\Sigma |\chi|^p \operatorname{div} S\,\mathrm{d}\mu = -\int_\Sigma S(\nabla|\chi|^p) \leq \varepsilon^{-1}\int_\Sigma |S|^2|\chi|^p + \varepsilon\tfrac{p^2}{4}\int_\Sigma |\chi|^{p-2}\big|\nabla|\chi|\big|^2\,\mathrm{d}\mu\,,
\end{aligned}$$

and

$$\int_\Sigma |\chi|^p\langle K^\Sigma,\chi\rangle\,\mathrm{d}\mu \leq \varepsilon\int_\Sigma |\chi|^{p+2}\,\mathrm{d}\mu + (4\varepsilon)^{-1}\int_\Sigma |K^\Sigma|^2|\chi|^p\,\mathrm{d}\mu\,.$$

Inserting these estimates in the original inequality, we arrive at the estimate

$$\begin{aligned}
\int_\Sigma |\chi|^{p+2} + |\chi|^p|S|^2\,\mathrm{d}\mu \leq \int_\Sigma \big((1-\varepsilon)^{-1}+\varepsilon\big)\tfrac{p^2}{4}|\chi|^{p-2}\big|\nabla|\chi|\big|^2 + \varepsilon|\chi|^{p+2}\,\mathrm{d}\mu \\
+ \int_\Sigma \varepsilon^{-1}\big(|S|^2 + \tfrac{1}{4}|K^\Sigma|^2\big)|\chi|^p + |\chi|^p|\Psi_M|\,\mathrm{d}\mu\,.
\end{aligned}$$

Now subtract $\varepsilon\int|\chi|^{p+2}\,\mathrm{d}\mu$ and divide by $(1-\varepsilon)$. This yields a term $\int_\Sigma|\chi|^{p+2}\,\mathrm{d}\mu$ on the left hand side of the equation. If $\varepsilon < \frac{1}{2}$ then the last term divided by $1-\varepsilon$ is at most double itself, and the factor in front of $\int\big|\nabla|\chi|\big|^2\,\mathrm{d}\mu$ is of the form $1+\varepsilon'$, where $\varepsilon' > 0$ can be as small as desired. Thus the estimate of the proposition follows. □

We now aim for an estimate on the gradient term on the right hand side of the estimate in proposition 6.4. The main tool will be the Simons identity from section 3. To avoid that the estimated depend on derivatives of curvature, we use similar techniques as in [Met04].

Proposition 6.5. *Let Σ be an L_M-stable MOTS. Then there exists $p_0 > 2$ such that for $2 \leq p \leq p_0$ we have the estimate*

$$\int_\Sigma |\chi|^{p-2}\big|\nabla|\chi|\big|^2\,\mathrm{d}\mu \leq C(p,\|K^\Sigma\|_\infty,\|Q\|_\infty,\|{}^L\mathrm{Rm}^\Sigma\|_\infty,\|S\|_\infty)\int_\Sigma |\chi|^p + |\chi|^{p-2}\,\mathrm{d}\mu\,.$$

If Σ is an L_--stable MOTS, then the same estimate holds, with $\|\Psi_-\|_\infty$ replacing $\|\Psi_M\|_\infty$ in the constant on the right hand side.

Before we can start the proof of the proposition, we state the following lemma. It states an impoved Kato's inequality similar to [SY81]. A general reference for such inequalities is [CGH00].

Lemma 6.6. *On a surface Σ with $\theta^+ = 0$ we have the estimate*

$$|\nabla\chi|^2 - \big|\nabla|\chi|\big|^2 \geq \tfrac{1}{33}\left(\big|\nabla|\chi|\big|^2 + |\nabla\chi|^2\right) - c\left(|Q|^2 + |S|^2|\chi|^2\right).$$

Here c is a purely numerical constant.

Proof. The proof goes along the lines of a similar argument in Schoen and Yau in [SY81, p. 237], but for the sake of completeness, we include a sketch of it here.

In the following computation we do not use the Einstein summation convention and work in a local orthonormal frame for $T\Sigma$. Let $T := |\nabla\chi|^2 - \big|\nabla|\chi|\big|^2$. We compute

$$\begin{aligned}
|\chi|^2 T &= |\chi|^2|\nabla\chi|^2 - \tfrac{1}{4}\big|\nabla|\chi|^2\big|^2 \\
&= \sum_{i,j,k,l,m} (\chi_{ij}\nabla_k\chi_{lm})^2 - \sum_k \Big(\sum_{ij}\chi_{ij}\nabla_k\chi_{ij}\Big)^2 \\
&= \tfrac{1}{2}\sum_{i,j,k,l,m}\Big(\chi_{ij}\nabla_k\chi_{lm} - \chi_{lm}\nabla_k\chi_{ij}\Big)^2.
\end{aligned}$$

In the last term consider only summands with $i = k$ and $j = m$. This gives

$$|\chi|^2 T \geq \tfrac{1}{2}\sum_{i,j,l}\Big(\chi_{ij}\nabla_i\chi_{jl} - \chi_{jl}\nabla_i\chi_{ij}\Big)^2 \geq \tfrac{1}{8}\sum_l\Big(\sum_{i,j}\chi_{ij}\nabla_i\chi_{jl} - \chi_{jl}\nabla_i\chi_{ij}\Big)^2.$$

