

Strong cosmic censorship for T^2 -symmetric cosmological spacetimes with collisionless matter

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Abstract

We prove strong cosmic censorship for T^2 -symmetric cosmological spacetimes (with spatial topology T^3 and vanishing cosmological constant Λ) with collisionless matter. Gowdy symmetric spacetimes constitute a special case. The formulation of the conjecture is in terms of generic C^2 -inextendibility of the metric. Our argument exploits a rigidity property of Cauchy horizons, inherited from the Killing fields.

1 Introduction

Strong cosmic censorship is one of the fundamental open problems of classical general relativity. Properly formulated [7], it is the conjecture that the maximal development of generic compact or asymptotically flat initial data for suitable Einstein-matter systems be inextendible as a suitably regular Lorentzian manifold.

In recent years, progress has been made when the initial data are restricted to symmetry classes, in particular, spherical [6, 11, 12] and Gowdy [9, 18] symmetry. The nature of the difficulties in these two classes is very different. In addition to the case of a horizon arising from a singular point on the centre, Cauchy horizons in spherical symmetry can arise on account of a global property of the causal geometry of the Lorentzian quotient manifold of group orbits. An example is provided by the Reissner-Nordström solution. The stability or instability of this phenomenon depends on what is essentially a completely global analysis.¹ In the Gowdy case with spatial topology T^3 , Cauchy horizon formation is a local phenomenon from the point of view of the quotient, and is related to the group orbits becoming null. The remarkable recent progress [18]

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¹This already indicates that strong cosmic censorship in its full generality can never be approached by a local analysis in the style of the so-called “BKL proposal”.

on the C^2 -inextendibility version of cosmic censorship in the Gowdy case for the vacuum equations rests on a detailed asymptotic analysis of the local behaviour of a solution near points of the past boundary of the maximal development.

Gowdy symmetric spacetimes are a sub-class of so-called T^2 -symmetric spacetimes, i.e. spacetimes which admit a torus action. The asymptotic analysis of [18] seems prohibitively difficult in this more general case, leaving to far in the future the task of pursuing this approach for proving strong cosmic censorship in this class.

The aim of the present paper is to show that this difficult analysis can in fact be completely circumvented if one is willing to couple the Einstein equations with the Vlasov equation, i.e. to consider spacetimes with collisionless matter. We will thus here give an elementary proof of strong cosmic censorship (in the C^2 -inextendibility formulation) for general T^2 -symmetric spacetimes (with spatial topology T^3) solving the Einstein-Vlasov system. The proof relies on a previous characterization of the boundary of the maximal development proven by Weaver [19].² Our method can be expected to apply when additional matter fields are also coupled, but the presence of the Vlasov field is essential.

The main idea of the method is quite simple: Let us suppose that our spacetime is C^2 -past extendible with Cauchy horizon \mathcal{H}^- . The characterization [19] of the past boundary of the maximal development allows us to deduce that there is a null vector in the span of the Killing fields for a dense open subset of \mathcal{H}^- . It turns out that this fact gives the Cauchy horizon considerable rigidity, in particular, $\text{Ric}(K, K) \leq 0$ where K denotes the null generator of \mathcal{H}^- at regular points, and thus, by the null convergence condition, $\text{Ric}(K, K) = 0$. On the other hand, we can bound the Ricci curvature away from 0 by following geodesics back to initial data, provided that the matter is supported initially on a suitably large portion of the mass shell, specifically, that its support intersects every open set. By exploiting conservation of the inner product of velocity with a Killing vector along any geodesic, we may weaken this to the condition that the matter intersects every open set of sufficiently small tangential momentum. The contradiction yields strong cosmic censorship.

The arguments of the present paper have been adapted from our recent work [14] on strong cosmic censorship in surface symmetry, in particular, the case of hyperbolic symmetry. In the context of the type of arguments employed here, the hyperbolic symmetric case is in fact considerably more complicated than the T^2 symmetric case, because its Lie algebra is non-abelian and Killing vectors can vanish. Moreover, the hyperbolic symmetric case also may admit Cauchy horizons on account of global phenomena, and these must be treated by a separate and very different method. The reader is strongly encouraged to look at [14].

