The asymptotics of stationary electro-vacuum metrics in odd space-time dimensions

Robert Beig*
Institut für Theoretische Physik
Universität Wien

Piotr T. Chruściel†
LMPT, Fédération Denis Poisson
Tours

December 1, 2006

Abstract

We show that stationary, asymptotically flat solutions of the electro-vacuum Einstein equations are analytic at i^0, for a large family of gauges, in odd space-time dimensions higher than seven. The same is true in space-time dimension five for static vacuum solutions with non-vanishing mass.

1 Introduction

There is currently interest in asymptotically flat solutions of the vacuum Einstein equations in higher dimensions [5, 9]. It is thus natural to enquire which part of our body of knowledge of $(3+1)$–dimensional solutions carries over to higher dimensions. In this note we study that question for asymptotic expansions at spatial infinity of stationary or static electro-vacuum metrics. We prove analyticity at i^0, up to a conformal factor, for a family of natural geometric gauges, in even dimensions $n \geq 6$. The same result is established in space-dimension $n = 4$ for static vacuum metrics with non-vanishing ADM mass.

2 Static vacuum metrics

We write the space-time metric in the form

$$ds^2 = -e^{2u}dt^2 + e^{-\frac{2u}{n-2}}\tilde{g}_{ij}dx^i dx^j \ ,$$

*E–mail robert.beig@univie.ac.at
†E-mail Piotr.Chrusciel@lmpt.univ-tours.fr, URL www.phys.univ-tours.fr/∼piotr

where $\tilde{g}$ is an asymptotically flat Riemannian metric, with $\partial_t u = \partial_t \tilde{g}_{ij} = 0$. The vacuum Einstein equations show that u is $\tilde{g}$–harmonic, with $\tilde{g}$ satisfying the equation

$$\tilde{R}_{ij} = \frac{n-1}{n-2}\partial_i u \partial_j u \; , \qquad (2.1)$$

where n is the space-dimension, and $\tilde{R}_{ij}$ is the Ricci tensor of $\tilde{g}$. We assume $n \geq 3$ throughout.

It is a standard consequence of those equations that, in harmonic coordinates in the asymptotically flat region, and whatever $n \geq 3$, both u and $\tilde{g}_{ij}$ have a full asymptotic expansion in terms of powers of $\ln r$ and inverse powers of r. Solutions without $\ln r$ terms are of special interest, because they can be used to construct smoothly compactifiable hyperboloidal initial surfaces. In even space-time dimension initial data sets containing such asymptotic regions, when close enough to Minkowskian data, lead to asymptotically simple space-times [1, 11]. It has been shown by Beig and Simon that logarithmic terms can always be gotten rid of by a change of coordinates when space-dimension equals three [4, 14].

From what has been said one can infer that the leading order corrections in the metric can be written in a Schwarzschild form, which in "isotropic" coordinates reads

$$\begin{aligned} g_m &= -\left(\frac{1-m/2|x|^{n-2}}{1+m/2|x|^{n-2}}\right)^2 dt^2 + \left(1+\frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \left(\sum_{1=1}^{n} dx_i^2\right) \\ &\approx -\left(1-\frac{m}{\tilde{r}^{n-2}}\right)^2 dt^2 + \left(1+\frac{m}{\tilde{r}^{n-2}}\right)^{\frac{2}{n-2}} \left(\sum_{1=1}^{n} dx_i^2\right) \; , \end{aligned} \qquad (2.2)$$

where m is of course a constant, and $\tilde{r} = |x|$ is a radial coordinate in the asymptotically flat region. This gives the asymptotic expansion

$$u = -\frac{m}{\tilde{r}^{n-2}} + O(\tilde{r}^{-n+1}) \; , \qquad (2.3)$$

Further we have

$$\tilde{g}_{ij} = \delta_{ij} + O(\tilde{r}^{1-n}) \; . \qquad (2.4)$$

Equation (2.3) shows that for $m \neq 0$ the function

$$\omega := (u^2)^{\frac{1}{n-2}} \qquad (2.5)$$

behaves asymptotically as $\tilde{r}^{-2}$, and can therefore be used as a conformal factor in the usual one-point compactification of the asymptotic region. Indeed, assuming that $m \neq 0$ and setting

$$g_{ij} := \omega^2 \tilde{g}_{ij} \; . \qquad (2.6)$$

one obtains a $C^{n-2,1}$ metric[1] on the manifold obtained by adding a point (which we denote by i^0) to the asymptotically Euclidean region.