Use the Codazzi equation (3.2) to swap indices in the gradient terms. We arrive at

$$|\chi|^2 T \geq \tfrac{1}{8}\sum_l\Big(\sum_{i,j}\left(\chi_{ij}\nabla_l\chi_{ij} + \chi_{ij}Q_{ilj} - \chi_{lj}Q_{iji}\right) + \sum_i\left(\theta S_i\chi_{il} - \chi_{il}\nabla_i\theta\right) - |\chi|^2 S_l\Big)^2.$$

By the fact that $(a-b)^2 \geq \frac{1}{2}a^2 - b^2$, this implies

$$\begin{aligned}
|\chi|^2 T \geq{}& \tfrac{1}{16}\sum_l\Big(\sum_{i,j}\chi_{ij}\nabla_l\chi_{ij}\Big)^2 \\
&- \tfrac{1}{8}\sum_l\Big(\sum_{i,j}\left(\chi_{ij}Q_{ilj} - \chi_{lj}Q_{iji}\right) + \sum_i\chi_{il}(\theta S_i - \nabla_i\theta) - |\chi|^2 S_l\Big)^2 \\
\geq{}& \tfrac{1}{16}|\chi|^2\big|\nabla|\chi|\big|^2 - c\,|\chi|^2\left(|Q|^2 + |S|^2|\chi|^2\right).
\end{aligned}$$

Dividing by $|\chi|^2$, we get

$$|\nabla\chi|^2 - \big|\nabla|\chi|\big|^2 \geq \tfrac{1}{16}\big|\nabla|\chi|\big|^2 - c\left(|Q|^2 + |S|^2|\chi|^2\right).$$

Adding $\frac{1}{32}\left(|\nabla\chi|^2 - \big|\nabla|\chi|\big|^2\right)$ to both sides of this inequality and multiplying by $\frac{32}{33}$ yields the desired estimate. ☐

Now we can prove proposition 6.5.

Proof. Of proposition 6.5. We will restrict to the proof of the first statement, since the second is proved similarly. Compute

$$\Delta|\chi|^2 = 2|\chi|\Delta|\chi| + 2\big|\nabla|\chi|\big|^2 .$$

On the other hand

$$\Delta|\chi|^2 = 2\chi_{ij}\Delta\chi_{ij} + 2|\nabla\chi|^2 .$$

Subtracting these equations yields

$$|\chi|\Delta|\chi| = \chi_{ij}\Delta\chi_{ij} + |\nabla\chi|^2 - \big|\nabla|\chi|\big|^2 .$$

In the case $\theta^+ = 0$, the Simons identity from lemma 3.1 gives

$$\begin{aligned}\chi_{ij}\Delta\chi_{ij} &= \chi_{ij}\big({}^L\mathrm{Rm}_{kilk}\chi_{lj} + {}^L\mathrm{Rm}_{kilj}\chi_{kl}\big) - |\mathrm{I\!I}|^2|\chi|^2 - P\chi^+_{ij}\chi^+_{jk}\chi^+_{ki} \\ &\quad + \chi_{ij}\nabla_k\big(Q_{kij} - \chi_{kj}S_i + \chi_{ij}S_k\big) + \chi_{ij}\nabla_i\big(Q_{kjk} + \chi_{jk}S_k\big) .\end{aligned}$$

Note that $\chi^+_{ij}\chi^+_{jk}\chi^+_{ki} = \mathrm{tr}(\chi^3)$, and the trace of a 2×2 matrix A satisfies the relation $\mathrm{tr}\, A^3 = \mathrm{tr}\, A(\mathrm{tr}\, A^2 - \det A)$. Since χ is traceless, this term vanishes. In addition $|\mathrm{I\!I}|^2 = \langle\chi^+,\chi^-\rangle = |\chi|^2 - 2\langle K^\Sigma,\chi^+\rangle$.

As we are not interested in the particular form of some terms, to simplify notation, we introduce the $*$-notation. For two tensors T_1 and T_2, the expression $T_1 * T_2$ denotes linear combinations of contractions of $T_1\otimes T_2$.

To remember that in the above equation we need to evaluate ${}^L\mathrm{Rm}$ only on vectors tangential to Σ, we use the projection of ${}^L\mathrm{Rm}$ to $T\Sigma$ and denote this by ${}^L\mathrm{Rm}^\Sigma$. Then the above equations combine to

$$-|\chi|\Delta|\chi| + |\nabla\chi|^2 - \big|\nabla|\chi|\big|^2 = |\chi|^4 + |\chi|^2 * \chi * K^\Sigma + \chi * \chi * {}^L\mathrm{Rm}^\Sigma + \chi * \nabla\big(Q + \chi * S\big) . \tag{6.1}$$

Multiply this equation by $|\chi|^{p-2}$ and integrate. This yields

$$\begin{aligned}&\int_\Sigma -|\chi|^{p-1}\Delta|\chi| + |\chi|^{p-2}\big(|\nabla\chi|^2 - \big|\nabla|\chi|\big|^2\big)\,\mathrm{d}\mu \\ &\quad = \int_\Sigma |\chi|^{p+2} + |\chi|^p\chi * K^\Sigma + |\chi|^{p-2}\chi * \chi * {}^L\mathrm{Rm}^\Sigma + |\chi|^{p-2}\chi * \nabla(Q + \chi * S)\,\mathrm{d}\mu .\end{aligned}$$