²Strictly speaking, the characterization of the boundary in [19] concerns a class of initial data too special, for in the present paper it will be assumed that the support of f on the mass shell is non-compact. These results can be easily adapted, however. See for instance [14] for this adaptation in the surface symmetric case.

2 The Einstein-Vlasov system

Let (\mathcal{M}, g) be a 4-dimensional spacetime with C^2 metric. Let $P \subset T\mathcal{M}$ denote the set of all future directed timelike vectors of length -1 . We will call P the *mass shell*. Let f denote a nonnegative function $f : P \rightarrow \mathbb{R}$. We say that $\{(\mathcal{M}, g), f\}$ satisfies the Einstein-Vlasov system (with vanishing cosmological constant Λ) if

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}, \quad (1)$$

$$p^\alpha \partial_{x^\alpha} f - \Gamma_{\beta\gamma}^\alpha p^\beta p^\gamma \partial_{p^\alpha} f = 0, \quad (2)$$

$$T_{\alpha\beta}(x) = \int_{\pi^{-1}(x)} p_\alpha p_\beta f, \quad (3)$$

where p^α define the momentum coordinates on the tangent bundle conjugate to x^α , where $\pi : P \rightarrow \mathcal{M}$ denotes the natural projection, and the integral in (3) is to be understood with respect to the natural volume form on $\pi^{-1}(x)$.

We call f the *Vlasov field*. The equation (2) is just the statement in coordinates that f be preserved along geodesic flow on P . In physical language, f describes thus the distribution of so-called *collisionless matter*, and sometimes we shall refer to solutions of the system (1)–(3) as collisionless matter spacetimes.

For any null vector V , in view of the condition $f \geq 0$, (3) and (1), one obtains the inequality

$$R_{\mu\nu} V^\mu V^\nu \geq 0. \quad (4)$$

Collisionless matter spacetimes thus satisfy the *null convergence condition*.

For a full discussion of the Einstein-Vlasov system, see [1, 17].

3 T^2 symmetry

We will say that a spacetime (\mathcal{M}, g) is T^2 symmetric if the Lie group T^2 acts differentiably on (\mathcal{M}, g) by isometry, and the group orbits are spatial. See [3] for a general discussion. The Lie algebra is spanned by two commuting Killing fields X and Y which are nonvanishing. We may normalise these so that the quantity

$$R = g(X, X)g(Y, Y) - g(X, Y)^2 \quad (5)$$

gives the area element of the group orbits when multiplied by $X \wedge Y$. In particular, X and Y are nowhere vanishing in the spacetime.

The Gowdy case studied in [19] is a special case of the above, when the so-called twists of X and Y vanish. Note that in the vacuum case, if (\mathcal{M}, g) is globally hyperbolic and spatially compact, then either it is Gowdy, or its spatial topology is T^3 .

4 The main theorem

Theorem 4.1. *Let (\mathcal{M}, g) be a globally hyperbolic T^2 -symmetric spacetime with C^2 metric, with compact Cauchy surface $\Sigma = T^3$ topologically, let X and Y be globally defined Killing fields spanning the Lie algebra, let R be as in (5). Assume*

1. *All past incomplete causal geodesics $\gamma(t)$ satisfy $R(t) \rightarrow 0$, and*
2. *$f : P \rightarrow \mathbb{R}$ is such that g, f satisfy the Einstein-Vlasov system, with $f \in C^0$, and*
3. *There exists a constant $\delta > 0$ such that for any open $\mathcal{U} \subset P \cap \pi^{-1}(\Sigma)$ we have that f does not vanish identically on $\mathcal{U} \cap \{p : g(p, X)^2 + g(p, Y)^2 < \delta\}$.*

Then (\mathcal{M}, g) is past inextendible as a C^2 Lorentzian manifold.

