How often does the Unruh-DeWitt detector click? Regularisation by a spatial profile.

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Abstract

We analyse within first-order perturbation theory the instantaneous transition rate of an accelerated Unruh-DeWitt particle detector whose coupling to a massless scalar field on four-dimensional Minkowski space is regularised by a spatial profile. For the Lorentzian profile introduced by Schlicht, the zero size limit is computed explicitly and expressed as a manifestly finite integral formula that no longer involves regulators or limits. The same transition rate is obtained for an arbitrary profile of compact support under a modified definition of spatial smearing. Consequences for the asymptotic behaviour of the transition rate are discussed. A number of stationary and nonstationary trajectories are analysed, recovering in particular the Planckian spectrum for uniform acceleration.

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1 Introduction

The fact that a uniformly accelerated observer in Minkowski space perceives the Minkowski vacuum as a thermal state with a temperature proportional to the acceleration is called the Unruh effect in honour of its discoverer [1]. It is the simplest among the phenomena linking thermal effects to spacetime horizons. Other examples of such phenomena are thermalisation of de Sitter space as seen by inertial observers [2] and the celebrated discovery by Hawking of the thermal radiation surrounding black holes [3].

A conceptually straightforward way to address the Unruh effect is in terms of a particle detector model, consisting of a quantum system with discrete energy levels and a weak coupling to the quantum field. In generic motion the detector will undergo transitions, which can be interpreted as due to absorption or emission of field quanta; the particle content of the field is thus defined operationally by reference to the measurable excitations or de-excitations of the detector. The simplest model, originated by Unruh [1] and DeWitt [4], involves a linear coupling of the detector's monopole moment to the field at the detector's position. For the uniformly accelerated detector in Minkowski vacuum this model exhibits the Planckian spectrum for the (infinite) total transition probability divided by the (infinite) total proper time [1, 4, 5], under certain technical assumptions on handling the infinities.

If the detector is allowed to move on an arbitrary (timelike) trajectory, the notion of transition probability becomes more subtle. As we will review in section 2, the formal first-order perturbation theory expression for the instantaneous transition rate involves the distribution-valued Wightman function of the quantum field in a way whose interpretation is ambiguous. Schlicht [6] observed that when the Wightman function is represented by the 'standard' $i\epsilon$ regularisation in a given Lorentz frame, the resulting instantaneous transition rate for a uniformly accelerated detector that has been switched on in the asymptotic past depends on the proper time of the detector; a result that breaks Lorentz invariance and appears thus physically incorrect. Schlicht also showed that when the detector response is regularised by first giving the detector a specific spatial profile and then taking the point-like limit, the result is equivalent to giving the Wightman function along the detector world line a non-standard regularisation. This nonstandard regularisation yields physically reasonable results for a number of trajectories, including the time-independent Planckian transition rate for the uniformly accelerated detector [6, 7]. P. Langlois [8] observed that this regularisation can be alternatively interpreted as an exponential frequency cut-off in the detector's instantaneous rest frame, rather than in a fixed Lorentz frame.

While Schlicht's regularisation of the detector by a spatial profile thus appears physically reasonable, the results in [6, 7] for non-inertial motion rely on a specific choice for the profile function: the Lorentzian function, given below in our (2.15). (For inertial motion, it was shown in [7] that the response in Minkowski vacuum is the same for any spherically symmetric profile.) Further, the final transition rate formula in [6, 7] involves an integral of the *regularised* correlation function, and the regulator may be taken to zero only after the integration. Such a formula can be readily applied to specific trajectories, at least numerically, but it does not provide a transparent starting point for extracting analytically properties of interest, such as asymptotic behaviour at large/small frequency or at early/late proper times.

The purpose of this paper is to address the above two questions. We first compute the zero size limit of Schlicht's transition rate explicitly and show that it can be written as a manifestly finite integral formula that no longer involves regulators or limits. The result holds both for a detector switched on at a finite time and for a detector switched on in the asymptotic past, in the latter case subject to certain asymptotic conditions on the trajectory. We also show that the transition rate obtained from the 'standard' $i\epsilon$ regularisation of the Wightman function in a given Lorentz frame differs by an additive, Lorentz-noninvariant term that is independent of the frequency and local as a function along the trajectory. This Lorentz-noninvariant term agrees with the analytic and numerical observations made in the special case of uniform acceleration in [6].

Second, we discuss a spatially extended detector model that uses a modified definition of spatial smearing. Subject to mild technical conditions on both the detector profile and the trajectory, we show that the transition rate in this model is *independent* of the spatial profile and coincides with that obtained from the (unmodified) smearing with the Lorentzian profile function.

Third, we use our transition rate formula to make certain observations on the parity and falloff properties of the detector response. Finally, we examine a number of specific trajectories, including all the stationary worldlines in Minkowski space and a selection of nonstationary worldlines with interesting asymptotics. We recover in particular the Planckian spectrum for uniform acceleration, and for a motion interpolating between inertial and uniformly accelerated motion we obtain an appropriately interpolating transition rate.

We begin in section 2 with a brief review of the Unruh-DeWitt particle detector and its regularisation by a spatial profile. The zero size limit of the transition rate regularised by the Lorentzian profile function is computed in section 3. The modified spatially extended detector model is analysed in section 4. Section 5 discusses the parity and falloff properties of the transition rate and presents the applications to specific trajectories. The results are summarised and discussed in section 6. The proofs of certain technical results are deferred to two appendices.

We work in four-dimensional Minkowski spacetime with metric signature $(-+++)$, and in units in which $\hbar = c = 1$. Boldface letters denote spatial three-vectors and sans-serif letters spacetime four-vectors. The Euclidean scalar product of three-vectors $\mathbf{k}$ and $\mathbf{x}$ is denoted by $\mathbf{k} \cdot \mathbf{x}$, and the Minkowski scalar product of four-vectors k and x is denoted by $\mathsf{k} \cdot \mathsf{x}$. $O(x)$ denotes a quantity for which $O(x)/x$ is bounded as $x \to 0$, $o(x)$ a quantity for which $o(x)/x \to 0$ as $x \to 0$, and $O(1)$ a quantity that remains bounded when the parameter under consideration approaches zero.

2 Particle detector models

We begin by considering a detector that consists of an idealised, pointlike atom with two energy levels, denoted by $|0\rangle_d$ and $|1\rangle_d$, which are eigenstates of the atomic Hamiltonian H_d with respective eigenvalues 0 and ω, $\omega \neq 0$. The detector is coupled to the real, massless scalar field ϕ at the position of the detector, with an interaction Hamiltonian of the form $H_{\text{int}} = c\chi(\tau)\mu(\tau)\phi\big(\mathsf{x}(\tau)\big)$, where c is a coupling constant, $\mu(\tau)$ is the atom's monopole moment operator and $\mathsf{x}(\tau)$ is the spacetime position of the atom parametrised by its proper time τ. $\chi(\tau)$ is a smooth switching function, positive during the interaction and vanishing before and after the interaction.

Suppose that before the interaction the detector is in the state $|0\rangle_d$ and the field in the state $|A\rangle$. After the interaction has taken place, the detector may have a nonzero probability to be in the state $|1\rangle_d$: for $\omega > 0$ (respectively $\omega < 0$), the transition of the detector is interpreted as the absorption (emission) of a particle of energy ω. In first-order perturbation theory in the coupling constant c, this probability is [1, 4, 5, 9, 10]

$$P(\omega) = c^2 \left|{}_d\langle 0|\mu(0)|1\rangle_d\right|^2 F(\omega) \ , \tag{2.1}$$

where the response function $F(\omega)$ is given by

$$F(\omega) = \int_{-\infty}^{\infty} \mathrm{d}\tau' \int_{-\infty}^{\infty} \mathrm{d}\tau'' \, \mathrm{e}^{-i\omega(\tau'-\tau'')} \, \chi(\tau')\chi(\tau'') \, \langle A|\phi\big(\mathsf{x}(\tau')\big)\phi\big(\mathsf{x}(\tau'')\big)|A\rangle \ . \tag{2.2}$$

The response function $F(\omega)$ encodes the part of the probability that depends on the trajectory but not on the detector's internal properties, while the prefactor that involves the matrix element ${}_d\langle 0|\mu(0)|1\rangle_d$ depends on the detector's internal properties but not on the trajectory.

We now specialise to four-dimensional Minkowski spacetime M^4 and take the quantum state $|A\rangle$ of the field to be the Minkowski vacuum, denoted by $|0\rangle$. The Wightman function $\langle 0|\phi(\mathsf{x})\phi(\mathsf{x}')|0\rangle$ is then a well-defined distribution on $M^4 \times M^4$, and its pull-back to the detector world line, the correlation function

$$W(\tau', \tau'') := \langle 0|\phi\big(\mathsf{x}(\tau')\big)\phi\big(\mathsf{x}(\tau'')\big)|0\rangle \ , \tag{2.3}$$

is a well-defined distribution on $\mathbb{R} \times \mathbb{R}$ [10]. As long as the switching function $\chi(\tau)$ is assumed smooth and of compact support, formula (2.2) gives therefore an unambiguous definition to the response function $F(\omega)$, and the probability (2.1) is that of observing the detector in the state $|1\rangle_d$ after the interaction has ceased.

We wish to address a related but subtly different question [6, 11, 12, 13]: What is the probability of finding the detector in the state $|1\rangle_d$ *while the interaction is still switched on?* Proceeding for the moment formally, the answer should be obtained from (2.1) and (2.2) by introducing in the switching function a sharp cut-off at the proper time τ at which the detector is observed, $\chi(\tau') \to \chi(\tau')\Theta(\tau - \tau')$, where Θ is the Heaviside function. The modified response function $F_\tau(\omega)$ then reads

$$F_\tau(\omega) = \int_{-\infty}^{\tau} \mathrm{d}\tau' \int_{-\infty}^{\tau} \mathrm{d}\tau'' \, \mathrm{e}^{-i\omega(\tau'-\tau'')} \, \chi(\tau')\chi(\tau'') \, W(\tau', \tau'') \ . \tag{2.4}$$

The physically interesting quantity in this situation is the instantaneous transition rate $\dot{F}_\tau(\omega)$, where the overdot denotes derivative with respect to τ. Up to a proportionality constant, $\dot{F}_\tau(\omega)$ gives the number of transitions to the state $|1\rangle_d$ per unit proper time in an ensemble of identical detectors following the trajectory $\mathsf{x}(\tau)$. Differentiating (2.4), using the identify $W(\tau,\tau') = \overline{W(\tau',\tau)}$ and performing a change of variables in the remaining integral, we find

$$\dot{F}_\tau(\omega) = 2\,\chi(\tau)\,\mathrm{Re}\int_0^\infty \mathrm{d}s\; \mathrm{e}^{-i\omega s}\, W(\tau,\tau-s)\,\chi(\tau-s)\ . \tag{2.5}$$

If the switching function is taken to have a sharp switch-on at the initial time τ_0 and to be unity thereafter, $\chi(\tau') \to \Theta(\tau'-\tau_0)$, (2.5) becomes

$$\dot{F}_\tau(\omega) = 2\,\mathrm{Re}\int_0^{\tau-\tau_0} \mathrm{d}s\; \mathrm{e}^{-i\omega s}\, W(\tau,\tau-s)\ , \quad \tau > \tau_0\ . \tag{2.6}$$

For a detector switched on in the distant past, $\tau_0 \to -\infty$, we obtain

$$\dot{F}_\tau(\omega) = 2\,\mathrm{Re}\int_0^\infty \mathrm{d}s\; \mathrm{e}^{-i\omega s}\, W(\tau,\tau-s)\ . \tag{2.7}$$

As stressed by Schlicht [6] in the context of formula (2.7), all the expressions (2.4)–(2.7) are *causal*: The probability and the instantaneous transition rate are given by integrals over the *past* of the detector trajectory. There is no need to specify what the trajectory will do after the moment the detector is read (cf. [14]).

