SPECTRA OF POSITIVE MATRICES AND THE MARKOV GROUP CONJECTURE

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1 Markov semigroups and groups

The survey paper [6] contained a reminder that the 1965 conjecture by D.G. Kendall [4], usually called the *Markov group conjecture* (MGC), remains unresolved, despite important contributions by Speakman [9], Williams [10] and Mountford [7]. The purpose of this paper is to relate the incomplete results around the MGC to other open questions in the theory of positive [1] matrices and that of power series with positive coefficients.

In the classical theory [1] of Markov processes in continuous time with a countable state space and stationary transition probabilities, a *Markov semigroup* is a family of functions

$$p_{ij} \ : \ (0, \infty) \longrightarrow [0, 1] \tag{1.1}$$

indexed by i and j running over some finite or countably infinite 'state space' S, and satisfying

$$\sum_{j \in S} p_{ij}(t) \leqslant 1 \qquad (i \in S)\,, \tag{1.2}$$

$$\lim_{t \downarrow 0} p_{ij}(t) = \delta_{ij} \qquad (i, j \in S)\,, \tag{1.3}$$

and the Chapman-Kolmogorov equation

$$p_{ij}(t + u) = \sum_{k \in S} p_{ik}(t) p_{kj}(u) \tag{1.4}$$

for $t, u > 0$, $i, j \in S$.

Such a family determines operators

$$P_t : l_1 \longrightarrow l_1 \qquad (t > 0) \tag{1.5}$$

[1] Throughout this paper words like 'positive' and 'increasing' are used in the weak sense; x is positive if $x \geqslant 0$.

on the space $l_1 = l_1(S)$ of real-valued sequences

$$x = (x_i\,; i \in S)$$

with

$$\|x\| = \sum_{i\in S} |x_i| < \infty \tag{1.6}$$

by the recipe

$$(xP_t)_i = \sum_{i\in S} x_i p_{ij}(t)\,. \tag{1.7}$$

Condition (1.2) shows that the operator norm of P_t satisfies

$$\|P_t\| = \sup_{i\in S} \sum_{j\in S} |p_{ij}(t)| \leqslant 1\,, \tag{1.8}$$

while (1.4) is the semigroup property

$$P_{t+u} = P_t P_u \qquad (t, u > 0)\,,. \tag{1.9}$$

Equation (1.3) is equivalent to

$$\lim_{t\downarrow 0} \|xP_t - x\| = 0 \qquad (x \in l_1) \tag{1.10}$$

and implies that, if P_0 is the identity operator I on l_1, the function

$$t \longmapsto xP_t \qquad ([0,\infty) \longrightarrow l_1) \tag{1.11}$$

is strongly continuous for all $x \in l_1$. It does not however imply the stronger condition

$$\lim_{t\downarrow 0} \|P_t - I\| = 0\,. \tag{1.12}$$

A Markov semigroup satisfying (1.2) is said to be *uniform* or *q-bounded*.

To the operator semigroup (P_t) it is possible to apply the powerful Hille-Yosida theory [3] of strongly continuous one-parameter semigroups. One result of that theory is that the semigroup is uniform if and only if it can be expressed in the form

$$P_t = \exp(tQ) = \sum_{n=0}^{\infty} \frac{t^n}{n!} Q^n \tag{1.13}$$

for some bounded operator $Q : l_1 \longrightarrow l_1$. Then Q is represented by a matrix $(q_{ij}\,; i, j \in S)$ with

$$q_{ii} \leqslant 0\,,\ q_{ij} \geqslant 0 (i \neq j)\,,\ \sum_{j\in S} q_{ij} \leqslant 0\,. \tag{1.14}$$

Since

$$\begin{aligned} \|Q\| &= \sup_{i\in S} \sum_{j\in S} |q_{ij}| \\ &= \sup \left(|q_{ii}| + \sum_{j\neq i} q_{ij} \right) \leqslant 2 \sup_i |q_{ii}|\,, \end{aligned}$$

we have

$$\sup_{i\in S}(-q_{ii}) \leqslant \|Q\| \leqslant 2\sup_{i\in S}(-q_{ii})\,. \tag{1.15}$$

A matrix Q satisfying (1.14) is called a *Q-matrix*; if $(-q_{ii})$ is bounded, the associated operator on l_1 is here called a *Q-operator*.

