

# Multidimensional wave-wave regular interactions and genuine nonlinearity: some remarks

Applications to a multidimensional class of exact gasdynamic solutions <sup>1</sup>

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# 1. Introduction. Systems of equations considered. Exceptional directions

■ ***Main topics*** : hyperbolic systems of conservation laws, regular gasdynamic interactions, geometrical methods. ■ ***AMS Classification*** : 35A30, 35L65, 76N10, 76N15. □

The present paper begins with the description of an “*algebraic*” duality (in the sense of M. Burnat) between the characteristic or exceptional nature in the physical space and a *characteristic nature in the hodograph space* (§2). This “algebraic” duality is then used to describe aspects of some possible relations between the physical space and the hodograph space.

A first relation, among those mentioned above, is that indicated by a significant class of solutions: the *simple waves solutions* (§3). ■ Section 3.1 constructively considers one-dimensional simple waves solutions. The importance of a *genuinely nonlinear* character of the characteristic fields which contribute in construction is observed in this respect (section 3.1.3). ■ An *analogue* of the genuinely nonlinear character of an one-dimensional simple waves solution is then identified and *essentially* used for the construction of some *multidimensional extensions* [simple waves solutions (sections 3.2, 3.3, 3.5), regular interactions of simple waves solutions (§§4–7)]. ■ Section 3.7 comparatively revisits (cf. Scheme 1) the scalar, non-scalar, one-dimensional, multidimensional approaches. ■ We end §3 by presenting a *classification*, from a multidimensional prospect, in the class of the simple waves solutions (section 3.8).

A Riemann restricted *one-dimensional* version of the method of characteristics is considered at the beginning of §4. In sections 4.2–4.4 this version is adapted to the “algebraic” duality considered in §2. Section 4.4 makes use of this Riemann restricted version to consider two remarkable types of one-dimensional constructions. ■ In section 4.5 a nonlinearity hierarchy is considered from a Riemann restricted prospect. ■ Incidentally, the two remarkable types of construction mentioned above can be adapted, in order to make their persistence possible, when the Riemann restricted context is no more available. For the first remarkable construction, associated to the simple waves solutions, such an adaptation consists in replacing the Riemann invariance by a Riemann–Lax invariance (section 4.6). For the second remarkable construction, associated with regular interactions of simple waves solutions, the details of a candidate adaptation are presented in section 5.3.4 (Example 5.1). The role of the Riemann invariance is played in this case by a Riemann–Burnat invariance.

In §5 we present a *multidimensional* extension, in the sense of M. Burnat, of the results in §4. Sections 5.4, 5.5 follow the work of Z. Peradzyński and respectively S.P. Tsarev and E. Ferapontov to characterize the class of regular multidimensional interactions of simple waves solutions as a *natural extension* of the class of the simple waves solutions. Section 5.6 relates this extension with the concept of solution with a *nondegenerate /degenerate* hodograph. This concept is revisited in [5] from a distinct “nonalgebraic” prospect. ■ Particularities of the multidimensional approach, induced by its complexity, are considered in sections 2.5, 5.2, 5.3 and §7 by means of a parallel with the one-dimensional approach.

A *class* of exact multidimensional gasdynamic solutions is constructed in §6 whose interactive elements are regular.

Two examples of interactive solutions belonging to the class presented in §6 are considered in §7 (sections 7.1, 7.2). ■ In sections 7.1, 7.2 it is indicated, in a multidimensional context, the possibility of *several* Riemann representations for an interactive solution. ■ An *admissibility criterion* is then formulated and exemplified for selecting regular interactive representations “of a genuinely nonlinear type” where other (“hybrid”) solutions are formally possible (section 7.2). ■ Section 7.3 presents an extension, on regular multidimensional interactions of simple waves solutions, of the genuinely nonlinear character.

The Riemann–Lax invariance, presented in section 4.6, and the construction in sections 5.2–5.5 anticipate some applications of the rank theory included in §8.

We consider in this paper the *homogeneous* quasilinear system

$$\sum_{j=1}^n \sum_{k=0}^m a_{ijk}(u) \frac{\partial u_j}{\partial x_k} = 0, \quad 1 \leq i \leq n \quad (1.1)$$

together with its one-dimensional version

$$\frac{\partial u}{\partial t} + a(u) \frac{\partial u}{\partial x} = 0, \quad (1.2)$$

its two-dimensional version

$$\frac{\partial u}{\partial t} + a(u)\frac{\partial u}{\partial x} + b(u)\frac{\partial u}{\partial y} = 0, \quad (1.3)$$

and its concrete gasdynamic forms

$$\begin{cases} \frac{\partial p}{\partial t} + v_x \frac{\partial p}{\partial x} + \rho(p, \psi) c^2(p, \psi) \frac{\partial v_x}{\partial x} = 0 \\ \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + \frac{1}{\rho(p, \psi)} \frac{\partial p}{\partial x} = 0 \\ \frac{\partial \psi}{\partial t} + v_x \frac{\partial \psi}{\partial x} = 0, \end{cases} \quad (1.4)$$

corresponding [cf. (1.2) with  $u = (p, v_x, \psi)^t$ ] to an anisentropic (strictly adiabatic) gasdynamic flow (in usual notations;  $\psi$  is the particle function,  $c(p, \psi)$  is an ad hoc anisentropic sound speed; see Appendix 1 for details),  $m = 1$ ; and

$$\begin{cases} \frac{\partial c}{\partial t} + v_x \frac{\partial c}{\partial x} + v_y \frac{\partial c}{\partial y} + \frac{\gamma - 1}{2} c \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) = 0 \\ \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + \frac{2}{\gamma - 1} c \frac{\partial c}{\partial x} = 0 \\ \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + \frac{2}{\gamma - 1} c \frac{\partial c}{\partial y} = 0 \end{cases} \quad (1.5)$$

corresponding to an isentropic description (in usual notations:  $c$  is the sound velocity,  $v_x, v_y$  are fluid velocities), with  $m = 2$ .

We end this section with some terminological mentions.

**TERMINOLOGY 1.1.** The space  $E = \mathbb{R}^{m+1}$  of the independent variables  $x_0 = t; x_1, \dots, x_m$  is called the *physical space*;  $m$  is the *codimension* [the number of the space independent variables]. The space  $H = \mathbb{R}^n$  of the entities [dependent variables] is said to be the *hodograph space*.  $\square$

**TERMINOLOGY 1.2.** The systems (1.1)–(1.5) whose coefficients do not depend on the independent variables are said to have a *gasdynamic type form*. We essentially consider such systems in our present approach.  $\square$

**TERMINOLOGY 1.3** ([17]). A direction  $\vec{\beta}$  at a point  $u^* \in H$  is said to be *exceptional* for (1.1) if a linear combination of equations (1.1) exists at  $u^*$  for which the derivatives of  $u_i$ ,  $1 \leq i \leq n$ , in the direction  $\vec{\beta}$  are missing. As it is well-known a planar (/linear) infinitesimal element centered at  $u^*$  and orthogonal to a nonexceptional /exceptional direction is said to be *noncharacteristic /characteristic*.  $\square$

**PROPOSITION 1.4** ([17]). A direction  $\vec{\beta}$  at a point  $u^* \in H$  is exceptional iff the restrictions

$$\det[\mathbf{a}_{ij}(u^*) \cdot \vec{\beta}] = 0 \quad \left[ \begin{array}{l} \vec{\beta} = (\beta_0, \beta_1, \dots, \beta_m) \\ \mathbf{a}_{ij}(u) = [a_{ij0}(u), \dots, a_{ijm}(u)] \end{array} \right] \quad i, j = 1, \dots, n \quad (1.6)$$

are fulfilled at  $u^*$ .  $\square$

We eventually put the condition (1.6) in the form

$$\det \left[ \sum_{k=0}^m a_{ijk}(u^*) \beta_k \right] = 0 \quad (i, j = 1, \dots, n). \quad (1.7)$$

Restriction (1.7) takes for the system (1.2) the form

$$\det[\beta_0 I + \beta_1 a(u^*)] = 0 \quad (1.8)$$

and for the system (1.5) the form

$$(\beta_0 + \beta_1 v_x^* + \beta_2 v_y^*) [(\beta_0 + \beta_1 v_x^* + \beta_2 v_y^*)^2 - c^{*2}(\beta_1^2 + \beta_2^2)] = 0. \quad (1.9)$$

## 2. An essential algebraic concept

### 2.1. Dual pairs of directions

TERMINOLOGY 2.1 (M. Burnat). For the system (1.1) we say that the vector  $\vec{\kappa} \in \mathbb{R}^n$  is a *hodograph dual* at a point  $u^* \in H$  of a real exceptional vector  $\vec{\beta}$  defined at that point, and write  $\vec{\kappa} \leftrightarrow \vec{\beta}$ , if this vector satisfies at  $u^*$  the duality condition

$$\sum_{j=1}^n \sum_{k=0}^m a_{ijk}(u^*) \beta_k \kappa_j = 0, \quad 1 \leq i \leq n. \quad (2.1)$$

This terminology, corresponding to Remark 2.4 here below, could be naturally extended to the case presented in Remark 2.5.b.  $\square$

We notice that a dual character in  $H$  is *essentially connected* with an exceptional character in  $E$ . In fact in order to make Terminology 2.1 active at a point  $u^* \in H$ , we have to verify first the reality of an exceptional vector  $\vec{\beta}$  at  $u^*$ .

In certain cases, for defining a dual vector  $\vec{\kappa}$  we could ignore, in a first step, the duality relation which is implicit in Terminology 2.1. Such a case corresponds to  $n = m + 1$  in (1.1). It is easy to be seen, cf. (2.1), that a dual direction  $\vec{\kappa}$  at a point  $u^* \in H$  satisfies in this case the condition

$$\det \left[ \sum_{j=1}^n a_{ijk}(u^*) \kappa_j \right] = 0 \quad (i, k = 1, \dots, n) \quad (2.2)$$

which is formally independent of (2.1). Also see section 2.2 for a similar issue.

TERMINOLOGY 2.2 (M. Burnat). A smooth curve in  $H$  is said to be a *hodograph characteristic* if it is tangent at each point of it to a characteristic vector  $\vec{\kappa}$ . Argument: for  $n = 2$  in (1.2) interchanging the pair of independent variables  $t, x$  and the pair of dependent variables  $u_1, u_2$  results in interchanging the characteristic character associated to the pair  $t, x$  and the hodograph characteristic character associated with the pair  $u_1, u_2$ .  $\square$

### 2.2. The hyperbolic one-dimensional case. Structure of a dual pair. Indices

In case of the system (1.2) the duality condition (2.1) takes, at a point of strict hyperbolicity  $u^* \in H$ , the form, similar to (2.2),

$$[\beta_0 I + \beta_1 a(u^*)] \vec{\kappa} = 0.$$

Therefore  $\vec{\kappa}$  appears to be, at the mentioned point  $u^*$ , a right eigenvector  $R$  of the matrix  $a$ , corresponding to a (real) eigenvalue  $\lambda$  of  $a$ . This indicates the following duality connection at the mentioned point  $u^* \in H$ :

$$\vec{\beta}_i = \Theta_i(u)[- \lambda_i(u), 1] \text{ [in the physical space]} \leftrightarrow \vec{\kappa}_i = R^i(u) \text{ [in the hodograph space]}, \text{ for each } i = 1, \dots, n. \quad (2.3)$$

REMARK 2.3 [*structure of a dual pair, number of dual pairs*]. (i) In case of a system (1.2) strictly hyperbolic in a region  $\mathcal{R} \subset H$  to each real value of the matrix  $a$  a single right eigenvector  $\vec{\kappa}$  corresponds at each  $u^* \in \mathcal{R}$ . Therefore, each dual pair associates [cf. (2.3)] in this case, at each  $u^* \in \mathcal{R}$ , to a vector  $\vec{\kappa}$  a *single* dual vector  $\vec{\beta}$ . (ii) We dispose at each  $u^* \in \mathcal{R}$  of a *finite* number of dual pairs with the structure  $\vec{\beta} \leftrightarrow \vec{\kappa}$ . (iii) To each pair an *index*  $i = 1, \dots, n$  is associated cf. (2.3).  $\square$

### 2.3. The hyperbolic one-dimensional case. Hodograph characteristics

In a region  $\mathcal{R} \subset H$  of strict hyperbolicity of (1.2) the hodograph characteristics appear to be field lines of the  $n$  vector fields  $R^i(u)$ ,  $i = 1, \dots, n$ . The hodograph characteristics corresponding to a given index  $i$

are given by the orbits  $u = \overset{i}{U}(\alpha)$  of the autonomous system

$$\frac{dU}{d\alpha} = \overset{i}{\Lambda}(U) \overset{i}{R}(U) \quad , \quad U \in \mathcal{R} \subset H . \quad (2.4)_i$$

REMARK 2.4.a. An *unique* hodograph characteristic curve of index  $i$  through a regular point  $u^* \in \mathcal{R}$  of (2.4) <sub>$i$</sub>  could be constructed as an orbit of (2.4) <sub>$i$</sub>  with the condition  $U(\alpha^*) = u^*$  [as an unique direction  $\overset{i}{R}(u^*)$  is associated to the point  $u^*$  for each index  $i$ ]. See section 4.6 for the details of construction.  $\square$

REMARK 2.4.b. A **finite** union of hodograph characteristics through a regular point  $u^* \in \mathcal{R}$  results when all indices  $i = 1, \dots, n$  are considered.  $\square$

## 2.4. The multidimensional case of the isentropic gas dynamics. Structure of a dual pair

In case of a system (1.5), for which  $n = m + 1 = 3$ , condition (2.2) takes, at a point  $u^* \in H$ , the form

$$c^2 \kappa_1 \left[ \left( \frac{2}{\gamma - 1} \right)^2 \kappa_1^2 - (\kappa_2^2 + \kappa_3^2) \right] = 0 \quad (2.5)$$

which will be connected with (1.9) in order to determine the duality relations:

(i) given  $\vec{\beta}$  [cf. (1.9)] we obtain from (2.1)

$$\vec{\beta} = (v_x \beta_1 + v_y \beta_2, -\beta_1, -\beta_2) \leftrightarrow \vec{\kappa} = (0, -\beta_2, \beta_1) \quad (2.6)$$

$$\vec{\beta} = [-(v_x \beta_1 + v_y \beta_2) - \varepsilon c, \beta_1, \beta_2] \leftrightarrow \vec{\kappa} = \left[ \varepsilon \frac{\gamma - 1}{2}, \beta_1, \beta_2 \right] \quad \varepsilon = \pm 1 \quad (2.7)$$

(ii) given  $\vec{\kappa}$  [cf. (2.5)] we get from (2.1)

$$\vec{\kappa} = (0, \kappa_2, \kappa_3) \leftrightarrow \vec{\beta} = (v_x \kappa_3 - v_y \kappa_2, -\kappa_3, \kappa_2) \quad (2.8)$$

$$\vec{\kappa} = \left[ \varepsilon \frac{\gamma - 1}{2}, \kappa_2, \kappa_3 \right] \leftrightarrow \vec{\beta} = [-(v_x \kappa_2 + v_y \kappa_3) - \varepsilon c, \kappa_2, \kappa_3] \quad \varepsilon = \pm 1 \quad (2.9)$$

where we assumed  $c \neq 0$  and we normalized the vectors  $\vec{\beta}$  or  $\vec{\kappa}$  by  $\beta_1^2 + \beta_2^2 = 1$  or, respectively,  $\kappa_2^2 + \kappa_3^2 = 1$ ; the orientations of these vectors have been chosen a priori.

REMARK 2.5.a. (Peradzyński [18]; **structure of a dual pair, number of dual pairs**). (i) In case of the **two-dimensional** system (1.5) each dual pair associates [cf. (2.8), (2.9)] at the mentioned  $u^*$  to a vector  $\vec{\kappa}$  a *single* dual vector  $\vec{\beta}$ . (ii) Cf. (1.9) we dispose in this case of an **infinite** number of dual pairs with the structure  $\vec{\kappa} \leftrightarrow \vec{\beta}$ .  $\square$

REMARK 2.5.b. (Peradzyński [18]; **structure of a dual pair, number of dual pairs**). (i) In the extended **three-dimensional** example of the isentropic gas dynamics with  $n = 4, m = 3$  [i.e. for  $u = (c, v_x, v_y, v_z)^t$  and three space dimensions in the extended version of (1.5)] each dual pair associates, at the mentioned  $u^*$ , to a vector  $\vec{\kappa}$  a *finite* [constant,  $\neq 1$ ] number of  $k$  independent exceptional dual vectors  $\vec{\beta}_j, j = 1, \dots, k$ . (ii) In this case we still dispose of an **infinite** number of dual pairs with the structure  $\vec{\kappa} \leftrightarrow (\vec{\beta}_1, \dots, \vec{\beta}_k)$ .  $\square$

## 2.5. Dual pairs of directions: a comparison between the one-dimensional approach and the multidimensional approach

- There is an **essential** difference between the case of  $m = 1$  and the case of  $m > 1$  in (1.1) concerning the structure of the set  $\mathcal{C}_m(u^*)$  of hodograph characteristic directions through a given point  $u^* \in H$ . Precisely, a nonvoid set  $\mathcal{C}_m(u^*)$  may be *infinite* in case of  $m > 1$  and is certainly *finite* in case of  $m = 1$  (see Remarks 2.3, 2.4). We may describe this by saying that a nonvoid (and finite) set  $\mathcal{C}_1(u^*)$  “bursts” into an infinite set  $\mathcal{C}_m(u^*)$  as  $m > 1$ . We notice at this point that the directions  $\vec{\kappa}$  of the generatrices

which rule the cone (2.5) centered at  $u^* \in H$  belong to the set  $\mathcal{C}_2(u^*)$  connected with (1.3) and that there is an infinite number of such directions.

- A natural consequence of this aspect is noticed in Remark 2.6.b here below.

## 2.6. The two-dimensional case of the isentropic gas dynamics. Hodograph characteristics: constructive details

REMARK 2.6.a. A hodograph characteristic arc through a point  $u^* \equiv (c^*, v_x^*, v_y^*)$  can be *constructed* for the system (1.5) as follows [Figure 1].

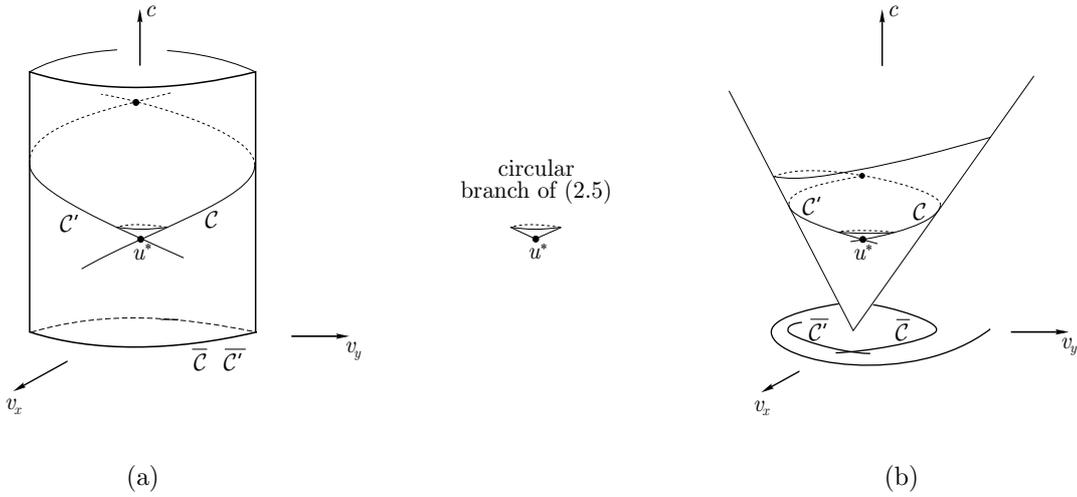


FIGURE 1

We consider in the plane  $c = c^*$  a smooth *arbitrary* arc

$$c = c^*, \quad v_x = v_x(\alpha), \quad v_y = v_y(\alpha) \quad \alpha \in \mathcal{I} \quad (\bar{\mathcal{C}})$$

through the point  $u^*$  corresponding to  $\alpha^* \in \mathcal{I}$ . The tangent vector associated, for each  $\alpha \in \mathcal{I}$ , to a point of  $\bar{\mathcal{C}}$  appears to be represented by  $[\kappa_2(\alpha), \kappa_3(\alpha)]$  where we calculate

$$\kappa_2(\alpha) = \frac{d}{d\alpha} v_x(\alpha), \quad \kappa_3(\alpha) = \frac{d}{d\alpha} v_y(\alpha).$$

Then we associate, for each  $\alpha \in \mathcal{I}$ , to the pair  $[\kappa_2(\alpha), \kappa_3(\alpha)]$  the value  $\kappa_1(\alpha) \neq 0$  obtained from (2.5) and integrate [taking into account the independence of the coefficients in (2.5) of the solution: a gasdynamic detail]

$$\frac{dc}{d\alpha} = \kappa_1(\alpha), \quad c(\alpha^*) = c^*, \quad \alpha \in \mathcal{I}$$

in order to complete the equation

$$c = c(\alpha), \quad v_x = v_x(\alpha), \quad v_y = v_y(\alpha), \quad \alpha \in \mathcal{I} \quad (\mathcal{C})$$

of the characteristic arc  $\mathcal{C}$  through  $(c^*, v_x^*, v_y^*)$ .

We notice that the characteristic curve which corresponds in this construction to  $\kappa_1 \neq 0$  in (2.5) is a *cylindrical or conical helix*. Indeed, a helix is a spatial curve whose tangents keep a constant inclination with respect to a fixed direction. Now, *primo*, at each point  $u^*$  of the mentioned characteristic curve the tangent is constructed by intersecting the circular branch corresponding to  $\kappa_1 \neq 0$  of the cone (2.5), considered at  $u^*$ , with the cylinder which includes  $\bar{\mathcal{C}}$  or with a suitable cone, and is parallel to the axis  $c$ , and *secundo*, the mentioned branch is independent of  $u^*$  [a gasdynamic fact]. The fixed direction is given by the axis  $c$ .

□

REMARK 2.6.b. An *infinity* of characteristic curves could be constructed through [an arbitrary]  $u^* \in \mathcal{R}$  in this way. This aspect suggests a *special complexity* of the multidimensional approach. This is in contrast with the *finite* character noticed in Remark 2.4.b. Some natural consequences of this aspect are presented in sections 5.2, 5.3 and §7.  $\square$

REMARK 2.7. Any smooth curve  $\bar{\mathcal{C}}$  placed in a plane  $c = \text{constant} \neq 0$  appears to be a hodograph characteristic curve corresponding to  $\kappa_1 = 0$  in (2.5).  $\square$

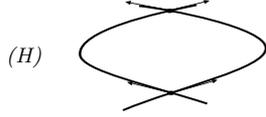


FIGURE 2

REMARK 2.8. Incidentally, Figure 1 presents cases for which the hodograph characteristics are self-intersecting; see Figure 2.  $\square$

### 3. Simple waves solution: definitions and constructive details. Genuine nonlinearity. Nature of a simple waves solution

#### 3.1. One-dimensional simple waves solution

##### 3.1.1. Constructive details

- Let  $\overset{i}{R}(u)$ ,  $\overset{i}{L}(u)$ , and  $\lambda_i(u)$ ,  $i = 1, \dots, n$ , be, respectively, the right eigenvectors, the left eigenvectors, and the eigenvalues of the matrix  $a(u)$ .
- Let  $U(\alpha)$  be an orbit of (2.4)<sub>*i*</sub> isolated with the condition  $U(\alpha^*) = u^*$ . We use this orbit to define the function

$$\zeta_i(\alpha) \equiv \lambda_i[U(\alpha)], \quad (3.1)_i$$

corresponding to the index  $i$  via the duality connection (2.3) [also see Remark 2.3], and *construct*, in a convenient neighbourhood of  $t = 0$ , a  $C^1$  smooth solution  $\alpha(x, t)$  of the Cauchy problem

$$\frac{\partial \alpha}{\partial t} + \zeta_i(\alpha) \frac{\partial \alpha}{\partial x} = 0, \quad -\infty < x < \infty, \quad t > 0 \quad (3.2)_i$$

$$\alpha(0, x) = \theta(x), \quad -\infty < x < \infty \quad (3.3)_i$$

where  $\zeta_i$  and  $\theta$  are assumed to be  $C^1(\mathbb{R})$  smooth functions. The mentioned construction depends on a simple remark.