[1] The differentiability class near i^0 can be established by examining Taylor expansions there.

From the fact that u is $\tilde{g}$–harmonic one finds

$$\Delta\omega = \mu \ , \tag{2.7}$$

where the auxiliary function μ is defined as

$$\mu := \frac{n}{2}\omega^{-1}g^{ij}\partial_i\omega\partial_j\omega \ . \tag{2.8}$$

Note that, in spite of the negative power of ω, this function can be extended by continuity to i^0, the extended function, still denoted by μ, being of $C^{n-2,1}$ differentiability class.

Let L_{ij} be the Schouten tensor of g_{ij},

$$L_{ij} := \frac{1}{n-2}\Big(R_{ij} - \frac{R}{2(n-1)}g_{ij}\Big) \ . \tag{2.9}$$

Using tildes to denote the corresponding objects for the metric $\tilde{g}$, from (2.1) one obtains

$$\tilde{L}_{ij} = \frac{1}{4}\omega^{n-4}\Big((n-1)\partial_i\omega\partial_j\omega - \frac{1}{2}g_{ij}g^{k\ell}\partial_k\omega\partial_\ell\omega\Big) \ . \tag{2.10}$$

We see that for $n \geq 3$ the tensor $\tilde{L}_{ij}$ is bounded on the one-point compactification at infinity, and for $n \geq 4$ it is as differentiable as $d\omega$ and the metric allow. This last property is not true anymore for $n = 3$, however the following objects are well behaved:

$$\tilde{L}_{ij}D^j\omega = \frac{2n-3}{4n}\omega^{n-3}\mu D_i\omega \ , \qquad \tilde{L}_{i[j}D_{k]}\omega = -\frac{1}{4n}\omega^{n-3}\mu g_{i[j}D_{k]}\omega \ . \tag{2.11}$$

3 Conformal rescalings

We recall the well-known transformation law of the Schouten tensor under the conformal rescaling (2.6)

$$L_{ij} = \tilde{L}_{ij} + \omega^{-1}D_iD_j\omega - \frac{1}{2}\omega^{-2}g_{ij}g^{k\ell}\partial_k\omega\partial_\ell\omega \ ; \tag{3.1}$$

we emphasize that this holds whether or not ω is related to $\tilde{L}$ as in (2.10). Taking a trace of (3.1) and using (2.7) one finds

$$R = \frac{(n-1)(n-2)}{2n}\omega^{n-3}\mu \ . \tag{3.2}$$

In our subsequent manipulations it is convenient to rewrite (3.1) as an equation for $D_iD_j\omega$,

$$D_jD_i\omega = \omega(L_{ij} - \tilde{L}_{ij}) + \frac{1}{n}\mu g_{ij} \ . \tag{3.3}$$

We note that the right-hand-side is well-behaved at $\omega = 0$ for all $n \geq 3$.

Let C_{ijk} denote the Cotton tensor,

$$C_{ijk} := D_kL_{ij} - D_jL_{ik} \ , \tag{3.4}$$

and let C_{ijkl} be the Weyl tensor of g. Note the identity

$$D^i C_{ijkl} = (3-n)C_{jkl} \ . \tag{3.5}$$

Applying D_k to (3.3) and anti-symmetrising over j and k one obtains

$$\omega C_{ijk} + C_{kji\ell} D^\ell \omega = \tilde{L}_{ijk} \ , \tag{3.6}$$

where

$$\tilde{L}_{ijk} := 2D_{[k}(\omega \tilde{L}_{j]i}) + 2g_{i[j}\tilde{L}_{k]\ell} D^\ell \omega \ . \tag{3.7}$$

Writing down the second term in (3.7) and using (2.11) and again (3.3), we find that the terms with ω^{n-4} drop out and there results

$$\tilde{L}_{ijk} = \frac{1}{2}\omega^{n-2}\Big(\underbrace{-(n-1)D_{[j}\omega(L_{k]i} - \tilde{L}_{k]i}) + g_{i[j}(L_{k]\ell} - \tilde{L}_{k]\ell})D^\ell \omega}\Big) \ . \tag{3.8}$$

Here $\tilde{L}_{ij}$ should be expressed in terms of ω, $d\omega$ and μ using (2.11). It should be emphasized that the underbraced expression is regular at $\omega = 0$.

