Geometrization of the Strong Novikov Conjecture for residually finite groups *

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Abstract

In this paper, we prove that the Strong Novikov Conjecture for a residually finite group is essentially equivalent to the Coarse Geometric Novikov Conjecture for a certain metric space associated to the group. As an application, we obtain the Coarse Geometric Novikov Conjecture for a large class of sequences of expanders.

1. Introduction

Let Γ be a finitely generated residually finite group, let $\{\Gamma_n\}_{n=1}^{\infty}$ be a sequence of finite index normal subgroups of Γ such that $\Gamma_n \supseteq \Gamma_{n+1}$ and $\bigcap_{n=1}^{\infty} \Gamma_n = \{e\}$. The purpose of this paper is to prove that the Strong Novikov Conjecture for Γ and $\{\Gamma_n\}_{n=1}^{\infty}$ is essentially equivalent to the Coarse Geometric Novikov Conjecture for the box metric space $\bigsqcup_{n=1}^{\infty} \Gamma/\Gamma_n$ (Theorem 5.2). As an application, we obtain the Coarse Geometric Novikov Conjecture for a large class of sequences of expanders.

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The Coarse Geometric Novikov Conjecture holds for bounded geometry metric spaces which are coarsely embeddable into Hilbert space [27]. More generally, Kasparov and Yu proved the Coarse Geometric Novikov Conjecture for bounded geometry metric spaces which are coarsely embeddable into uniformly convex Banach spaces [16]. Recall that if Γ is an infinite group with property T, then the box metric space is a sequence of expanders and therefore does not admit a coarse embedding into Hilbert space [18, 23]. Lafforgue has constructed residually finite property T groups whose associated sequences of expanders are not coarsely embeddable into any uniformly convex Banach space [17]. The Strong Novikov Conjecture holds for many infinite groups with property T [5, 6, 9, 10, 14, 15, 24, 26, 27]. As a consequence, our main result implies the Coarse Geometric Novikov Conjecture for a large class of sequences of expanders. In particular, we obtain the Coarse Geometric Novikov Conjecture for Lafforgue's sequences of expanders in [17], which are not coarsely embeddable into any uniformly convex Banach space.

2. Rips complex and its K-homology

In this section, we review the concept of Rips complex and prove a useful result about equivariant K-homology of Rips complexes.

2.1. Let Γ be a finitely generated discrete group with a finite generating set S. We assume that $S = S^{-1}$, that is, $g \in S$ if and only if $g^{-1} \in S$. Define the word length metric d on Γ by

$$d(x, y) = \min\{k \mid x^{-1}y = g_1 g_2 \cdots g_k,\ g_i \in S,\ i = 1, 2, \cdots, k\}.$$

In this paper, we use $|\Gamma|$ to denote the underlining metric space of a finitely generated group Γ endowed with the word length metric. The left multiplication of Γ gives an isometric Γ-action on $(|\Gamma|, d)$.

2.2. In this paper, all the discrete metric spaces X are assumed to have bounded geometry, i.e., for any $r > 0$, there exists $N > 0$, such that

$\#B_r(x) \leq N$, where $B_r(x) = \{y \in X : d(y, x) \leq r\}$. Note that if $X = |\Gamma|$, the underlying metric space of a finitely generated discrete group Γ, then X has bounded geometry.

2.3. Definition (Rips Complex). For any $d > 0$, the Rips complex $P_d(X)$ is the finite dimensional simplicial polyhedron defined as follows:

(1) the vertex set of $P_d(X)$ is X.

(2) any $q+1$ vertices $x_0, x_1, \cdots, x_q$ span a simplex of $P_d(X)$ if and only if

$$d(x_i, x_j) \leq d, \quad \forall i, j \in \{0, 1, 2, \cdots, q\}.$$

Since X has bounded geometry, for each fixed d, $P_d(X)$ is a locally finite simplicial complex, that is, each vertex belongs to finitely many simplices.

2.4. Endow $P_d(X)$ with the spherical metric. Recall that on each path connected component of $P_d(X)$, the spherical metric is the maximal metric whose restriction to each simplex $\{\sum_{i=0}^q t_i x_i | t_i \geq 0, \sum_{i=0}^q t_i = 1\}$ is the metric obtained by identifying the simplex with S_+^q via the map

$$\sum_{i=0}^q t_i x_i \mapsto \left(\frac{t_0}{\sum_{i=0}^q t_i^2}, \frac{t_1}{\sum_{i=0}^q t_i^2}, \cdots, \frac{t_q}{\sum_{i=0}^q t_i^2} \right)$$

where $S_+^q := \{(s_0, s_1, \cdots, s_q) \in \mathbb{R}^{q+1}, s_i \geq 0, \sum_{i=0}^q s_i = 1\}$ is endowed with the standard Riemannian metric. If y_0, y_1 belong to two different connected components Y_0, Y_1 of $P_d(X)$, we define

$$d(y_0, y_1) = \min\{d(y_0, x_0) + d_X(x_0, x_1) + d(x_1, y_1) | x_0 \in X \cap Y_0, x_1 \in X \cap Y_1\}.$$

The topology induced by the above metric is the same as the weak topology of the simplicial complex: a subset $S \subset P_d(X)$ is closed if and only if the intersection of S with each simplex is closed.

If $d < d'$, then $P_d(X)$ is a subcomplex of $P_{d'}(X)$. Denote the inclusion of $P_d(X)$ into $P_{d'}(X)$ by $i_{d',d}$. Let $P_\infty(X) = \bigcup_{d=1}^\infty P_d(X)$, with the topology of simplicial complex, that is, a set $A \subset P_\infty(X)$ is closed if and only if $A \cap P_d(X)$ is closed for each $d > 0$. Also, denote the embedding from $P_d(X)$ to $P_\infty(X)$

by $i_{\infty,d}$. Note that $P_\infty(X)$ is not a locally finite simplicial complex unless X is a finite set.

2.5. If Γ is a finitely generated discrete group, then there is a natural action of Γ on $P_\infty(\Gamma)$:

$$g(t_0x_0 + t_1x_1 + \cdots + t_qx_q) = t_0gx_0 + t_1gx_1 + \cdots + t_qgx_q.$$

This Γ-action is proper, and $P_\infty(\Gamma)$ is a model of the universal space $\underline{E}\Gamma$ of proper Γ-actions. We also have $g(P_d(\Gamma)) \subset P_d(\Gamma)$ for any $g \in \Gamma$ and $d > 0$. Note that the topology introduced in [3] is a little different from the above topology. However, up to weak Γ-homotopy, they are the same.

Note that for any compact subspace $C \subset P_\infty(\Gamma)/\Gamma$, there is a $d > 0$ such that $C \subset P_d(\Gamma)/\Gamma$.

2.6. Let Z be a universal space for proper Γ-actions, with the quotient map $\pi : Z \to Z/\Gamma$. One can define

$$K_*^\Gamma(Z) = \lim_{C\subset Z/\Gamma,\ C \text{ compact}} K_*^\Gamma(\pi^{-1}(C)).$$

It is straight forward to check that

$$K_*^\Gamma(P_\infty(\Gamma)) = \lim_{d\to\infty} K_*^\Gamma(P_d(\Gamma)).$$

If Γ' is a normal subgroup of Γ with Γ/Γ' finite, then $P_\infty(\Gamma)$ with Γ'-action can also be regarded as a classifying space of proper Γ' actions (see 1.9 of [3]). Furthermore,

$$K_*^{\Gamma'}(P_\infty(\Gamma)) = \lim_{d\to\infty} K_*^{\Gamma'}(P_d(\Gamma)).$$

The following proposition will be used in the proof of our main theorem.

Proposition 2.7. If the classifying space for proper Γ-actions has finite homotopy type, i.e., there is a model Z of locally finite CW complex with universal proper Γ-action such that Z/Γ is a compact CW complex, then for any $r > 0$, there is $R > 0$ such that the following is true: for any two elements $x, y \in K_*^{\Gamma'}(P_r(\Gamma))$, where Γ' is a subgroup of Γ with finite index, if $(i_{\infty,r})_*(x) = (i_{\infty,r})_*(y)$ in $K_*^{\Gamma'}(P_\infty(\Gamma))$, then $(i_{R,r})_*(x) = (i_{R,r})_*(y)$ in $K_*^{\Gamma'}(P_R(\Gamma))$.

Proof. By the universal property of Z and $P_\infty(\Gamma)$, there are Γ-equivariant map $\phi : P_\infty(\Gamma) \to Z$ and $\psi : Z \to P_\infty(\Gamma)$ such that $\phi \circ \psi \sim_h id_Z$ and $\psi \circ \phi \sim_h id_{P_\infty(\Gamma)}$, where the homotopy is within Γ-equivariant maps.

Since Z/Γ is a compact CW complex, there is d_0 such that $\psi(Z) \subset P_{d_0}(\Gamma)$. Let $r' = \max\{r, d_0\}$ and $\phi' = \phi|_{P_{r'}(\Gamma)}$. Then $\psi \circ \phi' : P_{r'}(\Gamma) \to P_\infty(\Gamma)$ is Γ-homotopy equivalent to the inclusion map $i_{\infty,r'}$. Let $F : P_{r'}(\Gamma) \times [0,1] \to P_\infty(\Gamma)$ be the Γ-homotopy path between $\psi \circ \phi'$ and $i_{\infty,r'}$. Since $P_{r'}(\Gamma) \times [0,1]/\Gamma$ is compact, there is an $R > 0$ such that $F(P_{r'}(\Gamma) \times [0,1]) \subset P_R(\Gamma)$. Obviously, $R \geq r' = \max\{r, d_0\}$. Note that Γ-equivariance implies Γ'-equivariance for any subgroup Γ'. We will prove that R satisfies the requirement. If $(i_{\infty,r})_*(x) = (i_{\infty,r})_*(y)$ in $K_*^{\Gamma'}(P_\infty(\Gamma))$, then $\phi_* \circ (i_{\infty,r})_*(x) = \phi_* \circ (i_{\infty,r})_*(y)$ in $K_*^{\Gamma'}(Z)$, and $\psi_* \circ \phi_* \circ (i_{\infty,r})_*(x) = \psi_* \circ \phi_* \circ (i_{\infty,r})_*(y)$ in $K_*^{\Gamma'}(P_{d_0}(\Gamma))$. Since $R > d_0$, $(i_{R,d_0})_* \circ \psi_* \circ \phi_* \circ (i_{\infty,r})_*(x) = (i_{R,d_0})_* \circ \psi_* \circ \phi_* \circ (i_{\infty,r})_*(y)$ in $K_*^{\Gamma'}(P_R(\Gamma))$. Note that, $(i_{R,d_0})_* \circ \psi_* \circ \phi_* \circ (i_{\infty,r})_* = (i_{R,d_0} \circ \psi \circ \phi' \circ i_{r',r})_*$ and $i_{R,d_0} \circ \psi \circ \phi' \circ i_{r',r}$ is Γ'-homotopic to $i_{R,r}$ within $P_R(\Gamma)$. Hence, $(i_{R,r})_*(x) = (i_{R,r})_*(y)$ in $K_*^{\Gamma'}(P_R(\Gamma))$, as desired. □

3. Maximal Roe algebras and quasi-representations

In this section, we introduce the concepts of maximal Roe algebras and quasi-representations. We also discuss the relationship between equivariant Roe algebras and group C^*-algebras.

3.1. Let X be a discrete metric space with bounded geometry. Let $\mathcal{K}(H)$ be the algebra of all compact operator on a separable infinite dimensional Hilbert space. The algebra $C^*_{alg}(X)$ is defined as follows [21]. An element $a \in C^*_{alg}(X)$ is a function $a : X \times X \to K(H)$ with the following properties:

(1) (finite propagation) there exists an $r > 0$ such that $a_{x,y} = 0$ if $d(x, y) \geq$

r (the smallest such r is defined to be the propagation of a);

(2) there is a constant c such that $\|a_{x,y}\| \leq c$ for all $x, y \in X$, where the norm is the operator norm in $\mathcal{K}(H)$.

One can define the multiplication by

$$(a \cdot b)_{x,y} = \sum_{z \in X} a_{x,z} \cdot b_{z,y}.$$

Since X has bounded geometry, the above sum is a finite sum for each pair (x, y) and it is easy to check that $a \cdot b$ is in the algebra. Define $(a^*)_{x,y} = (a_{y,x})^*$. Then $C^*_{alg}(X)$ is a $*$-algebra.

3.2. Let $\phi : C^*_{alg}(X) \to \mathcal{B}(\ell^2(X, H))$ be the faithful $*$-representation:

$$(\phi(a)\xi)_x = \sum_{y \in X} a_{x,y}\xi_y, \quad \forall \xi \in \ell^2(X, H).$$

It is easy to check that, for each $a \in C^*_{alg}(X)$, $\phi(a)$ is a bounded operator. Define $C^*_r(X)$ to be the closure of $C^*_{alg}(X)$ under operator norm [20]. $C^*_r(X)$ is called the reduced Roe algebra.