- REMARK 3.1. The characteristics of (3.2)<sub>*i*</sub>, described in the physical plane  $x, t$  by

$$\frac{dx}{dt} = \zeta_i[\alpha(x, t)], \quad (3.4)_i$$

are straight lines along which  $\alpha = \text{constant}$  (the constant depends on line generally).  $\square$

- We complementarily assume that (3.2)<sub>*i*</sub> is *genuinely nonlinear* [= strictly quasilinear]:

$$\frac{d\zeta_i}{d\alpha} \neq 0. \quad (3.5)_i$$

- For a *genuinely nonlinear* equation (3.2)<sub>*i*</sub> we define *the simple waves solution of index  $i$*  (also called *the  $i$ -simple waves solution*) by

$$u(x, t) = U \circ \alpha \equiv U[\alpha(x, t)]. \quad (3.6)_i$$

We carry (3.6)<sub>i</sub> into (1.2) for getting

$$\left[ \frac{\partial \alpha}{\partial t} + \zeta_i(\alpha) \frac{\partial \alpha}{\partial x} \right] \frac{dU}{d\alpha} = 0. \quad (3.7)_i$$

and the fulfilment of system (1.2) results from (3.2)<sub>i</sub> and (2.4)<sub>i</sub>.

- The orbit  $\overset{i}{U}(\alpha)$  contains the *hodograph half* of the simple waves solution and the solution  $\alpha$  of (3.2)<sub>i</sub> appears to be the *physical half* of the  $i$ - simple waves solution  $U \circ \alpha$ .
- We remark that (3.6)<sub>i</sub> and (3.7)<sub>i</sub> indicate a *factorization* which distinguishes between the physical half and the hodograph half of the  $i$ - simple waves solution. The duality character of the relation between these two mentioned halves results from (2.3) via (2.4)<sub>i</sub> and (3.1)<sub>i</sub>.
- From (3.1)<sub>i</sub> and (3.4)<sub>i</sub> it results that the characteristics of the equation (3.2)<sub>i</sub> [which is associated with an index  $i$ ] appears, concurrently, to be characteristics of index  $i$  of the system (1.2).
- From Remark 3.1 and (3.6)<sub>i</sub> we notice that a simple waves solution  $u(x, t)$  of index  $i$  of the system (1.2) is constant along each characteristic straightlined arc of index  $i$  of this system.
- As a consequence, in the physical plane  $x, t$  a simple waves solution of index  $i$  appears to be limited [to the left (or right) side] by a constant state [ $u_l$  (or  $u_r$ )] or by a simple waves solution of the same index; in particular such a solution may appear to connect two constant states [ $u_l$  (or  $u_r$ )] placed, as points in the hodograph space, on the one-dimensional hodograph of this solution. The separation between the domain of a simple waves solution of index  $i$  and an adjacent to it domain of constant state [on the left (/right)] appears to be made along a characteristic arc of index  $i$ .

### 3.1.2. Implicit form

Remark 3.1 leads to an implicit representation for the solution we are looking for [we shall ignore in the following the index  $i$  associated to  $U$  and  $\alpha$ ]; the solution has the form

$$u(x, t) = U[\alpha(x, t)]. \quad (3.8)_i$$

where we have to determine  $\alpha(x, t)$  from the implicit representation

$$\alpha = \theta(\xi), \quad \xi = x - \zeta_i(\alpha)t. \quad (3.9)_i$$

This representation can be read as

$$\mathcal{F}_i(t, x, \alpha) \equiv \alpha - \theta[x - \zeta_i(\alpha)t] = 0. \quad (3.10)_i$$

The requirement  $\frac{\partial \mathcal{F}_i}{\partial \alpha} \neq 0$  imposed to a smooth function  $\mathcal{F}_i$  in order to apply the implicit function theorem is equivalent with the condition

$$1 + \frac{d\zeta_i}{d\alpha}[\theta(\xi)] \frac{d\theta}{d\xi} t \neq 0. \quad (3.11)_i$$

Restriction (3.11)<sub>i</sub> is fulfilled, for example, at the point [ $t^* = 0, x^*, \alpha^* = \theta(x^*)$ ] of the domain of  $\mathcal{F}_i$  in (3.10)<sub>i</sub> and consequently the implicit function theorem is active around this point. In fact, this active character of the implicit function theorem appears to persist in a neighbourhood of  $t = 0$  for which (3.11)<sub>i</sub> is fulfilled.

In presence of restriction (3.11)<sub>i</sub> we can obtain *explicitly* the solution  $\alpha(x, t)$  of (3.2)<sub>i</sub>, (3.3)<sub>i</sub> [cf. (3.10)<sub>i</sub>]; this solution describes, via the construction (3.9)<sub>i</sub>, the manner in which the data (3.3)<sub>i</sub> evolve over the half-plane  $t > 0$  by constancy (cf. Remark 3.1) along the characteristic lines.

### 3.1.3. Genuine nonlinearity. Linear degeneracy. Types of quasilinearity

We compute from (2.4)<sub>i</sub>, (3.1)<sub>i</sub>

$$\frac{d\zeta_i}{d\alpha} \equiv \overset{i}{\Lambda}[U(\alpha)] \{ \overset{i}{R}[U(\alpha)] \cdot \text{grad}_u \lambda_i[U(\alpha)] \}, \quad \overset{i}{\Lambda} \neq 0. \quad (3.12)_i$$

This naturally leads to the Terminologies 3.2 which follow.

TERMINOLOGY 3.2.a [**nature of indices**] (Lax [16]). For a strictly hyperbolic system (1.2) we say that an index  $i$  is *genuinely nonlinear* if for it

$${}^i R(u) \cdot \text{grad}_u \lambda_i(u) \neq 0, \quad u \in \mathcal{R} \quad (3.13)_i$$

and it is *linearly degenerated* if for it

$${}^i R(u) \cdot \text{grad}_u \lambda_i(u) \equiv 0, \quad u \in \mathcal{R} \quad (3.14)_i$$

□

TERMINOLOGY 3.2.b [**types of genuine nonlinearity**]. The genuine nonlinearity associated to an *index* is said to be *hard* [see Terminology 3.2.a and restriction (3.13)<sub>i</sub>] and appears to be a priori associated to *any* hodograph characteristic arc of such an index. A *soft* genuine nonlinearity is associated with a *particular* hodograph characteristic arc [in a soft genuine nonlinearity the restriction (3.13)<sub>i</sub> is considered only along a particular hodograph characteristic arc – thus guaranteeing [cf. (3.12)<sub>i</sub>] the fulfilment of (3.5)<sub>i</sub> for a particular simple waves solution only]. □

TERMINOLOGY 3.2.c [**types of quasilinearity**] (Lax [16]). We say that the quasilinearity of the system (1.2) is

- *strong* if for this system all the indices are genuinely nonlinear,
- *medium* if this system has genuinely nonlinear indices concurrently with linearly degenerate indices,
- *weak* if for this system all the indices are linearly degenerate. □

REMARK 3.3. A *simple wave* corresponding to a genuinely nonlinear equation (1.2) is a planar front which evolves with a *constant* velocity and at the points of which the dependent variable  $\alpha$  in (3.6)<sub>i</sub> keeps a *constant* value during the (permitted) evolution. This is the *simplest* ingredient of a *nonlinear* continuous evolution. The *remarkable* fact (first time noticed in Poisson [23]) that a *simple wave* could be identified in a *nonlinear* evolution has, on the other hand, a *structuring* (hence an *essential*) character: a simple waves *solution* appears to be a *continuous distribution* of simple waves and, as is well known, the [genuinely nonlinear] simple waves solutions appear to be *essential* structuring ingredients of a quasilinear evolution description (pairing the shock waves of a similar index). In sections 3.2, 3.3 the genuinely nonlinear character of a simple waves solution will be considered for extension. □

REMARK 3.4. Restriction (3.5)<sub>i</sub> could be also presented cf. (2.3) by

$$\frac{d\vec{\beta}_i}{d\alpha} \neq 0. \quad (3.15)_i$$

This indicate a “fanning out” of the characteristic straightlines which structure the simple waves solution – as  $\vec{\beta}_i$  is orthogonal to a characteristic direction. □

REMARK 3.5. The requirement of genuine nonlinearity for an index  $i$  has the form

$$\vec{\kappa}_i(u) \diamond \vec{\beta}_i(u) \equiv {}^i R(u) \cdot \text{grad}_u \lambda_i(u) \neq 0$$

and appears as a restriction imposed to the dual pair  $[\vec{\kappa}_i(u), \vec{\beta}_i(u)]$  at each point of the hodograph of a simple waves solution. □

## 3.2. Multidimensional planar simple waves solution

### 3.2.1. Implicit form

EXAMPLE 3.6 (Peradzyński [19]). If to each characteristic vector  $\vec{\kappa}$  tangent to a characteristic curve in  $H$  a *single* exceptional dual  $\vec{\beta}$  corresponds [see Remark 2.5.a(i)] then, the same as in section 3.1 a simple waves solution which is constant on  $m$ -dimensional hyperplanes can be constructed according to

$$u = U[\alpha(x, t)] \quad (3.6)'$$

where we have to determine  $\alpha(x, t)$  from the implicit representation

$$\alpha = \theta(\xi), \quad \xi = \sum_{\nu=0}^m \beta_\nu [U(\alpha)] x_\nu = \sum_{\nu=0}^m \beta_\nu \{U[\theta(\xi)]\} x_\nu. \quad (3.9)'$$

Here  $U(\alpha)$  is a smooth hodograph characteristic curve isolated in  $H$  cf.  $u^* = U(\alpha^*)$ , for arbitrary  $u^* \in H$ ,  $\alpha^* \in \mathbb{R}$  (the details of isolating such a curve in the gasdynamic context of section 2.6 are presented in Remark 2.6.a) and  $\theta$  is a smooth arbitrary scalar function. The function (3.10) and the requirement (3.11) have in the present example the analogues

$$\mathcal{F}(x, \alpha) \equiv \alpha - \theta \left\{ \sum_{\nu=0}^m \beta_\nu [U(\alpha)] x_\nu \right\} = 0 \quad (3.10)'$$

and

$$1 - \left( \sum_{\nu=0}^m \frac{d\beta_\nu}{d\alpha} x_\nu \right) \frac{d\theta}{d\xi} \neq 0. \quad (3.11)'$$

The simple waves solution  $u(x, t)$ , represented cf. (3.6)', fulfils (1.1) in a region of  $E$  for which (3.11)' holds [for example, in a convenient neighbourhood  $\mathcal{N}$  around  $u = u^*$ ,  $x = 0$ , for  $\alpha^* = \theta(0)$ ].

It is easy to verify that  $u(x, t)$  given by (3.6)' is a solution of (1.1). Indeed, we compute at  $u^*$

$$\frac{\partial u_j}{\partial x_k} = \frac{dU_j}{d\alpha} \cdot \frac{\partial \alpha}{\partial x_k} = \Omega_j \omega \kappa_j \beta_k, \quad (3.16)'$$

because at  $u^*$  we get from (3.10)'

$$\frac{\partial \alpha}{\partial x_k} = \frac{\frac{d\theta}{d\xi}}{1 - \left( \sum_{\nu=0}^m \frac{d\beta_\nu}{d\alpha} x_\nu \right) \frac{d\theta}{d\xi}} \beta_k [U(\alpha)] = \omega \beta_k$$

and take into account the characteristic nature of the arc  $U(\alpha)$ . Finally, we carry (3.16)' in (1.1) and use (2.1).

At this point it is proper to notice from (3.10)' that, in the mentioned region  $\mathcal{N} \subset E$ , the simple waves solution (3.9)' is constant over hyperplanes

$$\sum_{\nu=0}^m \beta_\nu \{U[\alpha(\tilde{x})]\} (x_\nu - \tilde{x}_\nu) = 0 \quad , \quad \tilde{x} \in \mathcal{N}. \quad \square$$

TERMINOLOGY 3.7.a. A simple waves solution constructed cf. Example 3.6 will be said to be *planar* ([19]).  $\square$

### 3.2.2. Genuine nonlinearity / linear degeneracy: an ad hoc definition

REMARK 3.8.a ([4]). The construction (3.9)' shows that for a (local) “fanning out” of the hyperplanes which structure this kind of solution it is proper to require  $\left| \frac{d\tilde{\beta}}{d\alpha} \right| \neq 0$  along the [particular] hodograph characteristic arc of Example 3.6. Such a requirement appears as a planar *soft genuine nonlinearity* restriction and induces an *implicit* character in (3.10)'. The *explicit* form  $\alpha(x, t)$  of the physical half of a simple waves solution results therefore via the implicit function theorem from (3.10)'. Also a *soft linear degeneracy* restriction could be alternatively considered for (3.10)' in the evident form  $\left| \frac{d\tilde{\beta}}{d\alpha} \right| \equiv 0$  along a hodograph [particular] characteristic arc.  $\square$

## 3.3. Multidimensional non-planar simple waves solution

### 3.3.1. Implicit form

EXAMPLE 3.9 (Peradzyński [20]). If to each characteristic vector  $\vec{\kappa}$  tangent to a characteristic curve in  $H$  a *finite* (constant,  $\neq 1$ ) number of  $k$  independent exceptional dual vectors  $\tilde{\beta}_j$ ,  $1 \leq j \leq k$ , correspond [see Remark 2.4(iii)] then a simple waves solution can be constructed according to

$$u = U[\alpha(x, t)] \quad (3.6)''$$

where we have to determine  $\alpha(x, t)$  from the implicit representation

$$\alpha = \theta(\xi_1, \dots, \xi_k), \quad \xi_j = \sum_{\nu=0}^m \beta_{j\nu}[U(\alpha)]x_\nu, \quad 1 \leq j \leq k \quad (3.9)''$$

Again,  $U(\alpha)$  is assumed to be a smooth hodograph [particular] characteristic arc and  $\theta$  is a smooth arbitrary scalar function. The function (3.10)' and the requirement (3.11)' have in the present example the analogues

$$\mathcal{F}(x, \alpha) \equiv \alpha - \theta \left\{ \sum_{\nu=0}^m \beta_{1\nu}[U(\alpha)]x_\nu, \dots, \sum_{\nu=0}^m \beta_{k\nu}[U(\alpha)]x_\nu \right\} = 0 \quad (3.10)''$$

and

$$1 - \sum_{\mu=1}^k \left( \sum_{\nu=0}^m \frac{d\beta_{\mu\nu}}{d\alpha} x_\nu \right) \frac{d\theta}{d\xi_\mu} \neq 0. \quad (3.11)''$$

The simple waves solution  $u(x, t)$  represented cf. (3.6)'' fulfils (1.1) in a region of  $E$  for which (3.11)'' holds [for example, in a convenient neighbourhood  $\mathcal{N}$  around  $u = u^*, x = 0$ , for  $\alpha^* = \theta(0, \dots, 0)$ ].

It is easy to verify that  $u(x, t)$  given by (3.6)'' is a solution of (1.1). Indeed, we compute at  $u^*$

$$\frac{\partial u_j}{\partial x_l} = \frac{dU_j}{d\alpha} \cdot \frac{\partial \alpha}{\partial x_l} = \sum_{\mu=1}^k \Omega_j \omega_{\mu l} \kappa_j \beta_{\mu l}, \quad (3.6)''$$

because at  $u^*$  we get from (3.16)''

$$\frac{\partial \alpha}{\partial x_l} = \frac{\sum_{\mu=1}^k \frac{\partial \theta}{\partial \xi_\mu} \beta_{\mu l}[U(\alpha)]}{1 - \sum_{\mu=1}^k \left( \sum_{\nu=0}^m \frac{d\beta_{\mu\nu}}{d\alpha} x_\nu \right) \frac{\partial \theta}{\partial \xi_\mu}} = \sum_{\mu=1}^k \omega_{\mu l} \beta_{\mu l}[U(\alpha)]$$

and take into account the characteristic nature of the arc  $U(\alpha)$ . Finally, we carry (3.16)'' in (1.1) and use (2.1).  $\square$

REMARK 3.10.(i) In case of  $m > 1$  the constancy of a simple waves solution over  $m$ -dimensional hyperplanes, *in spite of being possible* (cf. Example 3.6), *is no more a rule* (cf. Example 3.9). (ii) In case of  $m > 1$  the representations (3.9)' and (3.9)'' indicate that the *factorization* (3.6)<sub>i</sub> already noticed for  $m = 1$ , which distinguishes between the physical half and the hodograph half of a simple waves solution, is seen to persist [cf. (3.6)', (3.6)'']. (iii) Examples 3.6, 3.9 essentially correspond to a *gasdynamic* context [see Remarks 2.5.a, b].  $\square$

TERMINOLOGY 3.7.b. A simple waves solution constructed cf. Example 3.9 will be said to be *nonplanar* ([20]).  $\square$

REMARK 3.11. Figures 2, 7c correspond to some cases of *multiple intersections* of two hodograph characteristic curves. We notice that this results in the possibility for two points (constant states) in  $H$  / two constant regions in  $E$  to be connected respectively by (at least) two hodograph characteristic curves in  $H$  / simple waves regions in  $E$ . Still, at a point of re-intersection we have to distinguish between the vectors  $\vec{\kappa}$  [or their dual vectors  $\vec{\beta}$ ] respectively associated to the two intersecting hodograph characteristic arcs in  $H$  / simple waves regions in  $E$ .  $\square$

### 3.3.2. Genuine nonlinearity / linear degeneracy: an ad hoc definition

REMARK 3.8.b. The construction (3.9)'' shows that for a (local) “fanning out” of the hyperplanes which structure this kind of solution it is proper to require  $\sum_{\mu=1}^k \left| \frac{d\vec{\beta}_\mu}{d\alpha} \right| \neq 0$  along the hodograph characteristic arc of Example 3.9. Such a requirement appears as a nonplanar *soft genuine nonlinearity* restriction and induces an *implicit* character in (3.10)''. The *explicit* form  $\alpha(x, t)$  of the physical half of a simple waves solution results therefore via the implicit function theorem from (3.10)''. Also a *soft linear degeneracy* restriction could be alternatively considered for (3.10)'' in the evident form  $\sum_{\mu=1}^k \left| \frac{d\vec{\beta}_\mu}{d\alpha} \right| \equiv 0$  along a [particular] hodograph characteristic arc.  $\square$

### 3.4. Genuine nonlinearity / linear degeneracy: some gasdynamic two-dimensional examples

- It is easy to show that, cf. (2.9), the hodograph characteristic curves along which, in a gasdynamic construction,  $\kappa_1 \neq 0$  have a *genuinely nonlinear* character.
- Any smooth curve  $\bar{c}$  placed in a plane  $c = \text{constant} \neq 0$  appears to be a hodograph characteristic curve corresponding to  $\kappa_1 = 0$  in (2.5).
- It is easy to be seen, cf. (2.8), that the hodograph characteristic curves corresponding to  $\kappa_1 = 0$  are *linearly degenerated* only if they are straightlines and have a *genuinely nonlinear* character if they do not include straightlined arcs.

### 3.5. Simple waves solution. A general definition

DEFINITION 3.12. A nonconstant continuous solution of the system (1.1) whose hodograph is a genuinely nonlinear arc of characteristic curve is said to be a *simple waves solution*.  $\square$

REMARK 3.13. Definition 3.12 *essentially associates* to the *one-dimensional* nature of a solution hodograph a *characteristic* and a *genuinely nonlinear* nature.  $\square$

### 3.6. Genuine nonlinearity: a comparison between the one-dimensional approach and the multidimensional approach

In the one-dimensional construction of a simple waves solution we could strenghten a *soft* genuine nonlinearity into a *hard* genuine nonlinearity [Terminology 3.2.b]. In a multidimensional construction of a simple waves solution only a *soft* genuine nonlinearity is available [Remarks 2.5, 3.8.a and 3.8.b].

### 3.7. The nature of a simple waves solution in its dependence on the number of equations and codimension: a review

We resume in this section, cf. Scheme 1, the facts connected with the dependence of the structure of a simple waves solution [considered, in the *smooth* context of §3] on the number  $n$  of the equations in the system considered and the codimension  $m$ .

### 3.8. Simple waves solutions: a Varley–Peradzyński classification

A suggestive classification of the simple waves solutions corresponding to (2.6), (2.8)/respectively (2.7), (2.9) is due to Peradzyński [21] and Varley [30]; see the terminology which follows.