Kendall [4] observed that, for a uniform semigroup, the formula (1.13) is meaningful even if t is negative, and implies an embedding into a group

$$(P_t\,; t \in \mathbb{R}) \tag{1.16}$$

of operators satisfying (1.9) for all $t, u \in \mathbb{R}$. However, except in the trivial case $P_t = I$, the P_t for $t < 0$ do not have all matrix elements positive, nor do they satisfy (1.8).

The best partial result is expressed in terms of the function

$$g(t) = \inf_{i\in S} p_{ii}(t)\,. \tag{1.17}$$

This is interesting because, for $t > 0$,

$$1 - g(t) \leqslant \|P_t - I\| \leqslant 2\{1 - g(t)\}\ , \tag{1.18}$$

so that (1.12) is equivalent to

$$\lim_{t\downarrow 0} g(t) = 1\,. \tag{1.19}$$

Mountford showed in [7] that, if the semigroup is not uniform, then

$$g(t) \leqslant \tfrac{1}{2} \qquad (t > 0)\,. \tag{1.20}$$

This result is best possible, because Williams [10] has an example with $g(t) = \frac{1}{2}$ for all $t > 0$.

The relation of Mountford's theorem to the MGC is that, if there is equality in (1.2) and so

$$\|P_t - I\| = 2\{1 - g(t)\}\ ,$$

(1.20) means that

$$\|P_t - I\| \geqslant 1\,. \tag{1.21}$$

The negation of (1.21) is a sufficient, but not a necessary, condition for the inevitability of P_t.

However, the most compelling evidence for the truth of the MGC is that 40 years have elapsed without the emergence of a counterexample.

2 The direct sum construction

Kendall [4] and Speakman [9] make use of special cases of the following construction. Suppose that, for each $n = 1, 2, 3, \ldots$, there is a Markov semigroup

$$P_t^{(n)} = (p_{ij}(t); i, j \in S_n) \tag{2.1}$$

on the countable set S_n. Let S be the disjoint union of the S_n, and define $p_{ij}(t)(i,j \in S)$ by

$$\begin{aligned} p_{ij}(t) &= p_{ij}^{(n)}(t) \qquad (i,j \in S_n\,, n \geqslant 1)\,, \\ p_{ij}(t) &= 0 \qquad (i \in S_m\,, j \in S_n\,, m \neq n)\,. \end{aligned} \tag{2.2}$$

It is immediate that these functions satisfy (1.2), (1.3) and (1.4), so that

$$P_t = (p_{ij}(t)\,;\ i,j \in S) \tag{2.3}$$

is a Markov semigroup on S, called the *direct sum* of the semigroups (2.1). The same construction holds, of course, if n runs over any countable set.

If each of the semigroups (2.1) is uniform, there is for each n a Q-operator Q_n with

$$P_t^{(n)} = \exp(tQ_n)\,. \tag{2.4}$$

This formula makes sense even if t is negative, and extends each $p_{ij}^{(n)}$ to a function (not usually positive) on $\mathbb{R}$. Equation (2.2) then defines p_{ij} as a function on $\mathbb{R}$, satisfying (1.4) for all real t and u.

However, (2.3) does not necessarily define a bounded operator P_t on l_1 for negative t. It is easy to prove the following result, implicit in [9].