Now, the difficulty with formulas (2.4)–(2.7) is that each of them involves the correlation function W in a way that is ambiguous. As W is a distribution on $\mathbb{R}\times\mathbb{R}$ [10], rather than a function, the problem first arises in (2.4) because W is integrated against a function that is not smooth but has a discontinuity at the observation time. In (2.6) and (2.7) the problem is compounded respectively by the discontinuity of the switching function at the sharp switch-on time and the noncompact support of the switching function.

To see the ambiguity explicitly, recall that the formal mode sum expression for the Wightman function reads

$$\langle 0|\phi(\mathsf{x})\phi(\mathsf{x}')|0\rangle = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3k}{2|\mathbf{k}|}\mathrm{e}^{i\mathsf{k}\cdot(\mathsf{x}-\mathsf{x}')}\ , \tag{2.8}$$

where $\mathsf{k} = (|\mathbf{k}|,\mathbf{k})$. The usual way to regularise this mode sum is by the frequency cut-off $\mathsf{x}-\mathsf{x}' \to \mathsf{x}-\mathsf{x}' - i\epsilon\partial_t$, where $\epsilon > 0$, leading to the representation [5]

$$\langle 0|\phi(\mathsf{x})\phi(\mathsf{x}')|0\rangle = \lim_{\epsilon\to 0_+}\frac{-1}{4\pi^2}\frac{1}{(t-t'-i\epsilon)^2 - |\mathbf{x}-\mathbf{x}'|^2}\ . \tag{2.9}$$

Assuming $\mathbf{x}-\mathbf{x}' \neq \mathbf{0}$, expanding (2.9) into partial fractions and taking the limit in the sense of one-dimensional Hilbert transforms yields

$$\langle 0|\phi(\mathsf{x})\phi(\mathsf{x}')|0\rangle = \frac{1}{8\pi^2|\Delta\mathbf{x}|}\left\{P\left(\frac{1}{\Delta t-|\Delta\mathbf{x}|}\right)+P\left(\frac{1}{\Delta t+|\Delta\mathbf{x}|}\right)\right. \\ \left. + i\pi\big[\delta(\Delta t+|\Delta\mathbf{x}|)-\delta(\Delta t-|\Delta\mathbf{x}|)\big]\right\} , \tag{2.10}$$

where $\Delta t := t-t'$, $\Delta\mathbf{x} := \mathbf{x}-\mathbf{x}'$ and P stands for the Cauchy principal value. However, inserting the correlation function given by (2.3) with (2.10) into (2.5) gives an integral whose interpretation is ambiguous because Δt and $\Delta\mathbf{x}$ both vanish at $s=0$.

One is therefore led to seek a meaning for the instantaneous transition rates in (2.5)–(2.7) by first regularising W, then doing the integral over s and at the end removing the regulator. However, there is now an issue as to which regularisation to use. The Wightman function is a well-defined, Lorentz-invariant distribution, and it can be defined as

$$\langle 0|\phi(\mathsf{x})\phi(\mathsf{x}')|0\rangle = \lim_{\epsilon\to 0_+}\frac{-1}{4\pi^2}\frac{1}{(t-t')^2-|\mathbf{x}-\mathbf{x}'|^2-i\epsilon\big[T(\mathsf{x})-T(\mathsf{x}')\big]-\epsilon^2} , \tag{2.11}$$

where T is *any* global time function that increases to the future [15]. The usual representation (2.9) is obtained with the choice $T(\mathsf{x})=t$ in a specific Lorentz frame. But when the Wightman function, or the correlation function W obtained from it, is used outside the distributional setting, there is no a priori guarantee for different time functions in (2.11) to lead to the same results, nor is there a guarantee for the results to be Lorentz invariant. The transition rate formulas (2.6) and (2.7) are a case in point: Schlicht [6] observed that when used in (2.7), the usual Wightman function regularisation (2.9) yields for the uniformly accelerated detector a result that is not invariant under the Lorentz boosts that leave the trajectory invariant. We show in appendix A that the only trajectories for which the usual Wightman function regularisation (2.9) yields a Lorentz-invariant result for the transition rates (2.6) and (2.7) are the inertial trajectories. If we insist on a Lorentz-invariant transition rate, the usual regularisation (2.9) will therefore not do.

Schlicht [6] proposed to regularise the transition rate by giving the detector a finite spatial extent in its instantaneous rest frame. The idea is that the detector is not coupled to the field operator on the detector world line, $\phi\big(\mathsf{x}(\tau)\big)$, but instead to the spatially smeared field operator,

$$\phi_f(\tau) := \int \mathrm{d}^3\xi\, f_\epsilon(\boldsymbol{\xi})\,\phi\big(\mathsf{x}(\tau,\boldsymbol{\xi})\big) , \tag{2.12}$$

where $(\tau,\boldsymbol{\xi}) = (\tau,\xi^1,\xi^2,\xi^3)$ is a set of Fermi-Walker coordinates associated with the trajectory: $\mathsf{x}(\tau,\boldsymbol{\xi}) = \mathsf{x}(\tau)+\xi^i\mathsf{e}_i = \mathsf{x}(\tau)+\xi^1\mathsf{e}_1+\xi^2\mathsf{e}_2+\xi^3\mathsf{e}_3$, where e_i are three unit vectors that together with the velocity $\dot{\mathsf{x}}$ form an orthonormal tetrad, Fermi-Walker transported with the motion [16]. Note that the $\boldsymbol{\xi}$-coordinates parametrise the hyperplane orthogonal to the velocity at each moment of proper time. The profile function

f_ϵ is assumed normalised so that $\int d^3\xi\, f_\epsilon(\boldsymbol{\xi}) = 1$, to depend on a positive parameter ϵ so that $\lim_{\epsilon\to 0} f_\epsilon(\boldsymbol{\xi}) = \delta(\boldsymbol{\xi})$, and to have the scaling property

$$f_\epsilon(\boldsymbol{\xi}) = \epsilon^{-3} f(\boldsymbol{\xi}/\epsilon) \ . \tag{2.13}$$

The function f thus gives the detector's "shape," which is rigid in the detector's instantaneous rest frame, and the positive parameter ϵ determines the detector's "size." If the transition rate of a detector smeared in this way is well defined and has a well-defined limit as $\epsilon \to 0$, this limit can then be understood as the transition rate of a pointlike detector.

The correlation function for the smeared field operator (2.12) is defined by

$$W_\epsilon(\tau, \tau') := \langle 0|\phi_f(\tau)\phi_f(\tau')|0\rangle \ . \tag{2.14}$$

When the function f in (2.13) is the Lorentzian function,

$$f(\boldsymbol{\xi}) = \frac{1}{\pi^2}\frac{1}{\left(|\boldsymbol{\xi}|^2 + 1\right)^2} \ , \tag{2.15}$$

Schlicht [6] shows that

$$W_\epsilon(\tau, \tau') = \frac{1}{4\pi^2}\frac{1}{\left(\mathsf{x} - \mathsf{x}' - i\epsilon(\dot{\mathsf{x}} + \dot{\mathsf{x}}')\right)^2} \ , \tag{2.16}$$

where the unprimed and primed quantities on the right-hand side are evaluated respectively at τ and τ'. Note that W_ϵ (2.16) is manifestly Lorentz covariant because of the four-velocities appearing with ϵ. Schlicht examines [6, 7] the transition rate (2.7) for a detector switched on in the asymptotic past using W_ϵ (2.16) for a number of trajectories and finds results that appear physically sensible. In particular, the transition rate of a uniformly accelerated detector has the expected τ-independent Planckian spectrum of the Unruh effect,

$$\dot{F}_\tau(\omega) = \frac{\omega}{2\pi}\frac{1}{\mathrm{e}^{2\pi\omega/a} - 1} \ , \tag{2.17}$$

where a is the acceleration. Schlicht's results have been generalised by P. Langlois [8, 17] to a variety of situations, including Minkowski space in an arbitrary number of dimensions, quotients of Minkowski space under discrete isometry groups, the massive scalar field, the massless Dirac field and certain curved spacetimes.

These results of Schlicht and Langlois rely on the choice of the Lorentzian profile function, given in four dimensions by (2.15) and in other dimensions by the appropriate generalisation [8]. For the inertial detector switched on in the asymptotic past, it is shown in [7] that all profile functions of the form (2.13) with a spherically symmetric f give the transition rate $-\omega\Theta(-\omega)/(2\pi)$: This agrees with the results established in [5]. We shall address the profile-dependence of the transition rate in section 4 under a modified definition of the spatial smearing.

3 Zero size limit with the Lorentzian profile function

In this section we evaluate the zero size limit of the instantaneous transition rate for the regularised correlation function (2.16), obtained in [6] from spatial smearing with the Lorentzian profile function (2.15). We first address a detector switched on at a finite proper time and then a detector switched on in the asymptotic past.

3.1 Sharp switch-on

We denote by τ_0 the moment of the sharp switch-on and by τ the moment of observation, assuming $\tau > \tau_0$. The trajectory is assumed to be C^9 in the closed interval $[\tau_0, \tau]$. We wish to evaluate the limit $\epsilon \to 0$ of the transition rate given by (2.6) with the correlation function (2.16).

Writing $\Delta\tau := \tau - \tau_0$, the regularised transition rate (2.6) reads

$$\dot{F}_\tau(\omega) = \frac{1}{2\pi^2} \,\mathrm{Re} \int_0^{\Delta\tau} \mathrm{d}s \, \frac{\mathrm{e}^{-i\omega s}}{\big(\mathsf{x} - \mathsf{x}' - i\epsilon(\dot{\mathsf{x}} + \dot{\mathsf{x}}')\big)^2} \ , \tag{3.1}$$

where we have suppressed the index ϵ on the left-hand side. Decomposing $\dot{F}_\tau(\omega)$ into its even and odd parts in ω as $\dot{F}_\tau(\omega) = \dot{F}_\tau^{\mathrm{even}}(\omega) + \dot{F}_\tau^{\mathrm{odd}}(\omega)$, we find

$$\dot{F}_\tau^{\mathrm{even}}(\omega) = \frac{1}{2\pi^2} \int_0^{\Delta\tau} \mathrm{d}s \, \frac{\big[(\Delta\mathsf{x})^2 - \epsilon^2\mathsf{q}^2\big] \cos(\omega s)}{\big[\epsilon^2\mathsf{q}^2 - (\Delta\mathsf{x})^2\big]^2 + 4\epsilon^2(\mathsf{q}\cdot\Delta\mathsf{x})^2} \ , \tag{3.2a}$$

$$\dot{F}_\tau^{\mathrm{odd}}(\omega) = \frac{1}{\pi^2} \int_0^{\Delta\tau} \mathrm{d}s \, \frac{\epsilon(\mathsf{q}\cdot\Delta\mathsf{x}) \sin(\omega s)}{\big[\epsilon^2\mathsf{q}^2 - (\Delta\mathsf{x})^2\big]^2 + 4\epsilon^2(\mathsf{q}\cdot\Delta\mathsf{x})^2} \ , \tag{3.2b}$$

where $\Delta\mathsf{x} := \mathsf{x}(\tau) - \mathsf{x}(\tau - s)$, $(\Delta\mathsf{x})^2 := (\Delta\mathsf{x})\cdot(\Delta\mathsf{x})$ and $\mathsf{q} := \dot{\mathsf{x}}(\tau) + \dot{\mathsf{x}}(\tau - s)$. We further decompose $\dot{F}^{\mathrm{even}} = \dot{F}_<^{\mathrm{even}} + \dot{F}_>^{\mathrm{even}}$ and $\dot{F}^{\mathrm{odd}} = \dot{F}_<^{\mathrm{odd}} + \dot{F}_>^{\mathrm{odd}}$, where the subscripts $<$ and $>$ refer respectively to the integration subintervals $s \in \big[0, \sqrt{\epsilon}\big]$ and $s \in \big[\sqrt{\epsilon}, \Delta\tau\big]$ and we have suppressed τ and ω.