Next, do a partial integration on the term including the Laplacian and on the last term on the second line. We find that

$$\begin{aligned}&\int_\Sigma (p-1)|\chi|^{p-2}\big|\nabla|\chi|\big|^2 + |\chi|^{p-2}\big(|\nabla\chi|^2 - \big|\nabla|\chi|\big|^2\big)\,\mathrm{d}\mu \\ &\quad \le \int_\Sigma |\chi|^{p+2}\,\mathrm{d}\mu \\ &\qquad + c\int_\Sigma |\chi|^{p+1}|K^\Sigma| + |\chi|^p|{}^L\mathrm{Rm}^\Sigma| + |\chi|^{p-2}\big(|\nabla\chi| + \big|\nabla|\chi|\big|\big)(|Q| + |\chi||S|)\,\mathrm{d}\mu .\end{aligned}$$

$$(6.2)$$

Here c is a purely numerical constant. For any $\varepsilon > 0$, we can estimate

$$c\int_\Sigma |\chi|^{p+1}|K^\Sigma|\,\mathrm{d}\mu \le \varepsilon\int_\Sigma |\chi|^{p+2}\,\mathrm{d}\mu + C(\varepsilon^{-1})\int_\Sigma |\chi|^p|K^\Sigma|^2\,\mathrm{d}\mu$$

as well as

$$\begin{aligned}
&c\int_\Sigma |\chi|^{p-2}\big(|\nabla\chi| + \big|\nabla|\chi|\big|\big)(|Q| + |\chi||S|)\,\mathrm{d}\mu \\
&\quad\le \varepsilon\int_\Sigma |\chi|^{p-2}\big(|\nabla\chi|^2 + \big|\nabla|\chi|\big|^2\big)\,\mathrm{d}\mu + C(\varepsilon^{-1})\int_\Sigma |\chi|^p|S|^2 + |\chi|^{p-2}|Q|^2\,\mathrm{d}\mu\,.
\end{aligned}$$

Inserting these estimates into the estimate (6.2), gives

$$\begin{aligned}
&\int_\Sigma (p-1)|\chi|^{p-2}\big|\nabla|\chi|\big|^2 + |\chi|^{p-2}\big(|\nabla\chi|^2 - \big|\nabla|\chi|\big|^2\big)\,\mathrm{d}\mu \\
&\quad\le (1+\varepsilon)\int_\Sigma |\chi|^{p+2}\,\mathrm{d}\mu + \varepsilon\int_\Sigma |\chi|^{p-2}\big(|\nabla\chi|^2 + \big|\nabla|\chi|\big|^2\big)\,\mathrm{d}\mu \\
&\qquad + C(\varepsilon^{-1}, \|K^\Sigma\|_\infty, \|{}^L\mathrm{Rm}^\Sigma\|_\infty, \|S\|_\infty)\int_\Sigma |\chi|^p\,\mathrm{d}\mu + C(\varepsilon, \|Q\|_\infty)\int_\Sigma |\chi|^{p-2}\,\mathrm{d}\mu\,.
\end{aligned}$$

We apply lemma 6.6 to estimate the second term on the left hand side from below by $\frac{1}{33}\big(\int_\Sigma \big|\nabla|\chi|\big|^2 + |\nabla\chi|^2\,\mathrm{d}\mu\big)$. In addition, use proposition 6.4 to estimate the first term on the right hand side. This yields

$$\begin{aligned}
&\int_\Sigma (p-1)|\chi|^{p-2}\big|\nabla|\chi|\big|^2 + (\tfrac{1}{33} - \varepsilon)|\chi|^{p-2}\big(|\nabla\chi|^2 + |\nabla|\chi||^2\big)\,\mathrm{d}\mu \\
&\quad\le \tfrac{p^2}{4}(1+\varepsilon)^2\int_\Sigma |\chi|^{p-2}\big|\nabla|\chi|\big|^2\,\mathrm{d}\mu \\
&\qquad + C(\varepsilon^{-1}, \|K^\Sigma\|_\infty, \|Q\|_\infty, \|{}^L\mathrm{Rm}^\Sigma\|_\infty, \|S\|_\infty)\int_\Sigma |\chi|^p + |\chi|^{p-2}\,\mathrm{d}\mu\,.
\end{aligned}$$

Choose $p_0 > 2$ close enough to 2 and ε small enough, such that for $2 < p < p_0$ the gradient term on the right hand side can be absorbed on the left hand side. This gives the desired estimate. □

Combining propositions 6.4 and 6.5 with the initial L^2-estimate in lemmas 6.1 or 6.3 gives the following L^p estimates for $|\chi|$.

Theorem 6.7. *There exists $p_0 > 2$ such that for all $2 \le p < p_0$ and all L_M-stable MOTS Σ, the shear χ along Σ satisfies the estimates*

$$\int_\Sigma |\chi|^{p+2}\,\mathrm{d}\mu \le C(p, |\Sigma|, \|\Psi_M\|_\infty, \|K^\Sigma\|_\infty, \|Q\|_\infty, \|{}^L\mathrm{Rm}^\Sigma\|_\infty, \|S\|_\infty)\,, \tag{6.3}$$

$$\int_\Sigma |\chi|^{p-2}\big|\nabla|\chi|\big|^2\mathrm{d}\mu \le C(p, |\Sigma|, \|\Psi_M\|_\infty, \|K^\Sigma\|_\infty, \|Q\|_\infty, \|{}^L\mathrm{Rm}^\Sigma\|_\infty, \|S\|_\infty) \tag{6.4}$$

and

$$\int_\Sigma |\nabla\chi|^2\,\mathrm{d}\mu \le C(p,|\Sigma|,\|\Psi_M\|_\infty,\|K^\Sigma\|_\infty,\|Q\|_\infty,\|{}^L\mathrm{Rm}^\Sigma\|_\infty,\|S\|_\infty)\,. \tag{6.5}$$

If Σ is an L_--stable MOTS, then the same estimate holds, with $\|\Psi_-\|_\infty$ replacing $\|\Psi_M\|_\infty$ in the constants on the right hand side.