Lemma 1 *For $t > 0$, the direct sum (2.3) of uniform Markov semigroups defines an invertible operator P_t on l_1 if and only if*

$$\|P_{-t}\| = \sup_n \|\exp(-tQ_n)\| < \infty\,. \tag{2.5}$$

The direct sum is a uniform semigroup if and only if

$$\sup_n \|Q_n\| < \infty\,. \tag{2.6}$$

Condition (2.6) implies (2.5) because

$$\begin{aligned} \|\exp(-tQ_n)\| &= \|\sum_{r=0}^{\infty}(-tQ_n)^r/r!\| \\ &\leqslant \sum_{r=0}^{\infty} t^r\|Q_n\|^r/r! = e^{t\|Q_n\|}\,. \end{aligned}$$

A counterexample to the MGC could be constructed if it were possible to find a sequence of Q-operators Q_n with $\|\exp(-Q_n)\|$ bounded but $\|Q_n\|$ unbounded. Attempts to find such Q_n have so far failed. For example, if each S_n has two elements and

$$Q_n = \begin{pmatrix} -\alpha_n & \alpha_n \\ \beta_n & -\beta_n \end{pmatrix} \tag{2.7}$$

with $\alpha_n > \beta_n > 0$, then $\|Q_n\| = 2\alpha_n$ and

$$\begin{aligned}\|\exp(-Q_n)\| &= 1 + \frac{2\alpha_n}{\alpha_n+\beta_n}\left(e^{\alpha_n+\beta_n}-1\right) \\ &\geqslant 1 + 2(e^{\alpha_n}-1) \longrightarrow \infty \end{aligned} \tag{2.8}$$

if $\alpha_n \longrightarrow \infty$. This is a special case of an observation by Speakman, that there is no counterexample to the MGC consisting of a direct sum in which the S_n are finite sets of bounded size.

Although the direct sum construction has not so far permitted a disproof of the MGC, it does enable existing results to be strengthened. For instance, Mountford's theorem leads at once to the following.

Theorem 1 *For any $\delta \in (0, \frac{1}{2})$ there exists a constant $M(\delta)$ such that any uniform semigroup $P_t = \exp(tQ)$ with*

$$g(t_0) = \inf_i p_{ii}(t_0) \geqslant \tfrac{1}{2} + \delta \tag{2.9}$$

for some $t_0 > 0$ satisfies

$$\|Q\| \leqslant M(\delta) t_0^{-1} \,. \tag{2.10}$$

Proof Suppose the theorem false for a particular value of δ. Then, for any $n \geqslant 1$, there is a uniform semigroup (2.4) and $t_n > 0$ such that

$$g_n(t_n) \geqslant \tfrac{1}{2} + \delta \,, \;\; \|Q_n\| > n t_n^{-1} \,. \tag{2.11}$$

The semigroup

$$\tilde{P}_t^{(n)} = P_{tt_n}^{(n)} \tag{2.12}$$

has

$$\tilde{Q}_n = t_n Q_n \,,$$

so that

$$\tilde{g}_n(1) \geqslant \tfrac{1}{2} + \delta \,, \;\; \|\tilde{Q}_n\| > n \,. \tag{2.13}$$

The direct sum of the semigroups (2.12) has

$$g(1) = \inf_n \tilde{g}_n(1) \geqslant \tfrac{1}{2} + \delta$$

and by Mountford's theorem is uniform. There is thus a contradiction between (2.6) and (2.13), which proves the theorem.

The proof does not of course yield any upper bound $M(\delta)$. Calculation using Q-operators of the form (2.7) gives the inequality

$$M(\delta) \geqslant -\tfrac{1}{2}\log(2\delta) \,, \tag{2.14}$$

but this must be very far from sharp.

3 Consequences of the Markov group conjecture

It was pointed out in [5] that the MGC is related to a number of unsolved problems about positive matrices, and the purpose of this section is to try to make that connection more precise. This is done by studying some important quantities denoted by the letter K (in honour of Kendall as M in the last section recognises Mountford).

For any countable S and any $m > 1$, define $K(m, S) \leqslant \infty$ as the supremum of $\|Q\|$ over all Q-operators with

$$\|\exp(-Q)\| \leqslant m\,. \tag{3.1}$$

Clearly

$$K(\cdot, S) \ : \ (1, \infty) \longrightarrow (0, \infty]$$

is an increasing function.

