DEFINABLE RELATIONS IN THE REAL FIELD WITH A DISTINGUISHED SUBGROUP OF THE UNIT CIRCLE

OLEG BELEGRADEK AND BORIS ZILBER

Abstract. We describe definable relations in the real field augmented by a binary relation which is an arbitrary multiplicative group of complex numbers contained in the divisible hull of a finitely generated subgroup of the unit circle. We give a complete axiom system for this structure which admits quantifier elimination down to Boolean combinations od existential formulas of a special simple form.

1. Introduction

The goal of this paper is to describe definable relations in the field of reals augmented by a binary relation which is a subgroup of the multiplicative group of complex numbers, under certain assumptions about the subgroup.

We identify complex numbers with pairs of reals in a usual way. Let

$$\mathbb{S} = \{(x, y) \in \mathbb{C} : x^2 + y^2 = 1\}.$$

The set $\mathbb{S}$, the unit circle on the complex plane, is a subgroup of $\mathbb{C}^*$, the multiplicative group of complex numbers. We consider subgroups Γ of $\mathbb{S}$ such that

(i) Γ is countable;
(ii) Γ/Γ^n is finite for each $n > 0$, where Γ^n denotes the subgroup $\{g^n : g \in \Gamma\}$;
(iii) for every nonconstant polynomial $p(X_1, \dots, X_n)$ over $\mathbb{Z}$ there exist
- a positive integer k,
- elements $g_1, \dots, g_k$ of Γ, and
- nonzero n-tuples of integers $(m_{i1}, \dots, m_{in})$, $i = 1, \dots, k$,

such that, whenever $z_1, \dots, z_n \in \Gamma$, we have $p(z_1, \dots, z_n) = 0$ if and only if $z_1^{m_{i1}} \dots z_n^{m_{in}} = g_i$ for some $i = 1, \dots, k$.

Proposition 1.1. *Any infinite subgroup of the divisible hull of a finitely generated subgroup of* $\mathbb{C}^*$ *has the properties* (i)–(iii).

Here the divisible hull of a subgroup G of $\mathbb{C}^*$ is the subgroup $\bar{G}$ of all $z \in \mathbb{C}^*$ such that $z^m \in G$ for some positive integer m.

Proof. Let G be a finitely generated subgroup of $\mathbb{C}^*$, and $\Gamma \leq \bar{G}$.

The property (i) follows from countability of G and the fact that any element of G has exactly n complex roots of degree n.

We show (ii) by proving that $|\Gamma/\Gamma^n| \leq n^{k+1}$ if G is k-generated.

There is a homomorphism β from the additive group $\mathbb{Z}^k$, the kth direct power of $\mathbb{Z}$, onto the muiltiplicative group G. Since the group $\bar{G}$ is divisible, β extends to

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a homomorphism $\gamma : \mathbb{Q}^k \to \bar{G}$. Let $\delta : \mathbb{Q}^{k+1} \to \bar{G}$ be the homomorphism defined by $\delta(\boldsymbol{r}, r) = \gamma(\boldsymbol{r}) \exp(2\pi r i)$.

The map δ is surjective. Indeed, let $z \in \bar{G}$. Then $z^m \in G$ for some positive integer m. Let $z^m = \beta(\boldsymbol{l})$, where $\boldsymbol{l} \in \mathbb{Z}^k$. There is $\boldsymbol{r} \in \mathbb{Q}^k$ with $m\boldsymbol{r} = \boldsymbol{l}$. We have $z^m = \gamma(\boldsymbol{l}) = \gamma(\boldsymbol{r})^m$. Hence $z\gamma(\boldsymbol{r})^{-1}$ is a complex root of unity, and so is $\exp(2\pi r i)$ for some $r \in \mathbb{Q}$. Then $z = \delta(\boldsymbol{r}, r)$.

Put $A = \delta^{-1}(\Gamma)$. As $A \leq \mathbb{Q}^{k+1}$, we have $|A/nA| \leq n^{k+1}$ (for a proof see [1], Proposition 0.5). Clearly, δ induces a homomorphism $a + nA \mapsto \delta(a)\Gamma^n$ from A/nA onto Γ/Γ^n. Therefore $|\Gamma/\Gamma^n| \leq n^{k+1}$.

The property (iii) is a deep result of diophantine geometry, see [4]. Since in the most general form it was first conjectured by S. Lang, we call (iii) Lang's property. □

The following is the main result of the paper.

Theorem 1.2. *Let Γ be a subgroup of $\mathbb{S}$ with the properties* (i)–(iii). *The definable relations of the structure*

$$(\mathbb{R}, <, +, \cdot, 0, 1, \Gamma)$$

are exactly the Boolean combinations of relations of the form

$$\exists x_1 y_1 \ldots x_n y_n \, (P(x_1, y_1, \ldots, x_n, y_n, \boldsymbol{v}) \wedge \bigwedge_{i=1}^{n} (x_i, y_i) \in \Gamma),$$

where P is a semi-algebraic relation on $\mathbb{R}$, and n may be equal to 0.

A special case of the result, where Γ was the group of all complex roots of unity, had been proven by the second author in [5].

A more precise version of the result is as follows.

Theorem 1.3. *Let Γ be a subgroup of $\mathbb{S}$ with the properties* (i)–(iii), *and Γ_{re}, Γ_{im} be the sets of real and imaginary components of all pairs in Γ, respectively. The* 0*-definable relations of the structure*

$$M_0 = (\mathbb{R}, <, +, \cdot, 0, 1, \Gamma, a)_{a \in \Gamma_{\mathrm{re}} \cup \Gamma_{\mathrm{im}}}$$

are exactly the Boolean combinations of relations of the form

$$(\star) \qquad \exists x_1 y_1 \ldots x_n y_n \, (P(x_1, y_1, \ldots, x_n, y_n, \boldsymbol{v}) \wedge \bigwedge_{i=1}^{n} (x_i, y_i) \in \Gamma),$$

where P is quantifier-free definable in the ordered field $\mathbb{R}$ with parameters in the set $\Gamma_{\mathrm{re}} \cup \Gamma_{\mathrm{im}}$, and n may be equal to 0.

The structure M^0 on $\mathbb{R}$ whose atomic relations are all the relations of the form $(\star)$ is definitionally equivalent to M_0. Indeed,

- every atomic relation of M^0 is 0-definable in M_0 by an $\exists$-formula;
- every atomic relation of M_0 is an atomic relation in M^0.

The first statement is obvious. The second one holds because n may be equal to 0, and, for any polynomials $s(\boldsymbol{u})$ and $t(\boldsymbol{v})$ over the subfield of $\mathbb{R}$ generated by $\Gamma_{\mathrm{re}} \cup \Gamma_{\mathrm{im}}$, the relation $(s(\boldsymbol{u}), t(\boldsymbol{v})) \in \Gamma$ is equivalent to

$$\exists xy \, (x = s(\boldsymbol{u}) \wedge y = t(\boldsymbol{v}) \wedge (x, y) \in \Gamma).$$

Thus, Theorem 1.3 says exactly that M^0 admits quantifier elimination. Let M be the expansion of M_0 by all relations of the form $(\star)$. Then M and M^0 have the same atomic relations. So Theorem 1.3 is equivalent to

Theorem 1.4. *The structure M admits quantifier elimination.*

2. Preliminaries

In this section we introduce notation and collect some notions and facts we will use in the proofs. We assume the reader to be familiar with basic model theory, a good reference is e.g. [3].