TERMINOLOGY 3.14. From (2.6) and (3.16)' we compute

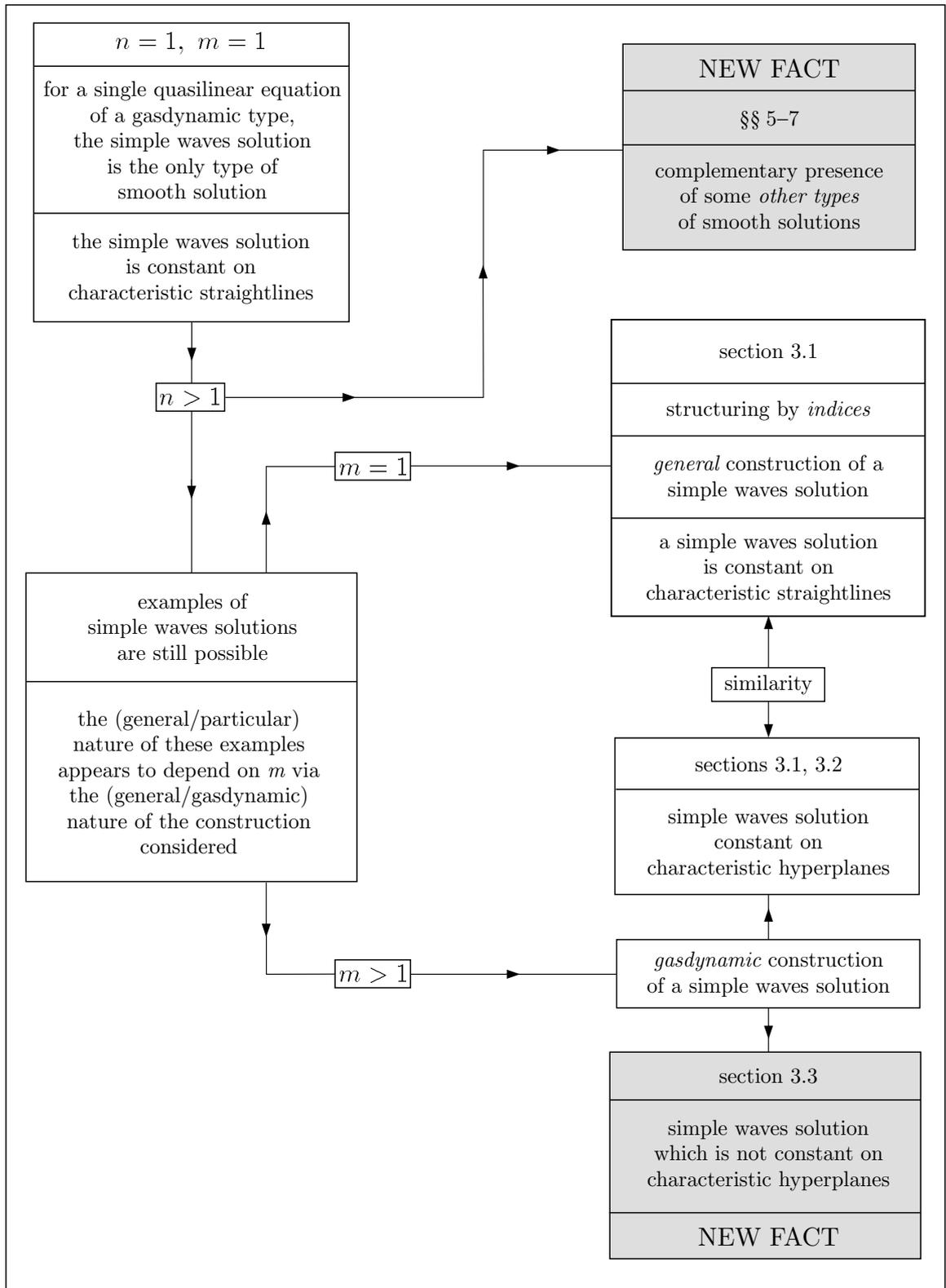
$$\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} = -\Omega\omega(\beta_1^2 + \beta_2^2) \neq 0 \quad (3.17)_r$$

while from (2.7) and (3.16)' we get

$$\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} = 0. \quad (3.17)_p$$

Therefore the simple waves associated cf. Example 3.6 to (2.6), (2.8) /respectively (2.7), (2.9) are said to be *rotational/potential*.  $\square$

REMARK 3.15. (i) The contact discontinuities of an one-dimensional flow described by (1.5) appear to be (degenerate) rotational (cf.  $\llbracket c \rrbracket = 0$ ,  $\llbracket v_x \rrbracket = 0$ ,  $\llbracket v_y \rrbracket \neq 0$  for  $\frac{\partial}{\partial y} \equiv 0$ ;  $\llbracket c \rrbracket = 0$ ,  $\llbracket v_x \rrbracket \neq 0$ ,  $\llbracket v_y \rrbracket = 0$  for  $\frac{\partial}{\partial x} \equiv 0$ ). (ii) As  $n = 3$ ,  $m = 2$ /respectively  $n = 4$ ,  $m = 3$  (evolution of  $v_z$  considered; three space dimensions) in (1.5) the rotational simple waves solutions are *planar*/respectively *nonplanar* while the potential simple waves solutions are planar in both these cases.  $\square$



SCHEME 1

## 4. Riemann invariants: one-dimensional details. Regular interactions of one-dimensional simple waves solutions

### 4.1. Introduction

In this paragraph we begin the presentation of an one-dimensional extension of the class of the simple waves solutions, constructed in §3, to a class related with it: the regular interactions of simple waves solutions. The nature of the relationship will be described in sections 4.2–4.4, 5.3.4, 7.3, 7.4 and in §8.

In sections 4.2–4.4 the mentioned extension is presented in a Riemann restricted context ([24]). Details of this context connected to a nonlinearity hierarchy are included in section 4.5.

Finally, for one-dimensional simple waves solutions or, respectively, regular interactions of one-dimensional simple waves solutions, the Riemann restricted context is relaxed into a Riemann–Lax context (section 4.6) or, respectively, a Riemann–Burnat context (section 5.3.4; see Example 5.1).

### 4.2. Riemann invariants (I). Frobenius restrictions

#### 4.2.1. Riemann entities. Frobenius restrictions

- For a strictly hyperbolic system (1.2) a Riemann restricted approach would begin with replacing the entities  $u$  by the Riemann entities  $v$  via a nonsingular *Riemann transform*

$$v = v(u) ; \quad v = (v_1, \dots, v_n), \quad u = (u_1, \dots, u_n), \quad (4.1)$$

given by

$$\frac{\partial v_k}{\partial u_l} = \alpha_k(u) L_l^k(u) \quad 1 \leq l \leq n . \quad (4.2)$$

Let

$$u = u(v) \quad (4.3)$$

the inverse of this transform.

- It results from (4.2) that the reality of a substructure  $v_k = v_k(u)$  of the Riemann transform depends on the integrable character of the Pfaff form

$$\sum_{l=1}^n L_l^k(u) du_l. \quad (4.4)$$

- Integrability restrictions for (4.4):

$$\frac{\partial L_i^k}{\partial u_j} - \frac{\partial L_j^k}{\partial u_i} = 0 \quad i, j = 1, \dots, n. \quad (4.5)_k$$

**THEOREM 4.1** (G. Frobenius). If  $L^k \in C^1(\mathcal{R})$  and in  $\mathcal{R}$  the restrictions (4.5)<sub>k</sub> are fulfilled, then for every pair  $(u^*, v_k^*)$ ,  $u^* \in \mathcal{R}$ ,  $v_k^* \in \mathbb{R}$  solution  $v_k(u; u^*, v_k^*)$  of the problem

$$dv_k - L^k(u) du = 0, \quad v_k(u^*) = v_k^* \quad (4.6)_k$$

exists (uniquely) in the whole region  $\mathcal{R}$  and  $v_k \in C^1(\mathcal{R})$ .  $\square$

- Integrability restrictions for (4.4) in presence of an integrating factor [cf. (4.2); here for  $n = 3$ ]:

$$L^k \cdot \text{rot}_u L^k = 0 \quad 1 \leq k \leq 3. \quad (4.7)_k$$

- There is an essential difference between the case of  $n > 2$  and the case of  $n \leq 2$ . Precisely, the form (4.4) is *unconditionally* integrable in case of  $n \leq 2$ . Restrictions (4.5) guarantee the integrability of (4.4) as  $n > 2$ ; see Example 4.2 here below for a *partial* fulfilment of (4.7).

- Here is an example for which only an *incomplete* set of Riemann invariants exists.

**EXAMPLE 4.2.** We consider the system (1.4) associated to an *anisentropic* (strictly adiabatic) gasdynamic flow. The eigenelements of this system are computed to be

$$\begin{aligned} \lambda_1(u) &= v_x + c(p, \psi), \quad \lambda_2(u) = v_x, \quad \lambda_3(u) = v_x - c(p, \psi), \\ \overset{1}{R}(u) &= \Lambda_1(u)[\zeta(p, \psi), 1, 0]^t, \quad \overset{2}{R}(u) = \Lambda_2(u)[0, 0, 1], \quad \overset{3}{R}(u) = \Lambda_3(u)[-\zeta(p, \psi), 1, 0]^t \\ \overset{1}{L}(u) &= \Theta_1(u)[1, \zeta(p, \psi), 0], \quad \overset{2}{L}(u) = \Theta_2(u)[0, 0, 1], \quad \overset{3}{L}(u) = \Theta_3(u)[-1, \zeta(p, \psi), 0] \end{aligned} \quad (4.8)$$

with

$$\zeta(p, \psi) = \rho(p, \psi)c(p, \psi).$$

We normalize the right eigenvectors  $R$  of a genuinely nonlinear index  $i$  by

$${}^i R(u) \cdot \text{grad}_u \lambda_i(u) = 1, \quad i = 1, 3,$$

and the right eigenvectors  $R$  of a linearly degenerate index by

$$\left\| {}^2 R(u) \right\| = 1.$$

Then we normalize the left eigenvectors  $L$  by

$${}^i R \cdot {}^i L = 1, \quad 1 \leq i \leq 3.$$

We find in the normalized expressions (4.8):

$$\begin{aligned} \Lambda_1(u) &= \left[ \zeta(p, \psi) \left( \frac{\partial c}{\partial p} \right)_\psi + 1 \right]^{-1} = \Lambda_3(u), \quad \Lambda_2(u) = 1, \\ \Theta_1(u) &= \frac{1}{2\zeta} \left[ \zeta(p, \psi) \left( \frac{\partial c}{\partial p} \right)_\psi + 1 \right] = \Theta_3(u), \quad \Theta_2(u) = 1. \end{aligned}$$

Finally we calculate

$$-\frac{1}{\Theta_1^2(u)} {}^1 L(u) \cdot \text{rot} {}^1 L(u) = \frac{1}{\Theta_3^3(u)} {}^3 L(u) \cdot \text{rot} {}^3 L(u) = \frac{\partial \zeta}{\partial \psi} \neq 0 \quad (4.9)$$

$${}^2 L(u) \cdot \text{rot} {}^2 L(u) = 0. \quad (4.10)$$

Only the substructure  $v_2 = v_2(u)$  of an (incomplete) Riemann transform is available in this case cf. (4.7), (4.10). In case of an **isentropic** flow, equation (1.4)<sub>3</sub> is identically fulfilled so that (1.4)<sub>1,2</sub> appears as a coherent system of two equations ( $n = 2$ ). This aspect is compatible with  $\frac{\partial \xi}{\partial \psi} \equiv 0$  in (4.9).  $\square$

#### 4.2.2. Connecting the physical space and the hodograph space in presence of a complete Riemann transform. Regular integrability

• Next, we use the entities  $v$  to structuring the connection between the hodograph space and the physical space. Precisely, on the complete fulfilment of integrability restrictions (4.5)/(4.7) we use (4.2) to compute for each  $k$ :

$$\alpha_k(u) {}^k L(u) \left[ \frac{\partial u}{\partial t} + a(u) \frac{\partial u}{\partial x} \right] = \sum_{j=1}^n \frac{\partial v_k}{\partial u_j} \cdot \frac{\partial u_j}{\partial t} + \lambda_k(u) \alpha_k(u) {}^k L(u) \frac{\partial u}{\partial x} = \frac{\partial v_k}{\partial t} + \bar{\lambda}_k(v) \frac{\partial v_k}{\partial x}, \quad 1 \leq k \leq n.$$

• The Riemann invariants  $v_1(x, t), \dots, v_n(x, t)$ , connecting the hodograph structure associated to the Riemann entities  $v$  of 4.2.1 with a physical structure, satisfy a *diagonal* system

$$\frac{\partial v_k}{\partial t} + \bar{\lambda}_k(v) \frac{\partial v_k}{\partial x} = 0, \quad 1 \leq k \leq n; \quad \bar{\lambda}_k(v) \equiv \lambda_k[u(v)] \quad (4.11)$$

associated to (1.2).

• A Riemann invariant  $v_k$  is constant, cf. (4.11), along each characteristic line of index  $k$  [the constant depends on line generally] in a given region of  $E$ .

• If the Frobenius restrictions (4.5) or (4.7) are fulfilled for each  $k = 1, \dots, n$  we are led [cf. (4.3)] to a Riemann representation

$$u = u[v_1(x, t), \dots, v_n(x, t)] \quad (4.12)$$

which will be associated with a *regular integrability* of (1.2) corresponding to some smooth initial data

$$u(x, 0) = u_0(x). \quad (1.2)_0$$

The arguments  $v_1(x, t), \dots, v_n(x, t)$  in (4.12) result from a Cauchy problem which associates [cf (4.1)] to the diagonal system (4.11) the initial data

$$v(x, 0) = v[u_0(x)]. \quad (4.1)_0$$

- We could transcribe (4.11) through

$$\left( \frac{\partial v_k}{\partial t}, \frac{\partial v_k}{\partial x} \right) = \eta_k \vec{\beta}_k[u(v)], \quad 1 \leq k \leq n. \quad (4.13)$$

### 4.3. Riemann invariants (II). Frobenius restrictions. Hodograph characteristic coordinates. One-dimensional indices. Structuring through indices

#### 4.3.1. Frobenius restrictions. Hodograph characteristic coordinates. One-dimensional indices. Riemann hodograph structures

- We notice from (4.2) that

$$R \cdot \text{grad}_u v_k = 0 \quad i \neq k \quad (4.14)$$

as

$$R \cdot L = 0 \quad \text{for } i \neq k \quad i, k = 1, \dots, n.$$

- From (4.14) it results the importance of the  $(n - 1)$ -dimensional hypersurfaces

$$v_k(u) = \text{constant} = v_k(u^*) \quad (4.15)_k$$

through an arbitrary point  $u^* \in \mathcal{R}$ .

■ From (4.14) we notice that if an arc of characteristic curve of index  $i$ ,  $1 \leq i \leq n$ ,  $i \neq k$  passes through a point of hypersurface (4.15) then this arc appears to entirely belonging to this hypersurface. This indicates, particularly, that a hodograph characteristic curve of index  $i$  appears to be an intersection of  $n - 1$  hypersurfaces  $v_k(u) = \text{constant}$ . ■ To each hypersurface (4.15) a characteristic system of  $n - 1$  coordinates will be associated via the intersections of this hypersurface with  $v_j(u) = \text{constant}$ ,  $j \neq k$ . ■ To each *intersection* of some hypersurfaces (4.15) a characteristic system of coordinates will be associated similarly. ■ A complete fulfilment [for each  $k = 1, \dots, n$ ] of Frobenius restrictions (4.5) or (4.7) (locally) results in existence of a hodograph characteristic system of coordinates. To each characteristic field of coordinates, in such a system, an *index* will be associated. ■ Hypersurfaces (4.15) or their intersections of various dimensions appear to be *Riemann hodograph structures*.

TERMINOLOGY 4.3. A hodograph hypersurface with a characteristic system of coordinates is said to be a *characteristic hypersurface*. □

REMARK 4.4. Riemann hodograph structures mentioned above are characteristic hypersurfaces. □

- A characteristic hodograph hypersurface is associated to a set of indices and reflects their genuinely nonlinear / linearly degenerate character. A characteristic hypersurface will be therefore regarded as *structured through indices* (also see section 7.3).

- In presence of Frobenius restrictions the circumstance depicted in Figure 3b is avoided: Figure 3a is selected.

- The hodograph characteristic coordinates have a *local* character generally.

EXAMPLE 4.5 (Smoller [27]). For the system

$$\frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial x} = 0, \quad \frac{\partial u_2}{\partial t} + \frac{\partial}{\partial x}[\exp(-u_1)] = 0 \quad (4.16)$$

the Riemann entities  $v$  are related with  $u$  cf.  $v_1(u) \equiv u_2 - 2 \exp(-\frac{1}{2}u_1) + 2$ ,  $v_2(u) \equiv u_2 + 2 \exp(-\frac{1}{2}u_1) - 2$ . At the points of the field line of index 1 through  $u_1^* = (0, \alpha_1) \in H$  we have  $u_2 - (\alpha_1 + 2) = -2 \exp(-\frac{1}{2}u_1) < 0$ . At the points of the field line of index 2 through  $u_2^* = (0, \alpha_2) \in H$  we have  $u_2 - (\alpha_2 - 2) = 2 \exp(-\frac{1}{2}u_1) > 0$ . Therefore an intersection of the two mentioned lines is not possible as  $\alpha_2 - \alpha_1 > 4$ . □

### 4.3.2. Types of Riemann hodograph structures.

#### Their formal relation with a solution hodograph

- A characteristic hypersurface (4.15) has the dimension  $n - 1$ . Characteristic hypersurfaces of a dimension  $p < n - 1$  result as intersections of hypersurfaces  $v_k(u) = \text{constant}$ . We notice in this respect that  $n - 1$  appears to be a maximal *possible* dimension of a hodograph characteristic hypersurface. We also notice that a smooth solution  $u = u(x_0, \dots, x_m)$  of (1.1) is described in the hodograph space by a hypersurface of a dimension  $p \leq m + 1$ . Therefore  $m + 1$  appears to be a maximal *effective* dimension for such a hypersurface.
- For  $n > 2$  we have  $n - 1 > 1$  coordinates on a characteristic hypersurface  $v_k(u) = \text{constant}$  through  $u^* \in \mathcal{R}$ .

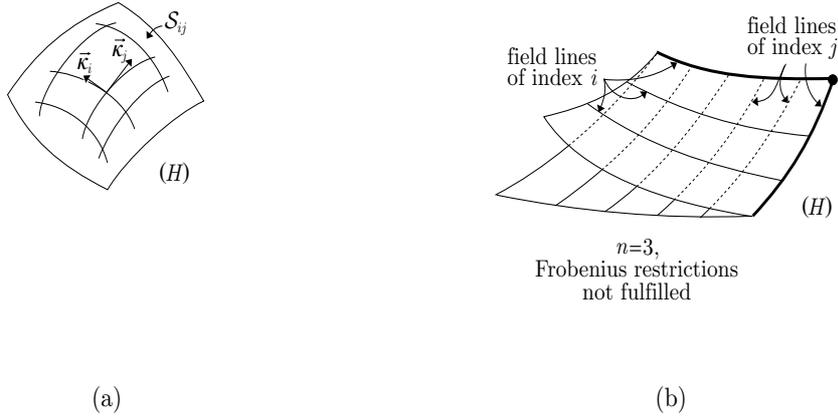


FIGURE 3

- The set of hypersurfaces  $v$  has a *specific* role – depending on  $n$  and  $m$ . For example,
  - for  $n = 2, m = 1$  a hypersurface (4.15) has the dimension  $n - 1 = 1$  which shows that it is reduced in this case to a characteristic line through  $u^*$ . In this case a two-dimensional solution hodograph is structured [cf. (4.12)] by hypersurfaces (4.15) which describe characteristic curves of indices  $k = 1, 2$ . Each such a characteristic curve corresponds to a physical characteristic curve; see section 4.7 here below.
  - For  $n = 3, m = 1$ , and in a Riemann restricted context, representation (4.12) indicates three types of solutions of (1.2) with a hodograph laid on a hypersurface (4.15); example: the hodograph of solution  $u = u[v_1(x, t), v_2^*, v_3(x, t)]$  is laid on  $v_2(u) = v_2^*$ .
  - For  $n > 3, m = 1$ , and in a Riemann restricted context, representation (4.12) indicates some types of solutions of (1.2) with a hodograph laid on an intersection of hypersurfaces (4.15). See 4.4.2 here below.

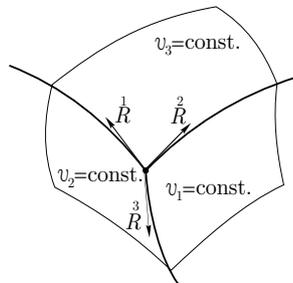


FIGURE 4

### 4.4. Two remarkable classes of constructions

- Two classes of smooth initial data can be identified as remarkable for (1.2) in presence of a Riemann restricted context: (i) those which evolve, in presence of a genuine nonlinearity, through a simple waves solution, and, (ii) those which evolve, in presence of an [adapted] genuine nonlinearity, through a regular interaction of simple waves solutions [see below].

#### 4.4.1. One-dimensional simple waves solutions

- For a genuinely nonlinear index  $j$  we structure in (4.12):

$$u = u[v_1^*, \dots, v_{j-1}^*, v_j(x, t), v_{j+1}^* \dots, v_n^*], \quad v_k^* = \text{constant}; \quad k \neq j. \quad (4.17)_1$$

with  $v_k[u_0(\tilde{x})] \equiv v_k^* = \text{constant}_k$  for  $k \neq j$  in (4.1)<sub>0</sub>.

- The hodograph of solution (4.17)<sub>1</sub> appears as an intersection of  $n-1$  hypersurfaces (4.15)<sub>k</sub>,  $k \neq j$ .
- If a Riemann restricted context is not available the construction of an analogue of (4.17)<sub>1</sub> is described in 4.6 in terms of a Riemann–Lax approach.

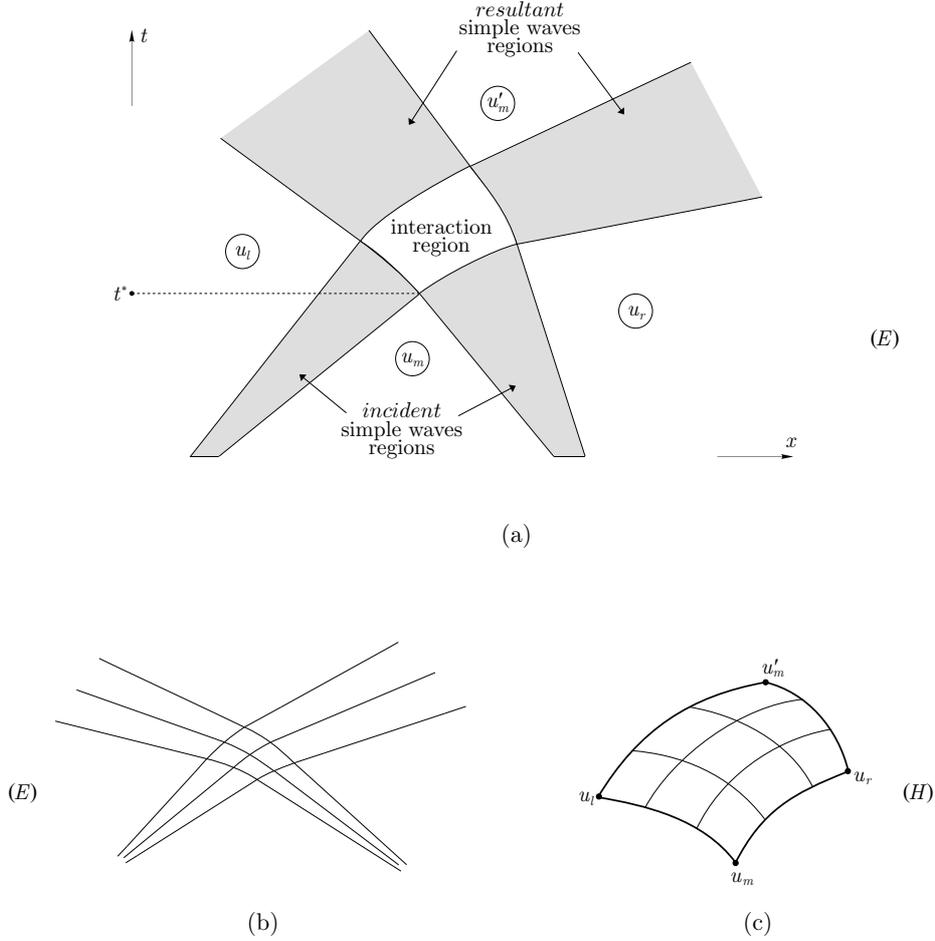


FIGURE 5

#### 4.4.2. Regular interactions of one-dimensional simple waves solutions

- As a smooth solution of (1.2) with a maximal rank has a two-dimensional hodograph we could *naturally* extend (4.17)<sub>1</sub> by the solution structure:

$$u = u[v_1^*, \dots, v_{i-1}^*, v_i(x, t), v_{i+1}^* \dots, v_{j-1}^*, v_j(x, t), v_{j+1}^*, \dots, v_n^*], \quad v_k^* = \text{constant}; \quad k \neq i, k \neq j. \quad (4.17)_2$$

with  $v_k[u_0(\tilde{x})] \equiv v_k^* = \text{constant}_k$  for  $k \neq i, k \neq j$  in (4.1)<sub>0</sub>.

- The hodograph of solution (4.17)<sub>2</sub> appears as an intersection of  $n-2$  hypersurfaces (4.15).
- If a Riemann restricted context is not available the construction of an analogue of (4.17)<sub>2</sub> is described in §5 in terms of a Riemann–Burnat approach.
- If the indices  $i$  and  $j$  are genuinely nonlinear such a solution describes a *regular interaction* of one-dimensional simple waves solutions [see Figures 3a and 5].