Suppose that $\phi : S \longrightarrow S'$ is an injection from one countable set into another. Then a Markov semigroup P_t on S induces a semigroup on S' by setting

$$p'_{i'j'}(t) = p_{ij}(t)$$

if $i' = \phi(i)$, $j' = \phi(j)$ for some $i, j \in S$, and

$$p'_{i'j'}(t) = \delta_{i'j'}$$

otherwise. It follows easily that

$$K(m, S) \leqslant K(m, S')\,. \tag{3.2}$$

Taking ϕ to be a bijection shows that $K(m, S)$ depends only on the cardinality of S. Thus there are increasing functions $K_1(\cdot)$, $K_2(\cdot)$, ..., $K(\cdot)$ on $(1, \infty)$ with

$$K_1(m) \leqslant K_2(m) \leqslant K_3(m) \leqslant \ldots \leqslant K(m)\,, \tag{3.3}$$

such that

$$K(m, S) = K_N(m) \tag{3.4}$$

if S is a finite set with N elements, and

$$K(m, S) = K(m) \tag{3.5}$$

if S is infinite.

It is true, but not quite obvious, that $K_N(m)$ is finite for all N, m. An explicit bound will be given in Theorem 3, justifying Speakman's observation. It is not known whether $K(m)$ is finite (for some or all m), and this is related to the MGC by the following theorem.

Theorem 2 *If the MGC is true, then $K(m)$ is finite for all $m > 1$, and any invertible Markov semigroup is expressible in the form (1.13) with*

$$t\|Q\| \leqslant K(\|P_t^{-1}\|) \tag{3.6}$$

for all $t > 0$.

Proof Suppose that $K(m)$ is infinite for some $m > 1$. Then, for any $n \geqslant 1$, there exists a Q-operator Q_n with

$$\| \exp(-Q_n) \| \leqslant m\,, \ \ \|Q_n\| > n\,.$$

Lemma 1 shows that the direct sum of the corresponding Markov semigroups is invertible but not uniform, contradicting the MGC. Thus the MGC implies that $K(m)$ is finite for all $m > 1$. Applying the definition of $K(m)$ with

$$m = \|P_t^{-1}\| = \| \exp(-tQ) \|$$

gives $\|tQ\| \leqslant K(m)$, proving (3.6).

Thus the MGC could be disproved by showing that $K(m) = \infty$ for some $m > 1$. This would follow from (3.3) if

$$\lim_{N \longrightarrow \infty} K_N(m) = \infty\,, \tag{3.7}$$

which is a statement solely about finite matrices. The following upper bound is consistent with (3.7), but does not prove it.

Theorem 3 *For any countable S, let $k(S)$ be the supremum of $\|Q\|$ over all Q-operators on S whose spectrum lies in*

$$H = \{\zeta \in \mathbb{C}\,; \ \ \mathrm{Re}\,\zeta \geqslant -1\} \tag{3.8}$$

Then $k(S)$ depends only on the cardinality of S; denote it by k_N if S is finite of size N, and k if S is infinite. Then

$$k_N \leqslant 2N^{1/2} \tag{3.9}$$

and

$$k_1 \leqslant k_2 \leqslant k_3 \leqslant \ldots \leqslant k\,. \tag{3.10}$$

For $m > 1$,

$$K_N(m) \leqslant k_N \log m\,, \ \ K(m) \leqslant k \log m\,. \tag{3.11}$$

If $k < \infty$ and the MGC is true, every invertible Markov semigroup satisfies

$$t\|Q\| \leqslant k \log(\|P_t^{-1}\|)\,. \tag{3.12}$$

Proof The same argument used to prove (3.2) shows that, if there is an injection $\phi : S \longrightarrow S'$, then

$$k(S) \leqslant k(S')\,.$$

This shows that $k(S)$ depends only on the cardinality of S, justifies the notation k_N, k, and proves (3.10).