Let B_{ij} denote the Bach tensor,

$$B_{ij} := D^k C_{ijk} - L^{k\ell} C_{kji\ell} \ . \tag{3.9}$$

Applying D^k to (3.6) and using (3.5) and (3.3) one obtains

$$B_{ij} - (n-4)\omega^{-1} C_{ijk} D^k \omega \ = \ \omega^{-1} D^k \tilde{L}_{ijk} - C_{kji\ell}\tilde{L}^{k\ell} \ . \tag{3.10}$$

Note that the factor ω^{-1} in front of the divergence $D^k\tilde{L}_{ijk}$ is compensated by ω^{n-2} in (3.7), so that for $n \geq 4$ the right-hand-side is a well-behaved function of the metric, of ω, and of their derivatives at zeros of ω. Alternatively we can, using (3.6), rewrite (3.10) as

$$B_{ij} + (n-4)\omega^{-2} C_{kijl} D^k \omega D^l \omega = \omega^{3-n} D^k(\omega^{n-4}\tilde{L}_{ijk}) - C_{kijl}\tilde{L}^{kl} \ . \tag{3.11}$$

Note that the right-hand-side of (3.11) is regular also for $n = 3$. Recall, now, the identity

$$\begin{aligned} B_{ij} &= \Delta L_{ij} - D_i D_j (\mathrm{tr} L) + \mathscr{F}_{ij} \\ &= \frac{1}{n-2}\Delta R_{ij} - \frac{1}{2(n-1)}\Big(\frac{1}{n-2}\Delta R\, g_{ij} + D_i D_j R\Big) + \mathscr{F}_{ij} \ , \end{aligned} \tag{3.12}$$

where $\mathscr{F}_{ij}$ depends upon the metric and its derivatives up to order two. We eliminate the Ricci scalar terms using (3.2). The terms involving derivatives of R will introduce derivatives of μ, which can be handled as follows. Differentiating (2.8) and using (3.3) one obtains

$$\begin{aligned} D_i \mu &= -n(L_{ij} - \tilde{L}_{ij})D^j \omega \\ &= -nL_{ij}D^j\omega + \frac{2n-3}{4}\omega^{n-3}\mu D_i \omega \ , \end{aligned} \tag{3.13}$$

which allows us to eliminate each derivative of μ in terms of μ, ω and $d\omega$.

3.1 Space-dimensions three and four

In dimension three the term involving $\omega^{-2}C_{kji\ell}L^{k\ell}$ on the left-hand-side of (3.11) goes away because the Weyl tensor vanishes. In dimension four its coefficient vanishes. In those dimensions one therefore ends up with an equation of the form

$$\Delta R_{ij} = F_{ij}(n,\omega,d\omega,\partial^2\omega,g,\partial g,\partial^2 g)\;. \tag{3.14}$$

with a tensor field F_{ij} which is well behaved at $\omega = 0$. Here we have used the expression of μ as a function of the metric, ∂g, $\partial\omega$ and $\partial^2\omega$ which follows from (2.7).

We can calculate the laplacian of μ by taking a divergence of (3.13) and eliminating again the second derivatives of ω in terms of μ, and the first derivatives of μ, as before. This leads to a fourth-order equation for ω of the form

$$\Delta^2\omega = F(n,\omega,d\omega,\partial^2\omega,g,\partial g,\mathrm{Ric})\;, \tag{3.15}$$

with F — well behaved at $\omega = 0$, where Ric stands for the Ricci tensor. Note that one should use the Bianchi identities to eliminate the term involving the divergence of L_{ij} which arises in the process:

$$D^j L_{ij} = \frac{1}{2(n-1)} D_i R\;.$$

In harmonic coordinates, Equations (3.14)-(3.15) can be viewed as a system of equations of fourth order for the metric g and the function ω, with diagonal principal part Δ^2. The system is elliptic so that usual bootstrap arguments show smoothness of all fields. In fact the solutions are real-analytic by [13], as we wished to show.