Languages. Let L be the language of ordered rings, and

$$L^+ = L \cup \{a : a \in \Gamma_{\mathrm{re}} \cup \Gamma_{\mathrm{im}}\}, \qquad L^+(\Gamma) = L^+ \cup \{\Gamma\},$$

where Γ is considered as a binary relation symbol, and elements of $\Gamma_{\mathrm{re}} \cup \Gamma_{\mathrm{im}}$ as constant symbols. We denote by $L^\star$ the language of the structure M; here $\Gamma(M)$ is Γ, and $L^\star \supseteq L^+(\Gamma)$.

Let N be an $L^\star$-structure. For $a \in \Gamma_{\mathrm{re}} \cup \Gamma_{\mathrm{im}}$ we denote by a_N the interpretation of the constant symbol a in N. For $g = (a, b) \in \Gamma$ we denote $g_N = (a_N, b_N)$.

Algebraic closure. For a subset X of a structure N we denote by $\mathrm{acl}(X)$ the *algebraic closure of X in N*, that is, the set of all elements in N algebraic over X in the sense of model theory. Here an element of N is called *algebraic over X* if it belongs to a finite subset definable in N with parameters from X.

An element a of N is called *definable over X* if the set $\{a\}$ is definable in N with parameters from X. The set of all elements in N definable over X is called the *definable closure of X in N*.

For a subset X of a field F we denote by $\mathrm{acl}_F(X)$ the set of all elements in F algebraic over X in the sense of field theory. Clearly, if F is a subfield of a field K then for $X \subseteq F$ we have $\mathrm{acl}_F(X) = \mathrm{acl}_K(X) \cap F$.

Some known facts about real closed fields. Any real closed field can be uniquely expanded to an ordered field; the positive elements in that ordered field are the nonzero squares, and so it is a definitional expansion. It follows that

- in real closed fields algebraic closure coincides with definable closure,
- for real closed fields R and R', any elementary map β from R to R' *uniquely* extends to an elementary bijection
$$\bar{\beta} : \mathrm{acl}(\mathrm{dom}(\beta)) \to \mathrm{acl}(\mathrm{rng}(\beta)).$$
- for real closed fields R and R', if β and γ are elementary maps from R to R' and $\beta \subseteq \gamma$ then $\bar{\beta} \subseteq \bar{\gamma}$.

The theory of ordered real closed fields is complete and admits quantifier elimination. This implies that

- the theory of ordered real closed fields is o-minimal;
- in real closed fields algebraic dependence in the sense of model theory is exactly algebraic dependence in the sense of field theory;
- elementary maps between two real closed fields are exactly partial isomorphisms of the ordered fields.

An algebraic closure of a real closed field. Let R be a real closed field, and C be R^2 equipped with the usual addition and multiplication

$$(x,y)+(x',y')=(x+x',y+y'),$$
$$(x,y)\cdot(x',y')=(xx'-yy',xy'+x'y).$$

Then $(C,+,\cdot)$ is a field, and $x\mapsto(x,0)$ is an embedding of R into C. We will identify x and $(x,0)$. Then C is an algebraic closure of R. As usual, (x,y) can be written $x+yi$, where $i=(0,1)$. The set

$$S=\{(x,y)\in C : x^2+y^2=1\}$$

is called the unit circle in C. It is a subgroup of the multiplicative group C^* of the field C, and for $z=(x,y)\in S$ we have $z^{-1}=(x,-y)$. For $z=(x,y)\in C$ put $z_{\mathrm{re}}=x$ and $z_{\mathrm{im}}=y$. We will repeatedly use the following observation:

if $z\in S$ then z, z_{re}, and z_{im} are pairwise inter-algebraic in C.

Indeed, suppose $z\in S$. Then $x^2+y^2=1$ and so x and y are inter-algebraic in C. Also, z and x are inter-algebraic in C. Indeed, as $z=x+yi$ and $i^2=-1$, we have $(z-x)^2+y^2=0$ and so $z^2-2xz+1=0$.

For $Z\subseteq C$, we denote $Z_{\mathrm{re}}=\{z_{\mathrm{re}} : z\in Z\}$.

Translation from C to R. For any polynomial $p(Z_1,\dots,Z_n)$ over $\mathbb{Z}$, there are polynomials

$$p_1(X_1,Y_1,\dots,X_n,Y_n) \quad\text{and}\quad p_2(X_1,Y_1,\dots,X_n,Y_n)$$

over $\mathbb{Z}$ such that for any $z_1,\dots,z_n\in C$ with $z_i=(x_i,y_i)$ we have

$$p(z_1,\dots,z_n)=(p_1(x_1,y_1,\dots,x_n,y_n),\ p_2(x_1,y_1,\dots,x_n,y_n)).$$

Moreover, any formula $\phi(w_1,\dots,w_n)$ in the language of rings can be translated to another formula $\phi^*(v_1,u_1,\dots,v_n,u_n)$ in the language of rings such that for every $z_1,\dots,z_n\in C$ with $z_i=(x_i,y_i)$ we have

$$C\models\phi(z_1,\dots,z_n) \quad\text{iff}\quad R\models\phi^*(x_1,y_1,\dots,x_n,y_n).$$

The translations do not depend on R.

Elementary maps. Let R, R' be real closed fields, and C, C' their algebraic closures. Let β be an elementary map from R to R'. Then

$$\mathrm{dom}(\bar\beta)=\mathrm{acl}(\mathrm{dom}(\beta)=\mathrm{acl}_R(\mathrm{dom}(\beta)).$$

The map

$$\hat\beta : \mathrm{dom}(\bar\beta)\times\mathrm{dom}(\bar\beta)\to C'$$

defined by

$$\hat\beta(x,y)=(\bar\beta(x),\bar\beta(y))$$

is elementary as a map from C to C'. Therefore, whenever H is a subgroup of C^* with

$$H\subseteq\mathrm{acl}_R(\mathrm{dom}(\beta))\times\mathrm{acl}_R(\mathrm{dom}(\beta)),$$

the map $\hat\beta$ embeds the group H into the group C'^*.

It follows that if H is a subgroup of S such that $H_{\mathrm{re}}\subseteq\mathrm{dom}(\beta)$ then

$$H\subseteq\mathrm{dom}(\beta)\times\mathrm{acl}_R(\mathrm{dom}(\beta))\subseteq\mathrm{dom}(\hat\beta),$$

and the map $\hat\beta$ embeds the group H into the group S'.