With the usual convention for time orientation, Assumption 1 above has been shown in [19] for maximal developments of all sufficiently regular T^2 initial data sets (with topology T^3) for (1)–(3), for which f does not vanish identically, provided that f is initially compactly supported in $P \cap \pi^{-1}(\Sigma)$. As in [14], this argument can easily be adapted to some class where the data remain compactly supported in the tangential momentum directions, but are allowed in other directions to decay sufficiently fast initially with respect to linear coordinates on P . Assumption 2 of course holds for such maximal developments by definition. Within this extended class, Assumption 3 can be viewed as a genericity assumption. Thus the above theorem implies strong cosmic censorship (in its C^2 -inextendibility formulation) in the past direction.

On the other hand, the future inextendibility requirement for strong cosmic censorship holds for maximal developments of arbitrary T^2 -symmetric initial data (with topology T^3) for (1)–(3) in view of the results of [2, 13]. The results of [13] are more elementary than those here, but also rest on the extendibility of the Killing vectors.

Thus, Theorem 4.1 implies strong cosmic censorship for T^2 symmetric spacetimes with collisionless matter and spatial topology T^3 .

5 Cauchy horizon rigidity

We show in this section that under the first assumption of Theorem 4.1, Cauchy horizons must inherit a certain rigidity, namely, at regular points, the Ricci curvature in the direction of the null generator must vanish.

Proposition 5.1. *Let (\mathcal{M}, g) be a globally hyperbolic T^2 symmetric spacetime as in Theorem 4.1, satisfying Assumption 1, but not necessarily Assumptions 2 and 3. Suppose (\mathcal{M}, g) is past extendible and let \mathcal{H}^- denote the past Cauchy horizon of Σ in the extension $(\tilde{\mathcal{M}}, \tilde{g})$. Then there exists a dense subset $\tilde{S} \subset \mathcal{H}^-$, at which $T_p \mathcal{H}^-$ is a hyperplane whose orthogonal complement is spanned by a null vector V , for which*

$$\text{Ric}(V, V) \leq 0.$$

Proof. \mathcal{H}^- is an achronal Lipschitz submanifold [16]. By the results of [8], it is differentiable on a dense subset S , on which its tangent plane must clearly then be null.

In what follows we will relate the null generator to $T_p\mathcal{H}^-$ at regular points to the span of the Killing vector fields.

By the results of [13], X and Y extend C^2 through \mathcal{H}^- . (That is to say, they can be extended to C^2 vector fields on $\tilde{\mathcal{M}}$, not necessarily Killing.)

Let $p \in \mathcal{H}^-$ be regular. The vectors X and Y must clearly be tangent to \mathcal{H}^- at p . This follows since the integral curves of X, Y through points of \mathcal{M} stay in \mathcal{M} .

Define

$$\mathcal{H}_2^- = \{X \wedge Y \neq 0\} \cap \mathcal{H}^-,$$

and

$$\mathcal{H}_1^- = \text{int}(\{X \wedge Y = 0, X \neq 0, Y \neq 0\}),$$

where int denotes the interior with respect to the topology of \mathcal{H}^- . These are clearly open subsets of \mathcal{H}^- .

Lemma 5.1. *If $p \in \mathcal{H}_2^-$, then $X(p), Y(p)$ span a null plane, tangent to \mathcal{H}^- if in addition $p \in S$. If $p \in \mathcal{H}_1^-$, then X and Y lie in a null direction, again tangent to \mathcal{H}^- if in addition $p \in S$. Finally,*

$$\mathcal{H}^- = \overline{\mathcal{H}_1^- \cup \mathcal{H}_2^-}. \quad (6)$$

Proof. By the assumption that $R \rightarrow 0$ along any causal geodesic approaching \mathcal{H}^- , it follows that R extends to a C^2 function vanishing along \mathcal{H}^- . From (5), it is clear that at points $p \in \mathcal{H}_2^-$, the plane spanned by $X(p)$ and $Y(p)$ is null. The first statement of the Lemma follows in view also of the fact that X and Y are tangent to \mathcal{H}^- at S .