We consider each of the four terms in turn. To begin we note the small s expansions

$$(\Delta\mathsf{x})^2 = -s^2 - \tfrac{1}{12}\ddot{\mathsf{x}}^2 s^4 + O(s^5) \ , \tag{3.3a}$$

$$\mathsf{q}\cdot\Delta\mathsf{x} = -2s - \tfrac{1}{3}\ddot{\mathsf{x}}^2 s^3 + O(s^4) \ , \tag{3.3b}$$

$$\mathsf{q}^2 = -4 - \ddot{\mathsf{x}}^2 s^2 + O(s^3) \ . \tag{3.3c}$$

Consider $\dot{F}_>^{\mathrm{even}}$ and $\dot{F}_>^{\mathrm{odd}}$. For fixed s, the integrand in (3.2a) tends to $\cos(\omega s)/(\Delta\mathsf{x})^2$. If we make this replacement under the integral, the resulting error in $\dot{F}_>^{\mathrm{even}}$ can be arranged into the form

$$\frac{1}{2\pi^2}\int_{\sqrt{\epsilon}}^{\Delta\tau} \mathrm{d}s \cos(\omega s) \frac{\epsilon^2}{\left[(\Delta\mathsf{x})^2\right]^2} \frac{\mathsf{q}^2 - 4\dfrac{(\mathsf{q}\cdot\Delta\mathsf{x})^2}{(\Delta\mathsf{x})^2} - \epsilon^2\dfrac{(\mathsf{q}^2)^2}{(\Delta\mathsf{x})^2}}{\left\{\left(1-\epsilon^2\dfrac{\mathsf{q}^2}{(\Delta\mathsf{x})^2}\right)^2 + 4\epsilon^2\dfrac{(\mathsf{q}\cdot\Delta\mathsf{x})^2}{\left[(\Delta\mathsf{x})^2\right]^2}\right\}} \ . \tag{3.4}$$

From (3.3) it follows that q^2, $(\mathsf{q}\cdot\Delta\mathsf{x})^2/(\Delta\mathsf{x})^2$ and $\epsilon/(\Delta\mathsf{x})^2$ are all bounded in absolute value over the interval of integration by constants independent of ϵ as $\epsilon\to 0$, uniformly in s. Since $\left|(\Delta\mathsf{x})^2\right| \ge s^2$, the absolute value of the integrand in (3.4) is thus bounded by a constant times ϵ^2/s^4 and the integral is of order $O(\sqrt{\epsilon})$. A similar estimate for the integrand in (3.2b) shows that $\dot{F}_{>}^{\mathrm{odd}} = O(\sqrt{\epsilon})$. Hence

$$\dot{F}_{>}^{\mathrm{even}} + \dot{F}_{>}^{\mathrm{odd}} = \frac{1}{2\pi^2}\int_{\sqrt{\epsilon}}^{\Delta\tau} \mathrm{d}s\, \frac{\cos(\omega s)}{(\Delta\mathsf{x})^2} \quad + O(\sqrt{\epsilon}) \ . \tag{3.5}$$

Consider then $\dot{F}_{<}^{\mathrm{even}}$ and $\dot{F}_{<}^{\mathrm{odd}}$. At $s=0$, the denominator in the integrands in (3.2) is of order ϵ^4. To control the denominator for $s\in\left[0,\sqrt{\epsilon}\right]$, we expand $(\Delta\mathsf{x})^2$ in s to order s^8, $\mathsf{q}\cdot\Delta\mathsf{x}$ to order s^5 and q^2 to order s^4: this gives the denominator accurately to order ϵ^5. Writing $s=\sqrt{\epsilon}r$, where $0\le r\le 1$, we find

$$\left[\epsilon^2\mathsf{q}^2 - (\Delta\mathsf{x})^2\right]^2 + 4\epsilon^2(\mathsf{q}\cdot\Delta\mathsf{x})^2 = \epsilon^2(4\epsilon+r^2)^2\left[1+\tfrac{1}{6}\ddot{\mathsf{x}}^2\epsilon r^2 + O(\epsilon^{3/2})\right] \ , \tag{3.6}$$

where taking out the factor $(4\epsilon+r^2)^2$ has allowed all the terms that depend on the higher derivatives of x to be grouped into the $O(\epsilon^{3/2})$ term, uniformly in r.

In $\dot{F}_{<}^{\mathrm{odd}}$, it suffices to keep in the denominator just the leading term in (3.6) and in the numerator just the leading power of s. We find

$$\begin{aligned}\dot{F}_{<}^{\mathrm{odd}} &= -\frac{2\omega\sqrt{\epsilon}}{\pi^2}\int_0^1 \mathrm{d}r\, \frac{r^2}{(4\epsilon+r^2)^2}\left[1+O(\epsilon)\right] \\ &= -\frac{\omega}{4\pi} + O(\sqrt{\epsilon}) \ ,\end{aligned} \tag{3.7}$$

where the integration is elementary. In $\dot{F}_{<}^{\mathrm{even}}$, we use (3.6) in the denominator and expand the numerator to next-to-leading order in s. We find

$$\begin{aligned}\dot{F}_{<}^{\mathrm{even}} &= \frac{1}{2\pi^2\sqrt{\epsilon}}\int_0^1 \mathrm{d}r\, \frac{(4\epsilon-r^2)\left[1-\left(\frac{1}{12}\ddot{\mathsf{x}}^2+\frac{1}{2}\omega^2\right)\epsilon r^2\right] + r^2 O(\epsilon^{3/2}) + O(\epsilon^{5/2})}{(4\epsilon+r^2)^2} \\ &= \frac{1}{2\pi^2\sqrt{\epsilon}} + O(\sqrt{\epsilon}) \ ,\end{aligned} \tag{3.8}$$

where the integrations are elementary.

Combining (3.5), (3.7) and (3.8), and writing $\epsilon^{-1/2} = (\Delta\tau)^{-1} + \int_{\sqrt{\epsilon}}^{\Delta\tau} s^{-2}\,\mathrm{d}s$, we obtain

$$\dot{F}_\tau(\omega) = -\frac{\omega}{4\pi} + \frac{1}{2\pi^2}\int_{\sqrt{\epsilon}}^{\Delta\tau} \mathrm{d}s \left(\frac{\cos(\omega s)}{(\Delta\mathsf{x})^2} + \frac{1}{s^2}\right) + \frac{1}{2\pi^2\Delta\tau} + O(\sqrt{\epsilon})\ . \tag{3.9}$$

As it follows from (3.3a) that the integrand in (3.9) has a small s expansion that starts with a constant term, taking the limit $\epsilon \to 0$ yields

$$\dot{F}_\tau(\omega) = -\frac{\omega}{4\pi} + \frac{1}{2\pi^2}\int_{0}^{\Delta\tau} \mathrm{d}s \left(\frac{\cos(\omega s)}{(\Delta\mathsf{x})^2} + \frac{1}{s^2}\right) + \frac{1}{2\pi^2\Delta\tau}\ . \tag{3.10}$$

Formula (3.10) is the promised result.

3.2 Switch-on in the asymptotic past

We now turn to a detector that is switched on in the asymptotic past. Two qualitatively different situations can arise here. One occurs for trajectories that are defined for arbitrarily negative proper times, the other for trajectories that come from infinity within finite proper time.

We note first that formula (3.10) has in both situations a well-defined limit. In the former case τ_0 is replaced by $-\infty$ and we obtain

$$\dot{F}_\tau(\omega) = -\frac{\omega}{4\pi} + \frac{1}{2\pi^2}\int_{0}^{\infty} \mathrm{d}s \left(\frac{\cos(\omega s)}{(\Delta\mathsf{x})^2} + \frac{1}{s^2}\right)\ . \tag{3.11}$$

In the latter case τ_0 in (3.10) is understood as the (asymptotic) value of the proper time at which the trajectory starts out at infinity. In either situation the integral is convergent in absolute value since $s^2 \le \left|(\Delta\mathsf{x})^2\right|$.

The only step that changes in the analysis of subsection (3.1) is that the estimates for $\dot{F}_>^{\mathrm{even}}$ and $\dot{F}_>^{\mathrm{odd}}$ need to control the integrand also as s approaches respectively ∞ or $\tau - \tau_0$. Inspection of the integrand in (3.4) and the similar arrangement of the integrand in $\dot{F}_>^{\mathrm{odd}}$ shows that a sufficient condition is to assume that the quantities

$$\frac{\mathsf{q}^2}{(\Delta\mathsf{x})^2}\ ,\quad \frac{\mathsf{q}\cdot\Delta\mathsf{x}}{(\Delta\mathsf{x})^2} \tag{3.12}$$

remain bounded as s approaches these asymptotic values. Taking the limit $\epsilon \to 0$ then yields the result (3.11) in the former case and the result (3.10) in the latter case.

We shall show in section 5 that the quantities (3.12) do remain bounded as $s \to \infty$ for a number of interesting trajectories that are defined for arbitrarily negative proper times, including in particular all stationary trajectories.

To summarise, we have obtained an explicit expression for the excitation rate of a particle detector regulated with the Lorentzian spatial profile once the zero-size limit has been taken and the regulator has disappeared. For detectors switched on at finite τ_0 the formula is (3.10) and is valid for arbitrary trajectories; for detectors switched on at the asymptotic past the formula is (3.11) (or again (3.10) in the special case of motion coming from infinity in a finite $\Delta\tau$) and is valid if quantities (3.12) are bounded over the trajectory.

In the next section we show that the same results can be obtained from more general spatial profiles, while in Appendix A we show that a similar calculation using the standard $i\epsilon$ regularisation gives the same result with an added Lorentz-noninvariant term.

4 Spatially extended detector model with a general profile

In this section we examine a spatially extended detector model that is motivated by the spatially smeared detector introduced in section 2. Although we will not be able to establish a precise connection between this model and the spatially smeared field, the interest of the model is that it does capture some of the contributions from spatial smearing and these contributions will be seen to be independent of the profile function, yielding the same zero-size limit as in section 3.

We again address first a detector switched on at a finite proper time and then a detector switched on in the asymptotic past.

4.1 Sharp switch-on

We again denote by τ_0 the moment of the sharp switch-on and by τ the moment of observation, assuming $\tau > \tau_0$. The trajectory is now assumed to be real analytic in the closed interval $[\tau_0, \tau]$.

Consider the spatially smeared detector model of section section 2, and assume that the function f in the profile (2.13) is smooth and has compact support. Note that f is not assumed to be spherically symmetric. Note also that our discussion will not cover the Lorentzian profile (2.15), which is not of compact support.

Substituting the smeared field operator (2.12) in (2.6) and formally interchanging the integrals, we obtain for the transition rate the formula

$$\dot{F}_\tau^{(\epsilon)}(\omega) = \int \mathrm{d}^3\xi \, \mathrm{d}^3\xi' \; f_\epsilon(\boldsymbol{\xi}) \, f_\epsilon(\boldsymbol{\xi}') \, G_{\tau,\tau_0}(\boldsymbol{\xi}, \boldsymbol{\xi}'; \omega) \; , \tag{4.1}$$

where

$$G_{\tau,\tau_0}(\boldsymbol{\xi}, \boldsymbol{\xi}'; \omega) := 2 \, \mathrm{Re} \int_0^{\Delta\tau} \mathrm{d}s \; \mathrm{e}^{-i\omega s} \, \langle 0| \phi\big(\mathsf{x}(\tau, \boldsymbol{\xi})\big) \phi\big(\mathsf{x}(\tau - s, \boldsymbol{\xi}')\big) |0\rangle \tag{4.2}$$

and $\Delta\tau := \tau - \tau_0$. The difficulty with (4.1) and (4.2) is that the latter formula suffers at $\boldsymbol{\xi} = \boldsymbol{\xi}'$ from the same ambiguity as the unsmeared transition rate (2.6). However, we shall show in subsection 4.2 that G_{τ,τ_0} is pointwise well defined by (4.2) whenever $|\boldsymbol{\xi}|$ and $|\boldsymbol{\xi}'|$ are sufficiently small and $\boldsymbol{\xi} \neq \boldsymbol{\xi}'$. We shall further show that the integral in (4.1) over the subset $\boldsymbol{\xi} \neq \boldsymbol{\xi}'$ is well defined for sufficiently small ϵ.