3.3. We need some preparations to define the maximal Roe algebra.

All the diagonal elements $a \in C^*_{alg}(X)$ (i.e., $a_{x,y} = 0$ if $x \neq y$) together form the C^*-algebra $C_b(X, \mathcal{K}(H))$ of all bounded, compact operator valued functions on X. For any $*$-representation $\phi : C_b(X, \mathcal{K}(H)) \to \mathcal{B}(H')$, where H' is a Hilbert space, we have $\|\phi(a)\| \leq \sup_{x \in X} \|a_{x,x}\|$. To define the maximum Roe algebra, we need the following lemma.

Lemma 3.4. For each element $a \in C^*_{alg}(X)$, there is a non-negative number c_a such that if $\phi : C^*_{alg}(X) \to \mathcal{B}(H')$ is a $*$-representation, then $\|\phi(a)\| \leq c_a$ for any $a \in C^*_{alg}(X)$.

Proof. Let r be a positive number larger than the propagation of a. That is, $a_{x,y} = 0$ for all x, y with $d(x, y) > r$. Since X has bounded geometry, there is an N such that for any $x \in X$, $\#B_{2r}(x) \leq N$. One can write $X = X_1 \cup X_2 \cup \cdots \cup X_{N+1}$ such that $X_i \cap X_j = \emptyset$ for $i \neq j$, and that

$d(x,y) > 2r$ if $x,y \in X_i$ for the same i. This can be done in the following way.

Consider $X_1, X_2, \cdots, X_{N+1}$ as $N+1$ boxes and we will put each element of X into those boxes. At the beginning, the boxes are empty. First, list all the elements of X as $x_1, x_2, \cdots, x_k, \cdots$. Put x_1 in X_1. Once each of $x_1, x_2, \cdots, x_k$ has been put into one of the boxes X_i, the element x_{k+1} should be put into box X_i for the smallest i such that

$$d(x_{k+1}, \quad X_i \cap \{x_1, x_2, \cdots, x_k\}) > 2r.$$

Here, we use the convention $d(x, \emptyset) = \infty$. Such i exists, since there are at most N elements in $B_{2r}(x_{k+1})$.

Let $E = \{(x,y) : d(x,y) \leq r\}$. Then $\mathrm{supp}(a) \subseteq E$, where $\mathrm{supp}(a) := \{(x,y) \in X \times X : a_{x,y} \neq 0\}$. Let $E_i = E \cap (X_i \times X)$, and let $x \in X_i$. Then there are at most N elements $y_1, y_2, \cdots, y_N$ such that $(x, y_j) \in E_i$ for any $j \in \{1, 2, \cdots, N\}$. So one can write $E_i = \cup_{j=1}^N E_{ij}$ such that, if $y_1 \neq y_2$, then (x, y_1) and (x, y_2) of E_i will be in different set E_{ij}. That is, if $(x, y_1), (x, y_2) \in E_{ij}$ then $y_1 = y_2$. Rename E_{ij} as G_i, $1 \leq i \leq (N+1)N$, we write

$$E = \bigcup_{i=1}^{(N+1)N} G_i$$

with the following property: if two different elements (x,y) and (x',y') are in G_i, then $d(x,x') > 2r$, and consequently, $y \neq y'$.

For any $a \in C^*_{alg}(X)$, let a_i be defined by

$$(a_i)_{x,y} = \begin{cases} a_{x,y}, & \text{if } (x,y) \in G_i, \\ 0, & \text{otherwise.} \end{cases}$$

Then $a = \sum a_i$, and

$$\begin{aligned} (a_i^* a_i)_{x,y} &= \sum (a_i^*)_{x,z} \cdot (a_i)_{z,y} \\ &= \sum ((a_i)_{z,x})^* \cdot (a_i)_{z,y} \\ &= \begin{cases} \sum_{z:\ (z,x) \in G_i} a^*_{z,x} \cdot a_{z,x}, & \text{if } x = y, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Furthermore, for each x, there is at most one z such that $(z,x) \in G_i$. Hence, $a_i^* a_i$ is a diagonal element such that each entry has norm at most C^2, where C is a number satisfying $\|a_{x,y}\| \leq C$ for all $x, y \in X$. From 3.3, we know that for each $*$-representation $\phi : C^*_{alg}(X) \to \mathcal{B}(H')$,

$$\begin{aligned} \|\phi(a)\| &\leq \textstyle\sum_{j=1}^{N(N+1)} \|\phi(a_j)\| \\ &\leq \textstyle\sum_{j=1}^{N(N+1)} \|\phi(a_j^* a_j)\|^{1/2} \\ &\leq C \cdot N(N+1) \end{aligned}$$

as desired. □

3.5. For each $a \in C^*_{alg}(X)$, define

$$\|a\|_{\max} := \sup_\phi \{\|\phi(a)\| : \ \ \phi : C^*_{alg}(X) \to \mathcal{B}(H'), \ \ \text{a } *\text{-representation}\}.$$

We define the maximal Roe algebra $C^*_{\max}(X)$ to be the completion of $C^*_{alg}(X)$ with respect to the maximum norm.

3.6. Next we introduce the concept of quasi-representations and study its properties. For any $l \geq 0$, let $C^*_{alg,\, l}(X)$ denote the subset of $C^*_{alg}(X)$ consisting of those elements whose propagation is at most l, that is, $a \in C^*_{alg,\, l}(X)$ if and only if $a_{x,y} = 0$ for all (x,y) with $d(x,y) > l$. Obviously, $(C^*_{alg,\, l}(X))^* = C^*_{alg,\, l}(X)$ and $(C^*_{alg,\, l_1}(X)) \cdot (C^*_{alg,\, l_2}(X)) \subseteq C^*_{alg,\, l_1+l_2}(X)$. In particular, $C^*_{alg,\, 0}(X) = C_b(X, \mathcal{K}(H))$ is a subalgebra of $C^*_{alg}(X)$.

An l-quasi-representation of $C^*_{alg,\, l}(X)$ is a linear map $\phi : C^*_{alg,\, l}(X) \to \mathcal{B}(H')$ such that

(1) if $a \in C^*_{alg,\, l}(X)$, then $\phi(a^*) = \phi(a)^*$;

(2) if $a, b, a \cdot b \in C^*_{alg,\, l}(X)$, then $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$.

We list the following trivial facts of l-quasi-representations:

(a) If $l' > l$, then any l'-quasi-representation is also an l-quasi-representation.

(b) A 0-quasi-representation is a $*$-representation of the subalgebra $C_b(X, K(H))$, the algebra of diagonal elements in $C^*_{alg}(X)$.

(c) A $*$-representation of $C^*_{alg}(X)$ is an l-quasi-representation for any l.

Lemma 3.7. For any $a \in C^*_{alg,\, l}(X)$, there is a number c_a such that if $\phi : C^*_{alg,\,m}(X) \to \mathcal{B}(H')$ is an m-quasi-representation with $m \geq l$, then $\|\phi(a)\| \leq c_a$.

Proof. Since X has bounded geometry, a can be decomposed as $a = \sum_{i=1}^{N(N+1)} a_i$ as in the proof of Lemma 3.4. Note that $\|\phi(a_i)\|^2 = \|\phi(a_i^*)\phi(a_i)\| = \|\phi(a_i^* a_i)\|$, the Lemma follows from the fact that $a_i^* a_i$ has propagation 0. □

3.8. For any element $a \in C^*_{alg,\,l}(X)$ and $m \geq l$, define

$$\|a\|_m = \sup_\phi \{\|\phi(a)\| : \phi \ \ m\text{-quasi-representation}\}.$$

By 3.7, $\|a\|_m < \infty$ for all $m > l$. By 3.6, $\|a\|_m \geq \|a\|_{m'}$ if $m \leq m'$. Define $\|a\|_\infty = \lim_{m\to\infty} \|a\|_m$. Then $\|a\|_\infty$ is well defined and is finite for all element $a \in C^*_{alg}(X)$.

Lemma 3.9. $\|a\|_\infty = \|a\|_{\max}$ for all $a \in C^*_{alg}(X)$.

Proof. By 3.6(c), $\|a\|_{\max} \leq \|a\|_m$ for any m. Hence, $\|a\|_{\max} \leq \|a\|_\infty$. On the other hand, it is straight forward to check that $\|\cdot\|_\infty$ satisfies the following conditions:

(i) $\|a+b\|_\infty \leq \|a\|_\infty + \|b\|_\infty$ and $\|\lambda a\|_\infty = |\lambda| \cdot \|a\|_\infty$ for any $\lambda \in \mathbb{C}$.
(ii) $\|a \cdot b\|_\infty \leq \|a\|_\infty \cdot \|b\|_\infty$.
(iii) $\|a\|^2_\infty = \|a^* a\|_\infty$.

Hence, the completion of $C^*_{alg}(X)$ with respect to the norm $\|\cdot\|_\infty$ is a C^*-algebra, denoted by A. Let $\psi : A \to \mathcal{B}(H')$ be a faithful representation. Then $\|a\|_\infty = \|\psi(a)\| \leq \|a\|_{\max}$ for all $a \in C^*_{alg}(X)$, as desired. □

3.10. In the rest of this section, we discuss the connection between equivariant Roe algebras and group C^*-algebras.

Let Γ be a finitely generated discrete group. There are two natural unitary representations $L, R : \Gamma \to \mathcal{B}(\ell^2(\Gamma))$ by $(L_\gamma \xi)(x) = \xi(\gamma^{-1}x)$ and $(R_\gamma \xi)(x) = \xi(x\gamma)$.

Recall that the group algebra $C^*_{alg}(\Gamma)$ is the set of all functions $a : \Gamma \to \mathbb{C}$ with finite support. The product and involution are defined by $(a \cdot b)_\gamma = \sum_{\delta\in\Gamma} a_\delta b_{\delta^{-1}\gamma}$ and $(a^*)_\gamma = \overline{a}_{\gamma^{-1}}$. We will regard $C^*_{alg}(\Gamma)$ as a subalgebra of

$\mathcal{B}(\ell^2(\Gamma))$ by the right $*$-representation defined by $(a \cdot \xi)_\gamma = \sum_{\delta \in \Gamma} a_\delta \xi_{\gamma\delta}$ for any $\xi \in \ell^2(\Gamma)$.

The above representation also induces a representation of $C^*_{alg}(\Gamma) \otimes \mathcal{K}(H)$ on $\ell^2(\Gamma, H) = \ell^2(\Gamma) \otimes H$ by the same formula. But this time, a_δ is a compact operator on H and $\xi_{\gamma\delta}$ is an element in H.

3.11. We identify $C^*_{alg}(|\Gamma|)$ with a $*$-subalgebra of $\mathcal{B}(\ell^2(\Gamma))$ through its natural faithful representation in 3.2. The natural left unitary representation of Γ on $\ell^2(\Gamma, H)$, still denoted by L, induces a Γ-action on the algebra $C^*_{alg}(|\Gamma|)$ by $\gamma(T) = L_\gamma \circ T \circ L_{\gamma^{-1}}$ for all $T \in C^*_{alg}(|\Gamma|)$. The entries of $\gamma(T)$ are given by

$$(\gamma(T))_{x,y} = T_{\gamma^{-1}x, \gamma^{-1}y}.$$

Let $C^*_{alg}(|\Gamma|)^\Gamma$ be the fixed point algebra of Γ-action on $C^*_{alg}(|\Gamma|)$, that is, $a \in C^*_{alg}(|\Gamma|)^\Gamma$ if and only if $a_{x,y} = a_{\gamma^{-1}x, \gamma^{-1}}y$ for any $\gamma \in \Gamma$. If Γ' is a normal subgroup of Γ with Γ/Γ' finite, then any Γ action induces a Γ' action. Denote by $C^*_{alg}(|\Gamma|)^{\Gamma'}$ the algebra of fixed points of the Γ' action on $C^*_{alg}(|\Gamma|)$.

3.12. Regard both $C^*_{alg}(|\Gamma|)^\Gamma$ and $C^*_{alg}(\Gamma) \otimes \mathcal{K}(H)$ as subalgebras of $\mathcal{B}(\ell^2(\Gamma, H))$. It is clear that $C^*_{alg}(|\Gamma|)^\Gamma = C^*_{alg}(\Gamma) \otimes \mathcal{K}(H)$. The correspondence $a \in C^*_{alg}(\Gamma) \otimes \mathcal{K}(H) \mapsto \tilde{a} \in C^*_{alg}(|\Gamma|)^\Gamma$ is given by

$$\tilde{a}_{x,y} = a_{x^{-1}y}.$$

The propagation of $\tilde{a}$ is

$$\max\{\text{length}(\gamma) : a_\gamma \neq 0\},$$

where the length is the word length of the group Γ with the given finite generating set.

3.13. Define the reduced equivariant Roe algebra $C^*_{r,\Gamma}(|\Gamma|)$ to be the closure of $C^*_{alg}(|\Gamma|)^\Gamma$ as a subalgebra of $\mathcal{B}(\ell^2(\Gamma, H))$. We have $C^*_{r,\Gamma}(|\Gamma|) = C^*_r(\Gamma) \otimes \mathcal{K}(H)$.