Lemma 2.1. *Let $Z \subseteq S$ and $H \leq S$. Suppose $Z_{\mathrm{re}} \subseteq H_{\mathrm{re}}$. Then $Z \subseteq H$.*

Proof. Suppose $(x, y) \in Z$. Then $x \in Z_{\mathrm{re}}$. Hence $x \in H_{\mathrm{re}}$. Therefore $(x, u) \in H$ for some u. Since Z, $H \subseteq S$, we have $x^2 + y^2 = x^2 + u^2 = 1$ and so $y = \pm u$. Then $(x, y) = (x, u)^{\pm 1} \in H$. □

We will repeatedly use the following

Lemma 2.2. *Let β be an elementary map from R to R'. Let G be a subgroup of the group S generated by a subset Z, and G' a subgroup of S'. Suppose $Z_{\mathrm{re}} \subseteq \mathrm{dom}(\beta)$ and $\beta(Z_{\mathrm{re}}) \subseteq G'_{\mathrm{re}}$. Then*

(i) $G \subseteq \mathrm{dom}(\hat{\beta})$,
(ii) $\hat{\beta}(G) \leq G'$,
(iii) $\bar{\beta}(G_{\mathrm{re}}) = \hat{\beta}(G)_{\mathrm{re}}$.

Proof. (i) Since $Z_{\mathrm{re}} \subseteq \mathrm{dom}(\beta)$, we have $Z \subseteq \mathrm{acl}_C(\mathrm{dom}(\beta))$, and therefore $G \subseteq \mathrm{acl}_C(\mathrm{dom}(\beta))$. Then

$$G_{\mathrm{re}} \subseteq \mathrm{acl}_C(\mathrm{dom}(\beta)) \quad \text{and} \quad G_{\mathrm{im}} \subseteq \mathrm{acl}_C(\mathrm{dom}(\beta)).$$

Hence

$$G_{\mathrm{re}} \subseteq \mathrm{acl}_R(\mathrm{dom}(\beta)) \quad \text{and} \quad G_{\mathrm{im}} \subseteq \mathrm{acl}_R(\mathrm{dom}(\beta)),$$

and so $G \subseteq \mathrm{dom}(\hat{\beta})$.

(ii) We have $\hat{\beta}(Z)_{\mathrm{re}} \subseteq G'_{\mathrm{re}}$. Indeed, any element of $\hat{\beta}(Z)$ is of the form $(\bar{\beta}(x), \bar{\beta}(y))$ where $(x, y) \in Z$. Then any element of $\hat{\beta}(Z)_{\mathrm{re}}$ is $\bar{\beta}(x)$ for some $x \in Z_{\mathrm{re}}$ and hence belongs to G'_{re} by our assumption. By Lemma 2.1, $\hat{\beta}(Z) \subseteq G'$, because $Z \subseteq S$ and so $\hat{\beta}(Z) \subseteq S'$. Since the restriction of $\hat{\beta}$ on G is a homomorphism to C'^*, we have $\hat{\beta}(G) \leq G'$.

(iii) Suppose $x \in G_{\mathrm{re}}$. Then $(x, y) \in G$, for some y. Let $z = (x, y)$. Then $(\bar{\beta}(x), \bar{\beta}(y)) = \hat{\beta}(z) \in \hat{\beta}(G)$ and so $\bar{\beta}(x) \in \hat{\beta}(G)_{\mathrm{re}}$.

Now suppose $x' \in \hat{\beta}(G)_{\mathrm{re}}$. Then $(x', y') \in \hat{\beta}(G)$ for some y'. We have $(x', y') = \hat{\beta}(z)$ for some $z = (x, y) \in G$. Then $x \in G_{\mathrm{re}}$ and $x' = \bar{\beta}(x)$. □

Some abelian group theory. We will need some facts about abelian groups. For an abelian group A we denote by A_d its greatest divisible subgroup, and by $A[n]$ the n-torsion subgroup of A. It is well-known that A_d has a direct complement in A (in general, not uniquely determined), which is a reduced group.

Proposition 2.3. *Let A be an abelian group such that $A[n]$ is finite for every positive integer n. Then*

(i) $A_d = \bigcap_{n>0} nA$;
(ii) *if A/nA is finite for all positive integers n then $|A : A_d| \leq 2^{\aleph_0}$.*

Proof. (i) Clearly, $A_d \leq \bigcap_{n>0} nA$. It suffices to show that the subgroup $\bigcap_{n>0} nA$ is divisible. Let $a \in \bigcap_{n>0} nA$, and k be a positive integer. Then $a \in kA$ and so the set $\{b : kb = a\}$ is not empty. Therefore it is a coset of the finite subgroup $A[k]$; let it be $\{b_1, \ldots, b_s\}$. We show that one of the b_i belongs to $\bigcap_{n>0} nA$. Suppose not. For each i choose a positive integer n_i so that $b_i \notin n_i A$. There is b such that $a = kn_1 \ldots n_s b$. Then $n_1 \ldots n_s b$ is one of the b_i, contrary to $b_i \notin n_i A$.

(ii) Suppose A/nA is finite for all positive integers n. Using (i), we have

$$A/A_d = A/\bigcap_{n>0} nA \hookrightarrow \prod_{n>0} A/nA,$$

and the result follows. □

We will apply Proposition 2.3 to subgroups of the multiplicative groups of fields; obviously, they satisfy the assumption of the proposition.

Proposition 2.4. *Let A be a pure subgroup of an abelian group G, and $A \leq B \leq G$. Suppose $|A : nA| = |B : nB| < \infty$ for any positive integer n. Then B is a pure subgroup of G.*

Proof. Let $b \in B \cap nG$. Let $a_1, \dots, a_k$ be representatives of all cosets of nA in A. Whenever $i \neq j$, we have $a_i - a_j \notin nA$ and hence $a_i - a_j \notin nG$ because A is pure in G; so $a_i - a_j \notin nB$. Then $a_1, \dots, a_k$ is a full system of representatives of the cosets of nB in B because $|B : nB| = k$. Then $b - a_i \in nB$ for some i. We have $a_i \in nG$ and so $a_i \in nA$, because A is pure in G. Therefore $b \in nB$. □

An abelian group is called *pure-injective* if it has a direct complement in every its pure extension.

Fact 2.5 (see [3], Section 10.7)**.** *Every $\aleph_1$-saturated abelian group is pure-injective.*

Fact 2.6 (see [2], Theorem 38.1)**.** *A direct summand of a pure-injective abelian group is pure-injective.*

3. Axioms for the theory of M

Our goal is to find a complete axiom system for the theory of the structure M introduced above and to show that it admits quantifier elimination.

Consider the class of all $L^\star$-structures N satisfying the conditions (1)–(7) below.