If Q is a Q-operator with $\|\exp(-Q)\| \leqslant m$, the spectral radius of $\exp(-Q)$ is at most m, so that the spectrum of Q lies in

$$\{\zeta \in \mathbb{C}; |e^{-\zeta}| \leqslant m\} = \{\zeta \in \mathbb{C};\ \mathrm{Re}\ \zeta \geqslant -\log m\}\,.$$

Hence the spectrum of $(\log m)^{-1}Q$ lies in H, so that

$$\|(\log m)^{-1}Q\| \leqslant k(S)\,.$$

This shows that

$$K(m,S) \leqslant k(S)\log m\,,$$

proving (3.11). In particular, if k is finite and the MGC is true, (3.12) follows from (3.6).

It remains to prove the inequality (3.9). Let Q be a Q-matrix on $\{1,2,\ldots,N\}$. The inequalities (1.14) show by standard Perron-Frobenius theory that the eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_N$ of Q have negative real part. If they lie in H, they satisfy

$$-1 \leqslant\ \mathrm{Re}\ \lambda_r \leqslant 0\,,$$

so that $\mathrm{Re}\ (\lambda_r^2) \leqslant (\ \mathrm{Re}\ \lambda_r)^2 \leqslant 1$. Hence

$$\begin{aligned} N &\geqslant \sum_{r=1}^{N} \mathrm{Re}\ (\lambda_r^2) = tr(Q^2) \\ &= \sum_{i,j=1}^{N} q_{ij}q_{ji} \geqslant \sum_{i=1}^{N} q_{ii}^2 \\ &\geqslant \max(q_{ii}^2) \geqslant \{\tfrac{1}{2}\|Q\|\}^2\,. \end{aligned}$$

Thus $\|Q\| \leqslant 2N^{1/2}$, proving (3.9) and completing the proof of the theorem.

4 Symmetric and other special matrices

Theorems 2 and 3 leave several important gaps in present knowledge. It is quite possible that $K(m) < \infty$ for all $m > 1$, but that the MGC is false. To prove otherwise would require the approximation of an invertible semigroup P_t by uniform semigroups

$$P_t^{(m)} = \exp(tQ_n) \tag{4.1}$$

in such a way that the boundedness of P_{-1} implies a bound for

$$\|\exp(-Q_n)\|\,. \tag{4.2}$$

A natural candidate would be

$$Q_n = n(P_{1/n} - I) \tag{4.3}$$

in view of Hille's exponential formula [3]

$$xP_t = \lim_{n \longrightarrow \infty} x \exp(tQ_n) \qquad (x \in l_1)\,, \tag{4.4}$$

but this does not necessarily imply that (4.2) is bounded.

It is also possible that, for each m, $K_N(m)$ is bounded but $K(m) = \infty$. Again, it is possible that $K(m)$ is finite but that $k = \infty$, or that k_N is bounded but $k = \infty$. This construction of counterexamples is made more difficult by the fact that Q-matrices which allow explicit calculation belong to classes of matrix that behave too well.

For instance, one familiar class is of matrices which are symmetric, or at least *symmetrisable* in the sense that there are constants $\alpha_i > 0 (i \in S)$ with

$$\alpha_i q_{ij} = \alpha_j q_{ji} \ (i \neq j)\,. \tag{4.5}$$

Such a matrix has a spectrum lying on the real axis, and the diagonal elements q_{ii} lie in the convex hull of the spectrum. This implies a strong version of (3.12):

$$t\|Q\| \leqslant 2 \log \|P_t^{-1}\|\,. \tag{4.6}$$

Thus symmetrisable matrices are no good for disproving the MGC.

E.B. Davies has shown me a similar result for Q-operators which are *bipartite* in the sense that S can be divided into subsets S_1 and S_2 such that $q_{ij} = 0$ if $i \neq j$ and i and j are in the same subset. Thus bipartite matrices are also ineligible as counterexamples.