#### 4.5. A Riemann restricted characterization of the quasilinearity hierarchy

- We also compute from (4.2)

$$\frac{\partial \lambda_i}{\partial u_j} = \sum_{k=1}^n \frac{\partial \bar{\lambda}_i}{\partial v_k} \cdot \frac{\partial v_k}{\partial u_j} = \sum_{k=1}^n \alpha_k(u) L_j^k(u) \frac{\partial \bar{\lambda}_i}{\partial v_k}$$

$${}^i R(u) \cdot \text{grad}_u \lambda_i(u) = \sum_{j=1}^n {}^i R_j(u) \sum_{k=1}^n \alpha_k(u) L_j^k(u) \frac{\partial \bar{\lambda}_i}{\partial v_k} = \bar{\alpha}_i(v) \frac{\partial \bar{\lambda}_i}{\partial v_i}$$

and respectively transcribe the restrictions of genuinely nonlinearity / linear degeneracy of an index  $i$  [see (3.3)/(3.4)] by

$$\frac{\partial \bar{\lambda}_i}{\partial v_i} \neq 0, \quad v \in \bar{\mathcal{R}} \quad (4.18)$$

or

$$\frac{\partial \bar{\lambda}_i}{\partial v_i} \equiv 0 \quad \text{in } \bar{\mathcal{R}}. \quad (4.19)$$

- At this point we shall use (4.18), (4.19) in order to characterize the quasilinearity hierarchy

$linear \prec semilinear \prec quasilinear \prec nonlinear].$

(4.20)

- As a complete Riemann transform always exists for  $n = 2$  we notice that a representative and most suggestive characterization of the mentioned hierarchy can be done for this case.

- So, for  $n = 2$ , a *strong* quasilinearity means

$$\frac{\partial \bar{\lambda}_1}{\partial v_1} \neq 0, \quad \frac{\partial \bar{\lambda}_2}{\partial v_2} \neq 0 \quad \text{in } \bar{\mathcal{R}}$$

a *medium* quasilinearity requires

$$\frac{\partial \bar{\lambda}_1}{\partial v_1} \neq 0, \quad \frac{\partial \bar{\lambda}_2}{\partial v_2} \equiv 0 \quad \text{or} \quad \frac{\partial \bar{\lambda}_1}{\partial v_1} \equiv 0, \quad \frac{\partial \bar{\lambda}_2}{\partial v_2} \neq 0 \quad \text{in } \bar{\mathcal{R}}$$

and a *weak* quasilinearity has the signification

$$\frac{\partial \bar{\lambda}_1}{\partial v_1} \equiv 0, \quad \frac{\partial \bar{\lambda}_2}{\partial v_2} \equiv 0 \quad \text{in } \bar{\mathcal{R}}. \quad (4.21)$$

- A nontrivial form of (4.21) is complementarily characterized by

$$\frac{\partial \bar{\lambda}_1}{\partial v_2} \neq 0, \quad \frac{\partial \bar{\lambda}_2}{\partial v_1} \neq 0 \quad \text{in } \bar{\mathcal{R}}. \quad (4.22)$$

As (4.21) and (4.22) hold we set

$$r = \bar{\lambda}_2(v_1), \quad s = \bar{\lambda}_1(v_2)$$

in order to transform the corresponding system (4.11) into

$$\frac{\partial r}{\partial t} + s \frac{\partial r}{\partial x} = 0, \quad \frac{\partial s}{\partial t} + r \frac{\partial s}{\partial x} = 0. \quad (4.23)$$

Then, we calculate from (4.23)

$$\frac{\partial r}{\partial t} \frac{\partial s}{\partial x} - \frac{\partial r}{\partial x} \frac{\partial s}{\partial t} = (r - s) \frac{\partial r}{\partial x} \frac{\partial s}{\partial x} \quad (4.24)$$

and

$$\frac{\partial r}{\partial t} + r \frac{\partial r}{\partial x} = (r - s) \frac{\partial r}{\partial x}, \quad \frac{\partial s}{\partial t} + s \frac{\partial s}{\partial x} = (s - r) \frac{\partial s}{\partial x}. \quad (4.25)$$

- Now, if we weaken the restriction (4.22), allowing for example that

$$\frac{\partial \bar{\lambda}_1}{\partial v_2} \equiv 0 \quad \left[ \text{yet } \frac{\partial \bar{\lambda}_2}{\partial v_2} \neq 0 \right] \quad \text{in } \bar{\mathcal{R}},$$

then (4.24) takes the form

$$\frac{\partial r}{\partial t} + \mathfrak{h} \frac{\partial r}{\partial x} = 0, \quad \frac{\partial s}{\partial t} + r \frac{\partial s}{\partial x} = 0, \quad \text{constant } \mathfrak{h}$$

which reduces to a *linear* equation:

$$\frac{\partial s}{\partial t} + r_0(x - \mathfrak{h}t) \frac{\partial s}{\partial x} = 0.$$

It appears that the restrictions (4.21) and (4.22) characterize the *lowest* level of nonlinearity in the weak quasilinearity connected with  $n = 2$ .

- We finally notice that a solution of (4.23) for which  $r \equiv s$  corresponds to a *degeneration* of the weakly quasilinear system (4.23). In this degeneration the two equations (4.23) become coincident in the genuinely nonlinear equation

$$\frac{\partial r}{\partial t} + r \frac{\partial r}{\partial x} = 0.$$

The mentioned degeneration implies a replacement of a  $n = 2$  linear degeneracy by a  $n = 1$  genuine nonlinearity.

#### 4.6. Riemann–Lax invariance. Riemann–Lax invariants

Construction of an one-dimensional simple waves solution begins [cf. 3.1.1] with isolating an orbit of the system  $(2.4)_i$ . We start this section by presenting this part of construction in terms of the first integrals of the system  $(2.4)_i$ . This part of construction will be seen to essentially contribute to the approach in section 8.2.

**TERMINOLOGY 4.6.** We say that a nonconstant function  $\varphi(u)$ , which is  $C^1$  in  $\mathcal{R}_0 \subset \mathcal{R}$ , is a *first integral* in  $\mathcal{R}_0$  for the system  $(2.4)_i$  if it keeps constant along each orbital arc of  $(2.4)_i$  included in  $\mathcal{R}_0$  (the constant depends on orbit).  $\square$

**TERMINOLOGY 4.7** (associated to the contribution in Lax [16]). We say that a nonconstant function  $\varphi(u)$ , which is  $C^1$  in  $\mathcal{R}_0 \subset \mathcal{R}$ , is a *Riemann–Lax characteristic invariance function of index  $i$*  (abbreviated  *$i$ -RLIF*) in  $\mathcal{R}_0$  if it satisfies in  $\mathcal{R}_0$  the equation

$$\overset{i}{R}(u) \cdot \text{grad}_u \varphi(u) = 0. \quad (4.26)$$

$\square$

- Terminology 4.7 is motivated by (4.14) and 4.3.1.
- The proposition which follows states the *equivalence* between the Terminologies 4.6 and 4.7.

**PROPOSITION 4.8.** *The function  $\varphi(u)$  is an  $i$ -RLIF in  $\mathcal{R}_0$  iff it is a first integral of  $(2.4)_i$  in  $\mathcal{R}_0$ .*  $\square$

This shows a hodograph invariance of each  *$i$ -RLIF* along an orbital arc (which appears, cf.  $(2.4)_i$ , to be a characteristic arc).

Since  $\mathcal{R}$  does not contain critical points of the system  $(2.4)_i$ , we have

**PROPOSITION 4.9.** (i) *There exist exactly  $n - 1$  independent  $i$ -RLIF,  $\overset{i}{\varphi}_1(u), \dots, \overset{i}{\varphi}_{n-1}(u)$ , in a neighbourhood  $\mathcal{U}(u^*)$  of every point  $u^* \in \mathcal{R}$  [see Figure 6]*<sup>1</sup>

(ii) *The general solution of (23) can be represented as*

$$\overset{i}{\varphi}(u) \equiv F[\overset{i}{\varphi}_1(u), \dots, \overset{i}{\varphi}_{n-1}(u)], \quad u \in \mathcal{W} \subset \mathcal{U}(u^*)$$

<sup>1</sup> We say that the functions  $g_i(u)$ ,  $1 \leq i \leq k$  are *independent* in a neighbourhood of  $u^*$

if  $\text{rank} \left\| \frac{\partial g_i}{\partial u_j} \right\|_{u=u^*} = k \leq n$  [ $1 \leq i \leq k$ ;  $1 \leq j \leq n$ ].

where  $F$  is an arbitrary  $C^1$  function defined in a neighbourhood  $\mathcal{V}$  of the point

$$[\dot{\varphi}_1(u^*), \dots, \dot{\varphi}_{n-1}(u^*)]. \quad \square$$

COROLLARY 4.10.

(i)  $\dot{R}(u)$ ,  $\text{grad}_u \dot{\varphi}_1(u), \dots, \text{grad}_u \dot{\varphi}_{n-1}(u)$  are independent in  $\mathcal{U}(u^*)$ ,

(ii)  $\dot{L}(u)$ ,  $\text{grad}_u \dot{\varphi}_1(u), \dots, \text{grad}_u \dot{\varphi}_{n-1}(u)$ ,  $k \neq i$ , are dependent in  $\mathcal{U}(u^*)$ ;  $L$  are left eigenvectors of  $a(u)$  in (1.2).  $\square$

REMARK 4.11. Each hypersurface  $\dot{\varphi}_j(u) = \text{constant} = \dot{\varphi}_j(u^*)$ ,  $1 \leq j \leq n-1$ , contains the characteristic curve of index  $i$  through  $u^* \in \mathcal{R}$ . Therefore, the (one-dimensional) intersection of these hypersurfaces is seen to consist in this, mentioned, characteristic curve. In fact, Proposition 4.9 (i) indicates a way of (locally) describing a characteristic curve through  $u^* \in \mathcal{R}$ .  $\square$

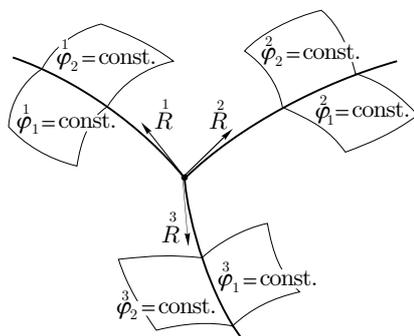


FIGURE 6

• We notice the local character of Proposition 4.9. Still, here is an example for which we have to replace  $\mathcal{U}$  by  $\mathcal{R}$  in Proposition 4.9 (i).

EXAMPLE 4.12. For the system (1.4) the explicit expressions of the Riemann–Lax characteristic invariance functions, valid in the whole  $\mathcal{R}$ , are  $[u = (p, v_x, v_y)^t$ , arbitrary  $p_0]$

$$\dot{\varphi}_1^1(u) = v_x - \int_{p_0}^p \frac{d\xi}{\rho(\xi, \psi)c(\xi, \psi)}, \quad \dot{\varphi}_2^1(u) = \psi, \quad (1\text{-RLIF})$$

$$\dot{\varphi}_1^2(u) = v_x, \quad \dot{\varphi}_2^2(u) = p, \quad (2\text{-RLIF})$$

$$\dot{\varphi}_1^3(u) = v_x + \int_{p_0}^p \frac{d\xi}{\rho(\xi, \psi)c(\xi, \psi)}, \quad \dot{\varphi}_2^3(u) = \psi. \quad (3\text{-RLIF})$$

$\square$

We conclude by remarking, cf. Proposition 4.8, that a hodograph characteristic curve can be constructed generally as a (local) intersection of hypersurfaces described by Riemann–Lax characteristic invariance functions (Figure 6; see comparatively Figure 4). A significant gasdynamic example of such a construction could take into account the Example 4.12.

TERMINOLOGY 4.13. It is proper to notice at this point that, in case of an  $i$ - simple waves solution  $u(x, t)$ , the Riemann–Lax hodograph invariance along a characteristic curve appears to characterize the hodograph range of the application  $u : \mathcal{D} \subset E \rightarrow H$ . This suggests a special terminology: we call

$$\dot{\varphi}_j^i(x, t) \stackrel{\text{def}}{=} \dot{\phi}_j^i[u(x, t)] \quad , \quad 1 \leq j \leq n, \quad j \neq i$$

to be  $i$ - **Riemann–Lax invariants** connected with the  $i$ - simple waves solution  $u$ .  $\square$

REMARK 4.14. A nonconstant solution  $u(x, t)$  of (1.2) for which  $\dot{\phi}_j^i[u(x, t)] = \text{constant}_j$ ,  $1 \leq j \leq n$ ,  $j \neq i$  has an one-dimensional characteristic hodograph and, cf. (3.6) $_i$  it is an  $i$ - simple waves solution. In section 4.4.1 we have presented the construction of a simple waves solution in a Riemann restricted context. If such a context is not available the mentioned construction persists in a Riemann–Lax version.  $\square$

#### 4.7. Correspondence between the physical characteristics and the hodograph characteristics

- Since the eigenvectors  $\vec{\kappa}_i$ ,  $1 \leq i \leq n$ , of the matrix  $a(u)$  in (1.2) generate  $\mathbb{R}^n$  we get at  $u^* \in H$  the representation

$$\frac{\partial u}{\partial t} = \sum_{k=1}^n \tilde{\eta}_k \vec{\kappa}_k, \quad \frac{\partial u}{\partial x} = \sum_{k=1}^n \eta_k \vec{\kappa}_k \quad (4.27)$$

where from (1.2) we obtain

$$\tilde{\eta}_k = -\lambda_k \eta_k, \quad 1 \leq k \leq n. \quad (4.28)$$

- Then for any solution of (1.2) we have [see (2.3)]

$$\left( \frac{\partial u_l}{\partial t}, \frac{\partial u_l}{\partial x} \right) = \sum_{k=1}^n \eta_k \kappa_{kl}(u) \vec{\beta}_k(u), \quad 1 \leq l \leq n. \quad (4.29)$$

- We shall notice in fact from (4.27) that

$$\frac{\partial u_l}{\partial t} + \lambda_j(u) \frac{\partial u_l}{\partial x} = \sum_{k=1}^n \eta_k \overset{k}{R}_l(\lambda_j - \lambda_k), \quad 1 \leq l \leq n. \quad (4.30)$$

- For  $n = 2$  the system (4.30) has the form

$$\begin{cases} \frac{\partial u}{\partial t} + \lambda_1(u) \frac{\partial u}{\partial x} = \eta_2 \overset{2}{R}(\lambda_1 - \lambda_2) \\ \frac{\partial u}{\partial t} + \lambda_2(u) \frac{\partial u}{\partial x} = \eta_1 \overset{1}{R}(\lambda_2 - \lambda_1) \end{cases} \quad (4.31)$$

which indicate that

$$u(\mathcal{C}_1) \subset \tilde{\mathcal{C}}_2, \quad u(\mathcal{C}_2) \subset \tilde{\mathcal{C}}_1 \quad (4.32)$$

where  $\mathcal{C}_1, \mathcal{C}_2$  and  $\tilde{\mathcal{C}}_1, \tilde{\mathcal{C}}_2$  are characteristic arcs in  $E$  and  $H$  respectively.

- A similar to (4.31) result does not hold for  $n \geq 3$  generally.

EXAMPLE 4.15. In case of  $n = 3$  the system (4.30) takes the form

$$\begin{cases} \frac{\partial u}{\partial t} + \lambda_1(u) \frac{\partial u}{\partial x} = \eta_2 \overset{2}{R}(\lambda_1 - \lambda_2) + \eta_3 \overset{3}{R}(\lambda_1 - \lambda_3) \\ \frac{\partial u}{\partial t} + \lambda_2(u) \frac{\partial u}{\partial x} = \eta_1 \overset{1}{R}(\lambda_2 - \lambda_1) + \eta_3 \overset{3}{R}(\lambda_2 - \lambda_3) \\ \frac{\partial u}{\partial t} + \lambda_3(u) \frac{\partial u}{\partial x} = \eta_1 \overset{1}{R}(\lambda_3 - \lambda_1) + \eta_2 \overset{2}{R}(\lambda_3 - \lambda_2). \end{cases} \quad (4.33)$$

- A regular interaction implying a hodograph characteristic system of coordinates with *three* [or *more*] fields is clearly not possible for  $m = 1$  [see Examples 4.15, 4.17]. Still, such a regular interaction could have a chance of being possible if  $n \geq 3$ ,  $m \geq 2$  [see §5 and section 8.4].

- A particular form of (4.30) corresponds to the case of a  $j$ -simple waves solution and it assumes (cf. section 3.1.1)  $\eta_k \equiv 0$  as  $k \neq j$ .  $\square$

EXAMPLE 4.16. For  $\eta_1 \equiv 0$ ,  $\eta_2 \equiv 0$ ,  $\eta_3 \neq 0$  in (4.33) we obtain

$$\begin{cases} \frac{\partial u}{\partial t} + \lambda_1(u) \frac{\partial u}{\partial x} = \eta_3 \overset{3}{R}(\lambda_1 - \lambda_3) \\ \frac{\partial u}{\partial t} + \lambda_2(u) \frac{\partial u}{\partial x} = \eta_3 \overset{3}{R}(\lambda_2 - \lambda_3) \\ \frac{\partial u}{\partial t} + \lambda_3(u) \frac{\partial u}{\partial x} = 0 \end{cases} \quad (4.34)$$

which indicates that

- $u(x, t) = \text{constant}$  along each characteristic arc  $\mathcal{C}_3$  [of index 3] in the domain of solution [the constant depends on the characteristic arc generally]; this shows that  $u(x, t)$  is a 3- simple waves solution;
- because, let  $u^*$  be a constant state adjacent on the left [/right] side to this simple waves solution in the physical plane. We consider in the hodograph space the characteristic arc  $\tilde{\mathcal{C}}_3^*$  through  $u^*$  to notice, cf. (4.34)<sub>1,2</sub> that  $u(\mathcal{C}_1) \subset \tilde{\mathcal{C}}_3^*$  and  $u(\mathcal{C}_2) \subset \tilde{\mathcal{C}}_3^*$ . In fact, the mentioned [3- simple waves] solution  $u(x, t)$  has  $\tilde{\mathcal{C}}_3^*$  as a [one-dimensional] hodograph;
- a point on  $\tilde{\mathcal{C}}_3^*$  is associated to an arbitrary characteristic arc  $\mathcal{C}_3$  in the domain of this solution; as [cf. (4.34)<sub>3</sub>]  $u(x, t) = \text{constant}$  along each arc  $\mathcal{C}_3$  [the constant depends on the characteristic arc generally].

- An other particular form of (4.29), (4.30) parallelling (4.31)] assumes  $\eta_k \equiv 0$  for  $k \neq i$ ,  $k \neq j$  in (4.29), (4.30); it corresponds to the case of a solution with a two-dimensional characteristic hodograph.  $\square$

EXAMPLE 4.17. For  $\eta_1 \equiv 0$ ,  $\eta_2 \neq 0$ ,  $\eta_3 \neq 0$  in (23) we get

$$\begin{cases} \frac{\partial u}{\partial t} + \lambda_1(u) \frac{\partial u}{\partial x} = \eta_2 \overset{2}{R} (\lambda_1 - \lambda_2) + \eta_3 \overset{3}{R} (\lambda_1 - \lambda_3) \\ \frac{\partial u}{\partial t} + \lambda_2(u) \frac{\partial u}{\partial x} = \eta_3 \overset{3}{R} (\lambda_2 - \lambda_3) \\ \frac{\partial u}{\partial t} + \lambda_3(u) \frac{\partial u}{\partial x} = \eta_2 \overset{2}{R} (\lambda_3 - \lambda_2) \end{cases} \quad (23)$$

which indicates that  $u(\mathcal{C}_2) \subset \tilde{\mathcal{C}}_3$  and  $u(\mathcal{C}_3) \subset \tilde{\mathcal{C}}_2$ . We still have, complementarily, that  $u(\mathcal{C}_1)$  is laid on the surface generated, in presence of Frobenius restrictions, by the families  $\tilde{\mathcal{C}}_2, \tilde{\mathcal{C}}_3$  [cf. Figure 3a; here for  $i = 2, j = 3$ ].  $\square$

## 5. Riemann invariants: multidimensional details. Regular interaction of multidimensional simple waves solutions

### 5.1. Introduction

This paragraph presents an extension (in the sense of M. Burnat) of the theory of regular interactions of simple waves solutions [see §4] with two valencies:

- (i) considering the one-dimensional case of an *incomplete* Riemann transform [for which the Riemann restricted context of §4 is not available], and,
- (ii) considering the *multidimensional* case.

In a multidimensional context construction of a regular interaction of simple waves solutions essentially assumes, in presence of Remark 2.6.b, that

- some (extended) indices are available; saving the indices depends on the details of an extended (adapted) form of the **Frobenius restrictions** (section 5.2);
- the Frobenius requirements are *completed* with some **restrictions of regular integrability** (sections 5.3–5.5);
- the two sets of restrictions mentioned before are finally *completed* by some **requirements of genuine nonlinearity** (§7; also see [4])

In fact, in contrast with the one-dimensional case, it can be shown that in the multidimensional case a fulfilment of the Frobenius restrictions does not lead to regular integrability generally (section 5.6). Incidentally, the multidimensional approach considered in this paragraph appears to be useful for the analysis of the case (i) mentioned above.

## 5.2. Hodograph details of Riemann–Burnat “algebraic” multidimensional construction. Frobenius restrictions. Multidimensional indices

- A hypersurface  $\mathcal{S}$  is considered, to begin with, in the hodograph space – to eventually support the image of a solution of the system (1.1).
- In the multidimensional case an *infinite* number of hodograph characteristics pass through a given point  $u^* \in H$  [Remark 2.6.b]. This is in contrast with the one-dimensional case where a *finite* number of hodograph characteristics pass through a given point  $u^* \in H$  [Remark 2.4.b] [this results in a finite number of indices available at that point]. Still, in the multidimensional case we have the possibility of identifying, on the considered hypersurface, of some (finite, local, nonunique) systems of characteristic coordinates. Then, a finite number of indices could be associated to such a system of characteristic coordinates – thus paralleling formally the one-dimensional issue.
- Some *Frobenius restrictions* are then taken into account at the points of this hypersurface to identify such a characteristic system of coordinates. The presence of such a system would have a *structuring* importance for an *eventual* solution with the hodograph on this mentioned hypersurface.
- Let  $\vec{\kappa}_1, \dots, \vec{\kappa}_k$  be, independent at each point, vector fields tangential to the hypersurface  $\mathcal{S}$  which span the tangent space  $T_u\mathcal{S}$  at each point  $u \in \mathcal{S}$ .  
Then the Frobenius conditions <sup>1</sup>

$$[\vec{\kappa}_i, \vec{\kappa}_j] \in \{\vec{\kappa}_i, \vec{\kappa}_j\}$$

are necessary and sufficient for the local existence of a coordinate system whose lines are tangent to the vector fields  $\vec{\kappa}_1, \dots, \vec{\kappa}_k$  [i.e. of a *characteristic* system of coordinates] <sup>2</sup>.

- In a multidimensional approach the indices appear concurrently with the identification, in presence of Frobenius restrictions, of a system of characteristic coordinates. The multidimensional indices essentially contribute to the structuring details of a regular interaction solution.

## 5.3. Connecting the physical space and the hodograph space. Riemann–Burnat invariants. Regular integrability. Regular interaction of multidimensional simple waves solutions (a formal approach)

### 5.3.1. A class of solutions of the system (1.1)

- Let  $R_1, \dots, R_p$  be characteristic coordinates on a given  $p$ -dimensional characteristic region  $\mathcal{R}$  of a hodograph hypersurface  $\mathcal{S}$ . In presence of a  $m > 1$  planar gasdynamic duality we are looking for (Burnat [2], [3]) solutions of the system

$$\frac{\partial u_l}{\partial x_s} = \sum_{k=1}^p \eta_k \kappa_{kl}(u) \beta_{ks}(u), \quad u \in \mathcal{R}; \quad 1 \leq l \leq n, \quad 0 \leq s \leq m \quad (5.1)$$

(for a certain labelling of the coordinate fields).