Motivated by these observations, we take the instantaneous transition rate in our smeared detector model to be *defined* for sufficiently small ϵ by

$$\dot{F}_\tau^{(\epsilon)}(\omega) := \int_{\boldsymbol{\xi} \neq \boldsymbol{\xi}'} \mathrm{d}^3\xi \, \mathrm{d}^3\xi' \, f_\epsilon(\boldsymbol{\xi}) \, f_\epsilon(\boldsymbol{\xi}') \, G_{\tau,\tau_0}(\boldsymbol{\xi}, \boldsymbol{\xi}'; \omega) \, . \tag{4.3}$$

We shall show in subsection 4.2 that the limit of $\dot{F}_\tau^{(\epsilon)}(\omega)$ (4.3) as $\epsilon \to 0$ exists, is indepedent of the profile function and given by (3.10).

If the passage from (2.6) with (2.12) to (4.3) can be justified in a sense in which G_{τ,τ_0} does not contain a distribution with support at $\boldsymbol{\xi} = \boldsymbol{\xi}'$, our model is equivalent to spatial smearing with a profile of compact support. As our model yields the same transition rate as spatial smearing with the Lorentzian profile function (which is not of compact support), we are led to suspect that the equivalence of our model to spatial smearing could be established for at least some classes of profile functions. We shall not pursue this question further in this paper.

Readers who wish to skip the technical estimates on G_{τ,τ_0} may prefer to proceed directly to subsection 4.3.

4.2 Estimates for G_{τ,τ_0}

As the profile function (2.13) has by assumption compact support, it suffices in (4.3) to define and examine G_{τ,τ_0} (4.2) at small $|\boldsymbol{\xi}|$ and $|\boldsymbol{\xi}'|$. To control the smallness, we introduce a positive parameter δ and assume $|\boldsymbol{\xi}| < \delta$, $|\boldsymbol{\xi}'| < \delta$ and $\boldsymbol{\xi} \neq \boldsymbol{\xi}'$. The limit of interest is $\delta \to 0$, where τ, τ_0 and ω are regarded as fixed.

As the singularity of the Wightman function is on the light cone, the integral over s in (4.2) has a singularity precisely when the vector

$$\mathsf{H} := \mathsf{x}(\tau) - \mathsf{x}(\tau - s) + \xi^i \mathsf{e}_i(\tau) - \xi'^i \mathsf{e}_i(\tau - s) \tag{4.4}$$

is null. For sufficiently small δ, it follows from the construction of the Fermi-Walker coordinates [16] that H is spacelike at $s = 0$, future timelike at $s = \Delta\tau$ and null at exactly one intermediate value of s, which we denote by s^*. This means that we can define the integral over s by representing the Wightman function as in (2.10): Decomposing H into its temporal and spatial components as $\mathsf{H} =: (H^0, \mathbf{H})$, we obtain

$$\begin{aligned} &\langle 0 | \phi\big(\mathsf{x}(\tau, \boldsymbol{\xi})\big) \phi\big(\mathsf{x}(\tau - s, \boldsymbol{\xi}')\big) | 0 \rangle \\ &= \frac{1}{8\pi^2} \frac{1}{|\mathbf{H}|} \left[P\left(\frac{1}{|\mathbf{H}| - H^0}\right) + P\left(\frac{1}{|\mathbf{H}| + H^0}\right) - i\pi\delta(|\mathbf{H}| - H^0) + i\pi\delta(|\mathbf{H}| + H^0) \right] , \end{aligned} \tag{4.5}$$

which contains a prescription for integrating over $s = s^*$. Note that since H is nonvanishing for all s, the integral over any zeroes of $\mathbf{H}$ is nonsingular despite the overall factor $1/|\mathbf{H}|$. Note also that this is the step where we need the assumption $\boldsymbol{\xi} \neq \boldsymbol{\xi}'$.

When $\boldsymbol{\xi} = \boldsymbol{\xi}'$, H vanishes at $s = 0$ and the integral over s faces the same ambiguity at $s \to 0$ as the unsmeared integral (2.6).

Since $|\mathbf{H}| + H^0 > 0$, the term proportional to $\delta(|\mathbf{H}| + H^0)$ in (4.5) gives a vanishing contribution to the integral, and in the term involving $P\big[1/(|\mathbf{H}| + H^0)\big]$ the principal value symbol is redundant and can be dropped. The contribution from the remaining principal value term can be converted into a contour integral by the identity

$$\int_0^{\Delta\tau} \mathrm{d}s\ P\big(g(s)\big) = -i\pi \mathrm{Res}\big(g(s)\big)_{s^*} + \int_C \mathrm{d}s\, g(s)\ , \tag{4.6}$$

where the contour C circumvents the pole at $s = s^*$ in the lower half of the complex s plane. The contribution from the residue will then cancel the contribution from the remaining delta-function, since $d(|\mathbf{H}| - H^0)/ds$ is negative at $s = s^*$: This is because H is spacelike for $s < s^*$ and *future* timelike for $s > s^*$. We thus obtain

$$G_{\tau,\tau_0}(\boldsymbol{\xi}, \boldsymbol{\xi}'; \omega) = \frac{1}{2\pi^2} \mathrm{Re} \int_C \mathrm{d}s\, \frac{\mathrm{e}^{-i\omega s}}{\mathsf{H}^2(s,\tau)}\ , \tag{4.7}$$

where the dependence of H on $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ has been suppressed.

We wish to compute G_{τ,τ_0} from (4.7) in the limit $\delta \to 0$. The technical subtlety in this computation is that although s^* is positive, it may be arbitrarily small compared with δ.

We first establish some facts about the small s behaviour of H^2. From (4.4), we obtain

$$\mathsf{H}^2 = (\Delta\mathsf{x})^2 + 2\,\Delta\mathsf{x} \cdot (\xi^i \mathsf{e}_i - \xi'^j \mathsf{e}'_j) + (\xi^i \mathsf{e}_i - \xi'^j \mathsf{e}'_j)^2\ , \tag{4.8}$$

where

$$\Delta\mathsf{x} := \mathsf{x}(\tau) - \mathsf{x}(\tau - s)\ , \tag{4.9a}$$
$$\mathsf{e}_i := \mathsf{e}_i(\tau)\ , \tag{4.9b}$$
$$\mathsf{e}'_i := \mathsf{e}_i(\tau - s)\ . \tag{4.9c}$$

Note that

$$(\xi^i \mathsf{e}_i - \xi'^j \mathsf{e}'_j)^2 = |\boldsymbol{\xi} - \boldsymbol{\xi}'|^2 + 2\xi^i \xi'^j \left(\delta_{ij} - \mathsf{e}_i \cdot \mathsf{e}'_j\right)\ . \tag{4.10}$$

Expanding $\mathsf{x}(\tau - s)$ and $\mathsf{e}(\tau - s)$ in powers of s gives the small s expansions (3.3a) and

$$\mathsf{H}^2 = s_0^2 + \sum_{j=2}^{\infty} H_j s^j \tag{4.11}$$

where $s_0 := |\boldsymbol{\xi} - \boldsymbol{\xi}'|$ and

$$\begin{aligned}
&H_2 = -1 - (\xi^i + \xi'^i)(\ddot{\mathsf{x}} \cdot \mathsf{e}_i) - \xi^i \xi'^j (\ddot{\mathsf{x}} \cdot \mathsf{e}_i)(\ddot{\mathsf{x}} \cdot \mathsf{e}_j) = -1 + O(\delta)\ ,\\
&H_3 = \tfrac{1}{3}(\xi^j + 2\xi'^j)(\dddot{\mathsf{x}} \cdot \mathsf{e}_j) + \tfrac{1}{3}\xi^i \xi'^j \big[(\dddot{\mathsf{x}} \cdot \mathsf{e}_i)(\ddot{\mathsf{x}} \cdot \mathsf{e}_j) + 2(\ddot{\mathsf{x}} \cdot \mathsf{e}_i)(\dddot{\mathsf{x}} \cdot \mathsf{e}_j)\big] = O(\delta)\ ,\\
&H_j = O(\delta^0)\ \ \text{for}\ \ j \geq 4\ .
\end{aligned} \tag{4.12}$$

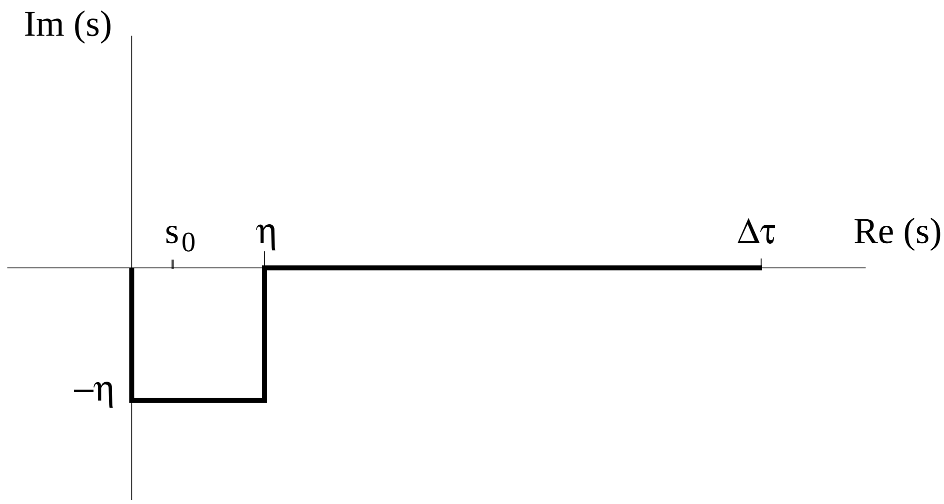

Figure 1: The contour C (4.13) in the complex s plane. We have written $\eta := \sqrt{\delta}$.

We have here used the normalisation condition $\dot{\mathsf{x}}^2 = -1$ and its consequences for the higher derivatives, the Fermi-Walker transport equation for the tetrad and the bounds $|\boldsymbol{\xi}| < \delta$ and $|\boldsymbol{\xi}'| < \delta$.

Next, we specify the contour. For sufficiently small δ, (4.11) and (4.12) show that H is timelike at $s = \sqrt{\delta}$, from which it follows that $s^* < \sqrt{\delta}$. We may thus take the contour C to consist of four straight lines C_i, $i = 1, 2, 3, 4$, as shown in Figure 1:

$$
\begin{aligned}
&C_1 : \ s = -ir, \quad 0 \le r \le \sqrt{\delta} \ , \\
&C_2 : \ s = -i\sqrt{\delta} + r, \quad 0 \le r \le \sqrt{\delta} \ , \\
&C_3 : \ s = \sqrt{\delta}(1 - i) + ir, \quad 0 \le r \le \sqrt{\delta} \ , \\
&C_4 : \ \sqrt{\delta} \le s \le \Delta\tau \ .
\end{aligned} \tag{4.13}
$$

Note that since $s_0 < 2\delta$, we have $s_0 < \sqrt{\delta}$ for sufficiently small δ.