(1) The L-reduct of N is an ordered real closed field R.
(2) The set $\Gamma(N)$ is a subgroup of S.
(3) The group $\Gamma(N)$ is elementarily equivalent to the group Γ.
(4) The set $\Gamma(N)_{\mathrm{re}}$ is dense in the interval $[-1, 1]$ of R.
(5) Whenever $f(X, \boldsymbol{Y}, \boldsymbol{Z})$ is a polynomial over $\mathbb{Z}$ of positive degree in X, for any tuple $\boldsymbol{c}$ in R every open interval in R contains an element a in R such that for every tuple $\boldsymbol{b}$ in $\Gamma(N)_{\mathrm{re}}$ we have $f(a, \boldsymbol{b}, \boldsymbol{c}) \neq 0$.
(6) Whenever
   - $p(X_1, \dots, X_n)$ is a nonconstant polynomial over $\mathbb{Z}$,
   - $(m_{11}, \dots, m_{1n}), \dots, (m_{k1}, \dots, m_{kn}) \in \mathbb{Z}^n - \{(0, \dots, 0)\}$,
   - $g_1, \dots, g_k \in \Gamma$,
   - for all $z_1, \dots, z_n \in \Gamma$

   $$\mathbb{C} \models p(z_1, \dots, z_n) = 0 \leftrightarrow \bigvee_{i=1}^{k} z_1^{m_{i1}} z_2^{m_{i2}} \dots z_n^{m_{in}} = g_i,$$

   we have for all $z_1, \dots, z_n \in \Gamma(N)$

   $$C \models p(z_1, \dots, z_n) = 0 \leftrightarrow \bigvee_{i=1}^{k} z_1^{m_{i1}} z_2^{m_{i2}} \dots z_n^{m_{in}} = (g_i)_N,$$

   where C is the algebraic closure of R.
(7) N satisfies all the quantifier-free $L^\star$-sentences that hold in M.

It is easy to see that there exists an infinite set T of first order $L^\star$-sentences such that an $L^\star$-structure N satisfies the conditions (1)–(7) if and only if N is a model of T.

Proposition 3.1. *The structure M is a model of T.*

Proof. Obviously, M satisfies Axioms 1–3 and 6–7.

To prove that Axiom 4 holds in M, it suffices to show that Γ is dense in $\mathbb{S}$, that is, whenever $0 \le a < b < 2\pi$, there exists $z \in \Gamma$ with $a < \arg(z) < b$. Choose n so that $b - a \le 2\pi/n$. As Γ is infinite, by the pigeon-hole principle there are $k \in \{0, \dots, n-1\}$ and $z, v \in \Gamma$ such that

$$2\pi k/n \le \arg(z) < \arg(v) < 2\pi(k+1)/n.$$

Let $u = vz^{-1}$ and $\phi = \arg(u)$. Then $u \in \Gamma$ and $0 < \phi < 2\pi/n$. There is a positive integer l such that $(l-1)\phi \le a < l\phi$. Clearly, $a < l\phi < b$. As $u^l \in \Gamma$, and $\arg(u) = l\phi$, the result follows.

Axiom 5 holds in M because any interval in $\mathbb{R}$ is uncountable, but Γ is countable and so for any finite subset A of $\mathbb{R}$ there are only countably many elements algebraic over $\Gamma_{\text{re}} \cup A$. □

We will prove

Theorem 3.2. *The theory T admits quantifier elimination.*

Since M is a model of T, Theorems 1.4 and 1.3 follow. Moreover, due to Axioms 7 and Theorem 3.2, a sentence holds in all models of T iff it holds in M. Therefore we will have

Corollary 3.3. *The theory T is complete.*

For any model N of T, let $f : \Gamma \to N \times N$ be the map defined by $f(g) = g_N$, where g_N is the element of $N \times N$ we defined in the first subsection of Section 2. As $g \in \Gamma$, we have $g_N \in \Gamma(N)$, by Axiom 7; so, in fact, $f : \Gamma \to \Gamma(N)$.

We will need later

Lemma 3.4. *The map f is a pure monomorphism from the group Γ to the group $\Gamma(N)$.*

Proof. For any quantifier-free formula $\phi(w_1, \dots, w_n)$ in the language of multiplicative groups there is a quantifier-free formula $\phi'(v_1, u_1, \dots, v_n, u_n)$ in the language of rings such that for any real closed field R and $z_1, \dots, z_n \in C^*$ with $z_i = (x_i, y_i)$ we have

$$C^* \models \phi(z_1, \dots, z_n) \quad \text{iff} \quad R \models \phi'(x_1, y_1, \dots, x_n, y_n).$$

Let R be the L-reduct of N, and C the algebraic closure of the real closed field R. For ϕ as above and $g_1, \dots, g_n \in \Gamma$ with $g_i = (a_i, b_i)$, we have

$$\begin{gathered}
\Gamma \models \phi(g_1, \dots, g_n) \Leftrightarrow \\
\mathbb{R} \models \phi'(a_1, b_1, \dots, a_n, b_n) \Leftrightarrow \quad \text{(by Axiom 7)} \\
\phi'(a_1, b_1, \dots, a_n, b_n) \in T \Leftrightarrow \\
R \models \phi'((a_1)_N, (b_1)_N, \dots, (a_n)_N, (b_n)_N) \Leftrightarrow \\
C^* \models \phi((g_1)_N, \dots, (g_n)_N) \Leftrightarrow \\
\Gamma(N) \models \phi((g_1)_N, \dots, (g_n)_N).
\end{gathered}$$

It follows that $f : \Gamma \to \Gamma(N)$ is a monomorphism. We show that f is pure. Let n be a positive integer, and ϕ be the formula $w_1^n = w_2$. For the corresponding $\phi'(v_1, u_1, v_2, u_2)$, whenever $g = (a, b) \in \Gamma$, we have

$$\begin{gathered}\Gamma \models \exists w(w^n = g) \Leftrightarrow \\ M \models \exists xy\,(\Gamma(x, y) \wedge \phi'(x, y, a, b)) \Leftrightarrow \quad \text{(by Axioms 7)} \\ \exists xy\,(\Gamma(x, y) \wedge \phi'(x, y, a, b)) \in T \Leftrightarrow \\ N \models \exists xy\,(\Gamma(x, y) \wedge \phi'(x, y, a_N, b_N)) \Leftrightarrow \\ \Gamma(N) \models \exists w(w^n = g_N).\end{gathered}$$

The lemma is proven. □

We denote the pure subgroup $f(\Gamma)$ of $\Gamma(N)$ by Γ_N.

4. Submodel completeness of T

To prove Theorem 3.2, it suffices to show that any finite partial isomorphism α between any two models N and N' of T is an elementary map.

We may assume that N and N' are $(2^{\aleph_0})^+$-saturated. Let N_0 and N_0' be the $L^+(\Gamma)$-reducts of N and N', respectively. Every elementary map from N_0 to N_0' is an elementary map from N to N', because N and N' are definitional expansions of N_0 and N_0', respectively. Therefore it suffices to prove that α extends to an elementary map from N_0 to N_0'. Thus, it suffices to prove the following

Proposition 4.1. *Let N and N' be $(2^{\aleph_0})^+$-saturated models of T. Then there exists a back-and-forth system $\mathcal{S}$ from N_0 to N_0' such that any finite partial isomorphism from N to N' extends to a member of $\mathcal{S}$.*

Here a *back-and-forth system* from N_0 to N_0' is defined to be a set $\mathcal{S}$ of partial isomorphisms from N_0 to N_0' such that for every $\beta \in \mathcal{S}$ and $a \in N$, $a' \in N'$ there exists $\gamma \in \mathcal{S}$ such that $\beta \subseteq \gamma$, $a \in \operatorname{dom}(\gamma)$, and $a' \in \operatorname{rng}(\gamma)$. It is well-known that any member of a back-and-forth system is an elementary map.