To prove (6), it is equivalent to prove that the set

$$\{X = 0\} \cup \{Y = 0\}$$

has empty interior in \mathcal{H}^- . This in turn will follow from the following statement: Let Z be a vector field in the span of X and Y , such that Z does not identically vanish on the spacetime. Since $R > 0$ in the spacetime, in fact, it follows that Z vanishes nowhere in the spacetime. Then $\{Z = 0\}$ has empty interior in \mathcal{H}^- .

Let Z be then as above, and let \mathcal{U} denote the interior of $\{Z = 0\}$.

We will first show that ∇Z vanishes identically in \mathcal{U} . For $q \in \mathcal{U} \cap S$, let $E_1(q), E_2(q), L(q), K(q)$ denote a null frame where $E_1(q), E_2(q), K(q)$ are tangent to \mathcal{H}^- at q . For any vector $W(q)$, we compute:

$$g(\nabla_{E_i} Z, W) = E_i g(Z, W) - g(Z, \nabla_{E_i} W) = 0,$$

$$g(\nabla_K Z, W) = K g(Z, W) - g(Z, \nabla_K W) = 0,$$

$$g(\nabla_L Z, E_i) = -g(\nabla_{E_i} Z, L) = 0,$$

$$g(\nabla_L Z, K) = -g(\nabla_K Z, L) = 0,$$

$$g(\nabla_L Z, L) = 0,$$

where we have used the Killing property of Z and its vanishing in \mathcal{U} . Thus ∇Z vanishes identically in $\mathcal{U} \cap S$, and thus, by density and continuity, identically on \mathcal{U} .

From the well known relation

$$\nabla_\alpha \nabla_\beta Z_\gamma = R_{\alpha\beta\gamma\delta} Z^\delta$$

which holds for any Killing vector field Z , by considering a family of timelike geodesics transverse to \mathcal{H}^- , it follows immediately that, if $\mathcal{U} \neq \emptyset$, then Z must vanish identically in a neighborhood of $S \cap \mathcal{U}$ in \mathcal{M} . Since Z does not vanish at any point of \mathcal{M} , and S is dense in \mathcal{U} , we must have $\mathcal{U} = \emptyset$. This shows (6).

We turn to show the second statement of the Lemma. Consider a point $p \in \mathcal{H}_1^- \cap S$. Completing X to a C^2 frame X, V_1, V_2, V_3 for the tangent bundle in a neighborhood of p , we may write

$$Y = \alpha X + \beta_1 V_1 + \beta_2 V_2 + \beta_3 V_3$$

where β_i, α are C^2 functions. Since \mathcal{H}_1^- is open, we have

$$\beta_1 = \beta_2 = \beta_3 = 0 \tag{7}$$

in a neighborhood in \mathcal{H}^- of p .

From $[X, Y] = 0$ we obtain

$$(X\alpha)X + (X\beta_1)V_1 + (X\beta_2)V_2 + (X\beta_3)V_3 = 0,$$

and thus α in particular is constant along integral curves of X . On the other hand, for V tangent to \mathcal{H}^- at p ,

$$\begin{aligned} 0 &= g(\nabla_V Y, V) \\ &= g(\nabla_V(\alpha X + \beta_1 V_1 + \beta_2 V_2 + \beta_3 V_3), V) \\ &= \alpha g(\nabla_V X, V) + (V\alpha)g(X, V) + \sum \beta_i g(\nabla_V V_i, V) + \sum_i (V\beta_i)g(V_i, V) \\ &= (V\alpha)g(X, V). \end{aligned}$$

If X is spacelike, then the orthogonal space of X is transverse to \mathcal{H}^- at p , and thus the above implies that $V\alpha = 0$ for a dense set of directions, and thus by continuity argument for all directions tangent to \mathcal{H}^- . By an additional continuity argument we obtain that Y is a constant multiple aX in the connected component of p in \mathcal{H}_1^- . Defining $Z = Y - aX$ and applying the previous result that $\{Z = 0\}$ has empty interior in \mathcal{H}^- , we obtain a contradiction. Thus X is null. \square

To proceed we will first need the following

Lemma 5.2. *Let $p \in S$ and let K denote a Killing vector field such that $K(p)$ is null. Then*

$$\nabla_K K(p) = \kappa(p)K(p), \quad (8)$$

for a real number κ .