3.2 Higher even dimensions

A natural generalisation of the Bach tensor in even dimensions $n \geq 6$ is the obstruction tensor $\mathscr{O}_{ij}$ of Fefferman and Graham [10, 12]. It is of the form

$$\mathscr{O}_{ij} = \Delta^{\frac{n-4}{2}}[\Delta L_{ij} - D_i D_j(\mathrm{tr}L)] + \mathscr{F}^n_{ij}\;, \tag{3.16}$$

where $\mathscr{F}^n_{ij}$ is a tensor constructed out of the metric and its derivatives up to order $n-2$. This leads us to expect that further differentiations of the equations above leads to a regular expression for $\Delta^{\frac{n-2}{2}}B_{ij}$ in terms of ω and its derivatives up to order $n-3$. However, we have not been able to conclude using this approach. Instead, we proceed as in [6]:

In coordinates x^i which are harmonic with respect to the metric $\tilde g$, (2.1) and the harmonicity condition for u lead to a set of equations for u and

$$f := (\tilde g_{ij} - \delta_{ij})$$

of the form

$$\tilde g^{ij}\partial_i\partial_j f = F(f)(\partial f)^2 + (\partial u)^2\;, \quad \tilde g^{ij}\partial_i\partial_j u = 0\;.$$

Setting

$$\Omega = \frac{1}{r^2} \ , \quad \tilde{f} = \Omega^{-\frac{n-2}{2}} f \ , \quad \tilde{u} = \Omega^{-\frac{n-2}{2}} u \ , \quad y^i = \frac{x^i}{r^2} \ ,$$

one obtains a set of regular elliptic equations in the coordinates y^i after a conformal rescaling $\delta_{ij} \to \Omega^2 \delta_{ij}$ of the flat metric, provided that $n \geq 6$. The reader is referred to [6] for a detailed calculation in a Lorentzian setting, which carries over with minor modifications (due to the quadratic rather than linear zero of Ω) to the current situation; note that n in the calculations there should be replaced by $n-1$ for the calculations at hand. We further note that the leading order behavior of $\tilde{f}$ is governed by the mass, which can be made arbitrarily small by a constant rescaling of the metric and of the original harmonic coordinates x^i; this freedom can be made use of to ensure ellipticity of the resulting equations. Finally we emphasise that this result, contrary to the one for n equal three or four, does not require the non-vanishing of mass.

4 Stationary vacuum solutions

We consider Lorentzian metrics ${}^{n+1}g$ in odd space-time-dimension $n+1 \geq 7$, with Killing vector $X = \partial/\partial t$. In adapted coordinates those metrics can be written as

$${}^{n+1}g = -V^2(dt + \underbrace{\theta_i dx^i}_{=\theta})^2 + \underbrace{g_{ij} dx^i dx^j}_{=g} \ , \tag{4.1}$$

$$\partial_t V = \partial_t \theta = \partial_t g = 0 \ . \tag{4.2}$$

The vacuum Einstein equations (with vanishing cosmological constant) read (see, e.g., [8])

$$\begin{cases} V\nabla^*\nabla V = \frac{1}{4}|\lambda|_g^2 \ , \\ \mathrm{Ric}(g) - V^{-1}\mathrm{Hess}_g V = \frac{1}{2V^2}\lambda \circ \lambda \ , \\ \mathrm{div}(V\lambda) = 0 \ , \end{cases} \tag{4.3}$$

where

$$\lambda_{ij} = -V^2(\partial_i \theta_j - \partial_j \theta_i) \ , \quad (\lambda \circ \lambda)_{ij} = \lambda_i{}^k \lambda_{kj} \ .$$

We consider metrics satisfying, for some $\alpha > 0$,

$$g_{ij} - \delta_{ij} = O(r^{-\alpha}) \ , \ \partial_k g_{ij} = O(r^{-\alpha-1}) \ , \quad V = O(r^{-\alpha}) \ , \ \partial_k V = O(r^{-\alpha-1}) \ . \tag{4.4}$$

As is well known [2], one can then introduce new coordinates, compatible with the above fall-off requirements, which are harmonic for g.