3.14. Recall that the maximum norm on $C^*_{alg}(|\Gamma|)^\Gamma$ is defined to be

$$\|a\|_{max} = \sup_\phi \{\|\phi(a)\| : \ \phi \ \ast\text{-representation of } C^*_{alg}(|\Gamma|)^\Gamma\}.$$

The completion of $C^*_{alg}(|\Gamma|)^\Gamma$ under this maximum norm will be called the maximal equivariant Roe algebra and denoted by $C^*_{\max,\Gamma}(|\Gamma|)$. The C^*-algebra $C^*_{\max,\Gamma}(|\Gamma|)$ is isomorphic to $C^*_{\max}(\Gamma)\otimes\mathcal{K}(H)$, where $C^*_{\max}(\Gamma)$ is the maximal group C^* -algebra. Similarly, one can define $C^*_{\max,\Gamma'}(|\Gamma|)$ for a normal subgroup $\Gamma'\subset\Gamma$ with Γ/Γ' finite (see 3.11). It is easy to see that $C^*_{\max,\Gamma'}(|\Gamma|)\cong C^*_{\max}(\Gamma')\otimes\mathcal{K}(H)$.

We caution that the restriction of the maximum norm of $C^*_{alg}(|\Gamma|)$ to its subalgebra $C^*_{alg}(|\Gamma|)^\Gamma$ might not be the maximum norm of $C^*_{alg}(|\Gamma|)^\Gamma$.

3.15. Similar to 3.6, for any $l\geq 0$, let $C^*_{alg,\,l}(|\Gamma|)^\Gamma$ be the subset of $C^*_{alg}(|\Gamma|)^\Gamma$ consisting of elements with propagation at most l. Furthermore, the l-quasi-representations of $C^*_{alg,\,l}(|\Gamma|)^\Gamma$ can be defined in a way similar to the corresponding case in 3.6. The following lemma is similar to Lemma 3.4 and Lemma 3.9, but the proof is much easier.

Lemma 3.16. For any $a\in C^*_{alg}(|\Gamma|)^\Gamma = C^*_{alg}(\Gamma)\otimes\mathcal{K}(H)$ with propagation l, there is a constant C_a such that for any m-quasi-representation $\phi: C^*_{alg,\,m}(|\Gamma|)^\Gamma\to\mathcal{B}(H')$ with $m\geq l$, it is true that $\|\phi(a)\|\leq C_a$.

Proof. Note that $a\in C^*_{alg}(\Gamma)\otimes\mathcal{K}(H)$ has finite support, and if $\gamma\in$ supp(a), then length$(\gamma)\leq l$. We write $a=\sum_\gamma a_\gamma$, where a_γ is supported only on a single point $\gamma\in\Gamma$. Then $a^*_\gamma a_\gamma$ is supported on the unit $e\in\Gamma$. So $a^*_\gamma a_\gamma$ corresponds to an element in $C_b(|\Gamma|,\mathcal{K}(H))$. In fact, it corresponds to a constant function in $C_b(|\Gamma|,\mathcal{K}(H))$. Hence,

$$\phi(a^*_\gamma a_\gamma)\leq\|a^*_\gamma a_\gamma\|,$$

where $\|\cdot\|$ is the operator norm in $\mathcal{K}(H)$. □

3.17. One can define a norm $\|\cdot\|_m$ for any element $a\in C^*_{alg,\,l}(|\Gamma|)^\Gamma$ and $m\geq l$ by $\|a\|_m=\sup_\phi\{\|\phi(a)\|\}$, where the sup is taken over all m-quasi-representations ϕ of $C^*_{alg}(|\Gamma|)^\Gamma$. Evidently, $\|a\|_m\geq\|a\|_{m'}$ if $m\leq m'$. Define $\|a\|_\infty=\lim_{m\to\infty}\|a\|_m$. The proof of the following lemma is similar to the proof of Lemma 3.9 and will be omitted.

Lemma 3.18. $\|a\|_{\max}=\|a\|_\infty$ for any $a\in C^*_{alg}(|\Gamma|)^\Gamma$.

Note that we use the same notations $\|\cdot\|_m$ and $\|\cdot\|_\infty$ for the norms on both $C^*_{alg}(X)$ and $C^*_{alg}(|\Gamma|)^\Gamma$. It will be clear from the context which one we will be using.

4. The Coarse Geometric Novikov Conjecture and the Strong Novikov Conjecture

In this section, we formulate a version of the Coarse Geometric Novikov Conjecture and recall two versions of the Strong Novikov Conjecture.

4.1. Let X be a locally compact metric space. An X-module H_X is a separable Hilbert space equipped with a faithful and non-degenerate $*$-representation π of $C_0(X)$ whose range contains no nonzero compact operators. When H_X is an X-module, for each $f \in C_0(X)$ and $h \in H_X$, we denote $(\pi(f))h$ by fh.

Definition 4.2. ([20]) (1) The support of a bounded linear operator $T : H_X \to H_X$ is defined to be the complement of the set of all points $(x, y) \in X \times X$ for which there exist $g, g' \in C_0(X)$ such that $g'Tg = 0$ but $g(x) \neq 0$, $g'(y) \neq 0$. (2) A bounded operator $T : H_X \to H_X$ is said to have finite propagation if

$$\sup\{d(x, y) : (x, y) \in \operatorname{supp}(T)\} < \infty.$$

And this number is called the propagation of T. (3) A bounded operator $T : H_X \to H_X$ is said to be locally compact if the operators gT and Tg are compact for all $g \in C_0(X)$.

4.3. Denote the algebra of all locally compact, finite propagation operators by $C^*_{alg}(X)$. It is easy to check that the definition of $C^*_{alg}(X)$ is independent of the choice of the X-module H_X. If X is a discrete metric space with bounded geometry, then the above definition of $C^*_{alg}(X)$ is the same as the definition given in subsection 3.1. One can see this by choosing X-module $H_X = \ell^2(X) \otimes H$, where H is a separable Hilbert space, and $C_0(X)$ acts on $\ell^2(X) \otimes H$ by multiplications on $\ell^2(X)$.

4.4. Let Y be a discrete subspace of X such that there are ε and r such that $d(x,y) > \varepsilon$ for any $x, y \in Y$, and $d(x, Y) \leq r$ for any $x \in X$. Then Y is coarsely equivalent to X and $C^*_{alg}(Y)$ is isomorphic to $C^*_{alg}(X)$. Let us describe a precise isomorphism between these two algebras. Take a regular measure μ on X such that for any compact set $A \subset X$, $\mu(A)$ is finite and for any non empty open set $U \subset X$, $\mu(U) > 0$. Choose $H_X = L^2(X, \mu) \otimes H$ to be the X-module in the definition of $C^*_{alg}(X)$. One can construct a partition $X = \bigcup_{y \in Y} A_y$, where each A_y is a Borel subset of X with nonzero measure such that for any $z \in A_y$, $d(y, z) \leq r$ and $A_y \cap A_{y'} = \emptyset$ if $y \neq y'$. We have $H_X = \bigoplus_{y \in Y} L^2(A_y, \mu) \otimes H$. We choose the Y-module in the definition of $C^*_{alg}(Y)$ to be $H_Y = \ell^2(Y) \otimes H'$, where H' is a separable Hilbert space. Choose a unitary $U : H_X \to H_Y$ by identifying each $L^2(A_y, \mu) \otimes H$ with H' via a unitary. Note that the unitary U intertwines the representations of the algebras $C^*_{alg}(Y)$ and $C^*_{alg}(X)$ on H_Y and H_X, i.e., $T \in C^*_{alg}(X) \subset \mathcal{B}(H_X)$ if and only if $UTU^{-1} \in C^*_{alg}(Y) \subset \mathcal{B}(H_Y)$.

4.5. Let X be a locally compact metric space. An element in $K_0(X)$ can be described by a triple (H_X, π, T) such that H_X is a Hilbert space with a $*$-representation π of $C_0(X)$ and $T \in \mathcal{B}(H)$, $T^*T - I$ and $TT^* - I$ are locally compact, and $\pi(f)T - T\pi(f)$ are compact for all $f \in C_0(X)$. We can always choose H_X to be an X-module. In this case, we use the pair (H_X, T) to denote the triple (H_X, π, T). In particular, we can assume $H_X = L^2(X, \mu) \otimes H$, where μ is a measure on X and H is a separable Hilbert space. (Note that each X-module H_X can be embedded into $L^2(X, \mu) \otimes H$, so that one can write $L^2(X, \mu) \otimes H = H_X \oplus H_X^{\perp}$, where $H_X^{\perp}$ is the orthogonal complement of H_X in $L^2(X, \mu) \otimes H$. Let $T' = T \oplus I_{H_X^{\perp}}$. Then (H_X, T) is equivalent to $(L^2(X, \mu) \otimes H, T')$.)

4.6. The assembly maps

$$\mu_{max} : K_0(X) \to K_0(C^*_{\max}(X)),$$

$$\mu_{red} : K_0(X) \to K_0(C_r^*(X))$$

are defined as below. Let (H_X, T) represent a cycle in $K_0(X)$. Let $\{U_i\}_i$ be a locally finite, uniformly bounded open cover of X and $\{\phi_i\}_i$ be a continuous partition of unity subordinate to the open cover $\{U_i\}_i$. Define $F = \sum_i \phi_i^{\frac{1}{2}} T \phi_i^{\frac{1}{2}}$, where the sum converges in the strong topology. It is not hard to see that (H_X, T) and (H_X, F) are equivalent in $K_0(X)$. Note that F has finite propagation, and $F^*F - I$, and $FF^* - I$ are in $C_{alg}^*(X)$. Let

$$W = \begin{pmatrix} I & F \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -F^* & I \end{pmatrix} \begin{pmatrix} I & F \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \in \mathcal{B}(H_X \oplus H_X).$$

Then

$$W \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} W^{-1} - \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \in C_{alg}^*(X) \otimes \mathcal{M}_2(\mathbb{C}),$$

since both W and W^{-1} have finite propagation. Hence

$$\left[W \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} W^{-1} \right] - \left[\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \right]$$

defines an element in $K_0(C_{\max}^*(X))$ by considering $C_{alg}^*(X)$ as a subalgebra of $C_{\max}^*(X)$, denoted by $\mu_{max}([(H_X, T)]) \in K_0(C_{\max}^*(X))$. One can also define an element $\mu_{red}([(H_X, T)]) \in K_0(C_r^*(X))$ by considering $C_{alg}^*(X)$ as a subalgebra of $C_r^*(X)$. Hence, we obtain two assembly maps $\mu_{max} : K_0(X) \to K_0(C_{\max}^*(X))$ and $\mu_{red} : K_0(X) \to K_0(C_r^*(X))$. Similarly, we can define $\mu_{max} : K_1(X) \to K_1(C_{\max}^*(X))$ and $\mu_{red} : K_1(X) \to K_1(C_r^*(X))$.

4.7. Let Y be a locally finite simplicial complex of finite dimension. There is a naturally defined Connes-Chern map

$$ch : K_0(Y) \to \bigoplus_{i=0}^{\infty} H_{2i}(Y, \mathbb{R})$$

where the homology group is the locally finite homology group. In particular, if Y is compact, then the Connes-Chern map is an isomorphism after tensoring with $\mathbb{R}$. We remark that this is not true when Y is noncompact.

Let X be a locally finite discrete metric space with bounded geometry, then by passing to inductive limit, we have a Connes-Chern map

$$ch : \lim_{d\to\infty} K_0(P_d(X)) \to \lim_{d\to\infty} \bigoplus_{i=0}^{\infty} H_{2i}(P_d(X), \mathbb{R}).$$

Similarly, we have a Connes-Chern map

$$ch : \lim_{d\to\infty} K_1(P_d(X)) \to \lim_{d\to\infty} \bigoplus_{i=0}^{\infty} H_{2i+1}(P_d(X), \mathbb{R}).$$

4.8. For any locally finite discrete metric space X of bounded geometry, we know that $C^*_{\max}(P_d(X))$ is isomorphic to $C^*_{\max}(X)$ for any $d > 0$, since X is a discrete subspace of $P_d(X)$ and is coarsely equivalent to the latter (see 4.4). Passing to inductive limit, the assembly map: $K_0(P_d(X)) \to K_0(C^*_{\max}(X))$ defines a map

$$\mu_{max} : \lim_{d\to\infty} K_0(P_d(X)) \to K_0(C^*_{\max}(X)).$$

We can similarly define

$$\mu_{max} : \lim_{d\to\infty} K_1(P_d(X)) \to K_1(C^*_{\max}(X)).$$

The Coarse Geometric Novikov Conjecture:

For any z in $\lim_{d\to\infty} K_*(P_d(X))$, if $\mu_{max}(z) = 0$ in $K_*(C^*_{\max}(X))$, then $ch(z) = 0$ in $\lim_{d\to\infty} \bigoplus_{i=0}^{\infty} H_{2i+*}(P_d(X), \mathbb{R})$.