Proof. We construct $\mathcal{S}$ satisfying the conditions of Proposition 4.1.

Let R and R' denote the ordered real closed fields that are the L-reducts of N and N', respectively. Let C and C' be their algebraic closures.

Let $\mathcal{E}$ be the set of all L^+-elementary maps from N to N'. Let $\mathcal{S}_0$ be the set of all $\beta \in \mathcal{E}$ such that there exist

- a finite subset A of R, and a finite subset A' of R',
- a subgroup H of $\Gamma(N)$ of cardinality at most $2^{\aleph_0}$, and a subgroup H' of $\Gamma(N')$ of cardinality at most $2^{\aleph_0}$

satisfying the following conditions:

(a) $\operatorname{dom}(\beta) = A \cup H_{\mathrm{re}}$, $\operatorname{rng}(\beta) = A' \cup H'_{\mathrm{re}}$, $\beta(A) = A'$, $\beta(H_{\mathrm{re}}) = H'_{\mathrm{re}}$;
(b) A is algebraically independent over $\Gamma(N)$ in C, and A' is algebraically independent over $\Gamma(N')$ in C';
(c) $\Gamma_N \leq H$ and $\Gamma_{N'} \leq H'$;
(d) H has a divisible torsion-free direct complement D in $\Gamma(N)$, and H' has a divisible torsion-free direct complement D' in $\Gamma(N')$.

Let $\mathcal{S} = \{\bar\beta : \beta \in \mathcal{S}_0\}$. Since $\beta \in \mathcal{E}$ implies $\bar\beta \in \mathcal{E}$, we have $\mathcal{S} \subseteq \mathcal{E}$. It suffices to prove the following three lemmas.

Lemma 4.2. *Any member of $\mathcal{S}$ is a partial isomorphism from N_0 to N'_0.*

Lemma 4.3. *Every finite partial isomorphism from N to N' extends to a member of $\mathcal{S}$.*

Lemma 4.4. *$\mathcal{S}$ is a back-and-forth system from N_0 to N'_0.*

Below we prove the lemmas. This completes the proof of Proposition 4.1 and hence of Theorem 3.2. □

Proof of Lemma 4.2. The following claim is crucial in the proof; it is where Lang's property of Γ and Axiom 6 of T is used.

Claim. *Let N be a model of T. Suppose $\Gamma(N)$ is the direct product of subgroups H and D such that $\Gamma_N \leq H$, and D is torsion-free. Let A be a subset of C algebraically independent over $\Gamma(N)$ in the field C. Then*

$$\mathrm{acl}_C(A, H) \cap \Gamma(N) = H.$$

Proof of the Claim. Clearly, $\mathrm{acl}_C(A, H) \cap \Gamma(N)$ contains H. We show that $z \in H$ assuming $z \in \mathrm{acl}_C(A, H) \cap \Gamma(N)$.

First we prove that $z \in \mathrm{acl}_C(H)$. Let A_0 be a minimal subset of A such that z belongs to $\mathrm{acl}_C(A_0, H)$. Then $A_0 = \emptyset$, because for $a \in A_0$ we would have $z \notin \mathrm{acl}_C(A_0 - \{a\}, H)$, and, by the Exchange Property of the algebraically closed field C,

$$a \in \mathrm{acl}_C(A_0 - \{a\}, z, H) \subseteq \mathrm{acl}_C(A_0 - \{a\}, \Gamma(N)),$$

contrary to algebraic independence of A over $\Gamma(N)$ in C.

Thus $p(z, h_1, \ldots, h_n) = 0$ for some polynomial $p(X_0, X_1, \ldots, X_n)$ over $\mathbb{Z}$ of positive degree in X_0, and some $h_1, \ldots, h_n \in H$. By the property (iii) of the group Γ, there exist

- a positive integer k,
- elements $g_1, \ldots, g_k$ of Γ, and
- nonzero $(n+1)$-tuples of integers $(m_{i0}, m_{i1}, \ldots, m_{in})$, $i = 1, \ldots, k$,

such that, whenever $z_0, z_1, \ldots, z_n \in \Gamma$, we have

$$\mathbb{C} \models p(z_0, z_1, \ldots, z_n) = 0 \leftrightarrow \bigvee_{i=1}^{k} z_0^{m_{i0}} z_1^{m_{i1}} \ldots z_n^{m_{in}} = g_i.$$

Then by Axioms (6), whenever $z_0, z_1, \ldots, z_n \in \Gamma(N)$,

$$C \models p(z_0, z_1, \ldots, z_n) = 0 \leftrightarrow \bigvee_{i=1}^{k} z_0^{m_{i0}} z_1^{m_{i1}} \ldots z_n^{m_{in}} = (g_i)_N.$$

The set of solutions in $\Gamma(N)$ of the equation $p(X_0, h_1, \ldots, h_n) = 0$ in X_0 is finite and nonempty. It follows that $m_{i0} \neq 0$ for at least one i, because otherwise this set would be either $\emptyset$ or $\Gamma(N)$. Thus, we have

$$z^m h_1^{m_1} \ldots h_n^{m_n} = g_N$$

for some $g \in \Gamma$ and integers $m, m_1, \ldots, m_n$, where $m \neq 0$. Since $g_N \in H$, it follows that $z^m \in H$. Let $z = hd$, where $h \in H$ and $d \in D$. So $h^m d^m \in H$, and therefore $d^m \in H \cap D = \{1\}$. Since D is torsion-free, we have $d = 1$ and hence $z \in H$, and we are done. The Claim is proven. □

Now we are ready to prove Lemma 4.2. Let $\beta \in \mathcal{S}_0$. We show that $\bar{\beta}$ is a partial isomorphism from N_0 to N_0'. Since $\bar{\beta} \in \mathcal{E}$, we need to prove only that $z \in \Gamma(N)$ iff $\hat{\beta}(z) \in \Gamma(N')$, for any $z \in \mathrm{dom}(\hat{\beta})$. We have

$$\mathrm{dom}(\bar{\beta}) = \mathrm{acl}_R(A, H_{\mathrm{re}}), \quad \mathrm{dom}(\hat{\beta}) = \mathrm{dom}(\bar{\beta}) \times \mathrm{dom}(\bar{\beta}).$$

Since $\beta(H_{\mathrm{re}}) = H'_{\mathrm{re}}$ by (a), we have $H \subseteq \mathrm{dom}(\hat{\beta})$ and $\hat{\beta}(H) = H'$ by Lemma 2.2. Therefore it suffices to observe that $\mathrm{dom}(\hat{\beta}) \cap \Gamma(N) = H$ and $\mathrm{rng}(\hat{\beta}) \cap \Gamma(N') = H'$. We show the first; the second is similar. Clearly, H is contained in $\mathrm{dom}(\hat{\beta}) \cap \Gamma(N)$. Then the resulf follows from the Claim, because $\mathrm{dom}(\hat{\beta}) \subseteq \mathrm{acl}_C(A, H)$: if $z \in \mathrm{dom}(\hat{\beta})$ then

$$z_{\mathrm{re}},\, z_{\mathrm{im}} \in \mathrm{dom}(\bar{\beta}) = \mathrm{acl}_R(A, H_{\mathrm{re}}) \subseteq \mathrm{acl}_C(A, H_{\mathrm{re}}) = \mathrm{acl}_C(A, H),$$

and hence $z \in \mathrm{acl}_C(A, H)$. Lemma 4.2 is proven.

