NONCOMMMUTATIVE BATALIN-VILKOVISKY GEOMETRY AND MATRIX INTEGRALS

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Abstract. We describe matrix model associated naturally with any solution to quantum master equation of noncommutative Batalin-Vilkovisky geometry. In the simplest example we get the Kontsevich model of 2-dimensional gravity.

1. Introduction.

We describe matrix model associated naturally with solution to quantum master equation of noncommutative Batalin-Vilkovisky geometry. The simplest example is the Kontsevich model of 2-dimensional gravity.

Notations.We work in the tensor category of super vector spaces, over an algebraically closed field k, $char(k) = 0$. Let $V = V^{1|0} \oplus V^{0|1}$ be a super vector space. We denote by $\overline{\alpha}$ the parity of an element α and by ΠV the super vector space with inversed parity. Sometimes we may also assume that V has a differential, we leave to the interested reader to make the appropriate modifications, if they are not mentioned explicitely. For a finite group G acting on a vector space U, we denote via U^G the space of invariants with respect to the action of G.

Acknowledgments.This work was done during my visits to Max Planck Institute (Bonn), Poncelet Laboratory (Moscow), Erwin Schrodinger Institute (Vienna) and Newton Institute (Cambridge). I'm greatful to these institutions for excellent working conditions. I'd like to acknowledge interesting discussions with A.Kapustin, S.Merkulov, M.Movshev, A.Schwartz, M.Wodzicki. This work was partly supported through ENIGMA europeen network.

2. Noncommutative Batalin-Vilkovisky geometry.

2.1. **Even inner products.** Let $B : V^{\otimes 2} \to k$ be an even symmetric inner product on V:

$$B(x \otimes y) = (-1)^{\overline{x}+\overline{y}} B(y \otimes x)$$

The inner prodcut B corresponds to an even symplectic structure on ΠV. We introduced in [B1] the space $F = \bigoplus_n F_n$

$$F_n = ((\Pi V)^{\otimes n} \otimes k[\mathbb{S}_n]')^{\mathbb{S}_n} \tag{2.1}$$

where $k[\mathbb{S}_n]'$ denotes the super $k-$vector space with the basis indexed by elements (σ, ρ_σ), where $\sigma \in \mathbb{S}_n$ is a permutation with i_σ cycles σ_α and $\rho_\sigma = \sigma_1 \wedge \ldots \wedge \sigma_{i_\sigma}$, $\rho_\sigma \in Det(Cycle(\sigma))$, $Det(Cycle(\sigma)) = Symm^{i_\sigma}(k^{0|i_\sigma})$, is one of the generators of

1991 *Mathematics Subject Classification.* Primary 14Q05; Secondary 05C25,20B30 ??

Key words and phrases. Batalin-Vilkovisky geometry, matrix integrals, Deligne-Mumford moduli spaces, Gromov-Witten invariants.

the one-dimensional determinant of the set of cycles of σ, i.e. ρ_σ is an order on the set of cycles defined up to even reordering, and $(\sigma, -\rho_\sigma) = -(\sigma, \rho_\sigma)$. The group $\mathbb{S}_n$ acts on $k[\mathbb{S}_n]'$ by conjugation.The space F carries the naturally defined operator

$$\Delta(a_1 \otimes \ldots \otimes a_n \otimes (\sigma, \rho_\sigma)) = B(a_{n-1} \otimes a_n)(a_1 \otimes \ldots \otimes a_{n-2} \otimes (\pi_{n-1,n}(s_{n-1,n}\sigma), \rho_\delta)$$

where $s_{n-1,n}$ is the transposition interchanging $n-1$ and n, $\pi_{n-1,n}$ is the operator of erasing of elements $n-1$ and n from the cycles of the representation, and ρ_δ is the natural ordering on the cycles of resulting permutation $\pi_{n-1,n}(\sigma s_{n-1,n})$, see section 9 from loc.cit. The space F can be identified with the symmetric power of the parity transposed space of cyclic invariant tensors on ΠV

$$F = Symm(\oplus_j \Pi(\Pi V^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}})$$