Proof. Complete $K(p)$ to a null frame $K(p), L(p), E_1(p), E_2(p)$ at $T_p\tilde{\mathcal{M}}$, such that E_1, E_2 are in $T_p\mathcal{H}^-$.

Note first that

$$E_1g(K, K)(p) = E_2g(K, K)(p) = 0,$$

since p is a local minimum of $g(K, K)$ along \mathcal{H}^- to which both E_1 and E_2 are tangent. Thus, by the Killing equation,

$$0 = E_1g(K, K)(p) = 2g(\nabla_{E_1}K, K)(p) = -g(\nabla_KK, E_1)(p),$$

$$0 = E_2g(K, K)(p) = 2g(\nabla_{E_2}K, K)(p) = -g(\nabla_KK, E_2)(p).$$

On the other hand $Kg(K, K) = 0$ on account of the Killing equation, and thus similarly we have $g(\nabla_KK, K) = 0$. Thus, $\nabla_KK(p)$ is in the direction K , and κ of (8) can be obtained from

$$\frac{1}{4}Lg(K, K)(p) = \frac{1}{2}g(\nabla_LK, K)(p) = -\frac{1}{2}g(\nabla_KK, L)(p) \doteq \kappa.$$

□

We will need a further partition of \mathcal{H}_1^- and \mathcal{H}_2^- . Applying Lemma 5.2 to points $p \in \mathcal{H}_1^- \cap S$ and the vector $K = X$, we deduce $\nabla_X X = \kappa(p)X$. By density and continuity, $\nabla_X X = \kappa X$ for all points of \mathcal{H}_1^- , where κ is a continuous function on \mathcal{H}^- . Define

$$\mathcal{H}_{1,0}^- = \{p \in \mathcal{H}_1^-, \kappa(p) = 0\}$$

and

$$\mathcal{H}_{1,reg}^- = \mathcal{H}_1^- \setminus \mathcal{H}_{1,0}^-.$$

The set $\mathcal{H}_{1,reg}^-$ is clearly open.

On the other hand, let $p \in \mathcal{H}_2^-$, and let L denote a C^2 null vector field transverse to the C^2 distribution spanned by X and Y in a neighborhood of p . Consider the C^1 function LR on $\tilde{\mathcal{M}}$. LR restricted to \mathcal{H}^- is continuous. Define

$$\mathcal{H}_{2,0}^- = \{p \in \mathcal{H}_2^-, LR = 0\}$$

and

$$\mathcal{H}_{2,reg}^- = \mathcal{H}_2^- \setminus \mathcal{H}_{2,0}^-.$$

Again, the set $\mathcal{H}_{2,reg}^-$ is clearly open. Note that $\mathcal{H}_{1,0}^- \cap S$ coincides with set of points p where $\kappa(p) = 0$ for any $\kappa(p)$ given by Lemma 5.2.

We have

$$\begin{aligned}\mathcal{H}^- &= \overline{\mathcal{H}_{1,reg}^- \cup \mathcal{H}_{2,reg}^-} \cup \mathcal{H}_{1,0}^- \cup \mathcal{H}_{2,0}^- \\ &= \overline{\mathcal{H}_{1,reg}^- \cup \mathcal{H}_{2,reg}^-} \cup (\mathcal{H}_{1,0}^- \cup \mathcal{H}_{2,0}^-) \cap S.\end{aligned}\tag{9}$$

We would like to do certain computations with frames “adapted” to the horizon \mathcal{H}^- . For this, we will need that the various parts of the horizon have sufficient regularity. Our first result in this direction is the following

Lemma 5.3. $\mathcal{H}_{1,reg}^-$ is a C^3 hypersurface.