4.9. Let us recall some facts about the Connes-Chern map. Assume that Y is a countable union of mutually disjoint path connected components $\{Y_j\}_j$, namely, $Y = \bigsqcup_{j=1}^{\infty} Y_j$ and let us assume that all Y_j are compact. Then

$$K_0(Y) = \prod_{j=1}^{\infty} K_0(Y_j),$$

$$H_{2i}(Y, \mathbb{R}) = \prod_{j=1}^{\infty} H_{2i}(Y_j, \mathbb{R})$$

and the Connes-Chern map

$$ch : K_0(Y) \longrightarrow \bigoplus_{i=0}^{m} H_{2i}(Y, \mathbb{R}) = \prod_{j=1}^{\infty} \Big(\bigoplus_{i=0}^{m} H_{2i}(Y_j, \mathbb{R}) \Big),$$

where $m = [\dim(Y)/2]$, satisfies

$$ch(x_1, x_2, \cdots, x_j, \cdots) = (ch(x_1), ch(x_2), \cdots, ch(x_j), \cdots).$$

Recall that if Y is compact, then a cycle in $K_0(Y)$ is represented by (H_Y, T) such that $T^*T - I$, $TT^* - I$ and $[f, T]$ are compact operators for all $f \in C(Y)$. The map $\pi : Y \to \{pt\}$ induces a map $\pi_* : K_0(Y) \to K_0(\{pt\}) = \mathbb{Z}$, which is given by

$$\pi_*(H_Y, T) = \mathrm{ind}(T),$$

where $\mathrm{ind}(T)$ is the Fredholm index of T. Let $Y = Y_1 \sqcup Y_2 \sqcup \cdots \sqcup Y_j \sqcup \cdots$, where each Y_j is a path connected compact space. Suppose that

$$((H_{Y_1}, T_1), (H_{Y_2}, T_2), \cdots, (H_{Y_j}, T_j), \cdots)$$

represents $(x_1, x_2, \cdots, x_j, \cdots) \in K_0(Y) = \prod_{j=1}^{\infty} K_0(Y_j)$, then

$$ch_0(x_1, x_2, \cdots, x_j, \cdots) = (\mathrm{ind}(T_1), \mathrm{ind}(T_2), \cdots, \mathrm{ind}(T_j), \cdots)$$
$$\in \prod_{j=1}^{\infty} \mathbb{Z} \subseteq \prod_{j=1}^{\infty} \mathbb{R} = \prod_{j=1}^{\infty} H_0(Y_j, \mathbb{R}).$$

4.10. Let $Y_1, Y_2, \cdots, Y_i, \cdots$ be a sequence of discrete metric spaces, each of which consists of finitely many elements. Let us assume that the metric d on Y_i satisfies the following conditions: $d(y, y')$ is an integer and there is a sequence $y = y_0, y_1, y_2, \cdots, y_m = y'$ such that $d(y_i, y_{i+1}) = 1$ for any two points $y, y' \in Y_i$. In particular, $P_d(Y_i)$ are path connected if $d \geq 1$. Furthermore, let us assume that for $r > 0$, there is an $N \geq 0$ such that for any Y_i and $y \in Y_i$

$$\#\{z \in Y_i : d(y, z) < r\} \leq N.$$

One can endow a metric d on $Y = \sqcup_{i=1}^{\infty} Y_i$ such that (i) $d|_{Y_i}$ is the metric on Y_i, and (ii) $\lim_{i+j\to\infty, i\neq j} d(Y_i, Y_j) = \infty$.

It is straight forward to check that for any two metrics d_1 and d_2 satisfying the conditions (i) and (ii), (Y, d_1) and (Y, d_2) are coarsely equivalent, and the coarse equivalence is implemented by id_Y. Without loss of generality, we assume that d satisfies the following conditions

$$d(Y_i, Y_n) > d(Y_m, Y_n), \quad d(Y_i, Y_n) > d(Y_i, Y_m), \quad d(Y_n, Y_{n+1}) > d(Y_m, Y_{m+1})$$

provided that $n > m > i$. Then for any $d \geq 1$, there is an integer $n(d) \in \mathbb{Z}_+$ such that $d(Y_{n(d)-1}, Y_{n(d)}) \leq d$ and $d(Y_{n(d)}, Y_{n(d)+1}) > d$. Let $Y^0 = \bigsqcup_{i=1}^{n(d)} Y_i$, then $P_d(Y) = P_d(Y^0) \sqcup \bigsqcup_{i=n(d)+1}^{\infty} P_d(Y_i)$, where each $P_d(Y^0)$ and $P_d(Y_i)$, $i \geq n(d) + 1$, is path connected and compact. Let $m = n(d) + 1$, and let $x \in K_0(P_d(Y))$. Then x can be written as $x = (x^0, x_m, x_{m+1}, \cdots)$, where $x^0 \in K_0(P_d(Y^0))$ and $x_i \in K_0(P_d(Y_i))$ for $i \geq m$. Assume that x is represented by

$$\left(H_{P_d(Y^0)} \oplus \bigoplus_{i=m}^{\infty} H_{P_d(Y_i)}, \quad T^0 \oplus \bigoplus_{i=m}^{\infty} T_i \right).$$

Then

$$\begin{aligned} ch_0(x) &= (\mathrm{ind}(T^0), \mathrm{ind}(T_m), \mathrm{ind}(T_{m+1}), \cdots) \\ &\in \mathbb{Z} \oplus \textstyle\prod_{i=m}^{\infty} \mathbb{Z} \\ &\subseteq \mathbb{R} \oplus \textstyle\prod_{i=m}^{\infty} \mathbb{R} \\ &= H_0(P_d(Y^0), \mathbb{R}) \oplus \textstyle\prod_{i=m}^{\infty} H_0(P_d(Y_i), \mathbb{R}). \end{aligned}$$

If $d' > d$, let $n(d')$ be the largest integer such that $d(Y_{n(d')-1}, Y_{n(d')}) \leq d'$. Let $m' = n(d') + 1$, $\widetilde{Y}^0 = \bigsqcup_{i=1}^{m'-1} Y_i$. Recall that the inclusion $i_{d',d} : P_d(Y) \to P_{d'}(Y)$ induces the map $(i_{d',d})_* : K_0(P_d(Y)) \to K_0(P_{d'}(Y))$. It is clear that $(i_{d',d})_*(x)$ can be written as $(\widetilde{x}^0, \widetilde{x}_{m'}, \widetilde{x}_{m'+1}, \cdots)$, where

$$\widetilde{x}^0 = (i_{d',d})_*(x^0 + x_m + x_{m+1} + \cdots + x_{m'-1})$$

and

$$\widetilde{x}_i = (i_{d',d})_*(x_i)$$

for all $i \geq m'$. In particular,

$$\begin{aligned} ch_0((i_{d',d})_*(x)) &= \left(\mathrm{ind}(T^0) + \textstyle\sum_{i=m}^{m'-1} \mathrm{ind}(T_i), \ \ \mathrm{ind}(T_{m'}), \ \ \mathrm{ind}(T_{m'+1}), \ \ \cdots \right) \\ &\in \mathbb{Z} \oplus \textstyle\prod_{i=m'}^{\infty} \mathbb{Z} \\ &\subseteq \mathbb{R} \oplus \textstyle\prod_{i=m'}^{\infty} \mathbb{R} \\ &= H_0(P_{d'}(\widetilde{Y}^0), \mathbb{R}) \oplus \textstyle\prod_{i=m'}^{\infty} H_0(P_{d'}(Y_i), \mathbb{R}). \end{aligned}$$

Lemma 4.11. Let Y be as in 4.10, and let $x \in \lim_{d\to\infty} K_0(P_d(Y))$. If $\mu_{max}(x) = 0$ in $K_0(C^*_{\max}(Y))$, then $ch_0(x) = 0$ in $\lim_{d\to\infty} H_0(P_d(Y), \mathbb{R})$.

Proof. For each Y_j, choose a point $w_j \in Y_j$. Let $W = \{w_1, w_2, \cdots, w_j, \cdots\}$. Let $i : W \to Y$ be the inclusion and $\pi : Y \to W$ be the map taking every point in Y_j to w_j. Then both i and π are proper, and $\pi \circ i = id_W$. The lemma follows from the Coarse Baum-Connes Conjecture for W and the isomorphism

$$\lim_{d\to\infty} H_0(P_d(Y), \mathbb{R}) \cong \lim_{d\to\infty} H_0(P_d(W), \mathbb{R}).$$

(Note that W has asymptotic dimension zero, hence the coarse Baum-Connes conjecture holds for W [26].) □

4.12. Let X be a locally compact metric space with proper Γ-action. Recall that $C^*_{alg}(X) \subset \mathcal{B}(L^2(X) \otimes H)$ consists of locally compact, finite propagation operators. Γ acts on $L^2(X) \otimes H$ by

$$(\gamma\xi)(x) = \xi(\gamma^{-1}x), \quad \forall \gamma \in \Gamma.$$

Similar to the discrete case in 3.11, there is a natural action of Γ on $C^*_{alg}(X)$ by

$$\gamma(T) = \gamma \cdot T \cdot \gamma^{-1}.$$

Denote by $C^*_{alg}(X)^\Gamma$ the algebra of all Γ-invariant elements in $C^*_{alg}(X)$. Similar to the discrete case again, one can define $C^*_{\max,\Gamma}(X)$ to be the completion of $C^*_{alg}(X)^\Gamma$ with respect to the maximum norm. To prove the existence of the maximum norm, first choose a Γ-invariant discrete subset Y which is coarsely equivalent to X. Then Y has bounded geometry and $C^*_{alg}(X)^\Gamma \cong C^*_{alg}(Y)^\Gamma$. The existence of the maximum norm follows from the following lemma.

Lemma 4.13. For any $a \in C^*_{alg}(Y)^\Gamma$, there exists $C_a > 0$ such that for any $*$-representation $\phi : C^*_{alg}(Y)^\Gamma \to \mathcal{B}(H')$, one has $\|\phi(a)\| \le C_a$.

Proof. The proof is similar to the proof of Lemma 3.4. The only difference is that we need to write a as the sum of Γ-invariant elements a_i such that $a_i^* a_i \in C_b(Y, \mathcal{K}(H))$. □

4.14. Let Γ be a finitely generated discrete group. Let X be a locally compact space with a proper Γ-action. In this subsection, we define the Baum-Connes map [1, 3, 22]

$$\mu : \; K_*^{\Gamma}(X) \to K_*(C^*_{max,\,\Gamma}(X)).$$

Recall that an equivariant K-cycle in $K_0^{\Gamma}(X)$ is described by a triple (H_X, π, T), where

(1) H_X is a Hilbert space endowed with a unitary representation of Γ.

(2) π is a covariant representation of $C_0(X)$ on H_X, i.e., $\pi : C_0(X) \to \mathcal{B}(H_X)$ is a $*$-homomorphism such that

$$\pi(\gamma(f)) = \gamma\pi(f)\gamma^{-1}, \quad \forall \gamma \in \Gamma, \; f \in C_0(X).$$

(3) $T \in \mathcal{B}(H_X)$ such that $[T, \pi(f)], \pi(f)(T^*T - I), \pi(f)(TT^* - I)$ and $\pi(f)[\gamma, T]$ are compact operators on H_X for any $f \in C_0(X)$ and $\gamma \in \Gamma$.

The Hilbert space H_X can always be chosen to be an X-module. In this case, we denote the triple (H_X, π, T) by the pair (H_X, T). Since the Γ-action is proper, one can assume that $[\gamma, T] = 0$. As in 4.5, one can also assume that $H_X = L^2(X, \mu) \otimes H$, where μ is a Γ-invariant measure, H is a separable Hilbert space, and $\gamma \in \Gamma$ acts on H_X by

$$(\gamma(\xi \otimes h))(x) = \xi(\gamma^{-1}x) \otimes h, \quad \forall \xi \otimes h \in L^2(X, \mu) \otimes H,$$

and, furthermore, $C_0(X)$ acts on H_X by multiplications on $L^2(X, \mu)$. We can choose a locally finite and uniformly bounded open cover $\{U_i\}_i$ such that, for each $\gamma \in \Gamma$ and each i, there exists j satisfying $\gamma U_i = U_j$. Let $\{\phi_i\}_i$ be a continuous partition of unity subordinate to $\{U_i\}_i$ such that, for each $\gamma \in \Gamma$ and each i, there exists j satisfying $\gamma(\phi_i) = \phi_j$. We define $F = \sum_i \phi_i^{\frac{1}{2}} T \phi_i^{\frac{1}{2}}$, where the sum converges in the strong topology. Note that F has finite propagation and is Γ-invariant. It is easy to see that $[(H_X, T)] = [(H_X, F)]$ in $K_*^{\Gamma}(X)$. Let

$$W = \begin{pmatrix} I & F \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -F^* & I \end{pmatrix} \begin{pmatrix} I & F \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Then W has finite propagation, and

$$W \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} W^{-1} - \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \in C^*_{alg}(X)^\Gamma.$$

Define the Baum-Connes map

$$\begin{aligned} \mu\Big([(H_X, T)]\Big) &= \left[W \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} W^{-1}\right] - \left[\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\right] \\ &\in K_0(C^*_{\max,\Gamma}(X)). \end{aligned}$$

Similarly, we can define

$$\mu: \ K_1^\Gamma(X) \to K_1(C^*_{max,\,\Gamma}(X)).$$

4.15. In this paper, we will use two versions of the Strong Novikov Conjecture [3, 14]. The first version is as follows.

The Strong Novikov Conjecture (I):
The Baum-Connes map

$$\mu: \lim_{d\to\infty} K_*^\Gamma(P_d(\Gamma)) \to \lim_{d\to\infty} K_*(C^*_{\max,\Gamma}(P_d(\Gamma))) \ \cong \ K_*(C^*_{\max}(\Gamma))$$

is rationally injective, i.e., if $x \in K_*^\Gamma(P_d(\Gamma))$ *such that* $\mu(x) = 0$ *in* $K_*(C^*_{\max}(\Gamma))$*, then there are* $d' \geq d$ *and* $n \in \mathbb{N}$ *such that* $nx = 0$ *in* $K_*^\Gamma(P_{d'}(\Gamma))$*.*

Note that $|\Gamma|$ and $P_d(\Gamma)$ are coarsely equivalent. Therefore,

$$C^*_{\max,\Gamma}(P_d(\Gamma)) \cong C^*_{\max,\Gamma}(|\Gamma|) \cong C^*_{\max}(\Gamma) \otimes \mathcal{K}(H).$$

4.16. The second version of the Strong Novikov Conjecture involves classifying space for free actions. Throughout this paper, all free actions are assumed to be proper. Namely, an action of Γ on X is said to be free if for any $x \in X$ there is a neighborhood $U \subset X$ of x such that $\gamma_1 U \cap \gamma_2 U = \emptyset$ for any $\gamma_1, \gamma_2 \in \Gamma$ with $\gamma_1 \neq \gamma_2$.