If one chooses a basis $\{a_i\}$ in ΠV, in which B has the form $B(a_i, a_j) = b_{ij}$ then operator Δ can be written as

$$\Delta = \sum_{ij} \gamma b_{ij} \frac{\partial^2}{\partial a_i \partial a_j}$$

where γ denotes the "dissection-gluing" operator acting on cycles, which corresponds to the to the symmetrisation of the operator $(\sigma, \rho_\sigma) \to (\pi_{n-1,n}(s_{n-1,n}\sigma), \rho_\delta)$ acting on pairs (σ, ρ_σ). The operator γ is odd since it changes the total number of cycles by one.

The operator Δ is of the second order with respect to the multiplication on F and by the general formula of Batalin-Vilkovisky geometry it defines the odd Lie bracket on F, which we described in loc.cit.,section 5, and which can be written as

$$\{x, y\} = \sum_{ij} \gamma b_{ij} \frac{\partial x}{\partial a_i} \frac{\partial x}{\partial a_j}$$

Let us denote by $\widetilde{\mathbb{S}}$ the $\mathbb{Z}/2\mathbb{Z}$-graded version of the twisted modular $\mathcal{K}-$operad $\widetilde{\mathfrak{s}\Sigma\mathbb{S}}[t]$. By (a $\mathbb{Z}/2\mathbb{Z}$-graded version) of the theorem 1 from [B1] the solutions of quantum master equation in F

$$d_V S + \hbar \Delta S + \frac{1}{2}\{S, S\} = 0,$$

$S = \sum_{2(g-1)+n>0} \hbar^g S_{g,n}$, are in one-to one correspondence with the structure of algebra over the $\mathbb{Z}/2\mathbb{Z}$-graded modular operad $\mathcal{F}_\mathcal{K}\widetilde{\mathbb{S}}$ on the vector space V with symetric inner product B. Its genus zero part corresponds to the structure of $A_\infty-$algebra with the even invariant inner product on V.

2.2. **Odd inner products.** Let $V = V^{1|0} \oplus V^{0|1}$ be a super vector space and $B : V^{\otimes 2} \to \Pi k$ be an odd symmetric inner product, $B(x \otimes y) = (-1)^{\overline{x}+\overline{y}} B(y \otimes x)$. By definition B is an odd symplectic structure on V. The space F in this situation has components

$$F_n = (V^{\otimes n} \otimes k[\mathbb{S}_n])^{\mathbb{S}_n} \tag{2.2}$$

where $k[\mathbb{S}_n]$ is the $k-$vector space with the basis indexed by $\sigma \in \mathbb{S}_n$ and $\mathbb{S}_n$ acts on $k[\mathbb{S}_n]$ by conjugation. The space F carries the naturally defined operator

$$\Delta(a_1 \otimes \ldots \otimes a_n \otimes \sigma = (-1)^{\sum_{i=1}^{n-2} \overline{a}_i} B(a_{n-1} \otimes a_n)(a_1 \otimes \ldots \otimes a_{n-2} \otimes (\pi_{n-1,n}(s_{n-1,n}\sigma))$$

where $s_{n-1,n}$ is the transposition interchanging $n-1$ and n, $\pi_{n-1,n}$ is the operator of erasing of elements $n-1$ and n from the cycles of the representation, see section

9 from loc.cit. The space F can be identified with the symmetric power of the space of cyclic invariant tensors on V

$$F = Symm(\oplus_i (V^{\otimes i})^{\mathbb{Z}/i\mathbb{Z}})$$

If one chooses a basis $\{a_i\}$ in V, in which B has the form $B(a_i, a_j) = b_{ij}$ then operator Δ can be written as

$$\Delta = \sum_{ij} \gamma b_{ij} \frac{\partial^2}{\partial a_i \partial a_j}$$

where γ denotes the "dissection-gluing" operator acting on cycles, which corresponds to the symmetrisation of the operator $\sigma \to \pi_{n-1,n}(s_{n-1,n}\sigma)$ acting on permutations. The operator γ is even in this case since in the definition of F the parity change on the space of cyclic invariant tensors is not involved.