Proof. First let $p=2$. Combine propositions 6.4, 6.5 and lemmas 6.1 or 6.3 to get L^4-estimates for $|\chi|$.

Then take any $2<p<p_0$ as in proposition 6.5. Proceed as before. The resulting L^p and L^{p-2}-norms of $|\chi|$ on the right hand side can now be estimated by combinations of the L^4-norm of $|\chi|$ and the area $|\Sigma|$.

To see the last estimate, note that in the proof of proposition 6.5, by appropriately choosing ε, we can retain a small portion of the term $\int_\Sigma |\chi|^{p-2}|\nabla\chi|^2\,\mathrm{d}\mu$ on the right hand side. □

For the next step – the derivation of sup-bounds on χ – we use the Hoffman-Spruck Sobolev inequality in the following form [HS74].

Lemma 6.8. *For (M,g) exist constants c_0^S, c_1^S, such that for all hypersurfaces $\Sigma\subset M$ and all functions $f\in C^\infty(\Sigma)$ with $|\mathrm{supp} f|\le c_0^S$ the following estimate holds:*

$$\left(\int_\Sigma |f|^2\,\mathrm{d}\mu\right)^{1/2} \le c_1^S \int_\Sigma |\nabla f| + |fH|\,\mathrm{d}\mu\,.$$

Here H is the mean curvature of Σ and the constants c_0^S, c_1^S depend only on a lower bound for the injectivity radius and an upper bound for the curvature of (M,g).

Remark 6.9. Replacing f by f^p in the above inequality and using Hölders inequality gives that for all $1<p<\infty$ and all f with $|\mathrm{supp} f|\le c_0^S$

$$\left(\int_\Sigma f^p\,\mathrm{d}\mu\right)^{2/p} \le c_p^S|\mathrm{supp} f|^{2/p}\int_\Sigma |\nabla f|^2 + |Hf|^2\,\mathrm{d}\mu\,.$$

The constant c_p^S only depends on c_1^S and p.

Theorem 6.10. *Let Σ be a stable MOTS, then the shear χ satisfies the estimate*

$$\sup_\Sigma |\chi| \le C(|\Sigma|,\|\Psi_M\|_\infty,\|K^\Sigma\|_\infty,\|Q\|_\infty,\|{}^L\mathrm{Rm}^\Sigma\|_\infty,\|S\|_\infty)$$

The constant on the right hand side in addition depends on the constants c_0^S and c_1^S in the Hoffman-Spruck-Sobolev inequality for M.

If Σ is an L_--stable MOTS, then the same estimate holds, with $\|\Psi_-\|_\infty$ replacing $\|\Psi_M\|_\infty$ in the constant on the right hand side.

Proof. We will restrict to the proof of the first statement, since the second is proved similarly.

We will proceed in a Stampacchia iteration. Let $u := |\chi|$ and for $k \geq 0$ set $u_k := \max\{u - k, 0\}$. In addition set $A(k) := \operatorname{supp} u_k$.

The L^2-bound for $|\chi|$ from lemma 6.1 implies that

$$k^2|A(k)| \leq \int_{A(k)} u^2 \, \mathrm{d}\mu \leq \int_\Sigma u^2 \, \mathrm{d}\mu \leq C(|\Sigma|, \|\Psi_M\|_\infty, \|K^\Sigma\|_\infty) \,.$$

Therefore there exists $k_0 = k_0(|\Sigma|, \|\Psi_M\|_\infty, \|K^\Sigma\|_\infty, c_0) < \infty$, such that $|A(k)| \leq c_0$ for all $k \geq k_0$. Here we want c_0^S to be the constant from lemma 6.8, to be able to apply the estimate from there for all functions with support in $A(k)$, with $k \geq k_0$.

To proceed, let $q > 2$. Multiply the Simons identity, in the form (6.1) from the proof of proposition 6.5, by u_k^q and integrate. This yields

$$\begin{aligned}
&\int_{A(k)} -u_k^q u\Delta u + u_k^q(|\nabla\chi|^2 - |\nabla u|^2) \, \mathrm{d}\mu \\
&\qquad \leq c \int_{A(k)} u_k^q u^4 + |K| u_k^q u^3 + |{}^L\mathrm{Rm}^\Sigma| u_k^q u^2 + u_k^q \chi * \nabla\big(Q + \chi * S\big) \, \mathrm{d}\mu \,.
\end{aligned}$$