Proof of Lemma 4.3. Let α be a finite partial isomorphism from N to N'. We need to construct $\beta \in \mathcal{S}_0$ such that $\alpha \subseteq \bar{\beta}$. We will use the following

Claim. *For any $X \subseteq \Gamma(N)_{\mathrm{re}}$ with $|X| \le 2^{\aleph_0}$ there exists $\gamma \in \mathcal{E}$ such that $\alpha \subseteq \gamma$, $\mathrm{dom}(\gamma) = \mathrm{dom}(\alpha) \cup X$, and $\gamma(X) \subseteq \Gamma(N')_{\mathrm{re}}$.*

Proof of the Claim. Let $X = (a_i : i < 2^{\aleph_0})$. For each $i < 2^{\aleph_0}$ choose a_i' such that $(a_i, a_i') \in \Gamma(N)$. Let p be the quantifier-free L^+-type over $\mathrm{dom}(\alpha)$ of the family $\{a_i, a_i' : i < 2^{\aleph_0}\}$, in the variables $\{x_i, x_i' : i < 2^{\aleph_0}\}$. Let αp stand for the quantifier-free L^+-type over $\mathrm{rng}(\alpha)$ induced by the map α.

We show that the set of formulas

$$\Delta = \alpha p \cup \{\Gamma(x_i, x_i') : i < 2^{\aleph_0}\}$$

realizes in N'. Since N' is $(2^{\aleph_0})^+$-saturated, it suffices to check that Δ is finitely satisfiable in N'. The latter is true because the map α preserves atomic $L^\star$-formulas.

Let $\{b_i, b_i' : i < 2^{\aleph_0}\}$ be a realization of Δ in N'. Put

$$\gamma = \alpha \cup \{(a_i, b_i) : i < 2^{\aleph_0}\}.$$

Clearly, $\mathrm{dom}(\gamma) = \mathrm{dom}(\alpha) \cup X$, and $\gamma(a_i) = b_i \in \Gamma(N')_{\mathrm{re}}$ for all i. Moreover, γ is a partial L^+-isomorphism and therefore is an L^+-elementary map because the theory of ordered real closed fields admits quantifier elimination. The Claim is proven. □

Choose a subset A of $\mathrm{dom}(\alpha)$ which is maximal among the subsets of $\mathrm{dom}(\alpha)$ algebraically independent in the field C over $\Gamma(N)$. Then any element of $\mathrm{dom}(\alpha)$ is algebraic over $\Gamma(N) \cup A$. Since $\mathrm{dom}(\alpha)$ is finite, there is a finite subset Z of $\Gamma(N)$ such that any element of $\mathrm{dom}(\alpha)$ is algebraic over $Z \cup A$.

Let U be a direct complement of $\Gamma(N)_d$ in $\Gamma(N)$. Since Z is finite, and Γ_N is countable, there is a countable divisible subgroup V of $\Gamma(N)_d$ such that $Z \cup \Gamma_N \subseteq UV$. Clearly, $H = UV$ is the direct product of the subgroups U and V. Let D be a direct complement of the divisible subgroup V in $\Gamma(N)_d$; clearly, D is divisible. Then $\Gamma(N)$ is the direct product of the subgroups U, V, and D, and so is the direct product of H and D.

Since $\Gamma \simeq \Gamma_N \le \Gamma(N)$ by Lemma 3.4, $\Gamma(N) \equiv \Gamma$ by Axioms 3, and Γ has at most n elements of order n, it follows that all elements of finite order in $\Gamma(N)$ belong to Γ_N, and so to H. Therefore D a torsion-free divisible group.

As Γ/Γ^n is finite and $\Gamma(N) \equiv \Gamma$, the group $\Gamma(N)$ is finite modulo n, for any $n > 0$. Therefore, by Proposition 2.3(ii), $|U| \le 2^{\aleph_0}$. Since V is countable,

$$|H| = |UV| \le 2^{\aleph_0}.$$

By the Claim applied to α and $X = H_{\mathrm{re}}$, we obtain $\gamma \in \mathcal{E}$ such that $\gamma \supseteq \alpha$, $\mathrm{dom}(\gamma) = \mathrm{dom}(\alpha) \cup H_{\mathrm{re}}$, and $\gamma(H_{\mathrm{re}}) \subseteq \Gamma(N')_{\mathrm{re}}$. We have

$$A \cup H_{\mathrm{re}} \subseteq \mathrm{dom}(\gamma) \subseteq \mathrm{acl}_R(A \cup H_{\mathrm{re}}).$$

Here the first inclusion is obvious. As $Z \subseteq H$, each element of $\mathrm{dom}(\alpha)$ is algebraic over $A \cup H$ in C and so over $A \cup H_{\mathrm{re}}$ in R. Thus we have the second inclusion. Hence $\mathrm{dom}(\bar{\gamma}) = \mathrm{acl}_R(\mathrm{dom}(\gamma)) = \mathrm{acl}_R(A \cup H_{\mathrm{re}})$.

Let β be the restriction of γ on $A \cup H_{\mathrm{re}}$. Then $\beta \in \mathcal{E}$. As $\beta \subseteq \gamma$ we have $\bar{\beta} \subseteq \bar{\gamma}$; moreover, $\bar{\beta} = \bar{\gamma}$ because

$$\mathrm{dom}(\bar{\gamma}) = \mathrm{acl}_R(A \cup H_{\mathrm{re}}) = \mathrm{dom}(\bar{\beta}).$$

Hence $\alpha \subseteq \gamma \subseteq \bar{\gamma} = \bar{\beta}$. We show that $\beta \in \mathcal{S}_0$.

Applying Lemma 2.2 with H as G and $\Gamma(N')$ as G', we have

$$H \subseteq \mathrm{dom}(\hat{\beta}), \quad \hat{\beta}(H) \le \Gamma(N'), \quad \bar{\beta}(H_{\mathrm{re}}) = \hat{\beta}(H)_{\mathrm{re}}.$$

Then for $A' = \beta(A)$ and $H' = \hat{\beta}(H)$ the condition (a) holds.

The set A was chosen to be algebraically independent in C over Γ^N. Since α preserves atomic $L^\star$-formulas, the set $A' = \alpha(A)$ is algebraically independent in C' over $\Gamma(N')$. So (b) holds.

By our construction $\Gamma_N \le H$. Therefore for all $a \in \Gamma_{\mathrm{re}} \cup \Gamma_{\mathrm{im}}$ we have

$$a_N \in \mathrm{acl}_R(H_{\mathrm{re}}) \subseteq \mathrm{dom}(\bar{\beta}).$$

As $\bar{\beta} \in \mathcal{E}$, we have $\bar{\beta}(a_N) = a_{N'}$. It follows that $\Gamma_{N'} = \hat{\beta}(\Gamma_N) \le H'$. Thus (c) holds.