- It easy to be shown [via (2.1)] that these solutions *appear to satisfy* the system (1.1). This indicates an “*algebraic*” importance of the concept of *dual pair*.
- We formally call these solutions *regular interactions* (“*of simple waves solutions*”). A more complete information would be available from the inspection of the nature of their hodographs [see §7 below].

### 5.3.2. Connecting the hodograph space and the physical space. Riemann invariants

- If a set of Riemann invariants exists, structuring the dependence of the solution  $u$  on  $x$ , we get for the

<sup>1</sup>  $[\vec{X}, \vec{Y}]$  is the *commutator* of the fields  $\vec{X}, \vec{Y}$

$$[\vec{X}, \vec{Y}] \stackrel{\text{def}}{=} (\vec{X} \cdot \text{grad})\vec{Y} - (\vec{Y} \cdot \text{grad})\vec{X}$$

and  $\{\vec{X}, \vec{Y}, \dots, \vec{Z}\}$  is the linear space generated by  $\vec{X}, \vec{Y}, \dots, \vec{Z}$ .

<sup>2</sup> See for example Sh. Sternberg, *Lectures on differential geometry*, Prentice Hall, Englewood Cliffs, 1964

solution mentioned in 5.3.1 a regular interaction representation

$$u_l = u_l[R_1(x_0, \dots, x_m), \dots, R_p(x_0, \dots, x_m)], \quad 1 \leq l \leq n. \quad (5.2)$$

We could compute (Burnat [2], [3])

$$\frac{\partial u_l}{\partial x_s} = \sum_{k=1}^p \eta_k \frac{\partial u_l}{\partial R_k} \cdot \frac{\partial R_k}{\partial x_s} = \sum_{k=1}^p \eta_k \kappa_{kl}(u) \frac{\partial R_k}{\partial x_s}, \quad 1 \leq l \leq n; \quad 0 \leq s \leq m. \quad (5.3)$$

Then we use the independence of the vectors  $\vec{\kappa}_k$ ,  $1 \leq k \leq n$  [of the coordinates  $R_k$ ] to obtain the (overdetermined and Pfaff) system

$$\frac{\partial R_k}{\partial x_s} = \eta_k \beta_{ks}[u(R)], \quad 1 \leq k \leq p, \quad 0 \leq s \leq m \quad (5.4)$$

for identifying the Riemann invariants  $\{R_k(x)\}_{1 \leq k \leq p}$  and the functions  $\{\eta_k(x)\}_{1 \leq k \leq p}$ .

- It is proper to remark, cf. (4.11) and (2.3), that in the one-dimensional case we similarly have [see (4.13)]

$$\left( \frac{\partial v_k}{\partial t}, \frac{\partial v_k}{\partial x} \right) = \eta_k \vec{\beta}_k[u(v)], \quad k = i \text{ or } k = j. \quad (5.5)$$

### 5.3.3. Regular integrability

- Existence of Riemann invariants could be connected with existence of a restricted integrability: we call it *regular integrability*.

### 5.3.4. One-dimensional regular interaction vs. multidimensional regular interaction

- System (5.4) is *determined* for  $p=1, 2$  [ $p=2$  may correspond to an **one-dimensional** regular interaction].

EXAMPLE 5.1. Example 4.2 considered the case of the system (1.4) of the anisentropic gas dynamics for which the Riemann transform (4.1) is incomplete [cf. (4.9); only a part of it is available in the form  $v_2 = v_2(u)$ , cf. (4.10)]. We look for a solution of (1.4) whose hodograph is laid on a surface  $v_2(u) = \text{constant} = v_2^* = v_2(u^*)$ . In our example this surface plays the role of the given  $p$ -dimensional hodograph hypersurface of 5.3.1 (here for  $p=2$ ). We notice that on this surface a characteristic system of coordinates exists and has two coordinate fields. For the mentioned solution we therefore follow the suggestions of 5.3.1 to pass the representation (5.2) into the regular interaction representation

$$u = u[v_1(x, t), v_2^*, v_3(x, t)] \quad (5.6)$$

where  $v_1(x, t)$  and  $v_3(x, t)$  are Riemann–Burnat invariants.

On eliminating  $\eta_k$  from (5.4) [in notations (5.5) with  $p=2$ ,  $m=1$ ] we get for  $v_1(x, t), v_3(x, t)$  the system

$$\begin{cases} \frac{\partial v_1}{\partial t} + \bar{\lambda}_1(v_1, v_2^*, v_3) \frac{\partial v_1}{\partial x} = 0 \\ \frac{\partial v_3}{\partial t} + \bar{\lambda}_3(v_1, v_2^*, v_3) \frac{\partial v_3}{\partial x} = 0. \end{cases} \quad (5.7)$$

Finally we have to solve (5.7) for some smooth data and carry its solution  $v_1(x, t), v_3(x, t)$  into (5.6) in order to make (5.6) active.  $\square$

- For a **multidimensional** regular interaction we have to solve an *overdetermined* system (5.4) generally and, in this case, some restrictions on the exceptional vectors  $\vec{\beta}$  could appear to be required at the points of  $\mathcal{S}$  (see sections 5.4–5.6). This reflects the Remark 2.6.b. Also see [21], [22].

- In the multidimensional case the Frobenius restrictions are associated with a *first step* of the approach only. This step must be completed with a *second step* needed to guarantee that a connection exists between the physical space and the hodograph space – via identifying a suitable set of Riemann invariants; in fact via identifying a regular integrability of (1.1). Example 5.4 here below indicates a case in which the fulfilment of the Frobenius restrictions does not lead to a regular integrability. This is in contrast with the one-dimensional approach which does not distinguish between the two mentioned steps [cf. 4.2.1].

Finally, we have to essentially complete the previously mentioned steps with a *third step* to guarantee a genuinely nonlinear character of the solution considered (§7; also see [4]).

#### 5.4. Cartan–Peradzyński approach

- A class of *involution* restrictions has been isolated in an analytical *planar* context in Peradzyński [19] by means of a variant of Cartan’s theorem <sup>1</sup> concerning the existence of an analytic solution for a system of Pfaff forms in a convenient neighbourhood of a regular point of this system (see D. Yang <sup>2</sup> for a non-analytic extension).

THEOREM 5.2 ([19]). *If, around a regular point  $(x^*, R^*, \eta^*)$  of it, the system (5.4) is analytic and fulfils the conditions of involution*

$$\vec{\beta}_k \wedge \left( \vec{\beta}_j \wedge \frac{\partial \vec{\beta}_k}{\partial R_j} \right) = 0, \quad k, j = 1, \dots, p, \quad k \neq j \quad (5.8)$$

(no summation) then, in a neighbourhood  $\mathcal{V}$  of the mentioned regular point a general analytic solution exists which depends on  $p$  arbitrary analytic functions of one variable.  $\square$

- If the Cartan–Peradzyński restrictions (5.8) are not fulfilled at the points of the hodograph characteristic hypersurface considered in section 5.3.1 it could be not possible to connect the hodograph and physical spaces [through a solution] even in presence of a suitable set of Frobenius restrictions guaranteeing a characteristic structure for the hodograph [see Example 5.4 below (section 5.6)].

#### 5.5. Tsarev–Ferafontov two-dimensional approach

- Another class of restrictions has been isolated in [7]–[12] following Tsarev [29] and Ferafontov (referred in [29]) for a *two-dimensional* approach. The steps of this concurrent approach follow.
- For  $m = 2$  we eliminate the functions  $\eta$  from the system (5.4) and put this system into the form

$$\frac{\partial R_k}{\partial t} = \lambda_k(R) \frac{\partial R_k}{\partial x}, \quad \frac{\partial R_k}{\partial y} = \mu_k(R) \frac{\partial R_k}{\partial x}; \quad \lambda_k(R) \equiv \frac{\beta_{k0}}{\beta_{k1}}, \quad \mu_k(R) \equiv \frac{\beta_{k2}}{\beta_{k1}}; \quad R = (R_1, \dots, R_p). \quad (5.9)$$

- The requirement of *commutativity* of the flows (5.9) is equivalent to the restrictions on their characteristic speeds:

$$\frac{\frac{\partial}{\partial R_j} \lambda_k}{\lambda_j - \lambda_k} = \frac{\frac{\partial}{\partial R_j} \mu_k}{\mu_j - \mu_k}, \quad j \neq k; \quad (5.10)$$

(no summation).

- Once these conditions are met, the general solution of (5.9) is given by Tsarev’s implicit “generalized hodograph” formula [29]:

$$\nu_k(R) = x + \lambda_k(R)t + \mu_k(R)y, \quad 1 \leq k \leq p; \quad R = (R_1, \dots, R_p) \quad (5.11)$$

where  $\nu_k(R)$  are characteristic speeds of the general flow commuting with (5.9), that is the general solution of the linear system

$$\frac{\frac{\partial}{\partial R_j} \nu_k}{\nu_j - \nu_k} = \frac{\frac{\partial}{\partial R_j} \lambda_k}{\lambda_j - \lambda_k} = \frac{\frac{\partial}{\partial R_j} \mu_k}{\mu_j - \mu_k}, \quad j \neq k. \quad (5.12)$$

- In the two-dimensional Tsarev–Ferafontov context the Cartan approach of section 5.4 is replaced with a Darboux approach (see [29]).

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<sup>1</sup> E. Goursat, *Leçons sur le problème de Pfaff*, Librairie Scientifique J. Hermann, Paris, 1922;  
M. Kuranishi, On Cartan’s prolongation theorem of exterior differential systems, *American J. of Math.*, **79**(1957), 1–47;  
M. Kuranishi, Lectures on involutive systems of partial differential equations, Publicações da Sociedade de Matematica de Sao Paulo, Sao Paulo, 1967;  
W. Ślebodziński, *Exterior forms and their applications*, PWN – Polish Scientific Publ., Warszawa, 1970.  
<sup>2</sup> Deane Yang, Involutive hyperbolic differential systems, *Memoirs of AMS*, No.370 (1987).

- Next, we take into account in (5.12) the expressions of  $\lambda$  and  $\mu$  in terms of  $\vec{\beta}$  [cf. (5.9)] in order to transcribe (5.10) by

$$\left(\vec{\beta}_k \wedge \vec{\beta}_j\right) \wedge \left(\vec{\beta}_k \wedge \frac{\partial \vec{\beta}_k}{\partial R_j}\right) = 0, \quad j \neq k. \quad (5.13)$$

- We notice the following relation between the Cartan–Peradzyński and Tsarev–Ferapontov restrictions:

$$\underbrace{\left(\vec{\beta}_k \wedge \vec{\beta}_j\right) \wedge \left(\vec{\beta}_k \wedge \frac{\partial \vec{\beta}_k}{\partial R_j}\right)}_{\text{Tsarev–Ferapontov}} = \vec{\beta}_k \wedge \underbrace{\left[\vec{\beta}_k \wedge \left(\vec{\beta}_j \wedge \frac{\partial \vec{\beta}_k}{\partial R_j}\right)\right]}_{\text{Cartan–Peradzyński}} + \left(\vec{\beta}_k \cdot \vec{\beta}_j\right) \left(\vec{\beta}_k \wedge \frac{\partial \vec{\beta}_k}{\partial R_j}\right), \quad j \neq k. \quad (5.14)$$

which suggests the missing of a hierarchy (/order) of these two sets of restrictions.

- The same as in 5.4, if the Tsarev–Ferapontov restrictions (5.10) are not fulfilled it could be not possible to connect the hodograph and physical spaces [through a solution] even in presence of a suitable set of Frobenius restrictions guaranteeing a characteristic structure for the hodograph [see section 5.6].

## 5.6. Importance of Cartan–Peradzyński or Tsarev–Ferapontov restrictions. A gasdynamic example of degeneracy

TERMINOLOGY 5.3. A region of a manifold in the hodograph space is said to be *nondegenerate* /*degenerate* with respect to the system (1.1) if a continuous (local) solution of (1.1) can/cannot be found whose hodograph is laid on this region.  $\square$

- The Cartan–Peradzyński or Tsarev–Ferapontov restrictions corresponding to the system (1.5) are not fulfilled at the points of the surface  $c = c^*$ . It is interesting to notice that a solution of (1.5) does not exist with a hodograph on this surface. A case of degeneracy is considered in the Example which follows.

EXAMPLE 5.4. The hodograph of a rank two solution of the system (1.5) cannot be laid on the plane  $c = c^* \neq 0$ . Indeed, for  $c \equiv c^* > 0$  in (1.5) we obtain

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0, \quad \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = 0, \quad \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = 0. \quad (5.15)$$

The last two equations (5.15) can be combined to give

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{A} &= \mathcal{A}^2 + v_x \frac{\partial}{\partial x} \mathcal{A} + v_y \frac{\partial}{\partial y} \mathcal{A} - 2\mathcal{B}, \quad \mathcal{C} = v_y \mathcal{B}, \quad \mathcal{D} = -v_x \mathcal{B} \\ \mathcal{A} &\equiv \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}, \quad \mathcal{B} \equiv \frac{D(v_x, v_y)}{D(x, y)} = \frac{\partial v_x}{\partial x} \frac{\partial v_y}{\partial y} - \frac{\partial v_x}{\partial y} \frac{\partial v_y}{\partial x}, \quad \mathcal{C} \equiv \frac{D(v_x, v_y)}{D(t, x)}, \quad \mathcal{D} \equiv \frac{D(v_x, v_y)}{D(t, y)}. \end{aligned}$$

We use the first equation of (5.15) and find that the rank of the Jacobian matrix associated to the considered solution is unity. Hence the only smooth solutions with a hodograph on  $c = c^*$  are the (rotational) simple waves solutions.  $\square$

- Incidentally, an infinity of characteristic coordinates can be constructed on  $c = c^*$ : genuinely nonlinear, linearly degenerate, hybrid. This indicates a fulfilment of Frobenius restrictions.
- The genuinely nonlinear characteristic coordinates could be regarded as hodographs of simple waves solutions.
- Still, as the Cartan–Peradzyński or Tsarev–Ferapontov restrictions corresponding to the system (1.3) are not fulfilled at the points of the surface  $c = c^*$ , this surface will not support the hodograph of a regular interaction of simple waves solutions. And this surface will not support the hodograph of a hybrid interaction.
- In other words: the Cartan–Peradzyński or Tsarev–Ferapontov restrictions are not fulfilled and it is not possible in this case to connect the hodograph and physical spaces [through a solution] even in presence of a suitable set of Frobenius restrictions guaranteeing a characteristic structure for the hodograph.

## 6. A class of exact multidimensional gasdynamic solutions

### 6.1. Constructive details

In order to obtain (local) solutions of the system (1.5) of the isentropic two-dimensional gas dynamics we put around the point  $(x_0, y_0, t_0)$  of the physical space  $E$

$$\xi = \frac{x - x_0}{t - t_0}, \quad \eta = \frac{y - y_0}{t - t_0} \quad (6.1)$$

and present the mentioned system in the form

$$\begin{cases} (v_x - \xi) \frac{\partial c^2}{\partial \xi} + (v_y - \eta) \frac{\partial c^2}{\partial \eta} + (\gamma - 1)c^2 \left( \frac{\partial v_x}{\partial \xi} + \frac{\partial v_y}{\partial \eta} \right) = 0 \\ \frac{\partial c^2}{\partial \xi} + (\gamma - 1)(v_x - \xi) \frac{\partial v_x}{\partial \xi} + (\gamma - 1)(v_y - \eta) \frac{\partial v_x}{\partial \eta} = 0 \\ \frac{\partial c^2}{\partial \eta} + (\gamma - 1)(v_x - \xi) \frac{\partial v_y}{\partial \xi} + (\gamma - 1)(v_y - \eta) \frac{\partial v_y}{\partial \eta} = 0. \end{cases} \quad (6.2)$$

We shall consider for the system (6.2) examples of local solutions for which

$$v_x = \mathbf{a}\xi + \mathbf{b}\eta + \mathbf{c}, \quad v_y = \bar{\mathbf{a}}\xi + \bar{\mathbf{b}}\eta + \bar{\mathbf{c}}, \quad \text{real constant } \mathbf{a}, \mathbf{b}, \mathbf{c}, \bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}. \quad (6.3)$$

### 6.2. The exhaustive list of elements of the class considered above

In §7 certain *representative* examples, cf. §5, will be taken from the following *exhaustive* list [see the Appendix 2 for details].

---


$$\text{[cf. (A2.12)]} \quad v_x \equiv \mathbf{c}, \quad v_y \equiv \bar{\mathbf{c}}, \quad c^2 \equiv K; \quad \text{arbitrary } K \quad (6.4)$$


---

$$\text{[cf. (A2.13)]} \quad v_x \equiv \mathbf{c}, \quad v_y \equiv \eta, \quad c^2 \equiv 0 \quad (6.5)$$


---

$$\text{[cf. (A2.14)]} \quad v_x \equiv \mathbf{c}, \quad v_y \equiv \frac{2}{\gamma + 1}\eta + \bar{\mathbf{c}}, \quad c^2 = \left( \frac{\gamma - 1}{\gamma + 1}\eta - \bar{\mathbf{c}} \right)^2 \quad (6.6)$$


---

$$\text{[cf. (A2.15)]} \quad v_x = \xi, \quad v_y \equiv \bar{\mathbf{c}}, \quad c^2 \equiv 0 \quad (6.7)$$


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$$\text{[cf. (A2.16)]} \quad v_x = \xi, \quad v_y = \eta, \quad c^2 \equiv 0 \quad (6.8)$$


---

$$\text{[cf. (A2.17)]} \quad v_x = \xi, \quad v_y = \frac{3 - \gamma}{\gamma + 1}\eta + \bar{\mathbf{c}}, \quad c^2 = \frac{3 - \gamma}{4} \left( 2\frac{\gamma - 1}{\gamma + 1}\eta - \bar{\mathbf{c}} \right)^2 \quad (6.9)$$


---

$$\text{[cf. (A2.18)]} \quad v_x = \frac{2}{\gamma + 1}\xi + \mathbf{c}, \quad v_y \equiv \bar{\mathbf{c}}, \quad c^2 = \left( \frac{\gamma - 1}{\gamma + 1}\xi - \mathbf{c} \right)^2 \quad (6.10)$$


---

$$\text{[cf. (A2.19)]} \quad v_x = \frac{3 - \gamma}{\gamma + 1}\xi + \mathbf{c}, \quad v_y = \eta, \quad c^2 = \frac{3 - \gamma}{4} \left( 2\frac{\gamma - 1}{\gamma + 1}\xi - \mathbf{c} \right)^2 \quad (6.11)$$


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$$\begin{aligned}
& v_x = \frac{1}{\gamma}\xi + \mathbf{c}, \quad v_y = \frac{1}{\gamma}\eta + \bar{\mathbf{c}}, \\
\text{[cf. (A2.20)]} \quad c^2 &= \frac{1}{2} \left[ \left( \frac{\gamma-1}{\gamma}\xi - \mathbf{c} \right)^2 + \left( \frac{\gamma-1}{\gamma}\eta - \bar{\mathbf{c}} \right)^2 \right]
\end{aligned} \tag{6.12}$$


---

$$\begin{aligned}
v_x &= \mathbf{a}\xi \pm \eta\sqrt{\mathbf{a}(1-\mathbf{a})} + K\sqrt{1-\mathbf{a}}, \quad K = \frac{\mathbf{c}}{\sqrt{1-\mathbf{a}}} = \mp \frac{\bar{\mathbf{c}}}{\sqrt{\mathbf{a}}} \\
\text{[cf. (A2.30)]} \quad v_y &= \pm \xi\sqrt{\mathbf{a}(1-\mathbf{a})} + \eta(1-\mathbf{a}) \mp K\sqrt{\mathbf{a}}, \\
c^2 &\equiv 0 \\
& 0 < \mathbf{a} < 1 \quad (6.13)
\end{aligned}$$


---

$$\begin{aligned}
v_x &= \sqrt{\mathbf{a}} \left( \xi\sqrt{\mathbf{a}} \pm \eta\sqrt{\frac{2}{\gamma+1} - \mathbf{a}} \right) + \mathbf{c} \\
\text{[cf. (A2.31)]} \quad v_y &= \pm \sqrt{\frac{2}{\gamma+1} - \mathbf{a}} \left( \xi\sqrt{\mathbf{a}} \pm \eta\sqrt{\frac{2}{\gamma+1} - \mathbf{a}} \right) + \bar{\mathbf{c}} \\
c^2 &= \frac{\gamma+1}{2} \left[ \frac{\gamma-1}{\gamma+1} \left( \xi\sqrt{\mathbf{a}} \pm \eta\sqrt{\frac{2}{\gamma+1} - \mathbf{a}} \right) - \left( \mathbf{c}\sqrt{\mathbf{a}} \pm \bar{\mathbf{c}}\sqrt{\frac{2}{\gamma+1} - \mathbf{a}} \right) \right]^2 \\
& 0 < \mathbf{a} < \frac{2}{\gamma+1} \quad (6.14)
\end{aligned}$$


---

$$\begin{aligned}
v_x &= \mathbf{a}\xi \pm \eta\sqrt{(1-\mathbf{a}) \left( \mathbf{a} - \frac{3-\gamma}{\gamma+1} \right)} + K\sqrt{1-\mathbf{a}}, \quad K = \frac{\mathbf{c}}{\sqrt{1-\mathbf{a}}} = \mp \frac{\bar{\mathbf{c}}}{\sqrt{\mathbf{a} - \frac{3-\gamma}{\gamma+1}}} \\
\text{[cf. (A2.32)]} \quad v_y &= \pm \xi\sqrt{(1-\mathbf{a}) \left( \mathbf{a} - \frac{3-\gamma}{\gamma+1} \right)} + \eta \left( \frac{4}{\gamma+1} - \mathbf{a} \right) \mp K\sqrt{\mathbf{a} - \frac{3-\gamma}{\gamma+1}} \\
c^2 &= \frac{(3-\gamma)(\gamma-1)}{2(\gamma+1)} \left( \xi\sqrt{1-\mathbf{a}} \mp \eta\sqrt{\mathbf{a} - \frac{3-\gamma}{\gamma+1}} - K \right)^2 \\
& \frac{3-\gamma}{\gamma+1} < \mathbf{a} < 1 \quad (6.15)
\end{aligned}$$


---

$$\text{[cf. (A2.36)]} \quad v_x \equiv \mathbf{c}, \quad v_y = \bar{\mathbf{a}}\xi + \eta + \bar{\mathbf{c}}, \quad c^2 \equiv 0 \tag{6.16}$$


---

$$\text{[cf. (A2.37)]} \quad v_x = \xi, \quad v_y = \bar{\mathbf{a}}\xi + \bar{\mathbf{c}}, \quad c^2 \equiv 0 \tag{6.17}$$


---

$$\text{[cf. (A2.38)]} \quad v_x = \mathbf{a}\xi + \mathbf{b}\eta + \mathbf{c}, \quad v_y = \frac{\mathbf{a}(1-\mathbf{a})}{\mathbf{b}}\xi + \eta(1-\mathbf{a}) - \frac{\mathbf{a}}{\mathbf{b}}\mathbf{c}, \quad c^2 \equiv 0 \tag{6.18}$$


---

REMARK 6.1. (i) Solution (6.13) tends to (6.5) in the limit  $\mathbf{a} \rightarrow 0$  and to (6.7) in the limits  $\mathbf{a} \rightarrow 1$ . Solution (6.14) tends to (6.6) in the limit  $\mathbf{a} \rightarrow 0$  and to (6.10) in the limit  $\mathbf{a} \rightarrow \frac{2}{\gamma+1}$ . Solution (6.15) tends

to (6.11) in the limit  $\mathbf{a} \rightarrow \frac{3-\gamma}{\gamma+1}$  and to (6.9) in the limit  $\mathbf{a} \rightarrow 1$ . (ii) The solution (6.16) tends to (6.5) in the limit  $\bar{\mathbf{a}} \rightarrow 0$ ,  $\bar{\tau} \rightarrow 0$ . Solution (6.17) tends to (6.7) in the limit  $\bar{\mathbf{a}} \rightarrow 0$ . Solution (6.18) tends to (6.13) in the limit  $\mathbf{b}^2 \rightarrow \mathbf{a}(1-\mathbf{a})$ .  $\square$

### 6.3. The nature of solutions on the exhaustive list. Identifying the nontrivial elements

Incidentally, and remarkably, *all* the solutions on the exhaustive list could be characterized according to the facts of §§3–5.