We show in appendix B that the contribution to G_{τ,τ_0} from $C_2 \cup C_3 \cup C_4$ reads

$$
G^{C_2 \cup C_3 \cup C_4}_{\tau,\tau_0}(\boldsymbol{\xi}, \boldsymbol{\xi}'; \omega) = -\frac{\omega}{4\pi} + \frac{1}{2\pi^2} \int_0^{\Delta\tau} \mathrm{d}s \left(\frac{\cos(\omega s)}{(\Delta \mathsf{x})^2} + \frac{1}{s^2} \right) + \frac{1}{2\pi^2 \Delta\tau} + O\big(\sqrt{\delta}\,\big) \ . \tag{4.14}
$$

It follows that $G^{C_2 \cup C_3 \cup C_4}_{\tau,\tau_0}(\boldsymbol{\xi}, \boldsymbol{\xi}'; \omega)$ has a limit as $(\boldsymbol{\xi}, \boldsymbol{\xi}') \to (\mathbf{0}, \mathbf{0})$ with $\boldsymbol{\xi} \ne \boldsymbol{\xi}'$, given by dropping the O-term from (4.14). We also show in appendix B that the contribution to G_{τ,τ_0} from C_1 is of the form

$$
G^{C_1}_{\tau,\tau_0}(\boldsymbol{\xi}, \boldsymbol{\xi}'; \omega) = O(\delta) \ln(s_0) + O\big(\sqrt{\delta}\,\big) \ . \tag{4.15}
$$

The logarithmic term in (4.15) implies that $G^{C_1}_{\tau,\tau_0}(\boldsymbol{\xi}, \boldsymbol{\xi}'; \omega)$ does not have a limit as $(\boldsymbol{\xi}, \boldsymbol{\xi}') \to (\mathbf{0}, \mathbf{0})$ with $\boldsymbol{\xi} \ne \boldsymbol{\xi}'$, but the coefficient of this term and the integrability of the logarithm imply that the contribution of $G^{C_1}_{\tau,\tau_0}$ to the transition rate (4.3) vanishes in the limit $\epsilon \to 0$. These estimates imply that the $\epsilon \to 0$ limit of the transition rate (4.3) exists and is given by (3.10).

4.3 Switch-on in the asymptotic past

For a detector switched on in the asymptotic past, we must again consider the situation in which the trajectory is defined for arbitrarily negative proper times and the situation in which the trajectory comes from infinity within finite proper time. We shall show in appendix B that in either case the analysis of subsection 4.2 generalises if the trajectory is sufficiently well-behaved in the asymptotic past. A sufficient condition is that the quantities

$$\frac{\mathsf{e}_i \cdot \mathsf{e}'_j}{(\Delta \mathsf{x})^2} \ , \quad \frac{\Delta \mathsf{x} \cdot \mathsf{e}_i}{(\Delta \mathsf{x})^2} \ , \quad \frac{\Delta \mathsf{x} \cdot \mathsf{e}'_i}{(\Delta \mathsf{x})^2} \ , \tag{4.16}$$

all remain bounded as s grows to its asymptotic value, be it $+\infty$ or $\Delta\tau$. We shall show in section 5 that this condition, just as the equivalent one for (3.12), is satisfied for many trajectories of interest and in particular for all stationary ones, whose associated particle spectra can therefore be calculated from (3.11).

5 Applications

In this section we shall first discuss some general properties of our transition rate formulas (3.10) and (3.11) and then apply these formulas to specific trajectories of interest.

5.1 Causality, parity and falloff

We have already noted that the integral formulas (3.10) and (3.11) have the technical advantage of no longer containing a regularisation parameter that would have to be taken to zero after integration. The formulas also show explicitly that the response is causal, as emphasised by Schlicht [6] in the context of formulas (2.7) and (2.16): The transition rate at time τ is a functional of the detector's trajectory at times prior to τ.

Formulas (3.10) and (3.11) give the spectrum as decomposed into its odd and even parts in ω. The odd part is always equal to $-\omega/4\pi$, and only the even part is affected by the past of the trajectory. Departures from inertial motion have thus an equal probability of inciting upward and downward transitions in the detector.

Another useful decomposition of the spectrum is into the inertial part and the non-inertial correction. This may be obtained by adding and subtracting $\cos(\omega s)/s^2$ under the integrals in (3.10) and (3.11). For a detector switched on in the asymptotic past of infinite proper time, (3.11) gives

$$\begin{aligned}\dot{F}_\tau(\omega) &= -\frac{\omega}{4\pi} + \frac{1}{2\pi^2}\int_0^\infty \mathrm{d}s \left[\frac{\cos(\omega s)}{(\Delta \mathsf{x})^2} + \frac{\cos(\omega s)}{s^2} + \left(\frac{1}{s^2} - \frac{\cos(\omega s)}{s^2}\right)\right] \\ &= -\frac{\omega}{2\pi}\Theta(-\omega) + \frac{1}{2\pi^2}\int_0^\infty \mathrm{d}s \cos(\omega s)\left(\frac{1}{(\Delta \mathsf{x})^2} + \frac{1}{s^2}\right) \ ,\end{aligned} \tag{5.1}$$

where the last expression is obtained by the method of residues. The first term in (5.1) is the spectrum of a detector in inertial motion, and the remaining integral term is thus the correction to it due to acceleration. As this correction is the cosine transform of a functional of the trajectory, and as the inertial response term vanishes for positive ω, we can extract from the transition rate information about the trajectory by inverting (5.1):

$$\frac{1}{(\Delta \mathsf{x})^2} + \frac{1}{s^2} = 4\pi \int_0^\infty \mathrm{d}\omega \, \dot{F}_\tau(\omega) \cos(\omega s) \ . \tag{5.2}$$

We can use (5.1) to obtain another interesting property of the spectrum. For sufficiently differentiable trajectories, one expects the non-inertial effects to become negligible at small length scales and the spectrum thus to approach the inertial spectrum at high frequencies. To verify that this indeed happens, we use the following theorem [18]: If the function h is C^∞ in $[a, \infty)$ and $h^{(n)}(s) = O(s^{-1-\epsilon})$ as $s \to \infty$ for some $\epsilon > 0$ and every $n \geq 0$, then

$$\int_a^\infty \mathrm{d}s \, h(s) \mathrm{e}^{ixs} \sim \mathrm{e}^{iax} \sum_{n=0}^\infty h^{(n)}(a) \left(\frac{i}{x}\right)^{n+1} \qquad \text{as } x \to \infty \ . \tag{5.3}$$

Taking $\cos(\omega s) = \mathrm{Re}\left(\mathrm{e}^{i\omega s}\right)$, we apply this theorem to (5.1) with $a = 0$ and $h(s) = 1/s^2 + 1/(\Delta \mathsf{x})^2$, in which case $\epsilon = 2$. The expansion will thus proceed in inverse powers of ω^2, with coefficients given by τ-derivatives of $\mathsf{x}(\tau)$. In the leading order we obtain

$$\dot{F}_\tau(\omega) = -\frac{\omega}{2\pi} \Theta(-\omega) + \frac{\ddot{\mathsf{x}} \cdot \dddot{\mathsf{x}}}{24\pi^2 \omega^2} + O\left(\omega^{-4}\right) \qquad \text{as } \omega \to \infty \ , \tag{5.4}$$

which shows that for a generic trajectory the first correction to the inertial response is of order ω^{-2}. There exist however trajectories for which the first correction is of higher order, owing to the vanishing of some of the coefficients in (5.3). An extreme example is the uniformly accelerated trajectory, for which $\mathsf{x}^{(m)} \cdot \mathsf{x}^{(n)} = 0$ whenever $m + n$ is odd and the coefficients of all inverse powers of ω^2 vanish: The asymptotic behaviour is in this case exponential, as seen from (2.17).

5.2 Stationary trajectories

We shall now examine the consequences of our transition rate formula (3.11) for the six families of *stationary* motions, classified in [19] and reviewed in [20, 21]. These motions have the property that $(\Delta \mathsf{x})^2$ depends only on the proper time difference between the two points along the trajectory. As a consequence, the transition rate of a detector switched on in the infinite past is independent of the proper time. We recall that as these motions are precisely the orbits of timelike Killing vectors in Minkowski space, they are the only motions in which an independent definition of "particles" is available via the positive and negative frequency decomposition with respect of a timelike Killing vector [22, 23, 24].

We consider each of the six families in turn. We shall in particular verify that in each case both (3.12) and (4.16) remain bounded in the asymptotic past, justifying the limiting procedure that led to (3.11) for both of our detector models.

The first stationary trajectory is the inertial motion, $\mathsf{x}(\tau) = \mathsf{u}\tau$, where u is a constant timelike unit vector. The associated Killing vector generates a timelike translation. The non-inertial correction term in (5.1) then vanishes and we obtain the expected result. It is evident that both (3.12) and (4.16) remain bounded at $s \to \infty$.

The second stationary trajectory is the uniformly accelerated motion, or Rindler motion. The associated Killing vector generates a boost. Denoting the proper acceleration by $a > 0$, the trajectory reads

$$\mathsf{x}(\tau) = \big(a^{-1}\sinh(a\tau), a^{-1}\cosh(a\tau), 0, 0\big) , \tag{5.5}$$

and we have $(\Delta\mathsf{x})^2 = -4a^{-2}\sinh^2(as/2)$. The tetrad vectors are

$$\mathsf{e}_1 = \big(\sinh(a\tau), \cosh(a\tau), 0, 0\big) , \quad \mathsf{e}_2 = (0, 0, 1, 0) , \quad \mathsf{e}_3 = (0, 0, 0, 1) . \tag{5.6}$$

The quantities (3.12) and (4.16) remain bounded at $s \to \infty$ since the numerator and denominator in each diverges proportionally to e^{as}.

To compute (3.11), we use the symmetry of the integrand under $s \to -s$ to write

$$\int_0^\infty \mathrm{d}s \left(-\frac{a^2\cos(\omega s)}{4\sinh^2(as/2)} + \frac{1}{s^2} \right) = \frac{1}{2}\int_{-\infty}^\infty \mathrm{d}s \left(-\frac{a^2\cos(\omega s)}{4\sinh^2(as/2)} + \frac{1}{s^2} \right) . \tag{5.7}$$

We then deform the contour to the horizontal line $\mathrm{Im}(s) = -\pi/a$. The second term in (5.7) gives a vanishing contribution by contour integration, while the contribution from the first term becomes [25]

$$\frac{a^2}{8}\cosh(\pi\omega/a)\int_{-\infty}^\infty \mathrm{d}s \frac{\cos(\omega s)}{\cosh^2(as/2)} = \frac{\pi\omega}{2}\coth(\pi\omega/a) . \tag{5.8}$$

Substituting this result in (3.11) yields the Planckian transition rate (2.17). We have thus recovered the Unruh effect.

The third stationary trajectory is the circular motion, given by

$$\mathsf{x}(\tau) = \big(\gamma\tau, R\cos(\gamma\Omega\tau), R\sin(\gamma\Omega\tau), 0\big) , \tag{5.9}$$

where $R > 0$, $\Omega \neq 0$, $|R\Omega| < 1$ and $\gamma = (1 - R^2\Omega^2)^{-1/2}$. The associated Killing vector is a linear combination of a time translation generator and a rotation generator. Solving the Fermi-Walker transport equations yields the tetrad

$$\begin{aligned}
\mathsf{e}_1(\tau) = \big(&-\gamma\Omega R\sin(\gamma^2\Omega\tau),\ \cos(\gamma\Omega\tau)\cos(\gamma^2\Omega\tau) + \gamma\sin(\gamma\Omega\tau)\sin(\gamma^2\Omega\tau),\\
&\sin(\gamma\Omega\tau)\cos(\gamma^2\Omega\tau) - \gamma\cos(\gamma\Omega\tau)\sin(\gamma^2\Omega\tau),\ 0\big) ,\\
\mathsf{e}_2(\tau) = \big(&\gamma\Omega R\cos(\gamma^2\Omega\tau),\ \cos(\gamma\Omega\tau)\sin(\gamma^2\Omega\tau) - \gamma\sin(\gamma\Omega\tau)\cos(\gamma^2\Omega\tau),\\
&\sin(\gamma\Omega\tau)\sin(\gamma^2\Omega\tau) + \gamma\cos(\gamma\Omega\tau)\cos(\gamma^2\Omega\tau),\ 0\big) ,\\
\mathsf{e}_3(\tau) = \big(&0, 0, 0, 1\big) .
\end{aligned} \tag{5.10}$$

At $s \to \infty$, $\Delta\mathsf{x}$ and $(\Delta\mathsf{x})^2$ grow respectively linearly and quadratically in s, while $\dot{\mathsf{x}}$ and each e_i is bounded. The quantities (3.12) and (4.16) therefore remain bounded at $s \to \infty$.