Let $E\Gamma$ with a free Γ-action be a universal space for free actions, and let $B\Gamma = E\Gamma/\Gamma$ be the classifying space. One can choose $B\Gamma$ to be a simplicial complex (not necessarily finite) and then $E\Gamma$ is a Γ-simplicial complex. Let $B_1 \subset B_2 \subset B_3 \subset \cdots$ be a sequence of finite sub-simplicial complex of $B\Gamma$ with $B\Gamma = \bigcup_{k=1}^{\infty} B_k$. Let $E_k\Gamma = \pi^{-1}(B_k) \subset E\Gamma$. Then

$$E_1\Gamma \subset E_2\Gamma \subset \cdots \subset E_k\Gamma \subset \cdots$$

is a sequence of locally finite simplicial complexes. One can endow each $E_k\Gamma$ with a Γ-invariant metric so that each $E_k\Gamma$ is a locally compact metric space with free Γ-action. By definition, we have

$$K_*^{\Gamma}(E\Gamma) = \lim_{k\to\infty} K_*^{\Gamma}(E_k\Gamma).$$

The Strong Novikov Conjecture (II):

The Baum-Connes map

$$\mu : \lim_{k\to\infty} K_*^{\Gamma}(E_k\Gamma) \to \lim_{k\to\infty} K_*(C^*_{\max,\Gamma}(E_k\Gamma)) \cong K_*(C^*_{\max}(\Gamma)))$$

is rationally injective.

Since the Γ-action is free, $E_k\Gamma$ is coarsely equivalent to Γ. Therefore,

$$C^*_{\max,\Gamma}(E_k\Gamma) \cong C^*_{\max,\Gamma}(|\Gamma|) \cong C^*_{\max}(\Gamma) \otimes \mathcal{K}(H).$$

Since a free action is also a proper action, there is a map

$$\Phi : \lim_{k\to\infty} K_*^{\Gamma}(E_k\Gamma) \to \lim_{d\to\infty} K_*^{\Gamma}(P_d(\Gamma))$$

such that the following diagram commutes

$$\begin{array}{ccc} \lim_{k\to\infty} K_*^{\Gamma}(E_k\Gamma) & \xrightarrow{\Phi} & \lim_{d\to\infty} K_*^{\Gamma}(P_d(\Gamma)) \\ & \searrow^{\mu} \quad \swarrow^{\mu} & \\ & K_*(C^*_{\max}(\Gamma)) & \end{array}.$$

It is well known that the map Φ is rationally injective [1, 3]. Hence, the Strong Novikov Conjecture (I) implies the Strong Novikov conjecture (II).

4.17. If Γ' is a normal subgroup of Γ with Γ/Γ' finite, then $E_k\Gamma/\Gamma'$ is a finite cover over $E_k\Gamma/\Gamma$. Hence, the map

$$K_*^{\Gamma}(E_k\Gamma) \to K_*^{\Gamma'}(E_k\Gamma)$$

is rationally injective.

5. The Main Theorem

In this section, we state and prove the main result of this paper.

5.1. Let Γ be a finitely generated residually finite group. We can assume that there is a sequence of normal subgroups of finite index

$$\Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_i \supseteq \cdots$$

such that

$$\bigcap_{i=1}^{\infty} \Gamma_i = \{e\}.$$

Endow Γ/Γ_i with the quotient metric, that is,

$$d(a\Gamma_i, b\Gamma_i) = \min\{d(a\gamma_1, b\gamma_2) : \quad \gamma_1, \gamma_2 \in \Gamma_i\}.$$

Let $X(\Gamma) = \bigsqcup_{i=1}^{\infty} \Gamma/\Gamma_i$ be the disjoint union of Γ/Γ_i. We endow a metric on $X(\Gamma)$ such that its restriction to each Γ/Γ_i is the quotient metric defined above and

$$\lim_{n+m\to\infty,\, n\neq m} d(\Gamma/\Gamma_n, \Gamma/\Gamma_m) = \infty.$$

The metric space $X(\Gamma)$ is called the box metric space [23].

The main theorem of this paper is the following

Theorem 5.2. Let Γ be a finitely generated residually finite group and let $X(\Gamma)$ be the space associated to Γ as in 5.1. Then the following statements hold:

(1) The Coarse Geometric Novikov Conjecture for $X(\Gamma)$ implies the Strong Novikov Conjecture (II) for Γ and all subgroups Γ_n, $n = 1, 2, \cdots$.

(2) If the classifying space $(\bigcup_{d=1}^{\infty} P_d(\Gamma))/\Gamma$ for proper Γ-actions has homotopy type of a compact CW complex, then the Strong Novikov Conjecture (I) for Γ and all subgroups Γ_n $(n = 1, 2, 3, \cdots)$ implies the Coarse Geometric Novikov Conjecture for $X(\Gamma)$.

(3) If the classifying space $E\Gamma/\Gamma$ for free Γ-actions has homotopy type of a compact CW complex, then the Strong Novikov Conjecture (II) for Γ

and all subgroups Γ_n $(n = 1, 2, 3, \cdots)$ implies the Coarse Geometric Novikov Conjecture for $X(\Gamma)$.

Lafforgue has constructed residually finite property T groups whose associated sequences of expanders are not coarsely embeddable into any uniformly convex Banach space [17]. Lafforgue's groups satisfy condition (2) of Theorem 5.2. By Theorem 5.2, we obtain the Coarse Geometric Novikov Conjecture for Lafforgue's sequences of expanders in [17].

Remark 5.3. (a) From Theorem 5.2 (1) and (3), we know that if $E\Gamma/\Gamma$ has homotopy type of a compact CW complex, then the Strong Novikov Conjecture (II) for Γ and all Γ_n $(n = 1, 2, 3, \cdots)$ is equivalent to the Coarse Geometric Novikov Conjecture for $X(\Gamma)$. This gives a geometrization of the Strong Novikov Conjecture for these groups.

(b) In part (2) of Theorem 5.2, we assume that the Strong Novikov Conjecture (I) holds not only for Γ, but also for all its subgroups Γ_n. We remark that, for all the known examples of groups satisfying the Strong Novikov Conjecture (I), their subgroups also satisfy the Strong Novikov Conjecture (I).

(c) Note that if $E\Gamma/\Gamma$ has homotpy type of a compact CW complex, then Γ is torsion free. In this case, the Strong Novikov Conjecture (I) and (II) are equivalent. Hence, statement (2) implies statement (3), and we need only to prove (1) and (2).

5.4. We need some preparations to prove Theorem 5.2. We shall prove Theorem 5.2 for the even case, i.e., when $* = 0$. The odd case can be proved in a similar way by a suspension argument.

For convenience, we also assume that, if $n > m > i$, then

$$d(\Gamma/\Gamma_i, \Gamma/\Gamma_n) > d(\Gamma/\Gamma_m, \Gamma/\Gamma_n),$$

$$d(\Gamma/\Gamma_i, \Gamma/\Gamma_n) > d(\Gamma/\Gamma_m, \Gamma/\Gamma_i),$$

$$d(\Gamma/\Gamma_n, \Gamma/\Gamma_{n+1}) > d(\Gamma/\Gamma_m, \Gamma/\Gamma_{m+1}).$$

The proof will occupy the rest of this section. In what follows, we will denote $X(\Gamma)$ by X. Let an element $\theta \in K_0(P_d(X))$ be represented by the pair

$$(L^2(P_d(X)) \otimes H,\ T),$$

where $T \in \mathcal{B}(L^2(P_d(X)) \otimes H)$ is an operator with finite propagation, and $(T^*T - I)f$, $(TT^* - I)f$ and $Tf - fT$ are compact for all $f \in C_0(X)$. We will denote its class in $K_0(P_d(X))$ by $[T]$.

We assume that the propagation of T is l. Let n be large enough such that

$$d_\Gamma(\gamma, e) > 2l + 2d, \quad \forall \gamma \in \Gamma_n,$$

and

$$d_X(\Gamma/\Gamma_n, \Gamma/\Gamma_{n+1}) > 2l + 2d.$$

Let $Y = \bigsqcup_{i=n}^{\infty} \Gamma/\Gamma_i \subset X$. Note that $P_d(Y)$ is a closed and open subset of X. Furthermore, $P_d(Y) = \bigsqcup_{i=n}^{\infty} P_d(\Gamma)/\Gamma_i$. We have

$$T|_{L^2(P_d(Y))\otimes H} = \mathrm{diag}\{T_n, T_{n+1}, \cdots\},$$

where $T_i \in \mathcal{B}(L^2(P_d(\Gamma)/\Gamma_i) \otimes H)$. The local compactness of the operator $(T^*T - I)$ is equivalent to the fact that the operators $(T_i^*T_i - I)$ for $i \geq n$ and

$$(T^*T - I)|_{L^2(P_d(\sqcup_{i=1}^{n-1}\Gamma/\Gamma_i)\otimes H)}$$

are all compact.

We shall lift each operator $T_i \in \mathcal{B}(L^2(P_d(\Gamma/\Gamma_i)) \otimes H)$ to a Γ_i-invariant operator $S_i \in \mathcal{B}(L^2(P_d(\Gamma)) \otimes H)$. Let B be the fundamental domain of $P_d(\Gamma)$ in the sense that $P_d(\Gamma) = \cup_{\gamma\in\Gamma}\, \gamma B$ and $\gamma_1 B \cap \gamma_2 B$ has measure zero if $\gamma_1 \neq \gamma_2 \in \Gamma$.

Such a fundamental domain can be obtained in the following way by using the barycentric subdivision of $P_d(\Gamma)$. Let B be the union of all simplices of the barycentric subdivision of $P_d(\Gamma)$ with the identity $e \in \Gamma \subset P_d(\Gamma)$ as a vertex. If $\gamma \neq e$, then any point $x \in \gamma B \cap B$ will be in a proper face of a simplex, which has e as a vertex and therefore has lower dimension. If we choose the measure careful enough, then such a set has measure zero.

Now we identify $L^2(P_d(\Gamma)/\Gamma)$ with $H_1 := L^2(B)$. Similarly, $L^2(P_d(\Gamma)/\Gamma_i)$ is identified with $\ell^2(\Gamma/\Gamma_i) \otimes H_1$, and $L^2(P_d(\Gamma))$ is identified with $\ell^2(\Gamma) \otimes H_1$. To define S_i in

$$\mathcal{B}(L^2(P_d(\Gamma)) \otimes H) \cong \mathcal{B}((\ell^2(\Gamma) \otimes H_1) \otimes H) \cong \mathcal{B}(\oplus_{x \in \Gamma}(H_1 \otimes H)),$$

one needs only to specify each entry $S_{i;x,y} \in \mathcal{B}(H_1 \otimes H)$ for $x, y \in \Gamma$. For each $x \in \Gamma$, let $[x] = x\Gamma_i \in \Gamma/\Gamma_i$ be the coset corresponding to x. We define

$$S_{i;x,y} = \begin{cases} T_{i;[x],[y]}, & \text{if } d(x,y) \leq l, \\ 0, & \text{otherwise,} \end{cases}$$

where, for $[x], [y] \in \Gamma/\Gamma_i$, the operator $T_{i;[x],[y]} \in \mathcal{B}(H_1 \otimes H)$ is the $([x],[y])$-entry in the matrix form of $T_i \in \mathcal{B}(\ell^2(\Gamma/\Gamma_i) \otimes H_1 \otimes H)$. It is straightforward to verify that S_i is Γ_i-invariant, with propagation at most l, and locally compact. Therefore, S_i defines an element in $K_0^{\Gamma_i}(P_d(\Gamma))$. (Another way to view S_i is to identify $L^2(P_d(\Gamma))$ with $\ell^2(\Gamma_i) \otimes L^2(P_d(\Gamma)/\Gamma_i)$ since Γ_i acts on $P_d(\Gamma)$ freely for $i \geq n$, and let $S_i = I_{\ell^2(\Gamma_i)} \otimes T_i$.)