It remains to check (d). We already checked that H has a divisible torsion-free direct complement D in $\Gamma(N)$. We prove that H' has a divisible torsion-free direct complement D' in $\Gamma(N')$.

First we show that H' is a pure subgroup of $\Gamma(N')$. By Lemma 3.4, $\Gamma_{N'}$ is a pure subgroup of $\Gamma(N')$. We have $\Gamma_{N'} \le H' \le \Gamma(N')$. Also, for any positive integer n we have

$$|H' : H'^n| = |H : H^n| = |\Gamma(N) : \Gamma(N)^n| = |\Gamma : \Gamma^n| = |\Gamma(N') : \Gamma(N')^n|.$$

Then the result follows from Proposition 2.4.

The group $\Gamma(N)$ is $\aleph_1$-saturated and so is pure-injective, by Fact 2.5. Being a direct summand of $\Gamma(N)$, the group H is pure-injective, too, by Fact 2.6. Then H' is pure-injective. Therefore H' has a direct complement D' in $\Gamma(N')$.

Since $\Gamma(N') \equiv \Gamma \simeq \Gamma_{N'}$, and $\Gamma(N')$ has at most n elements of order n for each n, all elements of finite order in $\Gamma(N')$ belongs to the subgroup $\Gamma_{N'}$, and so to H'. Therefore the group D' is torsion-free. We show that D' is divisible. Let $n > 0$. We have

$$|\Gamma_{N'} : (\Gamma_{N'})^n| = |H' : H'^n| \cdot |D' : D'^n|.$$

Since, as we already showed,

$$|\Gamma_{N'} : (\Gamma_{N'})^n| = |H' : H'^n|,$$

it follows that $|D' : D'^n| = 1$ and so $D' = D'^n$. This completes the proof of Lemma 4.3.

Proof of Lemma 4.4. By symmetry, it suffices to prove that if $\beta \in \mathcal{S}_0$ and $a \in N$ then there exists $\gamma \in \mathcal{S}_0$ such that $\beta \subseteq \gamma$ and $a \in \operatorname{dom}(\bar{\gamma})$.

Let A, A', H, H', D, and D' witness that $\beta \in \mathcal{S}_0$.

If $a \in \operatorname{dom}(\bar{\beta})$, we can take β for γ; so we assume that $a \notin \operatorname{dom}(\bar{\beta})$.

Case 1. $a \in \Gamma(N)_{\mathrm{re}}$.

In this case $a = c_{\mathrm{re}}$ for some $c \in \Gamma(N)$. Since $a \notin \operatorname{dom}(\bar{\beta}) = \operatorname{acl}_R(A \cup H_{\mathrm{re}})$, we have $c \notin \operatorname{acl}_C(A \cup H)$. Let $c = hd$, where $h \in H$ and $d \in D$. Then $d \notin \operatorname{acl}_C(A \cup H)$. As D is divisible, there exist $d_0, d_1, \dots$ in D such that

$$d_0 = d, \qquad d_n^n = d_{n-1} \quad \text{for all } n > 0.$$

Clearly, d_n is inter-algebraic with d in C, and so $d_n \notin \operatorname{acl}_C(A \cup H)$, for all n. Then for $e_n = (d_n)_{\mathrm{re}}$ we have $e_n \notin \operatorname{acl}_R(A \cup H_{\mathrm{re}})$, and the elements e_n pairwise inter-algebraic in R.

We will need the following

Claim. *For any $e \in \Gamma(N)_{\mathrm{re}} \setminus \operatorname{acl}_R(A \cup H_{\mathrm{re}})$ there is $e' \in \Gamma(N')_{\mathrm{re}}$ such that $\beta \cup \{(e, e')\} \in \mathcal{E}$.*

Proof of the Claim. Let $p(x)$ be the L^+-type of e over $A \cup H_{\mathrm{re}}$ in R. We need to prove that the set of formulas

$$\beta p(x) \cup \{\exists y \Gamma(x, y)\}$$

is realized in N'. As N' is $(2^{\aleph_0})^+$-saturated, and $|A \cup H_{\mathrm{re}}| \le 2^{\aleph_0}$, it suffices to show that whenever $\phi \in p$ the formula

$$\beta\phi(x) \wedge \exists y \Gamma(x, y)$$

has a solution e'_ϕ in N'. Since the ordered real closed field R is o-minimal, and e is not algebraic over $A \cup H_{\mathrm{re}}$ in R, there exist

$$b,\, b' \in \operatorname{acl}_R(A \cup H_{\mathrm{re}}) \cup \{\pm\infty\}$$

such that $b < e < b'$ and

$$R \models \forall x (b < x < b' \to \phi(x)).$$

It follows that

$$R' \models \forall x (\bar{\beta}(b) < x < \bar{\beta}(b') \to \beta\phi(x)).$$

Since $e \in \Gamma(N)_{\mathrm{re}}$, we have $-1 \le e \le 1$; so we may assume that

$$-1 \le b < b' \le 1,$$

and hence

$$-1 \le \bar{\beta}(b) < \bar{\beta}(b') \le 1$$

in R'. Since N' satisfies Axiom 4, there exists $e'_\phi \in \Gamma(N')_{\mathrm{re}}$ with

$$\bar{\beta}(b) < e'_\phi < \bar{\beta}(b').$$

Then e'_ϕ satisfies the required condition. The Claim is proven. □

Let $p(x_0, x_1, \dots)$ be the L^+-type of $(e_0, e_1, \dots)$ over $A \cup H_{\mathrm{re}}$ in N, and

$$\Delta = (\beta p)(x_0, x_1, \dots) \cup \{\exists y_n \Gamma(x_n, y_n) : n < \omega\}$$

Thus Δ is a set of formulas over $A' \cup H'_{\mathrm{re}}$ with free variables $(x_i : i < \omega)$.

Claim. *Δ is finitely satisfiable in N'.*

Proof of the Claim. It suffices to check that Δ_n is realizable in N' for all n, where Δ_n is the set of formulas in Δ in which no x_i with $i > n$ is involved.