Next, a redefinition $t \to t + \psi$, introduces a gauge transformation

$$\theta \to \theta + d\psi \ ,$$

and one can exploit this freedom to impose restrictions on θ. We will assume a condition of the form

$$g^{ij}\partial_i \theta_j = Q(\underbrace{g, V}_{p}; \underbrace{\partial g, \partial V, \theta}_{q}) \ , \tag{4.5}$$

where Q is a smooth function of the variables listed near $(\delta, 1; 0, 0, 0)$, with a zero of order two or higher *with respect to* q:

$$Q(p;0) = \partial_q Q(p;0) = 0 \ .$$

Examples include the *harmonic gauge*, $\Box_{{}^{n+1}g} t = 0$, which reads

$$\partial_i(\sqrt{\det g} V g^{ij}\theta_j) = 0 \ , \tag{4.6}$$

as well as the maximal gauge,

$$\partial_i(\frac{V^3\sqrt{\det g}\, g^{ij}}{\sqrt{1-V^2 g^{k\ell}\theta_k\theta_\ell}}\theta_j) = 0 \ . \tag{4.7}$$

Equation (4.6) can always be achieved by solving a linear equation for ψ, *cf.*, *e.g.*, [2, 7] for the relevant isomorphism theorems. On the other hand, (4.7) can always be solved outside of some large ball [3]. More generally, when non-linear in θ, equation (4.5) can typically be solved outside of some large ball using the implicit function theorem in weighted Hölder or weighted Sobolev spaces.

In harmonic coordinates, and in a gauge (4.5), the system (4.3) is elliptic and, similarly to the static case, standard asymptotic considerations show that g_{ij} is Schwarzschild in the leading order, and that there exist constants α_{ij} such that

$$\theta_i = \frac{\alpha_{ij}x^j}{r^n} + O(r^{-n}) \ .$$

To prove analyticity at i^0 one proceeds as in Section 3.2: thus, one first rewrites the second of equations (4.3) as an equation for

$$\tilde g_{ij} := e^{\frac{2u}{n-2}} g_{ij} \equiv V^{\frac{2}{n-2}} g_{ij} \ ,$$

which gets rid of the Hessian of V there. It should then be clear that, in coordinates which are harmonic for $\tilde g$, the first two equations in (4.3) have the right structure for the argument of Section 3.2. It remains to check the third one. For this we note that, in $\tilde g$–harmonic coordinates so that $\partial_i(\sqrt{\det \tilde g}\, \tilde g^{ij}) = 0$,

$$\begin{aligned}
\mathrm{div}(V\lambda)_k &= \frac{1}{\sqrt{\det g}}\partial_i\Big(\sqrt{\det g} V^3 g^{ij}(\partial_j\theta_k - \partial_k\theta_j)\Big) \\
&= \frac{V^{\frac{n}{n-2}}}{\sqrt{\det \tilde g}}\partial_i\Big(\sqrt{\det \tilde g} V^2 \tilde g^{ij}(\partial_j\theta_k - \partial_k\theta_j)\Big) \\
&= V^{\frac{n}{n-2}}\tilde g^{ij}\partial_i\Big(V^2(\partial_j\theta_k - \partial_k\theta_j)\Big) \\
&= V^{\frac{n}{n-2}}\Big(\tilde g^{ij}\partial_i\partial_j\theta_k + 2V\tilde g^{ij}\partial_i V(\partial_j\theta_k - \partial_k\theta_j) \\
&\quad - \underbrace{\tilde g^{ij}\partial_i\partial_k\theta_j}_{=V^{-\frac{2}{n-2}}(\partial_k(\underbrace{g^{ij}\partial_i\theta_j}_{Q}) + \partial_k g^{ij}\partial_i\theta_j)}\Big) \ .
\end{aligned}$$

If Q in (4.5) is zero, then the vanishing of $\mathrm{div}(V\lambda)$ immediately gives an equation of the right form for θ. Otherwise, ∂Q leads to nonlinear terms of the form $\partial_x^2 g\,\theta$, *etc.*, which are again of the right form, see the calculations in [6]. Note that such terms do not affect the ellipticity of the equations because of their off-diagonal character.