Proof. First we show that $\mathcal{H}_{1,reg}^-$ is C^2 . Consider the function

$$h = g(X, X).$$

This extends C^2 through \mathcal{H}^- . Clearly, if $\kappa \neq 0$, then $\nabla h \neq 0$, and thus, since then $\mathcal{H}_{1,reg}^- \subset h^{-1}(0)$, we have then that $\mathcal{H}_{1,reg}^-$ is C^2 .

Now consider the C^2 orthogonal distribution to the one-dimensional distribution spanned by X . Since $\mathcal{H}_{1,reg}^-$ has been shown to be C^1 (in fact C^2), and its normal coincides with X on S , then its normal is in the direction of X everywhere, i.e. its tangent space is the orthogonal complement of X . Thus $\mathcal{H}_{1,reg}^-$ is an integral manifold of the above mentioned C^2 distribution, and, consequently, is in fact C^3 . \square

For $\mathcal{H}_{2,reg}^-$ we similarly show

Lemma 5.4. $\mathcal{H}_{2,reg}^-$ is a C^3 hypersurface.

Proof. As before, first we show that $\mathcal{H}_{2,reg}^-$ is C^2 . Recall that the function R extends as a C^2 function through the boundary. Moreover,

$$\mathcal{H}_{2,reg}^- \subset \{R = 0\}.\tag{10}$$

Given $p \in \mathcal{H}_{2,reg}^-$, since by definition $LR(p) \neq 0$ for some L defined near p , it follows that $\nabla R \neq 0$. Thus $\{R = 0\}$ is a C^2 submanifold near p , which must thus coincide with the Lipschitz manifold $\mathcal{H}_{2,reg}^-$ in view of (10). Thus, $\mathcal{H}_{2,reg}^-$ is C^2 .

To show additional regularity, we shall construct a 3-dimensional C^2 distribution tangent to $\mathcal{H}_{2,reg}^-$.

The vectors X and Y span a two-dimensional distribution satisfying $XR = 0$, $YR = 0$. Let E be a C^2 section of the C^2 distribution orthogonal to that spanned by X and Y . Consider a regular point $p \in \mathcal{H}_2^- \cap S$. We may choose Killing fields \tilde{X}, \tilde{Y} so that $\tilde{X}(p)$ is null, normalised so that again $R = g(\tilde{X}, \tilde{X})g(\tilde{Y}, \tilde{Y}) - g(\tilde{X}, \tilde{Y})^2$. We have at p

$$\begin{aligned}ER(q) &= Eg(\tilde{X}, \tilde{X})g(\tilde{Y}, \tilde{Y}) + g(\tilde{X}, \tilde{X})Eg(\tilde{Y}, \tilde{Y}) - 2g(\tilde{X}, \tilde{Y})Eg(\tilde{X}, \tilde{Y}) \\ &= g(\nabla_{\tilde{X}} \tilde{X}, E)g(\tilde{Y}, \tilde{Y}) \\ &= g(\tilde{\kappa}\tilde{X}, E)g(\tilde{Y}, \tilde{Y}) \\ &= 0,\end{aligned}$$

where we have used $g(\tilde{X}, \tilde{X}) = 0$, $g(\tilde{X}, E) = 0$ and Lemma 5.2. Thus, $ER = 0$ on $\mathcal{H}_2^- \cap S$ and thus by continuity and density, on \mathcal{H}_2^- .

One easily sees that \mathcal{H}_2^- is an integral manifold of the distribution spanned by the C^2 vector fields E , X , and Y , and is thus C^3 . (We in fact only need the C^2 statement in what follows.) \square