Here c is a purely numerical constant. Partially integrate the Laplacian on the right hand side and the last term on the left hand side. This gives

$$\begin{aligned}
&\int_{A(k)} q u u_k^{q-1} |\nabla u|^2 + u_k^q |\nabla\chi|^2 \, \mathrm{d}\mu \\
&\quad \leq c \int_{A(k)} u_k^q u^4 + |K| u_k^q u^3 + |{}^L\mathrm{Rm}^\Sigma| u_k^q u^2 + \big(u_k^q|\nabla\chi| + u_k^{q-1}u|\nabla u|\big)\big(|Q| + u|S|\big) \, \mathrm{d}\mu.
\end{aligned}$$

Note that the term $\int q u u_k^{q-1} |\nabla u|^2 \, \mathrm{d}\mu$ on the left hand side controls $\int u_k^q |\nabla u|^2 \, \mathrm{d}\mu$. But before we use this estimate, we absorb the gradient terms on the right hand side. For example the term containing $|\nabla\chi|^2$:

$$c \int_{A(k)} u_k^q |\nabla\chi| (|Q| + u|S|) \, \mathrm{d}\mu \leq \int_{A(k)} u_k^q |\nabla\chi|^2 \, \mathrm{d}\mu + c \int_{A(k)} u_k^q |Q|^2 + u_k^q u^2 |S|^2 \, \mathrm{d}\mu \,.$$

The other term, which contains $|\nabla u|$, can be treated similarly, such that the resulting terms can be absorbed on the left. This yields an estimate of the form

$$\begin{aligned}
&\int_{A(k)} u_k^q |\nabla u|^2 \, \mathrm{d}\mu \\
&\qquad \leq C(q, \|K^\Sigma\|_\infty, \|Q\|_\infty, \|{}^L\mathrm{Rm}^\Sigma\|_\infty, \|S\|_\infty) \int_{A(k)} u_k^q u^4 + u u_k^{q-1} + u_k^{q-1} u^3 \, \mathrm{d}\mu \,.
\end{aligned} \tag{6.6}$$

Note that we used that $u_k \leq u$ and $u \leq u^2 + 1$ here to get rid of the extra terms. We begin estimating the terms on the right hand side of (6.6) using lemma 6.8. Rewrite and estimate the first term as follows:

$$\int_{A(k)} u_k^q u^4 \,\mathrm{d}\mu = \int_{A(k)} (u_k u^{4/q})^q \,\mathrm{d}\mu \leq |A(k)| \Big(\tilde{c}_q^S \int_{A(k)} \big|\nabla (u_k u^{4/q})\big|^2 + |H u_k u^{4/q}|^2 \,\mathrm{d}\mu \Big)^{q/2}. \tag{6.7}$$

To estimate the first term on the right hand side compute on $A(k)$, using $u_k/u \leq 1$,

$$\big|\nabla (u_k u^{4/q})\big| = u^{4/q} |\nabla u| + \tfrac{4}{q} u^{4/q} |\nabla u| \tfrac{u_k}{u} \leq c(q) u^{4/q} |\nabla u| \,.$$

Observe that if q is large enough, namely such that $2 + \frac{4}{q} < p_0$, then theorem 6.7 yields that

$$\int_\Sigma \big|\nabla (u_k u^{4/q})\big|^2 \,\mathrm{d}\mu \leq c(q) \int_\Sigma u^{4/q} |\nabla u|^2 \,\mathrm{d}\mu \leq C(q) \,.$$

Here, and for the remainder of the proof, $C(q)$ denotes a constant that depends on q and, in addition to that, on all the quantities the constant in the statement of this theorem depends on.

To address the second term in (6.7), recall that since $0 = \theta^+ = H + P$, we have $\|H\|_\infty = \|P\|_\infty \leq 2\|K^\Sigma\|_\infty$. Therefore

$$\int_{A(k)} H^2 u_k^2 u^{8/q} \,\mathrm{d}\mu \leq 4\|K^\Sigma\|_\infty^2 \int_\Sigma u^{2+\frac{8}{q}} \,\mathrm{d}\mu \leq C(q) \,,$$

where the last estimate also follows from theorem 6.7 if q is large enough. Summarizing these steps, we have

$$\int_{A(k)} u_k^q u^4 \,\mathrm{d}\mu \leq C(q) |A(k)| \,.$$

A similar procedure for the remaining terms in (6.6) finally yields the estimate

$$\int_{A(k)} u_k^q |\nabla u|^2 \,\mathrm{d}\mu \leq C(q) |A(k)| \,, \tag{6.8}$$

provided $q > q_0$ is large enough. Fix such a $q > q_0$ and let $f = u_k^{1+q/2}$. Then equation (6.8) implies that

$$\int_{A((k)} |\nabla f|^2 \,\mathrm{d}\mu \leq C(q) |A(k)| \,.$$

The Hoffman-Spruck-Sobolev inequality from lemma 6.8, combined with theorem 6.7, furthermore yields

$$\int_{A(k)} f^2 \,\mathrm{d}\mu = \int_{A(k)} u_k^{q+2} \,\mathrm{d}\mu \leq C(q) |A(k)| \left(\int_{A(k)} |\nabla u|^2 + H u^2 \,\mathrm{d}\mu \right)^{\frac{q+2}{2}} \leq C(q) |A(k)| \,.$$

Thus one further application of lemma 6.8 yields

$$\int_{A(k)} u_k^{q+2}\,\mathrm{d}\mu = \int_{A(k)} f^2\,\mathrm{d}\mu \le C(q)|A(k)|^2\,.$$