5 Essentially positive matrices

The conditions (1.14), which arise naturally in the theory of Markov semigroups, are rather special in the general context of matrix theory. The essential condition is that the non-diagonal $q_{ij} (i \neq j)$ are positive, because the diagonal elements can be made positive by adding multiples of I.

Call a matrix

$$A = (a_{ij}; i, j \in S) \tag{5.1}$$

essentially positive (EP) if

$$a_{ij} \geqslant 0 \ (i, j \in S, i \neq j)\,. \tag{5.2}$$

When S is infinite, we also impose the condition

$$\|A\| = \sup_i \sum_j |a_{ij}| < \infty \tag{5.3}$$

to ensure that A can be regarded as a bounded operator, with norm $\|A\|$, on l_1. (There are of course many other matrix operator norms that would work just as well.)

Condition (5.3) implies that $|a_{ij}|$ is bounded, so that for sufficiently large b the matrix

$$B = A + bI \tag{5.4}$$

is *positive* in the sense that

$$b_{ij} = a_{ij} + b\delta_{ij} \geqslant 0 \tag{5.5}$$

for all $i, j \in S$. The Perron-Frobenius theorem [8] shows that the spectral radius $r(B)$ is in the spectrum of B, and that there are $\xi_i > 0$ with

$$\sum_j b_{ij}\xi_i \leqslant r(B)\xi_i \tag{5.6}$$

for all i. The spectrum of B lies in

$$\{\zeta; |\zeta| \leqslant r(B)\} \subset \{\zeta;\ \mathrm{Re}\,\zeta \leqslant r(B)\}\,.$$

Thus

$$\rho = r(B) - b \tag{5.7}$$

lies in the spectrum of A, and that spectrum is contained in

$$\{\rho\} \cup \{\zeta;\ \mathrm{Re}\,\zeta < \rho\}\,. \tag{5.8}$$

The value of ρ is independent of the choice of (sufficiently large b), and the ξ_i can also be chosen independent of b, and satisfy

$$\sum_j a_{ij}\xi_j \leqslant \rho\xi_i\,. \tag{5.9}$$

It then follows that

$$q_{ij} = a_{ij}(\xi_j/\xi_i) - \rho\delta_{ij} \tag{5.10}$$

defines a Q-operator. Thus general results about Q-operators translate at once into results about EP matrices.

Suppose for example that we have been able to prove $k < \infty$. Then any Q-operator satisfies

$$\|Q\| \leqslant ks\,,$$

where $-s$ is the infimum of $\mathrm{Re}\,\zeta$ for ζ in the spectrum $sp(Q)$ of Q. In particular,

$$q_{ii} \geqslant -ks$$

for all i. Applying this to (5.10) shows that

$$a_{ii} - \rho \geqslant -ks = -k(\rho - \sigma)\,,$$

where $\sigma = \rho - s$ is the infimum of $\operatorname{Re} \zeta$ for $\zeta \in sp(A)$. We therefore have the inequality

$$\rho - k(\rho - \sigma) \leqslant a_{ii} \leqslant \rho\,, \tag{5.11}$$

where $[\sigma, \rho]$ is the convex hull of

$$\{\operatorname{Re} \zeta; \zeta \in sp(A)\}\,. \tag{5.12}$$

The inequality (5.11) only has force when we know that k is finite. However, if A is a finite $N \times N$ matrix, we can replace k by k_N, which is known to be finite. More explicitly, (3.9) shows that

$$\rho - 2N^{1/2}(\rho - \sigma) \leqslant a_{ii} \leqslant \rho\,. \tag{5.13}$$

This is certainly not best possible. The worst example I know is the 3×3 Q-matrix

$$\begin{pmatrix} -4 & 4 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \tag{5.14}$$

which has eigenvalues $-3, -3, 0$. This shows that

$$\tfrac{4}{3} \leqslant k_3 \leqslant 2\sqrt{3}\,. \tag{5.15}$$

6 The Feller identity

A powerful tool for the study of positive matrices, finite or infinite, is Feller's concept of a renewal sequence and the associated recurrent event identity [2]. Let