REMARK 6.2. • In this section we only notice that (6.6), (6.10) and (6.14) are one-dimensional simple waves solutions [respectively corresponding to  $\alpha(y, t) \equiv \eta$ ,  $\alpha(x, t) \equiv \xi$  or  $\alpha(x, y, t) \equiv \xi\sqrt{\bar{\mathbf{a}}} \pm \eta\sqrt{\frac{2}{\gamma+1} - \bar{\mathbf{a}}}$  in (3.11)<sub>i</sub>] and (6.4) is the trivial solution. Solution (6.14) appears to be one-dimensional after an evident replacement of the frame  $x, y$  by an orthogonal frame  $X, Y$  for which  $X = x\sqrt{\bar{\mathbf{a}}} \pm y\sqrt{\frac{2}{\gamma+1} - \bar{\mathbf{a}}}$ . • Solution (6.12) is characterized in section 7.1 to be a regular interaction of multidimensional simple waves solutions. • Solutions (6.9), (6.11) and (6.15) are characterized in section 7.2 as regular interactions of one-dimensional simple waves solutions. • Solutions (6.5), (6.7), (6.8), (6.13), (6.16), (6.17), (6.18) are constitutively inadmissible because of the requirement  $c^2 \equiv 0$ .  $\square$

We have finally to remove from the exhaustive list (6.4)–(6.18) the trivial and the constitutively inadmissible solutions and, cf. Remark 6.1(i), to close the intervals to which  $\mathbf{a}$  belongs in (6.13)–(6.15).

## 7. Two examples of regular interactions of multidimensional simple waves solutions

### 7.1. A first significant solution in the class mentioned above

- Let us consider in the hodograph space corresponding to the system (1.5) ( $n = 3$ ) the circular semi-cone

$$A^2 c^2 - B^2 (V_x^2 + V_y^2) = 0, \quad c > 0. \quad (7.1)$$

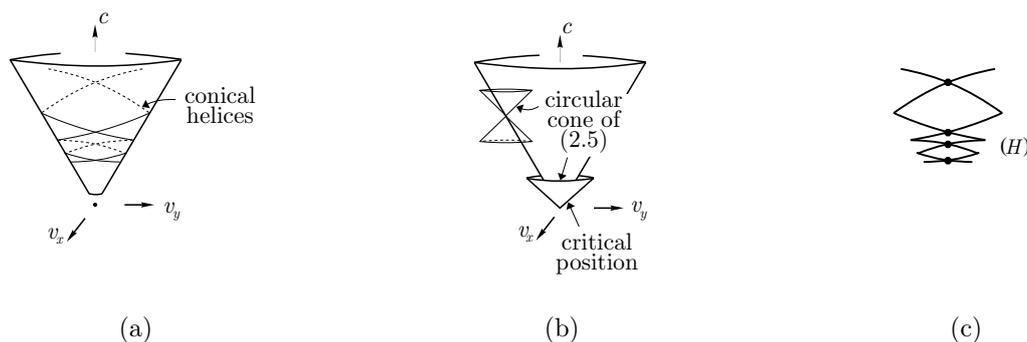


FIGURE 7

- A coordinate system results by considering two families of conical helices on this semi-cone. The equation of a conical helix on (7.1) is

$$c = B \exp hR, \quad V_x = A \exp hR \cos R, \quad V_y = A \exp hR \sin R. \quad (7.2)$$

We compute from (7.2) the tangent direction to this helix and notice that it is laid on the cone

$$(h^2 + 1)A^2(c - c^*)^2 = h^2B^2[(V_x - V_x^*)^2 + (V_y - V_y^*)^2] \quad (7.3)$$

with the vertex at the point  $U^* = (c^*, V_x^*, V_y^*)$  of (7.1). Therefore, the mentioned direction results at each  $U^*$  on (7.1) by intersecting this semi-cone with (7.3).

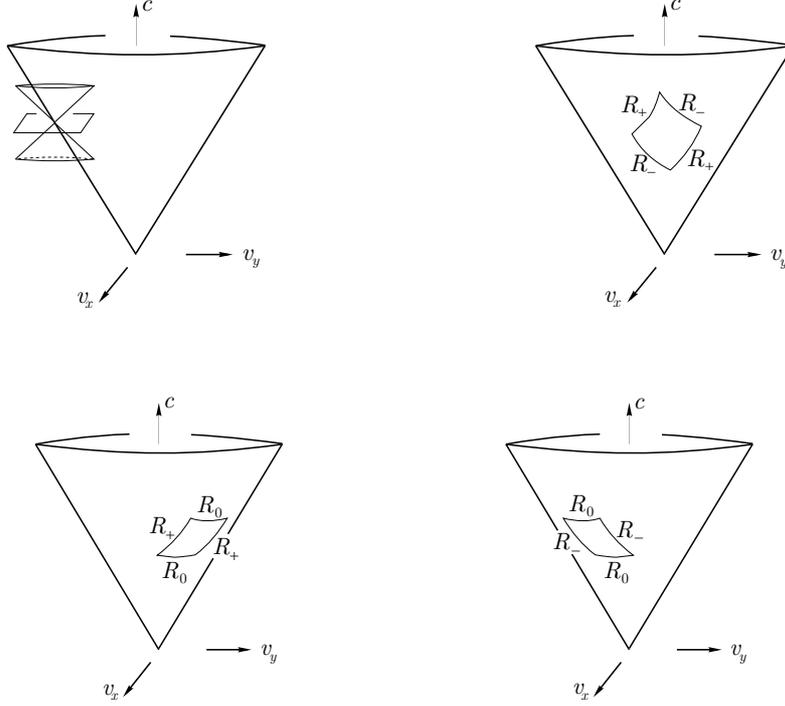


FIGURE 8

A helix is a spatial curve whose tangents keep a constant inclination with respect to a fixed direction. A particularity of the helix is the preserving of a constant quotient curvature/torsion at its points. For the construction of (7.2) the axis of the cone (7.3) appears to be a fixed direction.

A conical helix becomes a *characteristic* curve if (7.3) coincides with the circular branch of (2.5). Such a coincidence requires

$$\frac{B^2}{A^2} = \frac{(\gamma - 1)^2}{4} \cdot \frac{h^2 + 1}{h^2}. \quad (7.4)$$

To a particular choice of  $h$  in (7.2) a particular form of (7.1) corresponds cf. (7.4). For example, for  $h = \pm 1$  and  $A = 1$  in (7.4) we have to replace (7.1) by

$$c^2 = \frac{(\gamma - 1)^2}{2}(V_x^2 + V_y^2), \quad c > 0. \quad (7.5)$$

Two families of conical helices given, cf. (7.2), by

$$\begin{cases} c = \frac{\gamma - 1}{\sqrt{2}} \exp[-(R_+ + R_-)] \\ V_x = \exp[-(R_+ + R_-)] \cos(R_+ - R_-) \\ V_y = \exp[-(R_+ + R_-)] \sin(R_+ - R_-) \end{cases} \quad (7.6)$$

describe locally, around each point  $U^*$  of the semi-cone (7.1), a *characteristic* coordinate system  $R_+, R_-$ .

- We consider the solution (6.12)

$$v_x = \frac{1}{\gamma}\xi + \mathbf{c}, \quad v_y = \frac{1}{\gamma}\eta + \bar{\mathbf{c}}; \quad \text{arbitrary } \mathbf{c}, \bar{\mathbf{c}},$$

$$c^2 = \frac{1}{2} \left[ \left( \frac{\gamma-1}{\gamma}\xi - \mathbf{c} \right)^2 + \left( \frac{\gamma-1}{\gamma}\eta - \bar{\mathbf{c}} \right)^2 \right] \quad (7.7)$$

on the exhaustive list above. The hodograph of this solution is a characteristic conical surface.

- It results from (7.7) a concrete form (7.5) with

$$V_x = v_x - \mathbf{c}^*, \quad V_y = v_y - \bar{\mathbf{c}}^*; \quad \mathbf{c}^* = \frac{\gamma}{\gamma-1}\mathbf{c}, \quad \bar{\mathbf{c}}^* = \frac{\gamma}{\gamma-1}\bar{\mathbf{c}} \quad (7.8)$$

for the cone (7.1).

- Finally, we compare (7.7) and (7.8) with (7.6) to get the concrete form – connecting the hodograph and physical spaces – of the Riemann invariants  $R_+(\xi, \eta), R_-(\xi, \eta)$ . We include this form into (7.6) and take into account the *genuinely nonlinear* character of the hodograph characteristics, the two families of helices, to obtain a structured representation of a *regular interaction* of multidimensional simple waves solutions. The physical structure of such a regular interaction is presented in Figure 9 below.

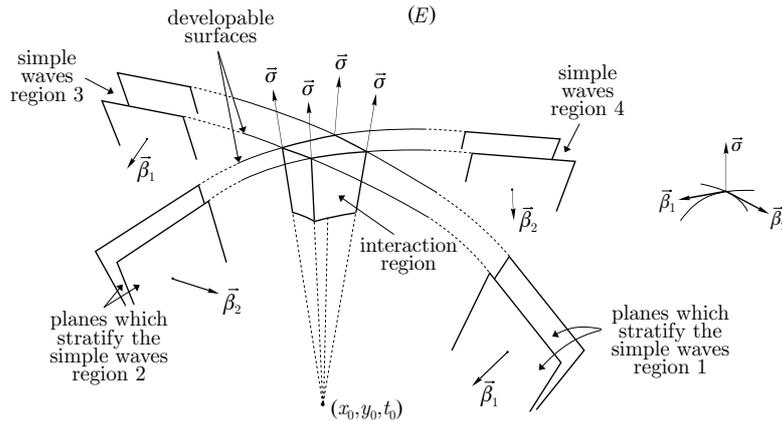


FIGURE 9

- In the one-dimensional case of Figure 5a the four simple waves solutions implied by a regular interaction will be classified to be *incident* or *resultant*. This classification appears to be useless in the two-dimensional case of Figure 9.
- A third family of hodograph characteristics on the semi-cone (7.5) will result from the intersection of this semi-cone with the planar branch of (2.5).
- As the characteristic curves of this family do not include straightlined arcs, these curves have a *genuinely nonlinear* character too and appear to be available for supporting a hodograph of (multidimensional) simple waves solution (cf. section 3.4).
- The solution (7.7), which is a *regular interaction of simple waves solutions* could be structured in *three distinct manners*. To each manner a pair of Riemann invariants contribute:

$$R_+(\xi, \eta), R_-(\xi, \eta); \quad R_+(\xi, \eta), R_0(\xi, \eta); \quad R_-(\xi, \eta), R_0(\xi, \eta),$$

which result when the solution (7.7) is compared with its Riemann invariance structure. The possibility of *several* Riemann structures appears to be a **new fact** of the multidimensional approach.

## 7.2 A second significant solution in the class mentioned above

### 7.2.1. Importance of the genuine nonlinearity

- Let us consider in the hodograph space corresponding to the system (1.5) ( $n = 3$ ) the plane

$$c - c^* = A(v_x - v_x^*) + B(v_y - v_y^*) \quad (7.9)$$

through the point  $u^* = (c^*, v_x^*, v_y^*)$ . Two families of characteristic straightlines could be drawn in this plane if the intersection of (7.9) with the circular branch of the cone (2.5),

$$(c - c^*)^2 = \frac{(\gamma - 1)^2}{4} [(v_x - v_x^*)^2 + (v_y - v_y^*)^2], \quad (7.10)$$

is real. The straightlines of these families appear to be the coordinate lines of a characteristic system of coordinates around  $u^*$ . The reality of the intersection of (7.9) and (7.10) is guaranteed by requiring

$$A^2 + B^2 - \frac{(\gamma - 1)^2}{4} > 0. \quad (7.11)$$

The vectors  $\vec{\kappa}^\pm$  are then normalized by  $\kappa_{v_x}^2 + \kappa_{v_y}^2 = 1$  and their orientations are a priori chosen.

- A coordinate system  $R_+, R_-$  on (7.9) around the point  $u^*$  is described by

$$\begin{cases} c - c^* = \kappa_c^+ R_+ + \kappa_c^- R_- \\ v_x - v_x^* = \kappa_{v_x}^+ R_+ + \kappa_{v_x}^- R_- \\ v_y - v_y^* = \kappa_{v_y}^+ R_+ + \kappa_{v_y}^- R_- \end{cases} \quad (7.12)$$

where the vectors  $\vec{\kappa}^\pm$  correspond to intersection directions of (7.9) and (7.10) [Figure 10].

We compute

$$(\kappa_c^\pm, \kappa_{v_x}^\pm, \kappa_{v_y}^\pm) = \left\{ \frac{\gamma - 1}{2} \left[ A \frac{\gamma - 1}{2} \pm B \sqrt{A^2 + B^2 - \frac{(\gamma - 1)^2}{4}} \right], - \left[ B^2 - \frac{(\gamma - 1)^2}{4} \right], \left[ AB \pm \frac{\gamma - 1}{2} \sqrt{A^2 + B^2 - \frac{(\gamma - 1)^2}{4}} \right] \right\}.$$

- For  $\frac{3-\gamma}{\gamma+1} < \mathfrak{a} < 1$  we consider the solution (6.15)

$$\begin{aligned} v_x &= \mathfrak{a}\xi \pm \eta \sqrt{(1 - \mathfrak{a}) \left( \mathfrak{a} - \frac{3 - \gamma}{\gamma + 1} \right)} + K \sqrt{1 - \mathfrak{a}}, \quad K = \frac{\mathfrak{c}}{\sqrt{1 - \mathfrak{a}}} = \mp \frac{\bar{\mathfrak{c}}}{\sqrt{\mathfrak{a} - \frac{3 - \gamma}{\gamma + 1}}} \\ v_y &= \pm \xi \sqrt{(1 - \mathfrak{a}) \left( \mathfrak{a} - \frac{3 - \gamma}{\gamma + 1} \right)} + \eta \left( \frac{4}{\gamma + 1} - \mathfrak{a} \right) \mp K \sqrt{\mathfrak{a} - \frac{3 - \gamma}{\gamma + 1}} \\ c &= \varepsilon \sqrt{\frac{(3 - \gamma)(\gamma - 1)}{2(\gamma + 1)}} \left( \xi \sqrt{1 - \mathfrak{a}} \mp \eta \sqrt{\mathfrak{a} - \frac{3 - \gamma}{\gamma + 1}} - K \right), \quad \varepsilon = \pm 1 \end{aligned} \quad (7.13)$$

on the exhaustive list above, with

$$K = \frac{\mathfrak{c}}{\sqrt{1 - \mathfrak{a}}} = \mp \frac{\bar{\mathfrak{c}}}{\sqrt{\mathfrak{a} - \frac{3 - \gamma}{\gamma + 1}}}.$$

The hodograph of this solution is a *double* characteristic (planar) surface.

- It results from (7.13) a concrete form for the plane (7.9):

$$\begin{aligned} A &= \varepsilon \sqrt{\frac{\gamma^2 - 1}{2(3 - \gamma)}} \sqrt{1 - \mathfrak{a}}, \quad B = \mp \varepsilon \sqrt{\frac{\gamma^2 - 1}{2(3 - \gamma)}} \sqrt{\mathfrak{a} - \frac{3 - \gamma}{\gamma + 1}}, \quad \varepsilon = \pm 1 \\ v_x^* &= K \sqrt{1 - \mathfrak{a}}, \quad v_y^* = \mp K \sqrt{\mathfrak{a} - \frac{3 - \gamma}{\gamma + 1}}, \quad c^* = -\varepsilon K \sqrt{\frac{(\gamma - 1)(3 - \gamma)}{2(\gamma + 1)}}. \end{aligned} \quad (7.14)$$

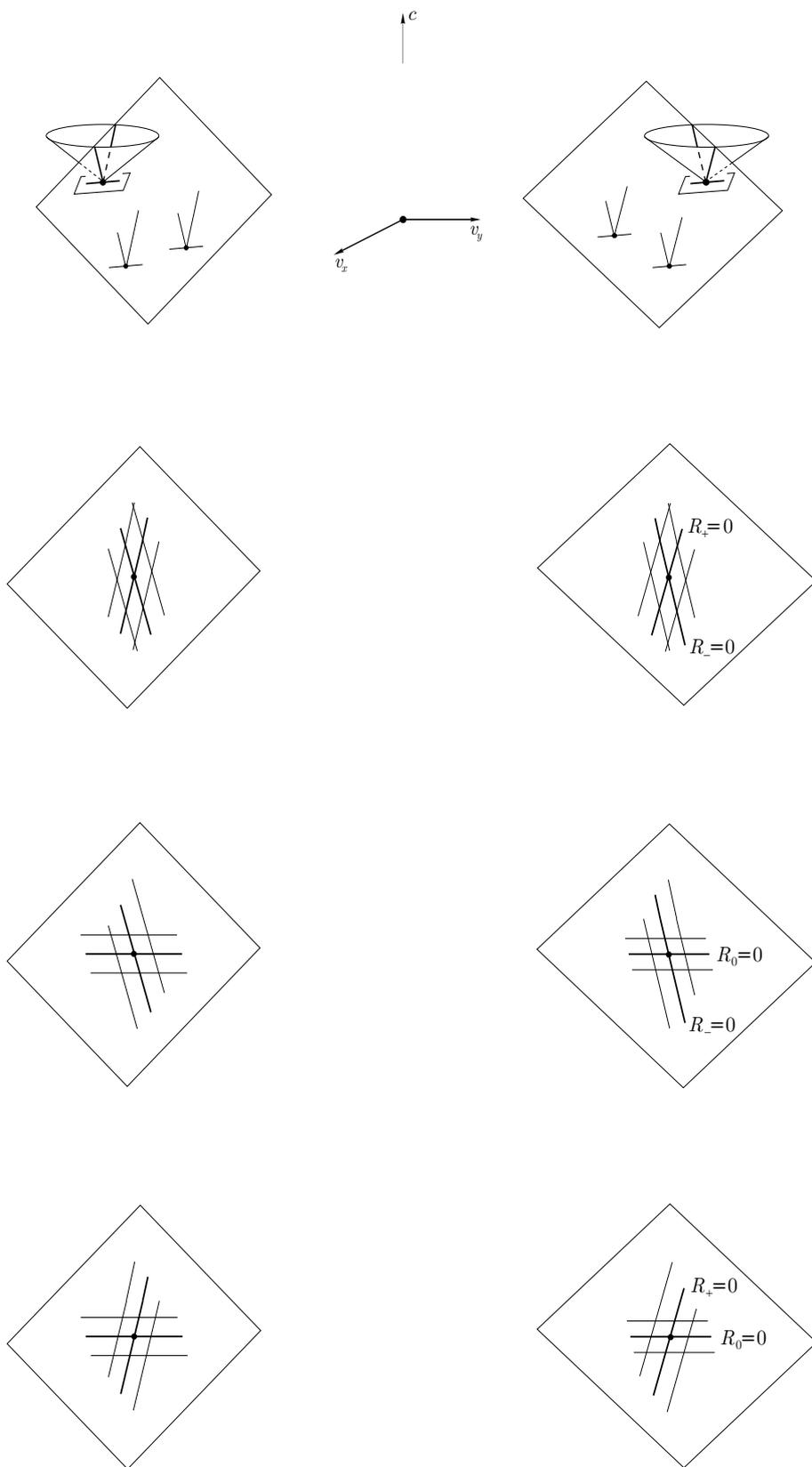


FIGURE 10

We compute

$$A^2 + B^2 - \frac{(\gamma - 1)^2}{4} = \frac{\gamma + 1}{3 - \gamma} \cdot \frac{(\gamma - 1)^2}{4} > 0.$$

to guarantee that a characteristic system exists whose coordinate lines are straightlines.

- Finally, we compare (7.12) and (7.13) to get the concrete form – connecting the hodograph and physical spaces – of the Riemann invariants  $R_+(\xi, \eta), R_-(\xi, \eta)$ . We include this form into (7.12) and take into account the *genuinely nonlinear* character of the characteristic straightlines supported by the vectors  $\vec{\kappa}_\pm$  to obtain a structured representation of a *regular interaction* of one-dimensional simple waves solutions <sup>1</sup> (see again Figure 9).

### 7.2.2. Possibility of a partial linear degeneracy

- A third family of hodograph characteristics on a plane (7.9) will result from the intersection of this plane with the *planar* branch of (2.5) [Figure 10].
- A coordinate system  $R_+, R_0$  in (7.9) around the point  $u^*$  is described by

$$\begin{cases} c - c^* = \kappa_c^+ R_+ + \kappa_c^0 R_0 \\ v_x - v_x^* = \kappa_{v_x}^+ R_+ + \kappa_{v_x}^0 R_0 \\ v_y - v_y^* = \kappa_{v_y}^+ R_+ + \kappa_{v_y}^0 R_0 \end{cases} \quad (7.15)$$

where

$$\vec{\kappa}^0 = (0, -B, A) \quad (7.16)$$

[as determined by the mentioned intersection].

- As the hodograph characteristic curves of the third family (index 0) are horizontal straightlined arcs, these curves have a *linearly degenerate* character (cf. section 3.4). We identify the index (0) to be a linearly degenerated index.
- The solution (7.13) could be again structured in *three distinct manners*. To each manner a pair of Riemann invariants contribute:

$$R_+(\xi, \eta), R_-(\xi, \eta) ; R_+(\xi, \eta), R_0(\xi, \eta) ; R_-(\xi, \eta), R_0(\xi, \eta) ,$$

which result when the solution (7.13) is compared with its Riemann invariance structure (7.12) or (7.15).

- The Riemann representation corresponding to the pair  $R_+(\xi, \eta), R_-(\xi, \eta)$  is associated with a regular interaction of simple waves solutions.
- The representations corresponding to the pairs  $R_+(\xi, \eta), R_0(\xi, \eta)$  or  $R_-(\xi, \eta), R_0(\xi, \eta)$  could be associated to a regular *hybrid* interaction. This possibility appears as an other essential **new fact** of the multidimensional approach <sup>1</sup>.