Note that Γ_i acts freely on $P_d(\Gamma)$ for $i \geq n$. Therefore,

$$K_0^{\Gamma_i}(P_d(\Gamma)) \cong K_0(P_d(\Gamma)/\Gamma_i).$$

This isomorphism takes $[T_i] \in K_0(P_d(\Gamma)/\Gamma_i)$ to $[S_i] \in K_0^{\Gamma_i}(P_d(\Gamma))$. Hence, $[T_i] = 0$ if and only if $[S_i] = 0$. The lifting defines a map

$$\alpha : K_0(P_d(X)) \to \prod_{i=1}^{\infty} K_0^{\Gamma_i}(P_d(\Gamma)) \Big/ \bigoplus_{i=1}^{\infty} K_0^{\Gamma_i}(P_d(\Gamma)).$$

Lemma 5.5. The map α in 5.4 satisfies the following condition: Given $d_0 > 0$, for any $[T] \in K_0(P_{d_0}(X))$, let $\{[S_i]\}_{i \geq n} \in \prod_{i=n}^{\infty} K_0^{\Gamma_i}(P_{d_0}(\Gamma))$ represent $\alpha([T])$. If $[S_i]$ are torsion elements except for finitely many i, then there is $d > d_0$ such that $ch_j((i_{d,d_0})_*([T])) \in H_{2j}(P_d(X), \mathbb{R})$ are zero for all $j \geq 1$.

We remark that $ch_0([T])$ may be different from zero.

Proof. Suppose that $[S_i]$ is a torsion element for every $i \geq n$. Without loss of generality, we can assume that n satisfies the conditions in 5.4, that is, $d_\Gamma(\gamma, e) > 2l + 2d$ for $\gamma \in \Gamma_n$, and $d_X(\Gamma/\Gamma_n, \Gamma/\Gamma_{n+1}) > 2l + 2d$ for S_i as defined in 5.4. Let $Z = \bigsqcup_{i=1}^{n-1} \Gamma/\Gamma_i$, and let $[T^0] \in K_0(P_{d_0}(Z))$ and $[T_i] \in K_0(P_{d_0}(\Gamma)/\Gamma_i)$ for $i \geq n$ be induced by $[T] \in K_0(P_d(X))$. Then $[T_i]$ are torsion elements for $i \geq n$. Hence,

$$ch([T_i]) = 0 \in H_{even}(P_{d_0}(\Gamma)/\Gamma_i, \ \mathbb{R}). \tag{$*$}$$

Of course, it will be zero, considered as an element in $H_{even}(P_d(\Gamma)/\Gamma_i, \ \mathbb{R})$ for any $d \geq d_0$. Choose d large enough such that $diameter(Z) < d$. Then the map $P_{d_0}(Z) \to P_d(X)$ is homotopic to a map $P_{d_0}(Z) \to \{x\}$, where $x \in P_d(X)$ is any chosen point. Hence,

$$ch((i_{d,d_0})_*([T^0])) \in H_{even}(P_d(X), \mathbb{R})$$

factors through $H_{even}(\{pt\}, \mathbb{R}) = H_0(\{pt\}, \mathbb{R})$. This implies $ch_j((i_{d,d_0})_*([T^0])) = 0$ for $j > 0$. Combining this with $(*)$, we obtain the lemma. $\square$

5.6. Next we shall define a homomorphism

$$\phi : C^*_{\max}(X) \to \prod_{i=1}^{\infty} C^*_{\max,\Gamma_i}(|\Gamma|) \Big/ \bigoplus_{i=1}^{\infty} C^*_{\max,\Gamma_i}(|\Gamma|).$$

Here, $C^*_{\max,\Gamma_i}(|\Gamma|)$ is the completion of the algebra $C^*_{alg}(|\Gamma|)^{\Gamma_i}$ of all Γ_i invariant elements in $C^*_{alg}(|\Gamma|)$, with respect to the maximum norm (see 3.14).

Let $T \in C^*_{alg}(X) \subset \mathcal{B}(\ell^2(X) \otimes H)$. Suppose that T has finite propagation l. Let n be the smallest positive integer such that $d(\gamma, e) > 2l$ for $\gamma \in \Gamma_n$ and $d_X(\Gamma/\Gamma_n, \Gamma/\Gamma_{n+1}) > 2l$. Let $Z = \bigsqcup_{i=1}^{n-1} \Gamma/\Gamma_i$, $Y = \bigsqcup_{i=n}^{\infty} \Gamma/\Gamma_i$. Evidently, T induces operators $T^0 \in \mathcal{B}(\ell^2(Z) \otimes H)$ and $T_i \in \mathcal{B}(\ell^2(\Gamma/\Gamma_i) \otimes H)$ for $i \geq n$. Let $S_i \in \mathcal{B}(\ell^2(\Gamma) \otimes H)$ be defined by

$$S_{i;x,y} = \begin{cases} T_{i;[x],[y]}, & \text{if } d(x,y) \leq l, \\ 0, & \text{otherwise,} \end{cases}$$

where, for $x, y \in \Gamma$, $S_{i;x,y}$ denotes the (x,y)-entry of the matrix form of S_i and, for $[x],[y] \in \Gamma/\Gamma_i$, the operator $T_{i;[x],[y]} \in \mathcal{B}(H)$ is the $([x],[y])$-entry in the matrix form of $T_i \in \mathcal{B}(\ell^2(\Gamma/\Gamma_i) \otimes H)$. Then $S_i \in C^*_{alg}(|\Gamma|)^{\Gamma_i} \subseteq C^*_{\max,\Gamma_i}(|\Gamma|)$, and the correspondence $T \mapsto \{S_i\}_{i \geq n}$ defines a map

$$\phi_l : C^*_{alg,\, l}(X) \to \prod_{i=n}^{\infty} C^*_{\max,\Gamma_i}(|\Gamma|)$$

which satisfies

$$\|\phi_l(T)\| \leq \|T\|_l,$$

where $C^*_{alg,l}(X)$ is defined as in 3.6 and $||T||_l$ is defined as in 3.8. Hence, by Lemma 3.9, let l go to infinity, one obtains a $*$-homomorphism

$$\phi : C^*_{alg}(X) \to \prod_{i=1}^{\infty} C^*_{\max,\Gamma_i}(|\Gamma|) \Big/ \bigoplus_{i=1}^{\infty} C^*_{\max,\Gamma_i}(|\Gamma|)$$

and $\|\phi(T)\| \leq \|T\|_\infty = \|T\|_{\max}$, where we used the fact that

$$\|(s_n, s_{n+1}, \cdots)\| = \overline{\lim}_{m\to\infty} \|s_m\|$$

for an element in

$$\prod_{i=1}^{\infty} C^*_{\max,\Gamma_i}(|\Gamma|) \Big/ \bigoplus_{i=1}^{\infty} C^*_{\max,\Gamma_i}(|\Gamma|)$$

represented by $(s_n, s_{n+1}, \cdots)$. Hence, ϕ can be extended to a $*$-homomorphism

$$\phi : C^*_{\max}(X) \to \prod_{i=1}^{\infty} C^*_{\max,\Gamma_i}(|\Gamma|) \Big/ \bigoplus_{i=1}^{\infty} C^*_{\max,\Gamma_i}(|\Gamma|).$$

Note that $C^*_{\max,\Gamma_i}(|\Gamma|) \cong C^*_{\max}(\Gamma_i) \otimes \mathcal{K}(H)$. So ϕ is a homomorphism from $C^*_{\max}(X)$ to

$$\left(\prod_{i=1}^{\infty} C^*_{\max}(\Gamma_i) \otimes \mathcal{K}(H)\right) \Big/ \left(\bigoplus_{i=1}^{\infty} C^*_{\max}(\Gamma_i) \otimes \mathcal{K}(H)\right).$$

5.7. Since every element $x \in K_0(C^*_{\max}(\Gamma_i) \otimes \mathcal{K}(H))$ can be realized as a formal difference of projections $[p] - [q]$ with $p, q \in C^*_{\max}(\Gamma_i) \otimes \mathcal{K}(H)$, we have

$$\begin{aligned} K_0\big(\prod_{i=1}^{\infty}(C^*_{\max}(\Gamma_i)\otimes\mathcal{K}(H))\big) &= \prod_{i=1}^{\infty} K_0(C^*_{\max}(\Gamma_i)\otimes\mathcal{K}(H)) \\ &= \prod_{i=1}^{\infty} K_0(C^*_{\max}(\Gamma_i)). \end{aligned}$$

Consequently,

$$\begin{aligned} & K_0\bigg(\prod_{i=1}^{\infty}(C^*_{\max}(\Gamma_i)\otimes\mathcal{K}(H))\Big/\bigoplus_{i=1}^{\infty}(C^*_{\max}(\Gamma_i)\otimes\mathcal{K}(H))\bigg) \\ \cong \; & \bigg(\prod_{i=1}^{\infty} K_0(C^*_{\max}(\Gamma_i))\bigg)\Big/\bigg(\bigoplus_{i=1}^{\infty} K_0(C^*_{\max}(\Gamma_i))\bigg). \end{aligned}$$

Hence, ϕ induces a map

$$\phi_*: K_0(C^*_{\max}(X)) \to \prod_{i=1}^{\infty} K_0(C^*_{\max}(\Gamma_i))\Big/\bigoplus_{i=1}^{\infty} K_0(C^*_{\max}(\Gamma_i)).$$

5.8. The proof of (2) of Theorem 5.2.

From 5.4, 5.6 and 5.7, there is a commuting diagram

$$\begin{array}{ccc} K_0(P_{d_0}(X)) & \xrightarrow{\alpha} & \Big(\prod_{i=1}^{\infty} K_0^{\Gamma_i}(P_{d_0}(\Gamma))\Big)\Big/\Big(\bigoplus_{i=1}^{\infty} K_0^{\Gamma_i}(P_{d_0}(\Gamma))\Big) \\ \mu_{max}\downarrow & & \downarrow \prod_{i=1}^{\infty}\mu_i \\ K_0(C^*_{\max}(X)) & \xrightarrow{\phi_*} & \Big(\prod_{i=1}^{\infty} K_0(C^*_{\max}(\Gamma_i))\Big)\Big/\Big(\bigoplus_{i=1}^{\infty} K_0(C^*_{\max}(\Gamma_i))\Big), \end{array}$$

where μ_i denotes the Baum-Connes map for Γ_i. Let $x \in K_0(P_{d_0}(X))$ and assume that

$$\mu_{max}(x) = 0 \in K_0(C^*_{\max}(X)).$$

We need to prove that there is a $d > 0$ such that

$$ch((i_{d,d_0})_*(x)) = 0 \in \bigoplus_{i=0}^{\infty} H_{2i}(P_d(X),\mathbb{R}).$$

Let

$$\alpha(x) = [(y_1, y_2, \cdots, y_n, \cdots)] \in \prod_{i=1}^{\infty} K_0^{\Gamma_i}(P_{d_0}(\Gamma))\Big/\bigoplus_{i=1}^{\infty} K_0^{\Gamma_i}(P_{d_0}(\Gamma)).$$

The assumption $(\prod_{i=1}^{\infty} \mu_i)(\alpha(x)) = 0$ implies that there is a positive integer n such that $\mu_i(y_i) = 0$ for all $i \geq n$. By the Strong Novikov Conjecture (I) for Γ and that for the subgroups Γ_i (this is the condition Theorem 5.2(2)), for each $i \geq n$, there is $R_i > d_0$ such that y_i is a torsion element in $K_0^{\Gamma_i}(P_{R_i}(\Gamma))$. By Proposition 2.7, one can choose R_i independent of i, denoted by R. By Lemma 5.5 applied to $(i_{R,d_0})_*(x)$, there is $d > R$ such that

$$ch_j((i_{d,d_0})_*(x)) = 0$$

for all $j \geq 1$. From Lemma 4.11, and the fact $\mu_{max}(x) = 0$, by increasing d, we also have

$$ch_0((i_{d,d_0})_*(x)) = 0 \in H_0(P_d(X), \mathbb{R}),$$

so we have

$$ch((i_{d,d_0})_*(x)) = 0 \in \bigoplus_{i=0}^{\infty} H_{2i}(P_d(X), \mathbb{R})$$

as desired. □

5.9. Let $E\Gamma$ be the classifying space of free Γ-actions. As in 4.16, we can write $E\Gamma = \cup_{k=1}^{\infty} E_k\Gamma$, where $E_k\Gamma$ are locally finite Γ-subsimplicial complex of $E\Gamma$. Recall that $\underline{E}\Gamma = \bigcup_{d=1}^{\infty} P_d(\Gamma)$ is the classifying space of proper Γ-actions. In particular, a free action is proper. For each $E_k\Gamma$, there is $d(k)$ depending on k, and a Γ-equivariant map $t_{d(k),k} : E_k\Gamma \to P_{d(k)}(\Gamma)$. This map induces a map

$$(t_{d(k),k})_* : K_0^{\Gamma}(E_k\Gamma) \to K_0^{\Gamma}(P_{d(k)}(\Gamma)).$$

Passing to inductive limit, we obtain a map

$$t : \lim_{k\to\infty} K_0^{\Gamma}(E_k\Gamma) \to \lim_{d\to\infty} K_0^{\Gamma}(P_d(\Gamma))$$

which relates to the two Baum-Connes maps as follows

$$\begin{array}{ccc} \lim_{k\to\infty} K_0^{\Gamma}(E_k\Gamma) & \xrightarrow{\quad t \quad} & \lim_{d\to\infty} K_0^{\Gamma}(P_d(\Gamma)) \,. \\ & {\scriptstyle\mu}\searrow \quad \swarrow{\scriptstyle\mu} & \\ & K_0(C^*_{\max}(\Gamma)) & \end{array}$$