$$B = (b_{ij}; i, j \in S) \tag{6.1}$$

be a positive matrix on the countable set S, with finite norm (5.3) if S is infinite. Select a 'home state' h in S, and write u_n for the (h, h) element of the matrix power B^n. Thus

$$u_n = \sum b(h, j_1) b(j, j_2) \ldots b(j_{n-1}, h)\,, \tag{6.2}$$

where we have for clarity written $b(i, j)$ for b_{ij}. The sum extends over all $j, j_2, \ldots, j_{n-1} \in S$, and is absolutely convergent because $\|B\|$ is finite.

Feller splits up the sum (6.2) according to the smallest value of r with $j_r = h$. This gives the identity

$$u_n = \sum_{r=1}^{n} f_r u_{n-r} \ (n \geqslant 1) \tag{6.3}$$

with the convention $u_0 = 1$, where f_n is the same sum as (6.2) but restricted to $j_1, j_2, \ldots, j_{n-1} \neq h$.

Because $u_n \leqslant \|B^n\|$ and $u_{m+n} \geqslant u_m u_n$, it is easy to see that

$$0 \leqslant f_n \leqslant u_n \leqslant r(B)^n\,, \tag{6.4}$$

so that the power series

$$U(\zeta) = \sum_{n=\infty}^{\infty} u_n \zeta^n\,,\ F(\zeta) = \sum_{n=1}^{\infty} f_n \zeta^n \tag{6.5}$$

converge in the disc $D = \{\zeta \in \mathbb{C};\ r(B)|\zeta| < 1\}$. The identity (6.3) translates into the power series identity

$$U(\zeta) = \frac{1}{1 - F(\zeta)} \qquad (\zeta \in D)\,. \tag{6.6}$$

This implies that $F(x) < 1$ for $0 \leqslant x < r(B)^{-1}$, so that

$$\sum_{n=1}^{\infty} f_n r(B)^{-n} \leqslant 1\,. \tag{6.7}$$

Thus F has a continuous extension to the closure of D, with $|F(g)| \leqslant 1$ there.

A sequence (u_n) satisfying (6.3) for some positive sequence (f_n) with

$$\sum_{n=1}^{\infty} f_n \leqslant 1 \tag{6.8}$$

is called a *renewal sequence*. Thus the equation (6.2) always generates a sequence such that, for some r, $(u_n r^{-n})$ is a renewal sequence.

The function $U(\zeta)$ is the (h, h) element of the operator-valued power series

$$\sum_{n=0}^{\infty} B^n \zeta^n = (I - \zeta B)^{-1} \qquad (\zeta \in D)\,. \tag{6.9}$$

Thus $U(\zeta)$ can be continued analytically to

$$\{\zeta \in \mathbb{C}\ ;\ \zeta^{-1} \neq sp(B)\}\,, \tag{6.10}$$

and so $U(\zeta)$ contains information about the spectrum of the parent matrix B. This raises the possibility of approaching the search for inequalities like (5.11) through a study of the analytic functions U and F. Note that the form of (5.11) is invariant under translations by multiples of the identity, so that the positive matrix B also gives information about the essentially positive matrix $A = B - bI$.

The situation is most clearest when S is finite. By Cramer's Rule, the function $U(\zeta)$ is a rational function with $U(\infty) = 0$, and (6.6) shows that $F(\zeta)$ is also rational. Indeed, (6.6) is then best regarded as a formal algebraic identity between the rational functions U and F. The problem is to make effective use of the conditions $f_n \geqslant 0$.

I can offer only one non-trivial use of these ideas, which is an inequality in the spirit of (5.11) for a matrix (finite or infinite) with all its elements positive except for one strictly negative diagonal element. The important aspect of this inequality is that it is independent of the size of the matrix.