The transition rate (3.11) appears not known in terms of elementary functions. An analytic approximation has been discussed in [26].

The fourth stationary trajectory is the orbit of a Killing vector that is a linear combination of a time translation generator and a null rotation generator. The spatial projection is the planar curve $y = \kappa\frac{\sqrt{2}}{3}x^{3/2}$, and the full trajectory reads

$$\mathsf{x}(\tau) = \bigl(\tau + \tfrac{1}{6}\kappa^2\tau^3,\, \tfrac{1}{2}\kappa\tau^2,\, \tfrac{1}{6}\kappa^2\tau^3,\, 0\bigr) \ , \tag{5.11}$$

where $\kappa > 0$. The tetrad is given by

$$\begin{aligned}
\mathsf{e}_1(\tau) &= \bigl(\kappa\tau\cos(\kappa\tau) + \tfrac{1}{2}\kappa^2\tau^2\sin(\kappa\tau),\, \cos(\kappa\tau) + \kappa\tau\sin(\kappa\tau),\\
&\qquad \kappa\tau\cos(\kappa\tau) + (-1 + \tfrac{1}{2}\kappa^2\tau^2)\sin(\kappa\tau),\, 0\bigr) \ ,\\
\mathsf{e}_2(\tau) &= \bigl(-\tfrac{1}{2}\kappa^2\tau^2\cos(\kappa\tau) + \kappa\tau\sin(\kappa\tau),\, \kappa\tau\cos(\kappa\tau) - \sin(\kappa\tau),\\
&\qquad (1 - \tfrac{1}{2}\kappa^2\tau^2)\cos(\kappa\tau) + \kappa\tau\sin(\kappa\tau),\, 0\bigr) \ ,\\
\mathsf{e}_3(\tau) &= (0,\, 0,\, 0,\, 1) \ ,
\end{aligned} \tag{5.12}$$

and it can be verified that the quantities (3.12) and (4.16) remain bounded at $s \to \infty$. The transition rate can be computed in closed form directly from (3.11) and equals

$$\dot{F}_\tau(\omega) = \frac{\omega}{2\pi}\Theta(-\omega) + \frac{\kappa}{8\sqrt{3}\pi}\, \mathrm{e}^{-2\sqrt{3}|\omega|/\kappa} \ , \tag{5.13}$$

which agrees with the result known from [19].

The fifth stationary trajectory is the orbit of a Killing vector that is a linear combination of a spatial translation generator and a boost generator. The spatial projection is the catenary $y = k\cosh(y/b)$, and the full trajectory reads

$$\mathsf{x}(\tau) = \frac{1}{a^2}\,(k\sinh(a\tau),\, k\cosh(a\tau),\, ba\tau, 0) \ , \tag{5.14}$$

where $a > 0$, $b > 0$ and $k = \sqrt{a^2 + b^2}$. The tetrad vectors are

$$\begin{aligned}
\mathsf{e}_1(\tau) &= \bigl(\sinh(a\tau)\cos(b\tau) + (b/a)\cosh(a\tau)\sin(b\tau),\\
&\qquad \cosh(a\tau)\cos(b\tau) + (b/a)\sinh(a\tau)\sin(b\tau),\, (k/a)\sin(b\tau),\, 0\bigr) \ ,\\
\mathsf{e}_2(\tau) &= \bigl(-\sinh(a\tau)\sin(b\tau) + (b/a)\cosh(a\tau)\cos(b\tau),\\
&\qquad -\cosh(a\tau)\sin(b\tau) + (b/a)\sinh(a\tau)\cos(b\tau),\, (k/a)\cos(b\tau),\, 0\bigr) \ ,\\
\mathsf{e}_3(\tau) &= (0,\, 0,\, 0,\, 1) \ .
\end{aligned} \tag{5.15}$$

As in the uniformly accelerated case, the quantities (3.12) and (4.16) remain bounded at $s \to \infty$ since the numerator and denominator in each diverges proportionally to e^{as}. The transition rate (3.11) appears not known in terms of elementary functions.

The sixth and last stationary trajectory is the orbit of a Killing vector that is a linear combination of a boost generator and a rotation generator. The trajectory reads

$$\mathsf{x}(\tau) = \left(\frac{\alpha}{R_+} \sinh(R_+\tau),\ \frac{\alpha}{R_+} \cosh(R_+\tau),\ \frac{\beta}{R_-} \cos(R_-\tau),\ \frac{\beta}{R_-} \sin(R_-\tau) \right) , \qquad (5.16)$$

where $R_\pm > 0$, $\beta > 0$ and $\alpha = \sqrt{1+\beta^2}$. Defining $\Omega := \sqrt{\alpha^2 R_-^2 + \beta^2 R_+^2}$, the tetrad vectors can be written as

$$\begin{aligned}
\mathsf{e}_1(\tau) = \Omega^{-2}\Big(& \sinh(R_+\tau) \left[\alpha^2 R_-^2 + \beta^2 R_+ \Omega \sin(\Omega\tau) + \beta^2 R_+^2 \cos(\Omega\tau)\right] , \\
& \cosh(R_+\tau) \left[\alpha^2 R_-^2 + \beta^2 R_+ \Omega \sin(\Omega\tau) + \beta^2 R_+^2 \cos(\Omega\tau)\right] , \\
& \alpha\beta R_+ \left[R_-\big(1 - \cos(\Omega\tau)\big) \cos(R_-\tau) - \Omega \sin(\Omega\tau) \sin(R_-\tau)\right] , \\
& \alpha\beta R_+ \left[R_-\big(1 - \cos(\Omega\tau)\big) \sin(R_-\tau) + \Omega \sin(\Omega\tau) \cos(R_-\tau)\right] \Big) , \\
\mathsf{e}_2(\tau) = \Omega^{-2}\Big(& \alpha\beta R_- \left[R_+\big(1 - \cos(\Omega\tau)\big) \sinh(R_+\tau) - \Omega \sin(\Omega\tau) \cosh(R_+\tau)\right] , \\
& \alpha\beta R_- \left[R_+\big(1 - \cos(\Omega\tau)\big) \cosh(R_+\tau) - \Omega \sin(\Omega\tau) \sinh(R_-\tau)\right] , \\
& \cos(R_-\tau)\big(\beta^2 R_+^2 + \alpha^2 R_-^2 \cos(\Omega\tau)\big) + \alpha^2 R_- \Omega \sin(R_-\tau) \sin(\Omega\tau), \\
& \sin(R_-\tau)\big(\beta^2 R_+^2 + \alpha^2 R_-^2 \cos(\Omega\tau)\big) - \alpha^2 R_- \Omega \cos(R_-\tau) \sin(\Omega\tau)\Big) , \\
\mathsf{e}_3(\tau) = \Big(& \beta \left[\cos(\Omega\tau) \cosh(R_+\tau) - (R_+/\Omega) \sin(\Omega\tau) \sinh(R_+\tau)\right] , \\
& \beta \left[\cos(\Omega\tau) \sinh(R_+\tau) - (R_+/\Omega) \sin(\Omega\tau) \cosh(R_+\tau)\right] , \\
& \alpha \left[- \cos(\Omega\tau) \sin(R_-\tau) - (R_-/\Omega) \sin(\Omega\tau) \cos(R_+\tau)\right] , \\
& \alpha \left[- \cos(\Omega\tau) \cos(R_-\tau) + (R_-/\Omega) \sin(\Omega\tau) \sin(R_+\tau)\right] \Big) .
\end{aligned} \qquad (5.17)$$

Once again the quantities (3.12) and (4.16) remain bounded at $s \to \infty$ since both the numerators and the denominators grow as $\mathrm{e}^{R_+ s}$. The transition rate (3.11) appears not known in terms of elementary functions.

5.3 From asymptotically inertial motion to asymptotically uniform acceleration

As a first example of nonstationary motion, we consider the trajectory

$$\mathsf{x}(\tau) = \left(\tau + (2a)^{-1}\left[\mathrm{e}^{a\tau} - \ln(\mathrm{e}^{a\tau}+1)\right] , (2a)^{-1}\left[\mathrm{e}^{a\tau} + \ln(\mathrm{e}^{a\tau}+1)\right] , 0 , 0\right) , \qquad (5.18)$$

where $a > 0$. The magnitude of the proper acceleration is $a/(1 + \mathrm{e}^{-a\tau})$. The trajectory is asymptotically inertial as $\tau \to -\infty$ and has asymptotically uniform acceleration a as $\tau \to \infty$. As the quantities (3.12) and (4.16) are clearly bounded as $s \to \infty$, we are justified to use the transition rate formulas (3.11) and (5.1) for a detector switched on in the asymptotic past.

At large negative τ, one expects the transition rate to asymptote to that of inertial motion. To examine the noninertial term in (5.1) in this limit, we rearrange the integrand as

$$\frac{1}{(\Delta \mathsf{x})^2} + \frac{1}{s^2} = \frac{-(\Delta \mathsf{x})^2 - s^2}{s^4} \left(1 + \frac{-(\Delta \mathsf{x})^2 - s^2}{s^2} \right)^{-1} \tag{5.19}$$

and note that (5.18) yields

$$-(\Delta \mathsf{x})^2 = \frac{1}{a^2} \left[as + \frac{g(1 - \mathrm{e}^{-as})}{(1-g)} \right] \left\{ as + \ln\left[1 - g(1 - \mathrm{e}^{-as})\right] \right\} , \tag{5.20}$$

where $g := 1/(1 + \mathrm{e}^{-a\tau})$. Expanding in (5.20) the logarithm as $\ln(1-x) = -x - \frac{1}{2}x^2 + O(x^3)$ shows that the second factor in (5.19) is of the form $1 + O(g^2)$, uniformly in s, and yields then in the first factor an estimate that can be applied under the integral over s. We find

$$\dot{F}_\tau(\omega) = -\frac{\omega}{2\pi}\Theta(-\omega) + \frac{a}{2\pi^2} h(\omega/a) \mathrm{e}^{2a\tau} + O(\mathrm{e}^{3a\tau}) , \quad \tau \to -\infty , \tag{5.21}$$

where

$$h(x) := \frac{1}{2} \int_0^\infty \mathrm{d}y \, \frac{(1 - \mathrm{e}^{-y})\left[(2+y)\mathrm{e}^{-y} - 2 + y\right] \cos(yx)}{y^4} . \tag{5.22}$$

An expression for h in terms of elementary functions can be found by repeated integration by parts and use of formulas 3.434 in [25]. $h(x)$ is even in x and decreasing in x^2, and it has the asymptotic expansions $h(x) = \frac{1}{3}(1 - \ln 2) + O\big(x^2 \ln(|x|)\big)$ as $x \to 0$ and $h(x) = 1/(12x^2) + O\big(x^{-4}\big)$ as $|x| \to \infty$. The transition rate thus asymptotes to that of inertial motion as $\mathrm{e}^{2a\tau}$ when $\tau \to -\infty$.

At large positive τ, one expects the transition rate to asymptote to that of uniformly accelerated motion. We have verified from (5.1) that this is the case, using a monotone convergence argument to take the limit under the integral, but we have not pursued an estimate for the error term.

5.4 From vanishing to diverging acceleration

As the last example, we consider the two trajectories given by

$$\mathsf{x}(\tau) = \left(\frac{\tau^3}{6} - \frac{1}{2\tau}, \frac{\tau^3}{6} + \frac{1}{2\tau}, 0, 0 \right) , \tag{5.23}$$

one with $\tau \in (0, \infty)$ and the other with $\tau \in (-\infty, 0)$. The acceleration has magnitude $2/|\tau|$. The latter case was discussed in [7].