The operator Δ is of the second order with respect to the multiplication on F and it defines the odd Lie bracket on F, which can be written as

$$\{x, y\} = \sum_{ij} \gamma b_{ij} \frac{\partial x}{\partial a_i} \frac{\partial x}{\partial a_j}$$

The odd analogue corresponds to the modular operad, which is natural to denote $\mathbb{S}$, the *untwisted* analogue of $\mathbb{Z}/2\mathbb{Z}$-graded version of $\widetilde{\mathfrak{s}\Sigma\mathbb{S}}[t]$. The spaces $\mathbb{S}((n))$ are simply the group algebras $k[\mathbb{S}_n]$ with the action of $\mathbb{S}_n$ by conjugation. The composition maps are defined by operators $\pi_{f,f'}(s_{f,f'}\sigma)$ as in section 9 of [B1]. The Feynman transform $\mathcal{F}\mathbb{S}[t]$ is a $\mathcal{K}-$twisted modular operad. The solutions of quantum master equation in F are in one-to one correspondence with the structure of algebra over the modular operad $\mathcal{F}\mathbb{S}[t]$ on the vector space V with *odd* symmetric inner product B. The genus zero part of $\mathcal{F}\mathbb{S}[t]$ action on V corresponds to the structure of $A_\infty-$algebra with the odd invariant inner product on V. It is straightforward generalisation of the theorem 2 from loc.cit. that the complexes $\mathcal{F}\mathbb{S}[t]((0, \gamma, \nu)))$ compute the cohomology of the compactified moduli spaces with coefficients in the local system $H^i(\overline{\mathcal{M}}'_{\gamma,\nu}/\mathbb{S}_\nu, \mathcal{L})$, $\mathcal{L} = Det(P_\Sigma)$, where P_Σ is the set of marked points on the Riemann surface Σ.

3. Matrix integrals.

We showed in [B1] that solutions to quantum master equation corresponding to a twisted modular operad $\mathcal{P}$ define natural cocycles in the graph complex which is the Feynmann transform of $\mathcal{P}$. In this section we show that for $\widetilde{\mathbb{S}}$ and $\mathbb{S}$ these cocycles can be naturally interpreted as the terms of Feynman diagram expansions of matrix integrals of $\exp(S/\hbar)$.

3.1. **Odd inner product.** Let us assume that

$$S(a_i, \hbar) = \sum_{2(g-1)+n>0} \hbar^g S_{n,g}(a_i)$$

$S_{n,g} \in F_n$, be a solution to quantum master equation in the space (2.2). Let us consider the super vector space

$$M = Hom(V, End_k(U))$$

where U is a super vector space $\dim U < \infty$. If $V = k^{(d|f)}$, then $M = (End_k(U))^{\times d} \times (\Pi End_k(U))^{\times f}$. The supertrace gives a natural even symmetric inner product on the space $End_k(U)$:

$$Tr(X \otimes Y) = \Sigma_{\alpha\beta}(-1)^{\overline{\alpha}} X^{\alpha}_{\beta} Y^{\beta}_{\alpha}$$

X, $Y \in End_k(U)$, it is invariant with respect to the action of the group $GL(U)$. It gives an extension to M of the odd symmetric inner product on $Hom(V,k)$ dual to B. In coordinates it is given by

$$B^{-1}(X,Y) = \sum_{i,j,\alpha,\beta} (B^{-1})^{ij}(-1)^{\overline{\alpha}} X^{\alpha}_{i,\beta} Y^{\beta}_{j,\alpha}$$

Let us extend S to a function S_U on M in the obvious way, so that instead of each cyclically symmetric tensor we put the supertrace of the product of the corresponding matrices from $End_k(U)$

$$\sum a_{i_1} \otimes \ldots \otimes a_{i_k} \to Tr(X_{i_1} \cdot \ldots \cdot X_{i_k})$$

and the product of cyclic words goes to the product of traces.