The map $E_k\Gamma \to P_{d(k)}(\Gamma)$ also induces a sequence of maps

$$K_0^{\Gamma_n}(E_k\Gamma) \to K_0^{\Gamma_n}(P_{d(k)}(\Gamma)),$$

$n = 1, 2, 3, \cdots$, which give a homomorphism π from

$$\lim_{k\to\infty}\left(\prod_{n=1}^{\infty} K_0^{\Gamma_n}(E_k\Gamma)\Big/\bigoplus_{n=1}^{\infty} K_0^{\Gamma_n}(E_k\Gamma)\right)$$

to

$$\lim_{d\to\infty}\left(\prod_{n=1}^{\infty} K_0^{\Gamma_n}(P_d(\Gamma))\Big/\bigoplus_{n=1}^{\infty} K_0^{\Gamma_n}(P_d(\Gamma))\right).$$

Lemma 5.10. π is an isomorphism.

Proof. We shall construct a commutative diagram:

$$\begin{array}{ccccc}
\prod_{n=1}^{\infty} K_0^{\Gamma_n}(E_{k_1}\Gamma)\Big/\bigoplus_{n=1}^{\infty} K_0^{\Gamma_n}(E_{k_1}\Gamma) & \xrightarrow{(i_{k_2,k_1})_*} & \prod_{n=1}^{\infty} K_0^{\Gamma_n}(E_{k_2}\Gamma)\Big/\bigoplus_{n=1}^{\infty} K_0^{\Gamma_n}(E_{k_2}\Gamma) & \longrightarrow & \cdots\cdots \\
\downarrow{\scriptstyle (t_{d_1,k_1})_*} & \nearrow{\scriptstyle (s_{k_2,d_1})_*} & \downarrow{\scriptstyle (t_{d_2,k_2})_*} & \nearrow & \downarrow \\
\prod_{n=1}^{\infty} K_0^{\Gamma_n}(P_{d_1}(\Gamma))\Big/\bigoplus_{n=1}^{\infty} K_0^{\Gamma_n}(P_{d_1}(\Gamma)) & \xrightarrow{(i_{d_2,d_1})_*} & \prod_{n=1}^{\infty} K_0^{\Gamma_n}(P_{d_2}(\Gamma))\Big/\bigoplus_{n=1}^{\infty} K_0^{\Gamma_n}(P_{d_2}(\Gamma)) & \longrightarrow & \cdots\cdots
\end{array}$$

where $k_1 \le k_2 \le \cdots$ and $d_1 \le d_2 \le \cdots$ will be chosen in the next paragraph, $i_{k_2,k_1} : E_{k_1}\Gamma \to E_{k_2}\Gamma$ and $i_{d_2,d_1} : P_{d_1}(\Gamma) \to P_{d_2}(\Gamma)$ are the standard embeddings, and $t_{d_1,k_1} : E_{k_1}\Gamma \to P_{d_1}(\Gamma)$ is given in 5.9 as $t_{d(k),k}$. In the following, we shall construct $s_{k_2,d_1} : P_{d_1}(\Gamma) \to E_{k_2}\Gamma$ which will be Γ_n-equivariant for n large enough.

Let $k_1 = 1$ and $d_1 = d(k_1)$ as in 5.9. In such a way, we obtain $(t_{d_1,k_1})_*$ as in the diagram. For such d_1, choose n_1 such that

$$d(\gamma, e) > 2d_1, \quad \forall \gamma \in \Gamma_{n_1}.$$

Then Γ_{n_1} acts freely on $P_{d_1}(\Gamma)$. Also, $E\Gamma = \cup_{k=1}^{\infty} E_k\Gamma$ can be regarded as the classifying space for free Γ_{n_1}-actions. Therefore, there is a $k_2' > k_1$ and a Γ_{n_1} equivariant map

$$s_{k_2',d_1} : P_{d_1}(\Gamma) \to E_{k_2'}\Gamma.$$

Consider two Γ_{n_1} equivariant maps i_{k_2',k_1} and $s_{k_2',d_1} \circ t_{d_1,k_1}$. Since $E\Gamma = \cup_{k=1}^{\infty} E_k\Gamma$ is also the classifying space for free Γ_{n_1}-actions, by universality, there exists $k_2 > k_2'$ such that, after composition with $i_{k_2,k_2'}$, the above two maps are Γ_{n_1}-homotopic to each other. Let $s_{k_2,d_1} = i_{k_2,k_2'} \circ s_{k_2',d_1}$, we obtain the following commuting diagram:

$$\begin{array}{ccc} K_0^{\Gamma_i}(E_{k_1}\Gamma) & \xrightarrow{(i_{k_2,k_1})_*} & K_0^{\Gamma_i}(E_{k_2}\Gamma) \\ \downarrow{\scriptstyle (t_{d_1,k_1})_*} & \nearrow{\scriptstyle (s_{k_2,d_1})_*} & \\ K_0^{\Gamma_i}(P_{d_1}(\Gamma)) & & \end{array}$$

for each $i \geq n_1$. Hence, we obtain the first piece of the desired diagram by passing to direct product. Let $d_2' = d(k_2)$ and consider the maps $t_{d_2',k_2} \circ s_{k_2,d_1}$ and $i_{d_2',d_1} : P_{d_1}(\Gamma) \to P_{d_2'}(\Gamma)$. Again $\cup_{d=1}^{\infty} P_d(\Gamma)$ is the classifying space for proper Γ_{n_1}-actions. By universality, there is $d_2 > d_2'$ such that $i_{d_2,d_2'} \circ t_{d_2',k_2} \circ s_{k_2,d_1}$ and $i_{d_2,d_2'} \circ i_{d_2',d_1} = i_{d_2,d_1}$ are Γ_{n_1} homotopic to each other. Let

$$t_{d_2,k_2} = i_{d_2,d_2'} \circ t_{d_2',k_2}.$$

We have the following diagram:

$$\begin{array}{ccc} & & K_0^{\Gamma_i}(E_{k_2}\Gamma) \\ & \nearrow{\scriptstyle (s_{k_2,d_1})_*} & \downarrow{\scriptstyle (t_{d_2,k_2})_*} \\ K_0^{\Gamma_i}(P_{d_1}(\Gamma)) & \xrightarrow[(i_{d_2,d_1})_*]{} & K_0^{\Gamma_i}(P_{d_2}(\Gamma)) \end{array}$$

for $i \geq n_1$. Passing to direct product, we obtain the second piece of the desired diagram. Let n_2 be such that $d(\gamma, e) > 2d_2$ for all $\gamma \in \Gamma_{n_2}$. Then Γ_{n_2} acts freely on $P_{d_2}(\Gamma)$, and we can repeat the above procedure with n_2 in the place of n_1 to obtain the next two diagrams. The whole diagram can be constructed inductively. The fact that π is an isomorphism follows from the commuting diagram. □

5.11. The forgetful map $f_i : K_0^{\Gamma}(E_k\Gamma) \to K_0^{\Gamma_i}(E_k\Gamma)$ and $f_i : K_0^{\Gamma}(P_d(\Gamma)) \to$

$K_0^{\Gamma_i}(P_d(\Gamma))$ give rise to the following commutative diagram

$$\begin{array}{ccc} \lim_{k\to\infty} K_0^{\Gamma}(E_k\Gamma) & \xrightarrow{\prod_{i=1}^{\infty} f_i} & \lim_{k\to\infty}\left(\prod_{i=1}^{\infty} K_0^{\Gamma_i}(E_k\Gamma)\Big/\bigoplus_{i=1}^{\infty} K_0^{\Gamma_i}(E_k\Gamma)\right) \\ \downarrow & & \downarrow \pi \\ \lim_{d\to\infty} K_0^{\Gamma}(P_d(\Gamma)) & \xrightarrow{\prod_{i=1}^{\infty} f_i} & \lim_{d\to\infty}\left(\prod_{i=1}^{\infty} K_0^{\Gamma_i}(P_d(\Gamma))\Big/\bigoplus_{i=1}^{\infty} K_0^{\Gamma_i}(P_d(\Gamma))\right). \end{array}$$

5.12. One can define a $*$-homomorphism

$$\psi : C^*_{\max}(\Gamma)\otimes\mathcal{K}(H)\to C^*_{\max}(X)$$

as below. First, note that $C^*_{alg}(\Gamma)\subset\mathcal{B}(\ell^2(\Gamma))$ is generated by the translations $\gamma\xi(x)=\xi(\gamma^{-1}x)$, where $\gamma\in\Gamma$ is considered as an element in $C^*_{alg}(\Gamma)$. For any $\gamma\in\Gamma$, we also define a translation on $\bigoplus_{i=1}^{\infty}\ell^2(\Gamma/\Gamma_i)$ by:

$$(\gamma\eta)([x])=\bigoplus_{i=1}^{\infty}\eta_i([\gamma^{-1}x]),$$

where $\eta=\oplus_{i=1}^{\infty}\eta_i\in\bigoplus_{i=1}^{\infty}\ell^2(\Gamma/\Gamma_i)$, and $[x]\in\Gamma/\Gamma_n$ is a coset. We obtain a map $C^*_{alg}(\Gamma)\otimes\mathcal{K}(H)\to C^*_{alg}(X)\subset C^*_{\max}(X)$, which gives rise to a $*$-homomorphism:

$$\psi : C^*_{\max}(\Gamma)\otimes\mathcal{K}(H)\to C^*_{\max}(X)$$

5.13. Let $\theta\in K_0^{\Gamma}(P_d(\Gamma))$. Then θ can be represented by $(L^2(P_d(\Gamma))\otimes H, T)$, where T is a Γ-invariant operator of finite propagation. Suppose the propagation of T is l. Let $n=n(d,l)$ be the integer (depending on d and l) such that $d(\gamma,e)>2l+2d$ for $\gamma\in\Gamma_n$, and $d_X(\Gamma/\Gamma_n,\Gamma/\Gamma_{n+1})>2l+2d$. Then for each $i\geq n$, one can define $S_i\in\mathcal{B}(L^2(P_d(\Gamma)/\Gamma_i)\otimes H)$ by

$$(S_i)_{[x],[y]}=\begin{cases} T_{x,y}, & \text{if } d(x,y)\leq l,\ \ x\in[x],\ y\in[y],\\ 0, & \text{otherwise.}\end{cases}$$

Then

$$\left(\bigoplus_{i=n}^{\infty}\ell^2(\Gamma/\Gamma_i)\otimes H,\ \bigoplus_{i=n}^{\infty}S_i\right)$$

defines an element in $K_0(P_d(\sqcup_{i=n}^{\infty}\Gamma/\Gamma_i)) \subseteq K_0(P_d(X))$. Let us denote this element by $\Psi(\theta) \in K_0(P_d(X))$. Obviously, the map $\theta \mapsto \Psi(\theta)$ depends on the choice of the integer n. However, the composition

$$\alpha \circ \Psi : \quad K_0^{\Gamma}(P_d(\Gamma)) \longrightarrow \prod_{i=1}^{\infty} K_0^{\Gamma_i}(P_d(\Gamma)) \Big/ \bigoplus_{i=1}^{\infty} K_0^{\Gamma_i}(P_d(\Gamma))$$

does not depend on the choice of n, and $\alpha \circ \Psi = \prod_{i=1}^{\infty} f_i$, where

$$f_i : \; K_0^{\Gamma}(P_d(\Gamma)) \longrightarrow K_0^{\Gamma_i}(P_d(\Gamma))$$

is as in 5.11.