Consider $h > k \ge k_0$, then on $A(h)$ we have that $u_k \ge h-k$ and therefore we derive the following iteration inequality

$$|h-k|^{q+2}|A(h)| \le \int_{A(h)} u_k^{q+2}\,\mathrm{d}\mu \le \int_{A(k)} u_k^{q+2}\,\mathrm{d}\mu \le C(q)|A(k)|^2\,.$$

The lemma of Stampacchia [Sta66, Lemma 4.1] now implies that $|A(k_0+d)| = 0$ for

$$d^{q+2} \le C(q)|A(k_0)| \le C(q)|\Sigma|$$

In view of the definition of $A(k) = \operatorname{supp}\max\{u-k,0\}$, this yields the desired estimate. □

Corollary 6.11. *Let $\Sigma \subset M$ be an L_M-stable MOTS. Then Σ satisfies the following estimates.*

$$\sup_\Sigma |\chi| \le C(|\Sigma|, \|K\|_\infty, \|\nabla K\|_\infty, \|{}^M\mathrm{Rm}\|_\infty, (\mathrm{inj}(M,g))^{-1})$$

In addition, we have an L^2-gradient estimate

$$\int_\Sigma |\nabla\chi|^2\,\mathrm{d}\mu \le C(|\Sigma|, \|K\|_\infty, \|\nabla K\|_\infty \|{}^M\mathrm{Rm}\|_\infty)\,.$$

Proof. The statement to prove is that the constants only depend on the stated quantities. This is due to the following reasons.

First, for L_M stable surfaces, we can prove the above theorems using lemma 6.2 instead of lemma 6.1. Thus instead of $\|\Psi_M\|_\infty$, the constants depend on $\|\tilde\Psi_M\|_\infty$. As we have seen in remark 1, we can estimate

$$|\tilde\Psi_M| \le c(|K|^2 + |\nabla K| + |{}^M\mathrm{Rm}|)\,,$$

where c is a numerical constant. Second since for all $X, Y, Z \in \mathcal{X}(\Sigma)$

$$Q(X,Y,Z) = {}^L\mathrm{Rm}(X,Y,n,Z) + {}^L\mathrm{Rm}(X,Y,\nu,Z)\,,$$

we can use the Gauss and Codazzi equations of the embedding $M \hookrightarrow L$ to estimate

$$|Q| + |{}^L\mathrm{Rm}^\Sigma| \le c(|K|^2 + |\nabla K| + |{}^M\mathrm{Rm}|)\,.$$

Third, obviously

$$|K^\Sigma|^2 + |S|^2 \le |K|^2\,.$$

Thus we see that all quantities are controlled by $\|K\|_\infty$, $\|\nabla K\|_\infty$ and $\|{}^M\mathrm{Rm}\|_\infty$, where the ∞-norms are computed on Σ. Note that the dependency on $\mathrm{inj}(M)$ comes from the fact that the constants c_0^S and c_1^S in the Hoffman-Spruck-inequality only depend on $\|{}^M\mathrm{Rm}\|_\infty$ and $\mathrm{inj}(M)$. □

We conclude with an estimate for the principal eigenfunction to L_M or L_-.

Theorem 6.12. *Let Σ be an L_M-stable MOTS. Let $\lambda \geq 0$ be the principal eigenvalue of L_M and $f > 0$ its corresponding eigenfunction. They satisfy the estimates*

$$\lambda|\Sigma| + \tfrac{1}{2}\int_\Sigma f^{-2}|\nabla f|^2 \,\mathrm{d}\mu \leq 4\pi + \int_\Sigma |S|^2 \,\mathrm{d}\mu - \int_\Sigma \tilde{\Psi}_M \,\mathrm{d}\mu$$

and

$$\int |\nabla^2 f|^2 \,\mathrm{d}\mu \leq C(|\Sigma|, \|K\|_\infty, \|\nabla K\|_\infty, \|{}^M\mathrm{Rm}\|_\infty, \mathrm{inj}(M,g)^{-1}) \int_\Sigma f^2 + |\nabla f|^2 \,\mathrm{d}\mu + \lambda^2 \int_\Sigma f^2 \,\mathrm{d}\mu\,.$$

The same estimates hold for L_--stable MOTS when f and λ are the principal eigenfunction and eigenvalue of L_- instead, then $\tilde{\Psi}_M$ has to be replaced by $\tilde{\Psi}_-$ in the first estimate.

Proof. The first estimate follows from a computation similar to the proof of lemma 6.1, but applied as in lemma 6.2.

The second estimate then follows from the first by using the identity

$$\int_\Sigma |\nabla^2 f|^2 \,\mathrm{d}\mu = \int_\Sigma (\Delta f)^2 + {}^\Sigma\mathrm{Rc}(\nabla f, \nabla f) \,\mathrm{d}\mu\,.$$

To estimate the terms on the right hand side, note that

$$-\Delta f = \lambda f - 2S(\nabla f) - f(\operatorname{div} S - \tfrac{1}{2}|\chi|^2 - |S|^2 + \tfrac{1}{2}{}^\Sigma\mathrm{Sc} - \tilde{\Psi}_M)$$

and as Σ is two-dimensional

$${}^\Sigma\mathrm{Rc}(\nabla f, \nabla f) = \tfrac{1}{2}{}^\Sigma\mathrm{Sc}|\nabla f|^2\,.$$

In view of the Gauss equation for $\Sigma \subset M$ and the bounds for χ, we find the claimed estimate. □

Corollary 6.13. *If Σ is an L_M-stable MOTS, then the principal eigenfunction $f > 0$ to L_M which is normalized such that $\|f\|_\infty = 1$ satisfies the estimate*

$$\int_\Sigma f^2 + |\nabla f|^2 + |\nabla^2 f|^2 \,\mathrm{d}\mu \leq C(|\Sigma|, |\Sigma|^{-1}, \|K\|_\infty, \|\nabla K\|_\infty, \|{}^M\mathrm{Rm}\|_\infty, \mathrm{inj}(M,g)^{-1})$$

The same estimate holds for L_--stable MOTS, when f is the principal eigenfunction to L_- instead.