Theorem 4 *Let*

$$A = (a_{ij}\ ;\ i, j \in S) \tag{6.11}$$

be a matrix of finite norm on the countable set S. Suppose that, for some $h \neq S$,

$$a_{ij} \geqslant 0 \qquad ((i,j) \neq (h,h))\ . \tag{6.12}$$

Then

$$-a_{hh} \leqslant 2\left(1+\sqrt{2}\right) + (A)\,. \tag{6.13}$$

Proof Start by observing that the proof of the Feller identity (6.3) is purely algebraic, and does not depend on positivity. It is only necessary to check that all the sums are absolutely convergent, and this follows easily from the fact that

$$\sum_{j\in S} |a_{ij}| \leqslant \|A\| \tag{6.14}$$

for all i. Thus the (h,h) element u_n of A^n satisfies (6.3), where f_n is given by the sum (6.2) with the restriction $j_r \neq h$. This restriction also means that, when $n \geqslant 2$, all the summands are positive, so that

$$f_n \geqslant 0 \qquad (n \geqslant 2)\,. \tag{6.15}$$

Of course,

$$f_1 = u_1 = a_{hh} = -\alpha \qquad \text{(say)} \tag{6.16}$$

and there is no loss of generality in supposing that $\alpha > 0$. From (6.14),

$$f_n \leqslant \|A\|^n\ ,\ |u_n| \leqslant \|A\|^n\,, \tag{6.17}$$

so that

$$U(\zeta) = \sum_{n=0}^{\infty} u_n\zeta^n\ ,\ G(\zeta) = \sum_{n=2}^{\infty} f_n\zeta^n \tag{6.18}$$

converge in $|\zeta| < \|A\|^{-1}$, and

$$U(\zeta)\{1 + \alpha\zeta - G(\zeta)\} = 1 \tag{6.19}$$

in that disc. Moreover, $U(\zeta)$ actually converges in the larger disc

$$D = \{\zeta\ ;\ |\zeta| < r(A)^{-1}\}\,. \tag{6.20}$$

Write

$$\kappa = 2\left(1+\sqrt{2}\right) = 4.828\ldots \tag{6.21}$$

and note that

$$\kappa^2 = 4(\kappa+1)\,. \tag{6.22}$$

To prove (6.13) by contradiction, suppose that

$$\alpha > \kappa r(A)\,, \tag{6.23}$$

so that the radius of convergence of $U(\zeta)$ is

$$r(A)^{-1} > \kappa\alpha^{-1}\,.$$

Thus $U(\zeta)$ converges in a disc (centre 0) of radius greater than $\kappa\alpha^{-1}$. The identity (6.19) defines an extension of $G(\zeta)$ to a mesomorphic function in that disc, with poles exactly at the zeros of $U(\zeta)$. Moreover, taking ζ real, (6.19) shows that

$$1+\alpha x - G(x) > 0 \qquad \left(0 \leqslant x \leqslant \kappa\alpha^{-1}\right)\,. \tag{6.24}$$

Because the coefficients $f_n (n \geqslant 2)$ in the power series of G are positive, this shows that the series converges in $|\zeta| \leqslant \kappa\alpha^{-1}$, and that

$$G(\kappa\alpha^{-1}) \leqslant 1+\kappa = \left(\tfrac{1}{2}\kappa\right)^2\,.$$

But $G(x)x^{-2}$ is increasing in x, so that

$$G\left(2\alpha^{-1}\right) \leqslant \left(2\kappa^{-1}\right) G\left(\kappa\alpha^{-1}\right) < 1\,.$$

Hence $|G(\zeta)| < 1$ on $|\zeta| = 2\alpha^{-1}$, and Rouché's theorem implies that

$$1+\alpha\zeta - G(\zeta) \qquad \text{and} \qquad 1+\alpha\zeta$$

have the same number of zeros inside this circle, namely 1. Thus $U(\zeta)$ has a pole in $|\zeta| < 2\alpha^{-1}$, contradicting (6.23). This completes the proof.

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