Each of these trajectories resides entirely in one Rindler quadrant. The trajectory with $\tau < 0$ approaches $\mathcal{I}^-$ as $\tau \to -\infty$ and $\mathcal{I}^+$ as $\tau \to 0_-$, having thus taken an infinite amount of proper time to come from $\mathcal{I}^-$ in the past but reaching $\mathcal{I}^+$ in finite proper time in the future. Note that this trajectory is not asymptotically inertial in the past even though the proper acceleration tends to zero. The trajectory with $\tau > 0$ is the time-reversed version, approaching $\mathcal{I}^-$ as $\tau \to 0_+$ and $\mathcal{I}^+$ as $\tau \to \infty$ and having taken a finite amount of proper time to come from $\mathcal{I}^-$ in the past.

The tetrad vectors are

$$\mathsf{e}_1(\tau) = \left(\frac{\tau^2}{2} - \frac{1}{2\tau^2}, \frac{\tau^2}{2} + \frac{1}{2\tau^2}, 0, 0\right), \quad \mathsf{e}_2(\tau) = (0,0,1,0), \quad \mathsf{e}_3(\tau) = (0,0,0,1) \ . \tag{5.24}$$

On the trajectory with $\tau > 0$, both of the quantities (3.12) and the first and third of the quantities (4.16) diverge as $s \to \tau$. The limiting arguments of subsections 3.2 and 4.3 do therefore not justify our formula for a detector turned on in the asymptotic past (of finite proper time) for this trajectory. We have not investigated whether an improved set of limiting arguments could be found.

On the trajectory with $\tau < 0$, the quantities (3.12) and (4.16) all remain bounded as $s \to \infty$, and we are justified to use formula (5.1) for a detector switched on in the asymptotic past. We find

$$\dot{F}_\tau(\omega) = -\frac{\omega}{2\pi}\Theta(-\omega) + \frac{1}{2\pi^2(-\tau)} \int_0^\infty \mathrm{d}y \, \frac{\cos(\omega\tau y)}{y^2 + 3y + 3} \ , \tag{5.25}$$

where we have introduced the new integration variable $y := s/(-\tau)$. For $\omega = 0$, the noninertial term in (5.25) equals $\left[6\pi\sqrt{3}(-\tau)\right]^{-1}$. For $\omega \neq 0$, the asymptotic behaviour at $\tau \to -\infty$ can be found using (5.3) and that at $\tau \to 0_-$ by techniques similar to those in section 3. The result is

$$\dot{F}_\tau(\omega) = -\frac{\omega}{2\pi}\Theta(-\omega) + \frac{3}{2\pi^2\omega^2(-\tau)^3}\left[1 + O\left(\frac{1}{\omega^2\tau^2}\right)\right] \ , \quad \tau \to -\infty \ , \tag{5.26a}$$

$$\dot{F}_\tau(\omega) = \frac{1}{6\pi\sqrt{3}(-\tau)} - \frac{\omega}{4\pi} - \frac{3|\omega|}{4\pi^2}\Big[|\omega\tau| \ln(|\omega\tau|) + O\big(|\omega\tau|\big)\Big] \ , \quad \tau \to 0_- \ . \tag{5.26b}$$

The noninertial contribution to the transition rate thus vanishes as $(-\tau)^{-3}$ when $\tau \to -\infty$ for $\omega \neq 0$ but diverges as $(-\tau)^{-1}$ when $\tau \to 0_-$. This is consistent with what one might have expected from the magnitude of the proper acceleration in these limits.

6 Conclusions

We have analysed a particle detector model whose coupling to a massless scalar field in four-dimensional Minkowski space is regularised by a spatial profile, rigid in the detector's instantaneous rest frame. When the profile is given by the Lorentzian function as in (2.13) and (2.15), we computed explicitly the zero-size limit of the instantaneous transition rate, obtaining a manifestly finite integral formula that no longer involves regulators or limits. We then considered a detector model with a modified definition of spatial smearing and showed, under certain technical conditions, that the instantaneous transition rate is independent of the choice of the profile function and agrees with that obtained from the (unmodified) smearing with the Lorentzian profile. The formulas for the transition rate in the cases of finite and infinite proper time of detection are, respectively,

$$\dot{F}_\tau(\omega) = -\frac{\omega}{4\pi} + \frac{1}{2\pi^2}\int_0^{\Delta\tau} \mathrm{d}s \left(\frac{\cos(\omega s)}{(\Delta\mathsf{x})^2} + \frac{1}{s^2}\right) \; + \frac{1}{2\pi^2 \Delta\tau} \tag{6.1}$$

and

$$\dot{F}_\tau(\omega) = -\frac{\omega}{4\pi} + \frac{1}{2\pi^2}\int_0^{\infty} \mathrm{d}s \left(\frac{\cos(\omega s)}{(\Delta\mathsf{x})^2} + \frac{1}{s^2}\right) \; . \tag{6.2}$$

For a detector switched on in the asymptotic past, we have justified these formulas from our detector models under the assumption that the quantities (3.12) or (4.16) remain bounded in the asymptotic past. We have not examined whether this boundedness condition could be relaxed.

We showed that the acceleration only affects the part of the transition rate that is even as a function of the frequency ω, and we obtained a method to compute the coefficients of all inverse powers of ω^2 in the asymptotic large ω^2 expansion of the transition rate. Finally, we applied our transition rate formula to a number of examples, including all stationary trajectories. We recovered in particular the Unruh effect for uniform acceleration, and we obtained an interpolating transition rate for a nonstationary trajectory that interpolates between asymptotically inertial and asymptotically uniformly accelerated motion. We believe that these results strongly support the use of (6.1) and (6.2) as defining the instantaneous transition rate of an Unruh particle detector.

We re-emphasise that the need for a spatial smearing arose because we chose to address the instantaneous transition rate of the detector *while the interaction is switched on*, rather than the total excitation probability after the interaction has been smoothly switched on and off. The formal first-order perturbation theory expression for the transition rate involves in this case the field's Wightman function in a way that is ill-defined without regularisation. As first observed by Schlicht for the Rindler trajectory [6], and as we have verified in appendix A for arbitrary noninertial trajectories, the conventional $i\epsilon$ regularisation of the Wightman function in a given Lorentz frame results into a Lorentz-noninvariant expression for the transition rate and is hence not viable. The regularisation by a spatial profile, by contrast, is manifestly Lorentz invariant.

Our modified detector model of section 4 was related to spatial smearing with a profile function of compact support, and we showed that the transition rate in this model is independent of the profile function. We did not demonstrate the model to be equivalent to spatial smearing with a profile function of compact support, owing to the possibility that the integration over the spatial surfaces as defined in (4.3) could miss a distributional part of the integrand. However, as the modified detector model yields in the zero size limit the same transition rate as spatial smearing with the Lorentzian profile function (which is not of compact support), we suspect the model to be equivalent to spatial smearing for at least some classes of profile functions. This question would deserve further study.

An alternative to spatial smearing could be to define the instantaneous transition rate by starting with a pointlike detector and smooth switching function of compact support and then taking a limit in which the switching function approaches the characteristic function of an interval. We are not aware of reasons to expect an unambiguous limit to exist, but might there be specific limiting prescriptions that reproduce the result obtained with spatial smearing?

With a sharp switch-on at the initial time τ_0, the transition rate (6.1) diverges as $(\tau - \tau_0)^{-1}$ when $\tau \to \tau_0$. The total transition probability, obtained by integrating the transition rate, is therefore infinite, owing to the violent switch-on event, regardless how small the coupling constant in the interaction Hamiltonian is. For the stationary trajectories the transition rate (6.2) of a detector switched on in the asymptotic past is constant in time, and the total transition probability is again infinite, now owing to the infinite amount of time elapsed in the past. In these situations one may therefore have reason to view our results, all of which were obtained within first-order perturbation theory, as suspect, and perhaps even to question the whole notion of the instantaneous transition rate. However, in situations where the detector is switched on in the asymptotic past of infinite proper time and the total probability of excitation ($\omega > 0$) is finite, the first-order perturbation theory result should be reliable at least for the excitation rate, although the total probability of de-excitation ($\omega < 0$) then still diverges. Two examples of this situation, with an acceleration that vanishes asymptotically in the past, were found in section 5.

When the total excitation probability is finite, it has a directly observable meaning as the fraction of detectors that have become excited in an ensemble that is initially prepared in the state $|0\rangle_d$ and follows the trajectory $\mathsf{x}(\tau)$. Note, however, that observing the ensemble changes the initial conditions for the subsequent dynamics, and a single ensemble can thus be used to measure the excitation probability at only one value of proper time. To measure the excitation probability at several values of the proper time requires a family of identically prepared ensembles, each of which will be used to read off the excitation probablility at one value of the proper time only. The excitation rate, proportional to $\dot{F}_\tau(\omega)$, is then the proper time derivative of this probability. Relating $\dot{F}_\tau(\omega)$ to the acceleration effects that may become practically observable in particle accelerators [27] remains thus a subtle issue [20, 21].[1]

It would be interesting to investigate to what extent our results can be generalised to the variety of situations to which Schlicht's Lorentzian profile detector was generalised in [8, 17]. For example, do the formulas (6.1) and (6.2) generalise to spacetime dimensions other than four, and if yes, what is the form of the subtraction term? Does the clean separation of the spectrum into its even and odd parts continue? Further, to what extent can the notion of spatial profile be employed to regularise the transition rate in a curved spacetime, presumably reproducing known results for stationary trajectories [28]

[1]We thank Hans Westman for discussions on this point.

but also allowing nonstationary motion? In particular, might there be a connection with the regularisation prescriptions of the classical self-force problem [29]? Finally, would a nonperturbative treatment be feasible?

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A Appendix: $i\epsilon$ regularisation in a given Lorentz frame

In this appendix we evaluate the $\epsilon \to 0$ limits of the instantaneous transition rates (2.6) and (2.7) for for the $i\epsilon$ regularisation (2.9) in a given Lorentz frame. The results will differ from (3.10) and (3.11) by an additive Lorentz-noninvariant term. This generalises observations of Schlicht in the special case of uniformly accelerated motion [6] and supports the view that the $i\epsilon$ regularisation is physically inappropriate in the context of instantaneous transition rate calculations.

The notation follows subsection 3.1.

We start with finite τ_0 and assume the trajectory to be C^9 in closed interval $[\tau_0, \tau]$. With the $i\epsilon$ regularised correlation function (2.9), the transition rate for a detector switched on at τ_0 reads

$$\dot{F}_\tau(\omega) = \frac{1}{2\pi^2} \operatorname{Re} \int_0^{\Delta\tau} \mathrm{d}s \, \frac{\mathrm{e}^{-i\omega s}}{|\mathbf{x}(\tau) - \mathbf{x}(\tau - s)|^2 - \left[t(\tau) - t(\tau - s) - i\epsilon\right]^2} \ . \tag{A.1}$$

The decomposition of $\dot{F}_\tau(\omega)$ into its even and odd parts in ω is obtained from formulas (3.2) with the replacements

$$\mathsf{q}^2 \to -1, \quad \mathsf{q} \cdot \Delta\mathsf{x} \to -\Delta t \ , \tag{A.2}$$

where $\Delta t := t(\tau) - t(\tau - s)$. After the further decomposition into contributions from the integration subintervals $s \in \left[0, \sqrt{\epsilon}\right]$ and $s \in \left[\sqrt{\epsilon}, \Delta\tau\right]$, the estimates of subsection 3.1 readily adapt to show that the contribution from the latter subinterval is again given by (3.5).