We claim that the tensors which we associated with $S(a_i, \hbar)$ via the statistical sum model construction to a graph G,equipped with the marking corresponding to basis elements from $\mathbb{MS}^{dual}((G))$, are obtained naturally from the Feynman diagrams corresponding to the asympthotic expansion of the following integral

$$\log \int \exp \frac{1}{\hbar}\left(B^{-1}(X,\Xi) - \frac{1}{2}B^{-1}(X,X) + S_U(X)\right) dX/(2\pi\hbar)^{\frac{N}{2}}$$

(compare with the formula (0.1) from [GK]). This is immediate from the standard rules of the Feynman diagrammatics, in particular the combinatorics of the terms in $S_U(X)$ reproduce exactly the data associated with vertices in the complex $\mathbb{MS}^{dual}((G))$, i.e. symmetric product of cyclic permutations and an integer number.

3.2. **Even inner product.** Let now $S(a_i, \hbar) = \sum_{2(g-1)+n>0} \hbar^g S_{n,g}(a_i)$ is a solution to quantum master equation in the Batalin-Vilkovisky algebra 2.1. In this case the space M is the super vector space

$$M = Hom_k(\Pi V, \Pi End_k(U) \oplus End_k(U))$$

where U is a purely even vector space.The analogue of trace defines on the space $\Pi End_k(U) \oplus End_k(U)$ an *odd* symplectic structure, which after choosing a basis $\{e_\alpha\}$ in U can be written as

$$Tr(X \otimes Y) = \sum_{\alpha\beta} X^{\alpha}_{\beta} Y^{\beta}_{\alpha}$$

$X \in \Pi End_k(U)$, $Y \in End_k(U)$. Notice that there are natural isomorphisms

$$\Pi End_k(U) = Hom_k(\Pi U, U) = Hom_k(U, \Pi U)$$

and also

$$Hom_k(\Pi U, \Pi U) = End_k(U)$$

Therefore elements of the space $\Pi End_k(U) \oplus End_k(U)$ can be naturally composed so that this space is in fact an associative algebra with odd invariant inner product.

We have an extension to M of the even symmetric inner product on $Hom(V,k)$ dual to B:

$$B^{-1}(X,Y) = \sum_{i,j,\alpha,\beta} (B^{-1})^{ij}(-1)^{\overline{\alpha}} X^{\alpha}_{i,\beta} Y^{\beta}_{j,\alpha}$$

X, $Y \in \Pi End_k(U) \oplus End_k(U)$.

We extend as above the solution S to a function S_U on M in the obvious way, so that instead of each cyclically symmetric tensor in $(\Pi V)^{\otimes k}$ we put the trace of the product of the corresponding elements from $(\Pi End_k(U) \oplus End_k(U))$

$$\sum a_{i_1} \otimes \ldots \otimes a_{i_k} \to Tr(X_{i_1} \cdot \ldots \cdot X_{i_k})$$

and the product of cyclic words goes to the product of traces.

Then the standard rules of the Feynman diagrammatics imply immediately that the asymptotic expansion of the integral

$$\log \int \exp \frac{1}{\hbar} \left(B^{-1}(X, \Xi) - \frac{1}{2} B^{-1}(X, X) + S_U(X) \right) dX/(2\pi\hbar)^{\frac{N}{2}}$$

$X, \Xi \in M$, give the statistical sum tensors on graphs which were associated with the solution S in [B1].

Example 1. *Let us consider the case of even one-dimensional space V. It is easy to see that in this case any linear combination of cyclic words of X^3, X^5,..., X^{2n+1} is a solution of quantum master equation. The corresponding matrix integral reproduces the matrix integrals of* [K].

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