From (7.15) we notice that for  $R_+ = 0$  we have  $c = c^*$  in (7.9). Therefore  $R_+ = 0$  corresponds to the hodograph straightline

$$c - c^* = 0, \quad A(v_x - v_x^*) + B(v_y - v_y^*) = 0 \quad (7.17)$$

whose pre-image in the physical space, cf. (7.12) and (7.13), is the hyperplane [ex. for  $\varepsilon = 1$  in (7.13)]:

$$\xi \sqrt{1 - \mathbf{a}} - \eta \sqrt{\mathbf{a} - \frac{3 - \gamma}{\gamma + 1}} = 0. \quad (7.18)$$

If the axes  $x, y$  are changed with the axes  $X, Y$  and  $V_X, V_Y$  denote, respectively, at each point  $(X, Y) \in \mathbb{R}^2$ , the projections of the velocity vector on these new axes, we get

$$\begin{aligned} X &= \alpha_{xx}x + \alpha_{xy}y, & Y &= \alpha_{yx}x + \alpha_{yy}y \\ \alpha_{xx} &= \cos \theta = \alpha_{yy}, & \alpha_{xy} &= \sin \theta = -\alpha_{yx}; & -\pi < \theta < \pi \end{aligned} \quad (7.19)$$

<sup>1</sup> We put, for example,  $R_- = 0$  in (7.12) and use the resulted  $c(R_+), v_x(R_+), v_y(R_+)$  to express at each  $R_+$  the dual vector  $\vec{\beta}(R_+)$  [cf. (2.9)]. We find  $\frac{d\vec{\beta}^+}{dR_+} \neq 0$ . Finally, we carry the computed  $\vec{\beta}(R_+)$  in the construction (3.9)' of a simple waves solution. Similarly, we put, for example,  $R_+ = 0$  in (7.15) and use the resulted  $c(R_0), v_x(R_0), v_y(R_0)$  to express at each  $R_0$  the dual vector  $\vec{\beta}(R_0)$  [cf. (2.8)]. We find in this case  $\frac{d\vec{\beta}^0}{dR_0} = 0$ .

and correspondingly,

$$V_X = \alpha_{xx}v_x + \alpha_{xy}v_y, \quad V_Y = \alpha_{yx}v_x + \alpha_{yy}v_y. \quad (7.20)$$

then the form (1.5) is seen to persist (see (A3.1) in Appendix 3).

We take in this case in (7.19)

$$\cos \theta = \frac{A}{A^2 + B^2}, \quad \sin \theta = \frac{B}{A^2 + B^2}, \quad (7.21)$$

and put

$$\tilde{\xi} = \frac{X - X_0}{t - t_0}, \quad \tilde{\eta} = \frac{Y - Y_0}{t - t_0}, \quad (7.22)$$

to find by (7.19)

$$\begin{cases} \tilde{\xi} = \xi \cos \theta - \eta \sin \theta = \sqrt{\frac{\gamma+1}{2(\gamma-1)}} \left[ \xi \sqrt{1-\alpha} - \eta \sqrt{\alpha - \frac{3-\alpha}{\gamma+1}} \right] \\ \tilde{\eta} = \xi \sin \theta + \eta \cos \theta = \sqrt{\frac{\gamma+1}{2(\gamma-1)}} \left[ \xi \sqrt{\alpha - \frac{3-\gamma}{\gamma+1}} - \eta \sqrt{1-\alpha} \right] \end{cases} \quad (7.23)$$

with  $\tilde{\xi} = 0$  at the points of (7.18).

Next we use (7.22), (7.23) to re-arrange (7.13) into

$$\begin{aligned} V_X &= \frac{3-\gamma}{\gamma+1} \tilde{\xi} + V_X^*, \quad V_Y = \tilde{\eta}, \quad c = \tilde{\xi} \frac{\gamma-1}{\gamma+1} \sqrt{3-\gamma} + c^* \\ V_X^* &= K \sqrt{2 \frac{\gamma-1}{\gamma+1}}, \quad V_Y^* = 0, \quad c^* = -K \sqrt{2 \frac{(\gamma-1)(3-\gamma)}{\gamma+1}}. \end{aligned} \quad (7.24)$$

At the points of (7.18) we have  $\tilde{\xi} = 0$  and we therefore obtain from (7.24)

$$c = c^*, \quad V_X = V_X^* \quad (7.25)$$

and

$$V_Y = \tilde{\eta}. \quad (7.26)$$

The details (7.25) together with the *linearly degenerated* character of the hodograph (7.17) suggest an *affinity* between solution (7.24) and an *one-dimensional* piecewise constant solution with a contact discontinuity associated to *the same* hodograph (7.17) (see Appendix 3).

We have to remark two significant aspects here. ■ The pre-image of (7.17) corresponding to solution (7.13) is  $\tilde{\xi} = 0$  [cf. (7.18)] while the pre-image of *the same* hodograph (7.17) is *different* [ $\tilde{\xi} = V_X^* \neq 0$ ] when the mentioned one-dimensional solution (Appendix 3) is used. ■ If (7.17) is associated with the two-dimensional solution (7.13) then each point of this hodograph will appear to be significant: running through the hodograph only depends on varying  $\tilde{\eta}$  in (7.26). This is in contrast with the case when (7.17) corresponds to the mentioned one-dimensional solution (Appendix 3). In this last case the passage between two points (states)  $u^*, u$  belonging to (7.17) is made by a *jump* (corresponding to a contact discontinuity), so that only the two ends of the segment of (7.17) limited by  $u^*, u$  will be significant. This indicates a *degeneracy* of the hodograph (7.17) as associated with an one-dimensional linearly degenerated index.

We have to notice in this respect that in a regular interaction with the hodograph (7.9) this hodograph *must* be associated to solution (7.13). Because, a degenerated character of this hodograph would induce a degenerated character of the interaction.

The possibility of (7.15) to reflecting a continuous (nondegenerated) solution, thus being compatible with the alternative representation (7.12), is guaranteed by the multidimensional context.

- Finally, we have to observe that the (multidimensional) representation (7.15) does not reflect an interaction of simple waves solutions.

### 7.3. Genuine nonlinearity / linear degeneracy: an ad hoc extension.

CRITERION 7.1. If *several* regular representations are possible for a given solution and among them there exist some genuinely nonlinear representations [structured by genuinely nonlinear hodograph characteristic fields] we shall select as **admissible** the genuinely nonlinear representations thus avoiding the *hybrid* representations [whose structure include linearly degenerated hodograph characteristic fields]. □

TERMINOLOGY 7.2. A hodograph characteristic hypersurface is said to be *genuinely nonlinear* if it is structured by genuinely nonlinear hodograph characteristic fields. The genuinely nonlinear character, naturally introduced in §3, as associated to a simple waves solution, is thus amplified to a natural extension of the class of simple waves solutions: the regular interactions of simple waves solutions.  $\square$

- Identifying elements of this extended class depends on completing the two anterior steps of construction described in §5 [identifying Frobenius restrictions, identifying regular integrability restrictions] with a third, final step: characterizing the nature of the structuring hodograph characteristic fields [genuinely nonlinear, hybrid, linearly degenerated]. Completing this third step would indicate, in presence of an implicit “algebraic” nondegeneracy, the genuinely nonlinear heredity of a regular interaction solution as essentially constructed from [genuinely nonlinear] simple waves solutions.
- In [5] a parallel [“differential”] form of nondegeneracy is described as distinct from the present “algebraic” one.

## 7.4. Bibliographical note

A pioneering study on the double waves solutions of the equations of steady two-dimensional potential isentropic gasdynamic flow is due to J. Giese [15]. In this study an analogue of results 7.1 is discussed. The conical solution corresponding to this analogue reflects the well-known Taylor–Maccoll flow ([28]). On this subject see also a paper of P. Germain [14]. Examples of *irregular* interactions of simple waves solutions from a numerical prospect are considered in [25], [26], [31], [32]. In these papers an essential classifying role is revealed for the *pseudo Mach number*

$$\widetilde{M} = \frac{1}{c} \sqrt{(v_x - \xi)^2 + (v_y - \eta)^2}. \quad (7.27)$$

For the example presented in section 7.1 we compute  $\widetilde{M} \equiv \sqrt{2} > 1$  and, for the example presented in section 7.2, we similarly calculate  $\widetilde{M} \equiv \text{constant} = \frac{2}{\sqrt{3-\gamma}} > 1$ . In this respect, our regular interactions, in an analytical form, correspond to two purely *pseudo supersonic* solutions.

## 8. Regular integrability and rank partition

### 8.1. Introduction

Some applications of the regular integrability are included in this paragraph to structuring and characterizing *rank partitions* of solutions in the class of regular interactions of simple waves solutions.

### 8.2. One-dimensional details. A theorem of K.O. Friedrichs

LEMMA 8.1. The boundary of a constant region contained in the domain  $\mathcal{D}$  of a continuous solution is a polygonal line whose sides are segments of characteristic straightlines.

*Argument.* For a hyperbolic system the characteristics are straightlines in a constant region. Let  $\mathcal{C}$  be a noncharacteristic arc of the boundary of a constant region. It appears that the constant solution can be continued outside the constant region, a contradiction.  $\square$

TERMINOLOGY 8.2 (Friedrichs [13]). We say that an open segment (a connected set which does not contain vertices) of the polygonal boundary of a constant region is *essentially isolated*; see Figure 12a.  $\square$

THEOREM 8.3 (Friedrichs [13]). Let  $\mathcal{D}$  be the domain of a continuous solution  $u$  and  $\mathcal{D}_0, \widetilde{\mathcal{D}}_1$  open subsets of  $\mathcal{D}$  adjacent along an open arc  $\mathcal{C}$ . Denote  $\bar{u}, \tilde{u}_1$  the restrictions of solution  $u$  to  $\mathcal{D}_0$  and respectively  $\widetilde{\mathcal{D}}_1$ .

If  $\bar{u}$  is constant,  $\tilde{u}_1$  is smooth and nonconstant and  $\mathcal{C}$  is essentially isolated then there exists a region  $\mathcal{D}_1 \subset \widetilde{\mathcal{D}}_1$  adjacent to  $\mathcal{D}_0$  along  $\mathcal{C}$  so that restriction  $u_1$  of  $u$  to  $\mathcal{D}_1$  is a simple waves solution ; see Figure 12b.

*Proof.* Here is a significant application of the Riemann–Lax approach (section 4.6). Let  $\bar{u} = u(\mathcal{D}_0)$  and let  $i$  be the index of the characteristic  $\mathcal{C}$ . According to Corollary 4.10(ii) we have in a neighbourhood  $U$

of  $\bar{u}$  in  $H$

$$L(u) = \sum_{j=1}^{n-1} \theta_{kj}(u) \operatorname{grad}_u \varphi_j^i(u), \quad 1 \leq k \leq n, \quad k \neq i. \quad (8.1)$$

By (1.2) and (8.1) we obtain in  $\tilde{\mathcal{D}}_1$  :

$$\begin{aligned} 0 &= L(u) \left[ \frac{\partial u}{\partial t} + a(u) \frac{\partial u}{\partial x} \right] = L(u) \left[ \frac{\partial u}{\partial t} + \lambda_k(u) \frac{\partial u}{\partial x} \right] \\ &= \sum_{j=1}^{n-1} \theta_{kj}(u) \left[ \frac{\partial \varphi_j^i(u)}{\partial t} + \lambda_k(u) \frac{\partial \varphi_j^i(u)}{\partial x} \right], \quad 1 \leq k \leq n \quad k \neq i. \end{aligned} \quad (8.2)$$

We put

$$\vec{\varphi} = \left[ \underbrace{\varphi_1^i(u), \dots, \varphi_k^i(u), \dots, \varphi_n^i(u)}_{n-1 \text{ components}} \right], \quad k \neq i$$

and transcribe (8.2) by

$$\Theta(u) \frac{\partial \vec{\varphi}(u)}{\partial t} + \Lambda(u) \Theta(u) \frac{\partial \vec{\varphi}(u)}{\partial x} = 0,$$

a system of  $n - 1$  equations which can be re-arranged finally into the form

$$\frac{\partial \vec{\varphi}(u)}{\partial t} + \tilde{\Lambda}(u) \frac{\partial \vec{\varphi}(u)}{\partial x} = 0, \quad \tilde{\Lambda} = \Theta^{-1} \Lambda \Theta. \quad (8.3)$$

We notice therefore that the matrix  $\tilde{\Lambda}(u)$  has the eigenvalues  $\lambda_j(u)$ ,  $j \neq i$ .

Let us consider in  $\tilde{\mathcal{D}}_1$  the Cauchy problem which consists in the linear hyperbolic system (associated to solution  $u$ )

$$\frac{\partial \vec{\Psi}(x, t)}{\partial t} + \tilde{\Lambda}[u(x, t)] \frac{\partial \vec{\Psi}(x, t)}{\partial x} = 0, \quad \dim \vec{\Psi} = n - 1 \quad (8.4)$$

and the data

$$\vec{\Psi}(x, t) = \text{constant} = \vec{\varphi}(\bar{u}), \quad 1 \leq k \leq n - 1, \quad \text{along } \mathcal{C}. \quad (8.5)$$

The characteristics of (8.4) have the slopes  $\lambda_j(u)$ ,  $j \neq i$  so that the straightline  $\mathcal{C}$  is not a characteristic of (8.4). Then the problem (8.4), (8.5) has an unique solution in  $\mathcal{D}_1 \subset \tilde{\mathcal{D}}_1$ . This solution can be continuously continued through  $\mathcal{C}$  with an identically constant contribution in  $\mathcal{D}_0$

$$\vec{\Psi}(x, t) \equiv \text{constant} = \vec{\varphi}(\bar{u}). \quad (8.6)$$

On the other hand, we have in  $\mathcal{D}_1$  [cf. (8.3)]  $\vec{\Psi}(x, t) \equiv \vec{\varphi}[u(x, t)]$ . We finally use Remark 4.14.  $\square$

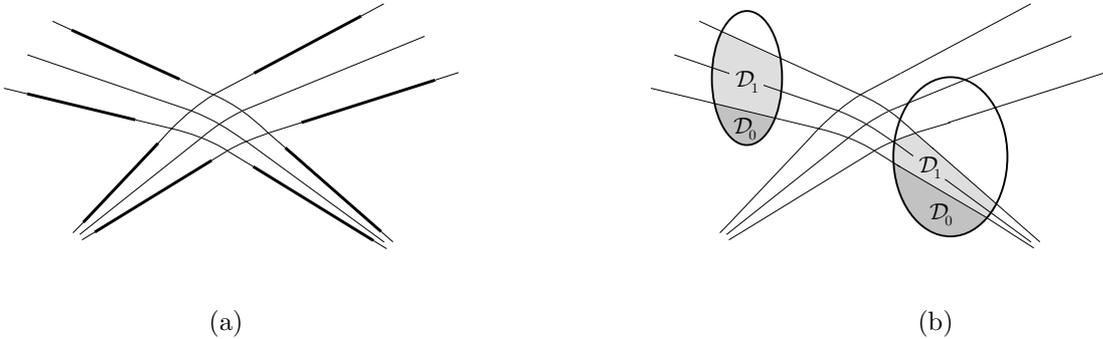


FIGURE 12

COROLLARY 8.4 (Friedrichs [13]). Let  $\mathcal{D}_1 \subset \mathcal{D}$  be a simple waves region. Region  $\mathcal{D}_1$  cannot be adjacent in  $\mathcal{D}$  along an essentially isolated segment of a characteristic straightline but to a constant region or a simple waves region of the same index.  $\square$

### 8.3. One-dimensional regular rank partition

Corollary 8.4 shows how can we characterize the rank of a smooth solution in its domain  $\mathcal{D}$  as this domain contains a constant region  $\mathcal{D}_0$ . So, let, in a **rank partition**,  $\mathcal{D}_j$  be a rank  $j$  region ( $j = 0, 1, 2$ ). We have to assume that the region  $\mathcal{D}_2$  corresponds to a *regular* interaction. Corollary 8.4 and the regular nature of  $\mathcal{D}_2$  imply that  $\mathcal{D}_1$  must be a simple waves region and that  $\mathcal{D}_2$  and  $\mathcal{D}_0$  cannot have in common but isolated points.

A typical example of Friedrichs rank partition is depicted in Figure 13 (we ignore the change of the simple waves index in the regions  $\mathcal{D}_1$  of this figure).

A natural context for which the rank partition description can be extended is presented in section 8.4.

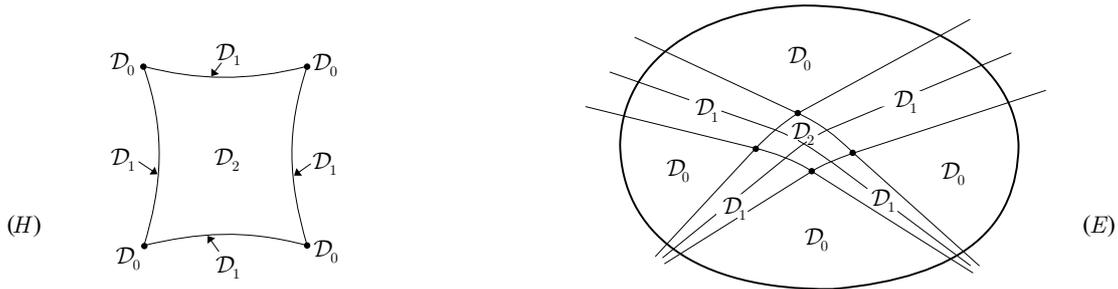


FIGURE 13

### 8.4. Multidimensional regular rank partition

PROPOSITION 8.5 (Peradzyński [19]). If a regular interaction of  $p$  simple waves solutions can be constructed [corresponding to a system of  $p$  Riemann–Burnat invariants  $\{R_j(x)\}_{1 \leq j \leq n}$ ] then regular interactions of  $s$  simple waves solutions,  $s < p$ , can be obtained when the values of  $p - s$  among the mentioned  $p$  invariants are fixed.

*Proof* ([19]). Cf. the hierarchy coherence of the Cartan involution restrictions (5.8).  $\square$

- Proposition 8.5 indicates a natural way of extending, in absence of a Riemann restricted context, the hierarchies of Riemann structures mentioned in sections 4.2–4.4.
- Representation (5.2) corresponds to a particular type [“algebraic”, with a characteristic hodograph] of solutions with a constant rank.
- Proposition 8.5 shows how can we put together, in presence of a regular integrability, solutions of constant rank – in order to build more ample structures. Such kind of structures will be called **regular rank partitions**. Examples of such structures, made of simple waves solutions and/or regular interactions of simple waves solutions have been already presented in sections 4.4.2, 7.1, 7.2; cf. Figure 5a and Figure 9. In [5] an example is included of “nonalgebraic” regular rank partition.

■ An interaction of one-dimensional simple waves solutions implies *four* simple waves solutions [divided into *incident* and *resultant*; section 4.4.2, Figure 5a; due to an even codimension]. ■ An interaction of two-dimensional simple waves solutions implies a *simultaneous* contribution of *four* interacting simple waves solutions [due to an odd codimension (sections 7.1, 7.2, Figure 9); in contrast with the previous case]. ■ A multidimensional approach has essentially a *local* character. ■ The ranks of two adjacent regions of constant rank are equal or differ by unity. This is a consequence of Proposition 8.5; cf. the constructive details of the adjacency. ■ A **new fact** concerning the *multidimensional* regular interactions is that, in the physical space  $E$ , an interaction region is *not necessarily* limited by adjacent simple waves solutions [as it happens in the one-dimensional case]. Examples could be produced, cf. Proposition 8.5, of such regions which are limited by regions of regular interaction of an inferior (by unity) rank.

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## Appendix

### 1. Anisentropic (strictly adiabatic) gasdynamic flow

An *anisentropic* (strictly adiabatic) gasdynamic flow results behind a shock discontinuity of nonconstant velocity which penetrates into a region of uniform flow. Such a flow is described, in usual notations, by the system of conservation laws

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} = 0 \\ \frac{\partial(\rho v_x)}{\partial t} + \frac{\partial}{\partial x}(\rho v_x^2 + p) = 0 \\ \frac{\partial(\rho S)}{\partial t} + \frac{\partial(\rho v_x S)}{\partial x} = 0, \quad S = S(p, \rho) \end{cases} \quad (A1.1)$$

where the entropy density  $S$  is constant along each particle line with the constant value depending on the particle line, cf. the evident transcription of (A1.1)<sub>3</sub>

$$\frac{\partial S}{\partial t} + v_x \frac{\partial S}{\partial x} = 0, \quad S = S(p, \rho). \quad (A1.2)$$

The particle function  $\psi$  introduced by (A1.1)<sub>1</sub> cf.

$$\rho = \frac{\partial \psi}{\partial x}, \quad \rho v_x = -\frac{\partial \psi}{\partial t}$$

fulfils (A1.2)

$$\frac{\partial \psi}{\partial t} + v_x \frac{\partial \psi}{\partial x} = 0 \quad (A1.3)$$

– which indicates a dependence  $S = f(\psi)$  [we assume that  $f$  is smooth and nonconstant]. This dependence will be put together with the structure  $S = S(p, \rho)$  of the entropy density into a relation

$$S(p, \rho) = f(\psi) \quad (A1.4)$$

which is used finally to express

$$\rho = \rho(p, \psi). \quad (A1.5)$$

We transcribe (A1.2) by

$$\frac{\partial S}{\partial p} \left( \frac{\partial p}{\partial t} + v_x \frac{\partial p}{\partial x} \right) + \frac{\partial S}{\partial \rho} \left( \frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho}{\partial x} \right) = 0 \quad \text{for each given } \psi \quad (A1.6)$$

and identify

$$c^2 = -\frac{\frac{\partial S}{\partial \rho}}{\frac{\partial S}{\partial p}} = c^2(p, \psi) = \left[ \left( \frac{\partial \rho}{\partial p} \right)_\psi \right]^{-1} \quad (A1.7)$$

as an ad hoc anisentropic sound speed to give (A1.6) the form

$$\left( \frac{\partial p}{\partial t} + v_x \frac{\partial p}{\partial x} \right) - c^2(p, \psi) \left( \frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho}{\partial x} \right) = 0 \quad \text{for each given } \psi. \quad (A1.8)$$

Finally we use (A1.8) and (A1.3) to put (A1.1) into the form

$$\begin{cases} \frac{\partial p}{\partial t} + v_x \frac{\partial p}{\partial x} + \rho(p, \psi) c^2(p, \psi) \frac{\partial v_x}{\partial x} = 0 \\ \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + \frac{1}{\rho(p, \psi)} \frac{\partial p}{\partial x} = 0 \\ \frac{\partial \psi}{\partial t} + v_x \frac{\partial \psi}{\partial x} = 0. \end{cases} \quad (A1.9)$$

In the *isentropic* case equations (A1.2) and (A1.3) [i.e. (A1.9)<sub>3</sub>] are identically satisfied so that (A1.9)<sub>1,2</sub> appears to make a coherent system of two equations ( $n = 2$ ). Because, in this case  $\rho$  in (A1.5) and  $c$  in (A1.7) do not depend on  $\psi$ .