In the subinterval $s \in \left[0, \sqrt{\epsilon}\right]$, the denominators have now the expansion

$$\left[\epsilon^2 + (\Delta\mathsf{x})^2\right]^2 + 4\epsilon^2(\Delta t)^2 = \epsilon^2 P \left[1 + \tfrac{1}{6}\ddot{\mathsf{x}}^2 \epsilon r^2 - 4\dot{t}\ddot{t}P^{-1}\epsilon^{3/2}r^3 + O(\epsilon^{3/2})\right] \ , \tag{A.3}$$

where $s = \sqrt{\epsilon} r$ and $P := \epsilon^2 + 2(2\dot{t}^2 - 1)\epsilon r^2 + r^4$. Note that the term $P^{-1}\epsilon^{3/2}r^3$ is of order $O(\epsilon)$, uniformly in r.

Keeping in $\dot{F}_<^{\text{odd}}$ just the leading term in (A.3) and in the numerator just the leading power of s, we find

$$\begin{aligned} \dot{F}_<^{\text{odd}} &= -\frac{\omega \dot{t}\sqrt{\epsilon}}{\pi^2}\int_0^1 \mathrm{d}r\, \frac{r^2}{P}\bigl[1 + O(\epsilon)\bigr] \\ &= -\frac{\omega}{4\pi} + O(\sqrt{\epsilon}) \ , \end{aligned} \tag{A.4}$$

where the integral is elementary. In $\dot{F}_<^{\text{even}}$ we use (A.3) in the denominator and expand the numerator to next-to-leading order in s, with the result

$$\begin{aligned} \dot{F}_<^{\text{even}} &= \frac{1}{2\pi^2\sqrt{\epsilon}}\int_0^1 \mathrm{d}r\, \frac{(\epsilon - r^2)\left[1 + 4\dot{t}\ddot{t}P^{-1}\epsilon^{3/2}r^3 + r^4 O(\epsilon) + r^3 O(\epsilon^{3/2}) + O(\epsilon^{5/2})\right]}{P} \\ &= \frac{1}{2\pi^2\sqrt{\epsilon}} - \frac{1}{4\pi^2}\frac{\ddot{t}}{(\dot{t}^2-1)^{3/2}}\left[\dot{t}\sqrt{\dot{t}^2-1} + \ln\Bigl(\dot{t} - \sqrt{\dot{t}^2-1}\Bigr)\right] + O(\sqrt{\epsilon}) \ , \end{aligned} \tag{A.5}$$

where the integrals are elementary, and in the last expression the term involving $\ddot{t}$ should for $\dot{t} = 1$ be understood as its limiting value 0. .

Combining these results and taking the limit $\epsilon \to 0$, we find that the transition rate differs from (3.10) by the additive term

$$-\frac{1}{4\pi^2}\frac{\ddot{t}}{(\dot{t}^2-1)^{3/2}}\left[\dot{t}\sqrt{\dot{t}^2-1} + \ln\Bigl(\dot{t} - \sqrt{\dot{t}^2-1}\Bigr)\right] \ , \tag{A.6}$$

understood for $\dot{t} = 1$ as its limiting value 0. The term (A.6) clearly vanishes for inertial trajectories. Given a point at which the proper acceleration is nonzero, (A.6) vanishes in Lorentz frames in which ∂_t is the velocity but is nonvanishing in Lorentz frames in which ∂_t is in the plane spanned by the velocity and the acceleration but not proportional to the velocity. The term (A.6) is therefore Lorentz invariant only for inertial trajectories.

Finally, these observations generalise to the switch-on in the asymptotic past provided the trajectory is asymptotically sufficiently well-behaved. From (3.12) and (A.2) it is seen that a sufficient condition is that $(\Delta t)/(\Delta \mathsf{x})^2$ remains bounded as s increases. This condition is in particular satisfied for uniformly accelerated motion. We have verified that the analytical and numerical results given in [6] for uniformly accelerated motion are consistent with the sum of the Planckian spectrum (2.17) and the Lorentz-noninvariant term (A.6).

B Appendix: Integral estimates for section 4

In this appendix we provide the required estimates for the integral (4.7) over the contour (4.13). We write $\eta := \sqrt{\delta}$.

B.1 C_1

We parametrise C_1 as in (4.13), $s = -ir$ with $0 \le r \le \eta$. Let

$$R(s) := s_0^2 + H_2 s^2 \ . \tag{B.1}$$

As it follows from (4.12) that $R > 0$ for sufficiently small η, we may write the contribution to (4.7) from C_1 as

$$G^{C_1}_{\tau,\tau_0}(\boldsymbol{\xi},\boldsymbol{\xi}';\omega) = \frac{1}{2\pi^2}\,\mathrm{Im}\int_0^\eta \mathrm{d}r\,\frac{\mathrm{e}^{-\omega r}}{R\big[1+(\mathsf{H}^2-R)/R\big]} \ . \tag{B.2}$$

As $r^2/R = O(\eta^0)$ and the small s expansion of $\mathsf{H}^2 - R$ starts with s^3, we may approximate the factor $\big[1+(\mathsf{H}^2-R)/R\big]^{-1}$ in (B.2) by the first two terms in its geometric series expansion, at the expense of an error of order $O(\eta)$ in the integral in (B.2). The contribution from the zeroth order term vanishes on taking the imaginary part. In the first order term we may replace $\mathsf{H}^2 - R$ by its s^3 term and the factor $\mathrm{e}^{-\omega r}$ by 1 at the expense of an error of order $O(\eta)$ in the integral. We thus obtain

$$G^{C_1}_{\tau,\tau_0}(\boldsymbol{\xi},\boldsymbol{\xi}';\omega) = -\frac{H_3}{2\pi^2}\int_0^\eta \mathrm{d}r\,\frac{r^3}{\big[s_0^2 - H_2 r^2\big]^2} \quad + O(\eta) \ . \tag{B.3}$$

Performing the elementary integral and using $s_0 = O(\eta^2)$, we find

$$G^{C_1}_{\tau,\tau_0}(\boldsymbol{\xi},\boldsymbol{\xi}';\omega) = \frac{H_3}{2\pi^2 H_2^2}\ln(s_0) \quad + O(\eta) \ . \tag{B.4}$$

As the coefficient of $\ln(s_0)$ in (B.4) is of order $O(\eta^2)$, (4.15) follows.

We note that for the uniformly accelerated trajectory the coefficient of $\ln(s_0)$ in (B.4) vanishes, since for this trajectory $\dddot{\mathsf{x}}$ is proportional to $\dot{\mathsf{x}}$ and H_3 vanishes by (4.12).

B.2 $C_2 \cup C_3 \cup C_4$

On $C_2 \cup C_3$, it follows from (4.13) and $0 < s_0 < 2\delta = 2\eta^2$ that $|-s^2 + s_0^2| > \frac{1}{2}\eta^2$ for sufficiently small η. Using (4.11) and (4.12), we may thus write the integrand in (4.7) as

$$\frac{\mathrm{e}^{-i\omega s}}{\mathsf{H}^2} = \frac{1 - i\omega s}{s_0^2 - s^2} + O(\eta^0) \ . \tag{B.5}$$

The integral of the first term in (B.5) is elementary and the integral of the second term is of order $O(\eta)$. We obtain

$$G^{C_2\cup C_3}_{\tau,\tau_0}(\boldsymbol{\xi},\boldsymbol{\xi}';\omega) = \frac{1}{4\pi^2 s_0}\ln\left(\frac{\eta+s_0}{\eta-s_0}\right) - \frac{\omega}{4\pi} + O(\eta) \ . \tag{B.6}$$

Consider then C_4 under the assumptions of subsection 4.2: τ_0 is finite and the trajectory is defined in the closed proper time interval $[\tau_0, \tau]$. We add and subtract $1/(s^2 - s_0^2)$ in the integrand in (4.7), obtaining

$$G^{C_4}_{\tau,\tau_0}(\boldsymbol{\xi}, \boldsymbol{\xi}'; \omega) = \frac{1}{2\pi^2} \int_\eta^{\Delta\tau} \mathrm{d}s \, \left(\frac{\cos(\omega s)}{\mathsf{H}^2} + \frac{1}{s^2 - s_0^2} \right) \, , \\ - \frac{1}{4\pi^2 s_0} \ln \left(\frac{\eta + s_0}{\eta - s_0} \right) + \frac{1}{4\pi^2 s_0} \ln \left(\frac{\Delta\tau + s_0}{\Delta\tau - s_0} \right) \, , \tag{B.7}$$

where we have evaluated the integral of the subtraction term. The second logarithm term in (B.7) is equal to $(2\pi^2 \Delta\tau)^{-1}$ plus a correction of order $O(\eta^4)$. Adding (B.6) and (B.7), the η-dependent logarithm terms cancel and we obtain

$$G^{C_2 \cup C_3 \cup C_4}_{\tau,\tau_0}(\boldsymbol{\xi}, \boldsymbol{\xi}'; \omega) = -\frac{\omega}{4\pi} + \frac{1}{2\pi^2} \int_\eta^{\Delta\tau} \mathrm{d}s \, \left(\frac{\cos(\omega s)}{\mathsf{H}^2} + \frac{1}{s^2 - s_0^2} \right) \; + \frac{1}{2\pi^2 \Delta\tau} + O(\eta) \, . \tag{B.8}$$

Let I denote $2\pi^2$ times the integral term in (B.8). Adding and subtracting in the integrand its limiting value as $(\boldsymbol{\xi}, \boldsymbol{\xi}') \to (\mathbf{0}, \mathbf{0})$, we obtain the rearrangement $I = I_1 + I_2 + I_3$, where

$$I_1 := \int_\eta^{\Delta\tau} \mathrm{d}s \, \left(\frac{\cos(\omega s)}{(\Delta\mathsf{x})^2} + \frac{1}{s^2} \right) \, , \tag{B.9a}$$

$$I_2 := \int_\eta^{\Delta\tau} \mathrm{d}s \, \left(\frac{1}{(s^2 - s_0^2)} - \frac{1}{s^2} \right) \, , \tag{B.9b}$$

$$I_3 := \int_\eta^{\Delta\tau} \mathrm{d}s \, \cos(\omega s) \left(\frac{1}{\mathsf{H}^2} - \frac{1}{(\Delta\mathsf{x})^2} \right) \, . \tag{B.9c}$$

In I_1, it follows from (3.3a) that the integrand has a small s expansion that starts with a constant term, and the lower limit can hence be replaced by zero at the expense of an error of order $O(\eta)$. I_2 is elementary and of order $O(\eta)$. In I_3, we rearrange the integrand as

$$-\frac{\cos(\omega s)}{(\Delta\mathsf{x})^2} \left[1 - \frac{1}{1 + \dfrac{\mathsf{H}^2 - (\Delta\mathsf{x})^2}{(\Delta\mathsf{x})^2}} \right] \, . \tag{B.10}$$

It follows from (3.3a), (4.11), (4.12) and the inequalities $s_0 < 2\eta^2$ and $s^2 \le \left|(\Delta\mathsf{x})^2\right|$ that the combination $\left[\mathsf{H}^2 - (\Delta\mathsf{x})^2\right] / \left[(\Delta\mathsf{x})^2\right]$ is of order $O(\eta^2)$. Hence (B.10) is of the form $O(\eta^2)/s^2$, from which we obtain $I_3 = O(\eta)$. Substituting these observations in (B.8), equation (4.14) follows.

Consider finally C_4 under the assumptions of subsection 4.3: Either the trajectory is defined for arbitrarily negative proper times and $\Delta\tau$ is replaced by infinity, or the trajectory has come from infinity in finite proper time and τ_0 is understood as the (asymptotic) value of the proper time at which the trajectory starts out at infinity. In either case the only nontrivial change occurs in that now there is a need to control the integrand in (B.9c), given by (B.10), also as s increases respectively to ∞ or to $\tau - \tau_0$. A sufficient condition for this control is to assume that the quantities (4.16) all remain bounded as s increases. Under this assumption it follows from (4.8)–(4.12) that $\mathsf{H}^2/\left[(\Delta\mathsf{x})^2\right] = 1 + O(\eta^2)$, uniformly in s. Hence (B.10) is again of the form $O(\eta^2)/s^2$, and we obtain $I_3 = O(\eta)$.

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