Proof. Since $\|f\|_\infty = 1$, we have $\int_\Sigma f^2 \,\mathrm{d}\mu \leq |\Sigma|$. Then since $f^{-2} \geq 1$, the first estimate from the previous theorem implies

$$\int_\Sigma |\nabla f|^2 \leq C(|\Sigma|, \|K\|_\infty, \|\nabla K\|_\infty, \|{}^M\mathrm{Rm}\|_\infty)\,.$$

Since $\lambda^2 \int_\Sigma f^2 \leq \lambda^2|\Sigma| \leq C|\Sigma|^{-1}$ the above estimates combined with the previous theorem imply the claim. □

7 Applications

The main application of the curvature estimates proved in this paper is the following compactness property of stable MOTS.

Theorem 7.1. *Let $F : M \times [t_0, t_1] \to L$ be a partial slicing of a space time by smooth space-like surfaces $M_t := F(M, t)$, $t \in [t_0, t_1]$. Let g_t and K_t be the first and second fundamental form of M_t. Let ${}^M\nabla_t$ and ${}^M\mathrm{Rm}_t$ denote the Levi-Civita connection and Riemannian curvature tensor of (M_t, g_t) respectively, and let $\mathrm{inj}(M_t, g_t)$ denote the injectivity radius of (M_t, g_t). Assume that the geometry of the M_t is uniformly bounded in the sense that there exists a constant $C > 0$ such that*

$$\|{}^M\mathrm{Rm}_t\|_\infty \leq C\,,$$
$$\|K_t\|_\infty + \|\,{}^M\nabla_t K_t\|_\infty \leq C \text{ and}$$
$$\mathrm{inj}(M_t, g_t) \geq C^{-1}\,.$$

Let $\Phi_n : \Sigma \to L$, $n \geq 1$ be a sequence of embeddings of marginally outer trapped surfaces $\Sigma_n = \Phi_n(\Sigma)$, such that
(i) there exists $t_n \in [t_0, t_1]$ with $\Sigma_n \subset M_{t_n}$,
(ii) there exists $\bar{C}$ such that the area $|\Sigma_n| \leq \bar{C}$ for all $n \geq 1$,
(iii) the union $\bigcup_{n\geq} \Sigma_n$ is precompact in L, and
(iv) every Σ_n is L_M-stable, or
(iv') every Σ_n is L_--stable.
Then there exists t_∞ and a smooth embedding $\Phi_\infty : \Sigma \to L$ such that $\Sigma_\infty = \Phi_\infty(\Sigma) \subset M_{t_\infty}$, Σ_∞ is a stable MOTS, and a sub sequence of reparameterizations of the surfaces Σ_n converge to Σ in $C^{1,\alpha} \cap W^{2,p}$ for any $0 < \alpha < 1$ and $1 \leq p < \infty$.

Proof. Since $[t_0, t_1]$ is compact, we can assume that the sequence t_n converges to some $t_\infty \in [t_0, t_1]$.

By the estimates in corollary 6.11, the above assumptions are sufficient to imply that the shear χ, and thus the second fundamental form A, of the Σ_n is uniformly bounded in $W^{1,2}$. Since all Σ_n are contained in a compact set, this implies that there exist parameterizations of the Σ_n which are uniformly bounded in $W^{3,2}$. By the Sobolev embedding the space $W^{3,2}$ is compactly embedded in $W^{2,p}$, for any fixed $1 < p < \infty$. Note that we can use the Sobolev inequality of a fixed metric on Σ. We conclude the existence of a convergent subsequence of the reparameterized Σ_n. Denote the limit surface by Σ. This limit is of class $W^{3,2}$, but the convergence is in $W^{2,p}$. Since θ^+ is a quasilinear differential operator of second order of the position, Σ satisfies $\theta^+ = 0$ strongly in the sense of $W^{2,p}$. Elliptic regularity therefore implies that Σ is smooth. Note that since $W^{2,p} \subset C^{1,\alpha}$ for all $0 < \alpha < 1$, we can apply the standard regularity theory, which can be found in [GT98, Chapter 8].

To prove stability of Σ, we use the parameterizations above, and pull-back the metrics of Σ_n to Σ, denote those by γ_n. The metric on Σ will be denoted by γ. Then define the operators L_n as the pull backs of the operator L_M on Σ_n to Σ. Let f_n be the principal eigenfunctions of L_n with eigenvalues λ_n and normalize such that $\|f_n\|_\infty = 1$. Since the area of the Σ_n is eventually bounded below by half of the area of Σ, theorem 6.12 implies that $0 \leq \lambda_n \leq C$, where $C = C(\bar{C}, \|K\|_\infty, \|\nabla K\|_\infty, \|{}^M\mathrm{Rm}\|_\infty)$. Thus we can assume that the λ_n converge to some λ with $0 \leq \lambda \leq C$.