Lemma 5.5. *For $q \in \mathcal{H}_{2,reg}^-$, let K denote a Killing vector field such that $K(q)$ is a null generator for $T_q\mathcal{H}_{2,reg}^-$. K can be completed to a C^2 frame K , L , E_1 , E_2 for the tangent bundle of $\tilde{\mathcal{M}}$ near q , such that at q , the vectors $K(q), N(q), E_1(q), E_2(q)$ constitute a null frame, the vector field E_1 is Killing in \mathcal{M} , E_2 is tangent to the Cauchy horizon and*

$$g(K, \nabla_{E_1} E_1)(q) = 0, \quad (11)$$

$$E_1 E_1 g(K, K)(q) = 0, \quad (12)$$

$$g(K, \nabla_{E_2} E_2)(q) = 0, \quad (13)$$

$$E_2 E_2 g(K, K)(q) = 8(g(\nabla_{E_2} K, E_1))^2(q). \quad (14)$$

Proof. Let E_1 denote a Killing field such that $g(E_1, E_1)(q) = 1$, and let E_2 denote (as in the proof of Lemma 5.4) a C^2 section (in a neighborhood of q) of the distribution orthogonal to that spanned by the Killing fields, with the additional restriction that $g(E_2, E_2)(q) = 1$.

To see (12), note first that $Yg(X, X) = 0$ everywhere in \mathcal{M} , and in addition, certainly $Xg(X, X) = 0$. Similarly $Xg(Y, Y) = 0$, $Yg(Y, Y) = 0$. Thus $Kg(E_1, E_1) = 0$ identically, in particular (11) holds, and $E_1 E_1 g(K, K) = 0$ identically, in particular (12).

For (13), just note

$$g(K, \nabla_{E_2} E_2) = E_2 g(K, E_2) - g(\nabla_{E_2} K, E_2) = E_2 g(K, E_2) = 0,$$

where we have used that $g(K, E_2) = 0$ identically, as well as the Killing property of K .

Note also that $E_2 g(K, K)(q) = 0$ since q is a local minimum of $g(K, K)$ restricted to \mathcal{H}^- . Now, by the regularity of Lemma 5.4, it follows in particular, that the integral curves of E_2 through points of $\mathcal{H}_{2,reg}^-$ stay on $\mathcal{H}_{2,reg}^-$ for short time, and thus, since

$$g(K, K)g(E_1, E_1) - g(K, E_1)^2 = 0$$

identically on \mathcal{H}^- , we may differentiate twice at q in the direction E_2

$$\begin{aligned} & E_2 E_2 g(K, K)g(E_1, E_1)(q) + 2E_2 g(K, K)E_2 g(E_1, E_1)(q) \\ & + g(K, K)E_2 E_2 g(E_1, E_1)(q) \\ & - 2(E_2 g(K, E_1))^2(q) - 2g(K, E_1)E_2 E_2 g(K, E_1) = 0 \end{aligned}$$

to obtain

$$E_2 E_2 g(K, K)(q) = 2(E_2 g(K, E_1))^2. \quad (15)$$

On the other hand,

$$\begin{aligned}
E_2g(K, E_1) &= g(\nabla_{E_2}K, E_1) + g(K, \nabla_{E_2}E_1) \\
&= g(\nabla_{E_2}K, E_1) - g(E_2, \nabla_K E_1) \\
&= g(\nabla_{E_2}K, E_1) - g(E_2, \nabla_{E_1}K) \\
&= g(\nabla_{E_2}K, E_1) + g(E_1, \nabla_{E_2}K) \\
&= 2g(\nabla_{E_2}K, E_1),
\end{aligned}$$

where we have used the Killing property of E_1 and K , as well as $[E_1, K] = 0$. The above and (15) gives (14). \square

Lemma 5.6. *Let $q \in \mathcal{H}_{1,reg}^-$, let K denote the Killing vector field X . Then K can be completed to a C^2 frame K, L, E_1, E_2 for the tangent bundle of $\tilde{\mathcal{M}}$ near q , such that at q , the vectors $K(q), L(q), E_1(q), E_2(q)$ constitute a null frame, and the relations (11), (12), (13) hold, as well as (14), where now both sides of the equality vanish.*

Proof. Choose unit vectors $E_1(q)$ and $E_2(q)$ tangent to \mathcal{H}^- at q and extend these as C^2 vector fields in $\tilde{\mathcal{M}}$ tangent to \mathcal{H}^- in a neighborhood of q . This is possible in view of Lemma 5.3. Define L to be an arbitrary extension of a null vector $L(q)$ orthogonal to $E_1(q)$ and $E_2(q)$, such that $g(L(q), K(q)) = -2$.