## 2. Details concerning the class mentioned above (§6)

From (6.2)<sub>2,3</sub> we obtain cf. (6.3)

$$\frac{\partial v_x}{\partial \xi} + \frac{\partial v_y}{\partial \eta} = \mathbf{a} + \bar{\mathbf{b}} \quad (\text{A2.1})$$

$$\begin{cases} -\frac{\partial c^2}{\partial \xi} = (\gamma - 1)\mathbf{a}[(\mathbf{a} - 1)\xi + \mathbf{b}\eta + \mathbf{c}] + (\gamma - 1)\bar{\mathbf{b}}[\bar{\mathbf{a}}\xi + (\bar{\mathbf{b}} - 1)\eta + \bar{\mathbf{c}}] \\ -\frac{\partial c^2}{\partial \eta} = (\gamma - 1)\bar{\mathbf{a}}[(\mathbf{a} - 1)\xi + \mathbf{b}\eta + \mathbf{c}] + (\gamma - 1)\bar{\mathbf{b}}[\bar{\mathbf{a}}\xi + (\bar{\mathbf{b}} - 1)\eta + \bar{\mathbf{c}}] \end{cases} \quad (2)$$

The requirement  $\frac{\partial c^2}{\partial \xi \partial \eta} = \frac{\partial c^2}{\partial \eta \partial \xi}$  takes, cf. (A2.2), the form

$$(\mathbf{b} - \bar{\mathbf{a}})(\mathbf{a} + \bar{\mathbf{b}} - 1) = 0. \quad (\text{A2.3})$$

Now, the expression of  $c^2$  could be calculated in two ways. On one hand, an expression of  $c^2$  results from (A2.2) and (A2.3). On the other hand, an expression of  $c^2$  results from (6.2)<sub>1</sub>, (A2.1) and (A2.2). Since the two expressions obtained for  $c^2$  are identical we get, by identifying the coefficients of  $\xi^2, \xi\eta, \eta^2, \xi, \eta$  respectively:

$$\frac{1}{2}[\mathbf{a}(\mathbf{a} - 1) + \mathbf{b}\bar{\mathbf{a}}][(\gamma + 1)\mathbf{a} + (\gamma - 1)\bar{\mathbf{b}} - 2] + \bar{\mathbf{a}}^2(\mathbf{a} + \bar{\mathbf{b}} - 1) = 0 \quad (\text{A2.4})$$

$$2\mathbf{a}\mathbf{b}(\mathbf{a} - 1) + (\mathbf{b} + \bar{\mathbf{a}})(\mathbf{a} - 1)(\bar{\mathbf{b}} - 1) + \mathbf{b}\bar{\mathbf{a}}(\mathbf{b} + \bar{\mathbf{a}}) + 2\bar{\mathbf{a}}\bar{\mathbf{b}}(\bar{\mathbf{b}} - 1) + (\gamma - 1)\mathbf{b}(\mathbf{a} + \bar{\mathbf{b}})(\mathbf{a} + \bar{\mathbf{b}} - 1) = 0 \quad (\text{A2.5})$$

$$\frac{1}{2}[\bar{\mathbf{b}}(\bar{\mathbf{b}} - 1) + \bar{\mathbf{a}}\bar{\mathbf{b}}][(\gamma - 1)\mathbf{a} + (\gamma + 1)\bar{\mathbf{b}} - 2] + \mathbf{b}^2(\mathbf{a} + \bar{\mathbf{b}} - 1) = 0 \quad (\text{A2.6})$$

$$2\mathbf{a}\bar{\mathbf{c}}(\mathbf{a} - 1) + \mathbf{b}\bar{\mathbf{c}}(\mathbf{a} - 1) + \mathbf{b}\bar{\mathbf{c}}\bar{\mathbf{a}} + \bar{\mathbf{c}}\bar{\mathbf{a}}^2 + \bar{\mathbf{a}}\bar{\mathbf{c}}(\bar{\mathbf{a}} - 1) + 2\bar{\mathbf{a}}\bar{\mathbf{b}}\bar{\mathbf{c}} + (\gamma - 1)(\mathbf{a} + \bar{\mathbf{b}})(\mathbf{a}\bar{\mathbf{c}} + \bar{\mathbf{b}}\bar{\mathbf{c}}) = 0 \quad (\text{A2.7})$$

$$2\bar{\mathbf{b}}\bar{\mathbf{c}}(\bar{\mathbf{b}} - 1) + \bar{\mathbf{c}}\bar{\mathbf{a}}(\bar{\mathbf{b}} - 1) + \mathbf{b}\bar{\mathbf{a}}\bar{\mathbf{c}} + \mathbf{b}^2\bar{\mathbf{c}} + \mathbf{b}\bar{\mathbf{c}}(\bar{\mathbf{b}} - 1) + 2\mathbf{a}\bar{\mathbf{b}}\bar{\mathbf{c}} + (\gamma - 1)(\mathbf{a} + \bar{\mathbf{b}})(\bar{\mathbf{c}}\bar{\mathbf{a}} + \bar{\mathbf{b}}\bar{\mathbf{c}}) = 0. \quad (\text{A2.8})$$

Therefore we have a nonlinear algebraic system (A2.3)–(A2.8) with six equations for the six coefficients  $\mathbf{a}, \bar{\mathbf{a}}, \mathbf{b}, \bar{\mathbf{b}}, \mathbf{c}, \bar{\mathbf{c}}$  in (6.3). We begin by presenting an exhaustive list of solutions for the system (A2.3)–(A2.8).

The requirement (A2.3) suggests the importance of two cases.

CASE 1. This case takes into account the circumstance

$$\mathbf{b} - \bar{\mathbf{a}} = 0 \quad (\text{A2.9})$$

in (A2.3). From (A2.4)–(A2.6) and (A2.9) we obtain the following system for  $\mathbf{a}, \bar{\mathbf{b}}, \bar{\mathbf{a}}$ :

$$\begin{cases} \mathbf{b}^2\{2[2(\mathbf{a} - 1) + \bar{\mathbf{b}}] + (\gamma - 1)(\mathbf{a} + \bar{\mathbf{b}})\} + \bar{\mathbf{a}}(\mathbf{a} - 1)[2(\mathbf{a} - 1) + (\gamma - 1)(\mathbf{a} + \bar{\mathbf{b}})] = 0 \\ \mathbf{b}\{2\mathbf{b}^2 + [2\mathbf{a}(\mathbf{a} - 1) + 2(\mathbf{a} - 1)(\bar{\mathbf{b}} - 1) + 2\bar{\mathbf{b}}(\bar{\mathbf{b}} - 1) + (\gamma - 1)(\mathbf{a} + \bar{\mathbf{b}})(\mathbf{a} + \bar{\mathbf{b}} - 1)]\} = 0 \\ \mathbf{b}^2\{2[2(\bar{\mathbf{b}} - 1) + \bar{\mathbf{a}}] + (\gamma - 1)(\mathbf{a} + \bar{\mathbf{b}})\} + 2\bar{\mathbf{b}}(\bar{\mathbf{b}} - 1)[2(\bar{\mathbf{b}} - 1) + (\gamma - 1)(\mathbf{a} + \bar{\mathbf{b}})] = 0. \end{cases} \quad (\text{A2.10})$$

Next, we have to distinguish cf. (A2.10)<sub>2</sub> between the possibilities  $\mathbf{b} = 0$  or  $\mathbf{b} \neq 0$ .

We begin our analysis with the subcase  $\mathbf{b} = 0$ . In this subcase, from (A2.10)<sub>1,3</sub> we obtain for  $\mathbf{a}, \bar{\mathbf{b}}$  the system

$$\begin{cases} \mathbf{a}(\mathbf{a} - 1)[(\gamma + 1)\mathbf{a} + (\gamma - 1)\bar{\mathbf{b}} - 2] = 0 \\ \bar{\mathbf{b}}(\bar{\mathbf{b}} - 1)[(\gamma - 1)\mathbf{a} + (\gamma + 1)\bar{\mathbf{b}} - 2] = 0. \end{cases} \quad (\text{A2.11})$$

Therefore we get the following exhaustive list of solutions of (A2.10) corresponding to the mentioned subcase [we complete this list with the information concerning  $\bar{\mathbf{a}}, \mathbf{c}, \bar{\mathbf{c}}$ ; cf. (A2.7), (A2.8), (A2.9)]

$$\mathbf{a} = 0, \quad \mathbf{b} = 0, \quad \bar{\mathbf{b}} = 0, \quad \bar{\mathbf{a}} = \mathbf{b}, \quad \text{arbitrary } \mathbf{c}, \bar{\mathbf{c}} \quad (\text{A2.12})$$

$$\mathbf{a} = 0, \quad \mathbf{b} = 0, \quad \bar{\mathbf{b}} = 1, \quad \bar{\mathbf{a}} = \mathbf{b}, \quad \text{arbitrary } \mathbf{c}; \bar{\mathbf{c}} = 0 \quad (\text{A2.13})$$

$$\mathbf{a} = 0, \quad \mathbf{b} = 0, \quad \bar{\mathbf{b}} = \frac{2}{\gamma + 1}, \quad \bar{\mathbf{a}} = \mathbf{b}, \quad \text{arbitrary } \mathbf{c}, \bar{\mathbf{c}} \quad (\text{A2.14})$$

$$\mathbf{a} = 1, \quad \mathbf{b} = 0, \quad \bar{\mathbf{b}} = 0, \quad \bar{\mathbf{a}} = \mathbf{b}, \quad \mathbf{c} = 0, \text{ arbitrary } \bar{\mathbf{c}} \quad (\text{A2.15})$$

$$\mathbf{a} = 1, \quad \mathbf{b} = 0, \quad \bar{\mathbf{b}} = 1, \quad \bar{\mathbf{a}} = \mathbf{b}, \quad \mathbf{c} = 0, \bar{\mathbf{c}} = 0 \quad (\text{A2.16})$$

$$\mathbf{a} = 1, \quad \mathbf{b} = 0, \quad \bar{\mathbf{b}} = \frac{3-\gamma}{\gamma+1}, \quad \bar{\mathbf{a}} = \mathbf{b}, \quad \mathbf{c} = 0, \text{ arbitrary } \bar{\mathbf{c}} \quad (\text{A2.17})$$

$$\mathbf{a} = \frac{2}{\gamma+1}, \quad \mathbf{b} = 0, \quad \bar{\mathbf{b}} = 0, \quad \bar{\mathbf{a}} = \mathbf{b}, \quad \text{arbitrary } \mathbf{c}, \bar{\mathbf{c}} \quad (\text{A2.18})$$

$$\mathbf{a} = \frac{3-\gamma}{\gamma+1}, \quad \mathbf{b} = 0, \quad \bar{\mathbf{b}} = 1, \quad \bar{\mathbf{a}} = \mathbf{b}, \quad \text{arbitrary } \mathbf{c}; \bar{\mathbf{c}} = 0 \quad (\text{A2.19})$$

$$\mathbf{a} = \frac{1}{\gamma}, \quad \mathbf{b} = 0, \quad \bar{\mathbf{b}} = \frac{1}{\gamma}, \quad \bar{\mathbf{a}} = \mathbf{b}, \quad \text{arbitrary } \mathbf{c}, \bar{\mathbf{c}}. \quad (\text{A2.20})$$

We extend our analysis by considering the subcase  $\mathbf{b} \neq 0$ . In this subcase we use (A2.10)<sub>2</sub> in order to eliminate  $\mathbf{b}^2$  from (A2.10)<sub>1,3</sub>. We notice that the equations (A2.10)<sub>1,3</sub> are not distinct in this subcase. In fact, we denote

$$X = \mathbf{a} - 1, \quad Y = \bar{\mathbf{b}} - 1, \quad Z = X + Y = \mathbf{a} + \bar{\mathbf{b}} - 2 \quad (\text{A2.21})$$

and obtain the following common form of equations (A2.10)<sub>1,3</sub>

$$(\gamma+1)^2 Z^3 + (\gamma+1)(5\gamma-1)Z^2 + 2(4\gamma^2 - \gamma - 1)Z + 4\gamma(\gamma-1) = 0 \quad (\text{A2.22})$$

with the roots

$$Z_1 = -\frac{2\gamma}{\gamma+1}, \quad Z_2 = -1, \quad Z_3 = -\frac{2(\gamma-1)}{\gamma+1}. \quad (\text{A2.23})$$

Finally we put (A2.23) in the form

$$\mathbf{a} + \bar{\mathbf{b}} = \frac{2}{\gamma+1} \quad [\text{cf. (A2.23)}_1] \quad (\text{A2.24})$$

$$\mathbf{a} + \bar{\mathbf{b}} = 1 \quad [\text{cf. (A2.23)}_2] \quad (\text{A2.25})$$

$$\mathbf{a} + \bar{\mathbf{b}} = \frac{4}{\gamma+1} \quad [\text{cf. (A2.23)}_3]. \quad (\text{A2.26})$$

For (A2.25) we obtain cf. (A2.10)<sub>2</sub>

$$\mathbf{b}^2 = \mathbf{a}(1-\mathbf{a})$$

and therefore

$$0 \leq \mathbf{a} \leq 1 \quad \text{and} \quad \mathbf{b} = \pm\sqrt{\mathbf{a}(1-\mathbf{a})}. \quad (\text{A2.27})$$

Similarly, we get, cf. (A2.10)<sub>2</sub>,

$$\mathbf{b}^2 = \mathbf{a} \left( \frac{2}{\gamma+1} - \mathbf{a} \right)$$

hence

$$0 \leq \mathbf{a} \leq \frac{2}{\gamma+1} \quad \text{and} \quad \mathbf{b} = \pm\sqrt{\mathbf{a} \left( \frac{2}{\gamma+1} - \mathbf{a} \right)} \quad (\text{A2.28})$$

for (A2.24), and

$$\mathbf{b}^2 = \left( \mathbf{a} - \frac{3-\gamma}{\gamma+1} \right) (1-\mathbf{a})$$

or, equivalently,

$$\frac{3-\gamma}{\gamma+1} \leq \mathbf{a} \leq 1 \quad \text{and} \quad \mathbf{b} = \pm\sqrt{\left( \mathbf{a} - \frac{3-\gamma}{\gamma+1} \right) (1-\mathbf{a})} \quad (\text{A2.29})$$

for (A2.26).

Consequently, we complete the list (A2.12)–(A2.20) which corresponds, for  $\mathbf{b} = 0$ , to the case (A2.9) with the following circumstances [which take into account (A2.7), (A2.8) and (A2.24)–(A2.26)]

$$0 < \mathbf{a} < 1, \quad \mathbf{b} = \pm\sqrt{\mathbf{a}(1-\mathbf{a})}, \quad \bar{\mathbf{a}} = \mathbf{b}, \quad \bar{\mathbf{b}} = 1-\mathbf{a}, \quad \text{arbitrary } \mathbf{c}; \quad \bar{\mathbf{c}} = \mp\mathbf{c}\sqrt{\frac{\mathbf{a}}{1-\mathbf{a}}} \quad (\text{A2.30})$$

$$0 < \mathbf{a} < \frac{2}{\gamma+1}, \quad \mathbf{b} = \pm\sqrt{\mathbf{a} \left( \frac{2}{\gamma+1} - \mathbf{a} \right)}, \quad \bar{\mathbf{a}} = \mathbf{b}, \quad \bar{\mathbf{b}} = \frac{2}{\gamma+1} - \mathbf{a}; \quad \text{arbitrary } \mathbf{c}, \bar{\mathbf{c}} \quad (\text{A2.31})$$

$$\frac{3-\gamma}{\gamma+1} < \mathbf{a} < 1, \quad \mathbf{b} = \pm \sqrt{\left(\mathbf{a} - \frac{3-\gamma}{\gamma+1}\right)(1-\mathbf{a})}, \quad \bar{\mathbf{a}} = \mathbf{b}, \quad \bar{\mathbf{b}} = \frac{4}{\gamma+1} - \mathbf{a}, \quad \text{arbitrary } \mathbf{c}; \quad \bar{\mathbf{c}} = \mp \mathbf{c} \sqrt{\frac{\mathbf{a} - \frac{3-\gamma}{\gamma+1}}{1-\mathbf{a}}}. \quad (\text{A2.32})$$

□

CASE 2. This case considers in (A2.3) the circumstance

$$\mathbf{a} + \bar{\mathbf{b}} - 1 = 0. \quad (\text{A2.33})$$

We use (A2.33) in order to give to (A2.4)–(A2.6) the form

$$\begin{cases} [2(\mathbf{a}-1) + (\gamma-1)][\mathbf{b}\bar{\mathbf{a}} + \mathbf{a}(\mathbf{a}-1)] = 0 \\ (\mathbf{b} + \bar{\mathbf{a}})[\mathbf{b}\bar{\mathbf{a}} + \mathbf{a}(\mathbf{a}-1)] = 0 \\ [2\mathbf{a} - (\gamma-1)][\mathbf{b}\bar{\mathbf{a}} + \mathbf{a}(\mathbf{a}-1)] = 0 \end{cases} \quad (\text{A2.34})$$

of a system for  $\mathbf{a}, \mathbf{b}, \bar{\mathbf{a}}$ .

A single relation results from (A2.34) for  $\mathbf{a}, \mathbf{b}, \bar{\mathbf{a}}$ :

$$\mathbf{b}\bar{\mathbf{a}} + \mathbf{a}(\mathbf{a}-1) = 0. \quad (\text{A2.35})$$

Now, the circumstance (A2.33) could be completely described by the following list of possibilities [which also considers the contribution of equations (A2.7), (A2.8) for  $\mathbf{c}, \bar{\mathbf{c}}$ ]:

$$\mathbf{a} = 0, \quad \mathbf{b} = 0; \quad \text{arbitrary } \bar{\mathbf{a}}; \quad \bar{\mathbf{b}} = 1; \quad \text{arbitrary } \mathbf{c}; \quad \bar{\mathbf{c}} = -\bar{\mathbf{a}}\mathbf{c} \quad (\text{A2.36})$$

$$\mathbf{a} = 1, \quad \mathbf{b} = 0; \quad \text{arbitrary } \bar{\mathbf{a}}; \quad \bar{\mathbf{b}} = 0; \quad \mathbf{c} = 0, \quad \text{arbitrary } \bar{\mathbf{c}} \quad (\text{A2.37})$$

$$\text{arbitrary } \mathbf{a}; \quad \text{arbitrary } \mathbf{b} \neq 0; \quad \bar{\mathbf{a}} = \frac{\mathbf{a}(\mathbf{a}-1)}{\mathbf{b}}; \quad \bar{\mathbf{b}} = 1 - \mathbf{a}; \quad \text{arbitrary } \mathbf{c}; \quad \bar{\mathbf{c}} = -\frac{\mathbf{a}}{\mathbf{b}}\mathbf{c}. \quad (\text{A2.38})$$

□

We notice that (A2.12)–(A2.20), (A2.30)–(A2.32), (A2.36)–(A2.38) represents an *exhaustive* list of possibilities. An exhaustive list of local solutions of the form (6.3) for the system (6.2) of the isentropic gas dynamics results from the above mentioned list: see section 6.2.

### 3. Details corresponding to section 7.2.2

• In the system (1.5)  $v_x, v_y$  denote, at each point  $(x, y) \in \mathbb{R}^2$ , the projections of the velocity vector on the axes  $x, y$  respectively. If the axes  $x, y$  are changed by the axes  $X, Y$  and  $V_X, V_Y$  denote, respectively, at each point  $(X, Y) \in \mathbb{R}^2$ , the projections of the velocity vector on these new axes then the form (1.5) is seen to persist cf.

$$\begin{cases} \frac{\partial c}{\partial t} + V_X \frac{\partial c}{\partial X} + V_Y \frac{\partial c}{\partial Y} + \frac{\gamma-1}{2} c \left( \frac{\partial V_X}{\partial X} + \frac{\partial V_Y}{\partial Y} \right) = 0 \\ \frac{\partial V_X}{\partial t} + V_X \frac{\partial V_X}{\partial X} + V_Y \frac{\partial V_X}{\partial Y} + \frac{2}{\gamma-1} c \frac{\partial c}{\partial X} = 0 \\ \frac{\partial V_Y}{\partial t} + V_X \frac{\partial V_Y}{\partial X} + V_Y \frac{\partial V_Y}{\partial Y} + \frac{2}{\gamma-1} c \frac{\partial c}{\partial Y} = 0. \end{cases} \quad (\text{A3.1})$$

In particular, (A3.1) results by considering in (1.5) the change of variables

$$\begin{aligned} X &= \alpha_{xx}x + \alpha_{xy}y, & Y &= \alpha_{yx}x + \alpha_{yy}y \\ \alpha_{xx} &= \cos \theta = \alpha_{yy}, & \alpha_{xy} &= \sin \theta = -\alpha_{yx}; & -\pi < \theta < \pi \end{aligned} \quad (\text{A3.2})$$

and correspondingly,

$$V_X = \alpha_{xx}v_x + \alpha_{xy}v_y, \quad V_Y = \alpha_{yx}v_x + \alpha_{yy}v_y. \quad (\text{A3.3})$$

The restriction  $\frac{\partial}{\partial Y} \equiv 0$  in (A3.1) leads to a system of the form (1.2) for which we get [independent variables:  $t, X$ ]

$$\begin{aligned} \vec{\beta}_{tX}^{(\sigma)} &= \Theta_{tX}^{(\sigma)}[-\lambda_{tX}^{(\sigma)}, 1, 0] \leftrightarrow \vec{\kappa}_{tX}^{(\sigma)} = R_{tX}^{(\sigma)} = \left[ \varepsilon \frac{\gamma-1}{2}, |\varepsilon|, 1 - |\varepsilon| \right]; & \left( \begin{array}{l} \varepsilon = -1, 0, 1 \\ \sigma = \text{sign } \varepsilon = -, 0, + \end{array} \right). \end{aligned} \quad (\text{A3.4})$$

$$\lambda_{tX}^{(\sigma)} = V_X + \varepsilon c$$

Next, we use (A3.2), (A3.3) to obtain for the vectors (A3.4) representations corresponding respectively to the frames  $(t, x, y)$  [for  $\vec{\beta}$ ] and  $(c - c^*, v_x - v_x^*, v_y - v_y^*)$  [for  $\vec{\kappa}$ ]. These representations have the forms

$$\vec{\beta}^{(0)} = \Theta^{(0)}[-(v_x \cos \theta + v_y \sin \theta), \cos \theta, \sin \theta] \leftrightarrow \vec{\kappa}^{(0)} = (0, -\sin \theta, \cos \theta), \quad (\text{A3.5})$$

and

$$\vec{\beta}^{(\pm)} = \Theta^{(\pm)}[-(v_x \cos \theta + v_y \sin \theta) \mp c, \cos \theta, \sin \theta] \leftrightarrow \vec{\kappa}^{(\pm)} = \left( \pm \frac{\gamma - 1}{2}, \cos \theta, \sin \theta \right). \quad (\text{A3.6})$$

We finally notice that the vectors  $\vec{\kappa}^{(\pm)}$  are laid in the plane

$$V_Y - V_Y^* \equiv -(v_x - v_x^*) \sin \theta + (v_y - v_y^*) \cos \theta = 0$$

through the point  $u^* \in H$  and the vector  $\vec{\kappa}^{(0)}$  is placed along the axis  $V_Y$  through  $u^*$  [we have to compare  $\vec{\kappa}^{(0)}$  of (A3.5), (7.21), with  $\vec{\kappa}$  of (7.16)].

• In the *one-dimensional* case mentioned by (A3.4) [independent variables:  $t, X$ ] the linearly degenerated field [of index 0] is associated to the eigenvalue  $\lambda = V_X$ . We notice (see for example [17]) that a piecewise constant solution with a *contact discontinuity* could be associated to this linearly degenerated eigenvalue with the properties:

$$c = c^*, \quad V_X = V_X^*$$

[see (7.25)], and, along the characteristic line  $\tilde{\xi} = V_X^* \neq 0$ <sup>1</sup> associated [in the physical plane  $t, X$ ] to a linearly degenerated index, a *jump* in  $V_Y$  is allowed:

$$[[V_Y]] = V_Y - V_Y^* \neq 0.$$

In the physical spaces  $t, X, Y$  or  $t, x, y$ ,  $\tilde{\xi} = V_X^* \neq 0$  represents a plane, distinct from (7.18) and parallel with axis  $Y$  [cf.  $\frac{\partial}{\partial Y} \equiv 0$ ].

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<sup>1</sup> We have  $\frac{dX}{dt} = \lambda = V_X = V_X^*$ . This leads to  $d[X - V_X^* t] = 0$  which could be transcribed as  $X - X_0 = V_X^*(t - t_0)$  or  $\tilde{\xi} = V_X^* \neq 0$ .

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