By corollary 6.13 the $W^{2,2}$-norm of the f_n taken with respect to the metrics γ_n is uniformly bounded. Recall that the difference of the Hessian of f with respect to γ^n and γ is of the form

$$\big(\nabla^2_{\gamma_n} - \nabla^2_\gamma\big) f = \big(\Gamma_{\gamma_n} - \Gamma_\gamma\big) * df$$

where Γ_γ and Γ_{γ_n} denote the connection coefficients of γ and γ_n. Furthermore ∇f is bounded in any L^p and by $W^{1,p}$ convergence of the metrics $\Gamma_{\gamma_n} - \Gamma_\gamma \to 0$ in L^p. Thus we find that also $\|f_n\|_{W^{2,2}} \leq C$, where the norm is taken with respect to the metric γ on Σ. Hence we can assume that $f_n \to f$ in $W^{1,p}$. The Sobolev embedding $W^{1,p} \hookrightarrow C^0$, implies that $f \geq 0$, and $\|f\|_\infty = 1$, so $f \not\equiv 0$.

The next step is to take the equation $L_n f_n = \lambda_n f_n$ to the limit. Since $f_n \to f$ only in $W^{1,p}$, we have to use the weak version of this equation, namely that for all $\phi \in C^\infty(\Sigma)$

$$\int_\Sigma \gamma_n^{ij} (df_n)_i d\phi_j + B_n^i (df_n)_i \phi + C_n f \phi \,\mathrm{d}\mu = \lambda_n \int_\Sigma f_n \phi \,\mathrm{d}\mu\,,$$

where B_n and C_n are the coefficients of the operator L_n. By the $W^{2,p}$-convergence of the surfaces, we find that γ_n converges to γ in $W^{1,p}$, and B_n^i and C_n converge in L^p to the coefficients B^i and C of L_M on Σ. Thus, since f_n converges in $W^{1,p}$ to f, we can choose p large enough to infer that the limit of the above integrals converges to the corresponding integral on Σ, that is f satisfies

$$\int_\Sigma \langle \nabla f, \nabla \phi\rangle + \langle B, \nabla f\rangle \phi + C f \phi \,\mathrm{d}\mu = \lambda \int_\Sigma f \phi \,\mathrm{d}\mu\,.$$

Thus f is a weak eigenfunction of L_M on Σ. Elliptic regularity implies that f is smooth and satisfies $L_M f = \lambda f$. Since $\lambda \geq 0$ and $f \geq 0$, $f \not\equiv 0$, we conclude that Σ is stable. □

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References

[AG05] A. Ashtekar and G. J. Galloway, *Some uniqueness results for dynamical horizons*, arXiv:gr-qc/0503109, 2005.

[AK03] A. Ashtekar and B. Krishnan, *Dynamical horizons and their properties*, Phys. Rev. D (3) **68** (2003), no. 10, 104030, 25.

[AMS05] L. Andersson, M. Mars, and W. Simon, *Local existence of dynamical and trapping horizons*, Phys. Rev. Lett. **95** (2005), 111102, arXiv:gr-qc/0506013.

[Bar84] R. Bartnik, *Existence of maximal surfaces in asymptotically flat spacetimes*, Comm. Math. Phys. **94** (1984), no. 2, 155–175.

[Bra01] H. L. Bray, *Proof of the Riemannian Penrose inequality using the positive mass theorem*, J. Differential Geom. **59** (2001), no. 2, 177–267.

[CGH00] D. M. J. Calderbank, P. Gauduchon, and M. Herzlich, *On the Kato inequality in Riemannian geometry*, Global analysis and harmonic analysis (Marseille-Luminy, 1999), Sémin. Congr., vol. 4, Soc. Math. France, Paris, 2000, pp. 95–113.

[GT98] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Rev. 3. printing. Second ed., Springer-Verlag Berlin, Heidelberg, New York, 1998.

[HI01] G. Huisken and T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differential Geom. **59** (2001), no. 3, 353–437.

[HS74] D. Hoffman and J. Spruck, *Sobolev and isoperimetric inequalities for Riemannian submanifolds*, Comm. Pure Appl. Math. **27** (1974), 715–727.

[Met04] J. Metzger, *Foliations of asymptotically flat 3-manifolds by 2-surfaces of prescribed mean curvature*, arXiv:math.DG/0410413, 2004.

[New87] R. P. A. C. Newman, *Topology and stability of marginal* 2*-surfaces*, Class. Quant. Grav. **4** (1987), no. 2, 277–290.

[Pen65] R. Penrose, *Gravitational collapse and space-time singularities*, Phys. Rev. Lett. **14** (1965), 57–59.

[Sim68] J. Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math. (2) **88** (1968), 62–105.

[SSY75] R. Schoen, L. Simon, and S. T. Yau, *Curvature estimates for minimal hypersurfaces*, Acta Math. **134** (1975), no. 3-4, 275–288.

[Sta66] G. Stampacchia, *Ѐquations elliptiques du second ordre à coefficients discontinus*, Séminaire de Mathématiques Supérieures, No. 16 (Été, 1965), Les Presses de l'Université de Montréal, Montreal, Que., 1966.

[SY81] R. Schoen and S.-T. Yau, *Proof of the positive mass theorem. II*, Comm. Math. Phys. **79** (1981), no. 2, 231–260.