5.14. Note that $\mathcal{K}(\oplus_{i=1}^{\infty}\ell^2(\Gamma/\Gamma_i) \otimes H)$ is an ideal of $C^*_{\max}(X)$. Let $H_1 = \oplus_{i=1}^{\infty}\ell^2(\Gamma/\Gamma_i) \otimes H$. We have the following short exact sequence

$$0 \to \mathcal{K}(H_1) \to C^*_{\max}(X) \to C^*_{\max}(X)/\mathcal{K}(H_1) \to 0.$$

We shall prove that $i_* : K_0(\mathcal{K}(H_1)) \to K_0(C^*_{\max}(X))$ is injective. Let $Z = \sqcup_{i=1}^{n}\Gamma/\Gamma_i$. Then

$$\mathcal{K}(\ell^2(Z) \otimes H) \subseteq \mathcal{K}(\oplus_{i=1}^{\infty}\ell^2(\Gamma/\Gamma_i) \otimes H) \subseteq C^*_{\max}(X).$$

Let $i : \mathcal{K}(\ell^2(Z) \otimes H) \to C^*_{\max}(X)$. We have the following lemma.

Lemma 5.15. $i_* : K_0(\mathcal{K}(\ell^2(Z) \otimes H)) \to K_0(C^*_{\max}(X))$ is injective.

Proof. Let $\pi : C^*_{\max}(X) \to C^*_r(X)$ be the quotient map. We only need to prove that

$$\pi_* \circ i_* : K_0(\mathcal{B}(\ell^2(Z)) \otimes \mathcal{K}(H)) \to K_0(C^*_r(X))$$

is injective. Note that $\mathcal{B}(\ell^2(Z)) \otimes \mathcal{K}(H) \cong \mathcal{K}(\ell^2(Z) \otimes H)$ since Z is a finite set. Let p_0, p_1 be two projections in $\mathcal{B}(\ell^2(Z)) \otimes \mathcal{K}(H)$. Then p_0, p_1 can be considered as projections in

$$C^*_{alg}(X) \subseteq C^*_r(X) \subseteq \mathcal{B}(\ell^2(X) \otimes H).$$

We have $\pi_* \circ i_*([p_0]) = \pi_* \circ i_*([p_1]) \in K_0(C^*_r(X))$. This implies that $p_0 \sim_h p_1$ in $C^*_r(X)$. Let $p(t)$ be the homotopy path of projections with $p(0) = p_0$ and $p(1) = p_1$. Choose

$$0 = t_0 < t_1 < t_2 < \cdots < t_m = 1$$

such that $\|p(t) - p(s)\| \leq \frac{1}{100}$ if $t, s \in [t_{k-1}, t_k]$.

There exist self adjoint elements $q(t_i) \in C^*_{alg}(X)$ such that $q(0) = p(0)$ and $q(1) = p(1)$ and

$$\|q(t_i) - p(t_i)\| \leq \frac{1}{100}, \quad \forall i \in \{0, 1, \cdots, m\}.$$

Define

$$q(t) = \frac{t - t_{k-1}}{t_k - t_{k-1}} q(t_k) + \frac{t_k - t}{t_k - t_{k-1}} q(t_{k-1}), \quad \forall t_{k-1} \leq t \leq t_k.$$

Then

$$\|q(t) - p(t)\| \leq \frac{5}{100}, \quad \forall 0 \leq t \leq 1.$$

Each $q(t_k)$ has finite propagation, so there is $l > 0$ such that all $q(t_k)$ have propagation at most l. Hence, all $q(t)$ have propagation at most l, since they are linear combinations of elements of propagation at most l. Let m be the least integer such that

$$d(\Gamma/\Gamma_m, \Gamma/\Gamma_{m+1}) > 2l.$$

Let $W = \sqcup_{i=1}^{m-1} \Gamma/\Gamma_i$ and $Y = \sqcup_{i=m}^{\infty} \Gamma/\Gamma_i$. Then $d(W, Y) > 2l$. Hence, $\ell^2(W) \otimes H$ and $\ell^2(Y) \otimes H$ are reducing subspaces for each $q(t)$, that is,

$$q(t) \in \Big(\mathcal{B}(\ell^2(W)) \otimes \mathcal{K}(H)\Big) \bigoplus \Big(\mathcal{B}(\ell^2(Y) \otimes H) \cap C^*_{alg}(X)\Big).$$

Note that the spectrum of $q(t)$ is contained in $[-5/100, 5/100] \cup [1 - 5/100, 1 + 5/100]$. Let

$$\chi : [-5/100, 5/100] \cup [1 - 5/100, 1 + 5/100] \to \{0, 1\}$$

be the function sending $[-5/100, 5/100]$ to 0 and $[1-5/100, 1+5/100]$ to 1. Then $p'(t) = \chi(q(t)) \in \mathcal{B}(\ell^2(W)) \otimes \mathcal{K}(H)$ and $p'(t)$ is a path of projections connecting p_0 and p_1. Hence, $[p_0] = [p_1] \in K_0(\mathcal{B}(\ell^2(W)) \otimes \mathcal{K}(H))$. Note that $Z \subset W$; Z and W are finite. Therefore,

$$K_0(\mathcal{B}(\ell^2(Z) \otimes \mathcal{K}(H))) \cong K_0(\mathcal{B}(\ell^2(W)) \otimes \mathcal{K}(H)) \cong \mathbb{Z},$$

and the isomorphism is induced by the inclusion $\mathcal{B}(\ell^2(Z)) \to \mathcal{B}(\ell^2(W))$. So $[p_0] = [p_1] \in K_0(\mathcal{B}(\ell^2(Z)) \otimes \mathcal{K}(H))$ as desired. ☐

5.16. From 5.14 and 5.15, we have the following exact sequence

$$0 \to K_0(\mathcal{K}(H_1)) \to K_0(C^*_{\max}(X)) \to K_0(C^*_{\max}(X)/\mathcal{K}(H_1)) \to 0.$$

Denote the above quotient map by π. Recall that $\mu_{max} : K_0(P_d(X)) \to K_0(C^*_{\max}(X))$ is the assembly map defined in §4. Again the map $\theta \to \mu_{max}(\Psi(\theta))$ depends on the choice of n in 5.13. However, the homomorphism

$$\pi \circ \mu_{max} \circ \Psi : K_0^\Gamma(P_d(\Gamma)) \to K_0(C^*_{\max}(X)/\mathcal{K}(H))$$

does not depend on the choice of n. Furthermore, we have

$$\pi \circ \mu_{max} \circ \Psi = \pi \circ \psi_* \circ \mu,$$

where $\mu : K_0^\Gamma(P_d(\Gamma)) \to K_0(C^*_{\max}(\Gamma) \otimes \mathcal{K}(H))$ is the Baum-Connes map and $\psi_* : K_0(C^*_{\max}(\Gamma) \otimes \mathcal{K}(H)) \to K_0(C^*_{\max}(X))$ is induced by ψ defined in 5.12.

5.17. Since Γ acts on $E_k\Gamma$ freely and Γ_n are normal subgroups of Γ, $E_k\Gamma/\Gamma_n$ is a finite cover over $E_k\Gamma/\Gamma$. Therefore,

$$f_n : K_0^\Gamma(E_k\Gamma) \to K_0^{\Gamma_n}(E_k\Gamma)$$

is rationally injective. In particular, for any $\theta \in K_0^\Gamma(E_k\Gamma)$, if $f_n(\theta)$ is a torsion element, then θ is a torsion element.

5.18. Proof of (1) of Theorem 5.2.

Note that for every subgroup Γ_n ($n = 1, 2, \cdots$) the box metric space $X(\Gamma_n) = \bigsqcup_{i=n+1}^{\infty} \Gamma_n/\Gamma_i$ is coarsely equivalent to $X(\Gamma) = \bigsqcup_{i=1}^{\infty} \Gamma/\Gamma_i$. Hence, the Coarse Geometric Novikov Conjecture for the box metric space $X(\Gamma)$ implies the Coarse Geometric Novikov Conjecture for the box metric space $X(\Gamma_n)$. So, it suffices to prove that the Coarse Geometric Novikov Conjecture for $X(\Gamma)$ implies that

$$\mu : \lim_{k\to\infty} K_0^{\Gamma}(E_k\Gamma) \to K_0(C^*_{\max}(\Gamma))$$

is rationally injective.

In this proof, $X(\Gamma)$ will be denoted by X. Let $\theta \in K_0^{\Gamma}(E_k\Gamma)$ be such that $\mu(\theta) = 0$. We need to prove that θ is a torsion element in $\lim_{k\to\infty} K_0^{\Gamma}(E_k\Gamma)$. Let $\theta' = t(\theta) \in K_0^{\Gamma}(P_d(\Gamma))$ for certain d, where t is defined as in 5.9. Then $\mu(\theta') = \mu(\theta) = 0$ in $K_0(C^*_{\max}(\Gamma))$. Let $\eta = \Psi(\theta') \in K_0(P_d(X))$. Then

$$\pi \circ \mu_{max}(\eta) = \pi \circ \psi_* \circ \mu(\theta') = 0$$

in $K_0(C^*_{\max}(X)/\mathcal{K}(H))$. One can choose an element $\eta' \in K_0(P_d(\Gamma/\Gamma_1))$ such that $\mu_{max}(i_*(\eta')) = \mu_{max}(\eta)$, where $i_* : K_0(P_d(\Gamma/\Gamma_1)) \to K_0(X)$ is induced by the embedding $i : P_d(\Gamma/\Gamma_1) \to P_d(X)$. Hence, $\mu_{max}(\eta - i_*(\eta')) = 0$. By the coarse geometric Novikov conjecture for X, there is $d_1 > d$ such that

$$ch(\eta - i_*(\eta')) = 0 \in \bigoplus_{i=0}^{\infty} H_{2i}(P_{d_1}(X), \mathbb{R}),$$

where we use the same notation for η and $(i_{d_1,d_2})_*(\eta)$. We assume that $\eta - i_*(\eta') \in K_0(P_{d_1}(X))$ is represented by an operator with propagation l, and let m be the integer satisfying

$$d(\gamma, e) > 2l + 2d_1, \quad \forall \gamma \in \Gamma/\Gamma_m$$

and

$$d_X(\Gamma/\Gamma_m, \Gamma/\Gamma_{m+1}) > 2l + 2d_1.$$

Then $\eta - i_*(\eta')$ defines $\eta_m, \eta_{m+1}, \cdots$, where $\eta_i \in K_0(P_{d_1}(\Gamma)/\Gamma_i)$ and

$$ch(\eta_i) = 0 \in \bigoplus_{i=0}^{\infty} H_{2i}(P_{d_1}(\Gamma)/\Gamma_i, \mathbb{R}),$$

for every $i \geq m$. Hence, η_i is a torsion element. Let $\theta_i \in K_0^{\Gamma_i}(P_{d_1}(\Gamma))$ be the corresponding element of $\eta_i \in K_0(P_{d_1}(\Gamma)/\Gamma_i)$ under the isomorphism $K_0^{\Gamma_i}(P_{d_1}(\Gamma)) \cong K_0(P_{d_1}(\Gamma)/\Gamma_i)$ (note that Γ_i acts freely on $P_{d_1}(\Gamma)$ for $i \geq m$). Then

$$\begin{aligned} [(0, \cdots, 0, \theta_m, \theta_{m+1}, \cdots)] &= \alpha((0, \cdots, 0, \eta_m, \eta_{m+1}, \cdots)) \\ &\in \textstyle\prod_{i=1}^{\infty} K_0^{\Gamma_i}(P_d(\Gamma)) / \bigoplus_{i=1}^{\infty} K_0^{\Gamma_i}(P_d(\Gamma)). \end{aligned}$$

Note that $\alpha(i_*(\eta')) = 0$. So

$$\alpha(\eta) = [(0, \cdots, 0, \theta_m, \theta_{m+1}, \cdots)].$$

Hence,

$$\begin{aligned} (\textstyle\prod_{i=1}^{\infty} f_i)(\theta) &= [(0, \cdots, 0, \theta_m, \theta_{m+1}, \cdots)] \\ &\in \textstyle\prod_{i=1}^{\infty} K_0^{\Gamma_i}(P_d(\Gamma)) / \bigoplus_{i=1}^{\infty} K_0^{\Gamma_i}(P_d(\Gamma)) \end{aligned}$$

with each θ_i being a torsion element. By using the following commutative diagram

$$\begin{array}{ccc} \lim_{k\to\infty} K_0^{\Gamma}(E_k\Gamma) & \xrightarrow{\prod_{i=1}^{\infty} f_i} & \lim_{k\to\infty} \left(\prod_{i=1}^{\infty} K_0^{\Gamma_i}(E_k\Gamma) \Big/ \bigoplus_{i=1}^{\infty} K_0^{\Gamma_i}(E_k\Gamma) \right) \\ \downarrow & & \downarrow \pi \\ \lim_{d\to\infty} K_0^{\Gamma}(P_d(\Gamma)) & \xrightarrow{\prod_{i=1}^{\infty} f_i} & \lim_{d\to\infty} \left(\prod_{i=1}^{\infty} K_0^{\Gamma_i}(P_d(\Gamma)) \Big/ \bigoplus_{i=1}^{\infty} K_0^{\Gamma_i}(P_d(\Gamma)) \right). \end{array}$$

and the isomorphism of π, we know that for k_1 large enough and n large enough, $f_n(\theta)$ is a torsion element in $K_0^{\Gamma_n}(E_{k_1}\Gamma)$. By 5.17., θ is a torsion element in $K_0^{\Gamma}(E_{k_1}\Gamma)$. This completes the proof. □

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