5 Einstein-Maxwell equations

The above considerations immediately generalise to the stationary Einstein-Maxwell equations, with a Killing vector which approaches a time-translation in the asymptotically flat region. Indeed, the calculations of Section 4 carry over to this setting, as follows:

Stationary Maxwell fields can be described by a time-independent scalar field $\varphi = A_0$ and a vector potential $A = A_i dx^i$, again time-independent. Here one needs to assume that, in addition to (4.4), one has

$$A_\mu = O(r^{-\alpha}) \ , \ \partial_k A_\mu = O(r^{-\alpha-1}) \ .$$

Maxwell fields lead to supplementary source terms in the right-hand-sides of (4.3) which are quadratic in the first derivatives of φ and A, hence of the right form for the argument so far. Next, if we write the Maxwell equations as

$$\frac{1}{\sqrt{{}^{n+1}g}}\,\partial_\mu\Big(\sqrt{{}^{n+1}g}\ {}^{n+1}g^{\mu\rho}\ {}^{n+1}g^{\nu\sigma}\partial_{[\nu}A_{\sigma]}\Big) = 0 \ ,$$

and impose the Lorenz gauge,

$$\frac{1}{\sqrt{{}^{n+1}g}}\partial_\mu\Big(\sqrt{{}^{n+1}g}\ {}^{n+1}g^{\mu\nu}A_\nu\Big) = 0 \ ,$$

the equations $\partial_t A_\mu = 0$ allow one to rewrite the above as

$$g^{ij}\partial_i\partial_j a = H(f, V, \theta; \partial f, \partial V, \partial\theta; \partial a) \ ,$$

where $a = (\varphi, A_i)$, with a function H which is bilinear in the second and third groups of arguments. This is again of the right form, which finishes the proof of analyticity of $\tilde f, \tilde\varphi, \tilde A$ and $\tilde\theta$ at i_0 for even $n \geq 6$, where the original fields are related to the tilde-ones via a rescaling by $\Omega^{\frac{n-2}{2}}$, *e.g.* $\varphi = \Omega^{\frac{n-2}{2}}\tilde\varphi$, and so on.

References

[1] M.T. Anderson and P.T. Chruściel, *Asymptotically simple solutions of the vacuum Einstein equations in even dimensions*, Commun. Math. Phys. **260** (2005), 557–577, gr-qc/0412020. MR MR2183957

[2] R. Bartnik, *The mass of an asymptotically flat manifold*, Commun. Pure and Appl. Math. **39** (1986), 661–693.

[3] R. Bartnik, P.T. Chruściel, and N. Ó Murchadha, *On maximal surfaces in asymptotically flat space-times.*, Commun. Math. Phys. **130** (1990), 95–109.

[4] R. Beig and W. Simon, *Proof of a multipole conjecture due to Geroch*, Commun. Math. Phys. **78** (1980), 75–82.

[5] P. Bizoń, T. Chmaj, and B.G. Schmidt, *Critical behavior in vacuum gravitational collapse in 4+1 dimensions*, Phys. Rev. Lett. **95** (2005), 071102, gr-qc/0506074.

[6] Y. Choquet-Bruhat, P.T. Chruściel, and J. Loizelet, *Global solutions of the Einstein-Maxwell equations in higher dimension*, Class. Quantum Grav. (2006), 7383–7394.

[7] D. Christodoulou and Y. Choquet-Bruhat, *Elliptic systems in $H_{s,\delta}$ spaces on manifolds which are Euclidean at infinity*, Acta. Math. **146** (1981), 129–150.

[8] R. Coquereaux and A. Jadczyk, *Riemannian geometry, fiber bundles, Kaluza-Klein theories and all that*, World Sci. Lect. Notes Phys. **16** (1988), 1–345.

[9] R. Emparan and H.S. Reall, *Black rings*, (2006), hep-th/0608012.

[10] C. Fefferman and C.R. Graham, *Conformal invariants*, Élie Cartan et les mathématiques d'aujourd'hui, The mathematical heritage of Élie Cartan, Sémin. Lyon 1984, Astérisque, No.Hors Sér. 1985, 95-116.

[11] H. Friedrich, *On the existence of n–geodesically complete or future complete solutions of Einstein's field equations with smooth asymptotic structure*, Commun. Math. Phys. **107** (1986), 587–609.

[12] C.R. Graham and K. Hirachi, *The ambient obstruction tensor and Q-curvature*, AdS/CFT correspondence: Einstein metrics and their conformal boundaries, IRMA Lect. Math. Theor. Phys., vol. 8, Eur. Math. Soc., Zürich, 2005, pp. 59–71.

[13] C.B. Morrey, *Multiple integrals in the calculus of variation*, Springer Verlag, Berlin, Heidelberg, New York, 1966.

[14] W. Simon and R. Beig, *The multipole structure of stationary space–times*, Jour. Math. Phys. **24** (1983), 1163–1171.