Let Δ^n be the set of formulas in Δ in which no x_i with $i \neq n$ is involved. By the previous Claim Δ^n is realizable in N' by some element e'_n; then

$$\delta = \beta \cup \{(e_n, e'_n)\} \in \mathcal{E}.$$

We have

$$(H \cup \{d_n\})_{\text{re}} = H_{\text{re}} \cup \{e_n\} \subseteq \text{dom}(\delta),$$
$$\delta((H \cup \{d_n\}_{\text{re}}) = H'_{\text{re}} \cup \{e'_n\} \subseteq \Gamma(N')_{\text{re}}.$$

Let G be the subgroup of $\Gamma(N)$ generated by $H \cup \{d_n\}$. Applying Lemma 2.2, we obtain $G \subseteq \text{dom}(\hat{\delta})$ and $\hat{\delta}(G) \leq \Gamma(N')$. For $i \leq n$ we have $d_i \in G$. Hence $\hat{\delta}(d_i) \in \Gamma(N')$ and $e_i \in \text{dom}(\bar{\delta})$. Put $e'_i = \bar{\delta}(e_i)$. Then $e'_i \in \Gamma(N')_{\text{re}}$. Since $\bar{\delta} \in \mathcal{E}$, it follows that the tuple $(e'_0, \ldots, e'_n)$ realizes Δ_n. The Claim is proven. □

As the structure N' is $(2^{\aleph_0})^+$-saturated and $|A' \cup H'_{\text{re}}| \leq 2^{\aleph_0}$, the set Δ is realized in N'; let $(e'_0, e'_1, \ldots)$ be a realization. Then

$$\tau = \beta \cup \{(e_n, e'_n) : n < \omega\} \in \mathcal{E},$$

and $e'_n \in \Gamma_{\text{re}}^{N'}$ for all n. Let P be the subgroup of D generated by all d_n. Clearly, $HP \subseteq \text{acl}_C(\text{dom}(\tau))$ and so

$$A \cup (HP)_{\text{re}} \subseteq \text{acl}_R(\text{dom}(\tau)) = \text{dom}(\bar{\tau}).$$

Let γ be the restriction of $\bar{\tau}$ on $A \cup (HP)_{\text{re}}$. Clearly, $\beta \subseteq \gamma$ and $\gamma \in \mathcal{E}$. We have $a = e_0 \in P_{\text{re}} \subseteq \text{dom}(\gamma)$. We prove that $\gamma \in \mathcal{S}_0$, and A and HP witness this. Since P is countable, $|HP| \leq 2^{\aleph_0}$.

It is easy to see that the group P is divisible. Let D_0 be a direct complement of P in D. Clearly, $\Gamma(N)$ is the direct product of HP and D_0, and D_0 is torsion-free and divisible. As

$$(HP)_{\text{re}} \subseteq \text{dom}(\gamma) \quad \text{and} \quad \gamma((HP)_{\text{re}} \subseteq \Gamma_{\text{re}}^{N'},$$

by Lemma 2.2 we have

$$HP \subseteq \text{dom}(\hat{\gamma}), \quad H'P' \leq \Gamma(N'), \quad \gamma((HP)_{\text{re}}) = (H'P')_{\text{re}},$$

where $P' = \hat{\gamma}(P)$. It remains to show that $H'P'$ has a torsion-free divisible direct complement D' in $\Gamma(N')$.

Remember that $\Gamma(N')$ is the direct product of H' and D'; we denote by P'_0 the D'-projection of P'. Then P'_0 is divisible and $H'P' = H'P'_0$. Let D'_0 be a direct complement of P'_0 in D'. Then $\Gamma(N')$ is the direct product of H', P'_0, and D'_0, and hence is the direct product of $H'P'$ and D'_0. As D' is torsion-free and divisible, so is D'_0.

Thus, in Case 1 we are done. We reduce to this the following more general case.

Case 2. $a \in \text{acl}_R(A, \Gamma(N)_{\text{re}})$.

In this case we have $a \in \text{acl}_R(A, a_1, \ldots, a_n)$ for some $a_i \in \Gamma(N)_{\text{re}}$. Repeating the arguments of Case 1 n times we can find $\gamma \in \mathcal{S}_0$ such that $\beta \subseteq \gamma$ and all a_i belong to $\text{dom}(\bar{\gamma})$. As $A \subseteq \text{dom}(\bar{\gamma})$ and the set $\text{dom}(\bar{\gamma})$ is algebraically closed, we have $a \in \text{dom}(\bar{\gamma})$. It remains to consider

Case 3. $a \notin \text{acl}_R(A \cup \Gamma(N)_{\text{re}})$.

Let $\boldsymbol{a}$ be an enumeration of the set A, and $q(x)$ the set of all sentences θ_f of the form

$$\forall u_1 v_1 \dots u_n v_n (\bigwedge_{i=1}^{n} \Gamma(u_i, v_i) \to f(x, \boldsymbol{u}, \boldsymbol{a}) \neq 0),$$

where $f(X, \boldsymbol{Y}, \boldsymbol{Z})$ is a polynomial over $\mathbb{Z}$ of positive degree in X. Clearly, a satisfies $q(x)$ in N. Let $p(x)$ be the L^+-type of a over $A \cup H_{\mathrm{re}}$ in R.

Claim. *The set of formulas $\Delta = \beta p(x) \cup \beta q(x)$ is finitely satisfiable in N'.*

Proof of the Claim. We show that if ϕ is a formula in p and $f_i(X, \boldsymbol{Y}, \boldsymbol{Z})$ are polynomials over $\mathbb{Z}$ of positive degree in X, where $i = 1, \dots, k$, then the formula

$$\beta\phi(x) \wedge \beta\theta_{f_1}(x) \wedge \dots \wedge \beta\theta_{f_k}(x)$$

has a solution in N'. Let $f = f_1 \dots f_k$. It suffices to show that the formula $\beta\phi(x) \wedge \beta\theta_f(x)$ has a solution d' in N'.

As the ordered field R is o-minimal, and a is not algebraic over $A \cup \Gamma(N)_{\mathrm{re}}$ in R, there are b, $b' \in \mathrm{acl}(A \cup \Gamma(N)_{\mathrm{re}}) \cup \{\pm\infty\}$ such that $b < a < b'$ and

$$R \models \forall x (b < x < b' \to \phi(x)).$$

It follows that $\bar{\beta}(b) < \bar{\beta}(b')$ in R', and

$$R' \models \forall x (\bar{\beta}(b) < x < \bar{\beta}(b') \to \beta\phi(x))$$

(where $\bar{\beta}(\pm\infty) = \pm\infty$). As N' satisfies Axiom 5, there is $d' \in N'$ such that $\bar{\beta}(b) < d' < \bar{\beta}(b')$ and $N' \models \beta\theta_f(d')$. Then d' satisfies the required condition. The Claim is proven. □

Since N' is $(2^{\aleph_0})^+$-saturated, and $|A \cup H_{\mathrm{re}}| \leq 2^{\aleph_0}$, the set of formulas Δ is realized in N' by some element a'. Since a' satisfies $\beta q(x)$, we have $a' \notin \mathrm{acl}_{R'}(A', \Gamma(N')_{\mathrm{re}})$. Put $\gamma = \beta \cup \{(a, a')\}$. As a' satisfies $\beta p(x)$, we have $\gamma \in \mathcal{E}$. Since $A \cup \{a\}$ is algebraically independent over $\Gamma(N)$ in C, and $A' \cup \{a'\}$ is algebraically independent over $\Gamma(N')$ in C', we have $\gamma \in \mathcal{S}_0$.

This completes the proof of Lemma 4.4.

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Department of Mathematics, Istanbul Bilgi University, 80370 Dolapdere–Istanbul, Turkey
E-mail address: olegb@bilgi.edu.tr

Mathematical Institute, University of Oxford, 24–29 St Giles, Oxford, OX1 3LB, United Kingdom
E-mail address: zilber@maths.ox.ac.uk