Recall that $g(K, K) = 0$ identically on $\mathcal{H}_{1,reg}^-$. Since the integral curves of the E_i remain on $\mathcal{H}_{1,reg}^-$ for short time, we have (12) and $E_2E_2g(K, K) = 0$. Again, since the E_i are tangent to $\mathcal{H}_{1,reg}^-$ in a neighborhood of q , we have that $g(K, E_i) = 0$ on this neighborhood, and thus in particular

$$g(K, \nabla_{E_i}E_i)(q) = E_i g(K, E_i)(q) - g(\nabla_{E_i}K, E_i)(q) = E_i g(K, E_i)(q) = 0$$

where we have also used the Killing property of K . This gives (11) and, in view also of $E_2E_2g(K, K) = 0$ derived earlier, (14), with both sides equal to 0. \square

For the case of $(\mathcal{H}_{1,0}^- \cup \mathcal{H}_{2,0}^-) \cap S$, it is not clear that one can obtain the regularity necessary to consider adapted C^2 (or even C^1) frames as above. The highly degenerate nature of this case, however, and the nature of the arguments that follow mean that the following result will suffice for us:

Lemma 5.7. *Let $q \in (\mathcal{H}_{1,0}^- \cup \mathcal{H}_{2,0}^-) \cap S$, and let K denote a Killing vector field such that $K(p)$ is null, and consider any C^2 frame K, L, E_1, E_2 for the tangent bundle near q , extending K , where $K(q), L(q), E_1(q), E_2(q)$ is a null frame such that $E_1(q)$ and $E_2(q)$ are tangent to \mathcal{H}^- at q . Then*

$$E_1E_1g(K, K)(q) \geq 0,$$

$$E_2E_2g(K, K)(q) \geq 0.$$

Proof. Let K, L, E_1, E_2 be any frame as in the statement of the Lemma. (There are no obstructions to the construction of such a frame. In the case of $q \in \mathcal{H}_{1,0}^-$, we may take $K = X$.)

In the case of $q \in \mathcal{H}_{1,0}^- \cap S$ recall κ defined by $\nabla_X X = \kappa X$. For $q \in \mathcal{H}_{2,0}^- \cap S$, consider the κ of Lemma 5.2 applied to K . Consider an arbitrary vector field W transverse to \mathcal{H}^- at q . By the condition $\kappa(q) = 0$, we have that $Wg(K, K)(q) = 0$. On the other hand, we have that $g(K, K) > 0$ in the spacetime. Thus $W(Wg(K, K))(q) \geq 0$. Since any E_i is a limit of transversal vectors, it follows that $E_i E_i g(K, K)(q) \geq 0$, as desired. \square

We may now complete the proof of Proposition 5.1. Let

$$q \in \mathcal{H}_{1,reg}^- \cup \mathcal{H}_{2,reg}^- \cup (\mathcal{H}_{1,0}^- \cup \mathcal{H}_{2,0}^- \cap S),$$

and let K, L, E_1, E_2 be a frame as in Lemma 5.5, Lemma 5.6, or Lemma 5.7. Recall now the identity

$$\square g(K, K) = -2\text{Ric}(K, K) + 2g(\nabla K, \nabla K). \quad (16)$$

We evaluate (16) at q . In view of the properties of the frame we obtain

$$\begin{aligned} \square g(K, K)(q) &= -2\nabla_{L,K}^2 g(K, K) + \nabla_{E_1,E_1}^2 g(K, K) + \nabla_{E_2,E_2}^2 g(K, K) \\ &= -2L(K(g(K, K))) + 2\nabla_{\nabla_L K} g(K, K) + E_1(E_1(g(K, K))) \\ &\quad + E_2(E_2(g(K, K))) - \nabla_{\nabla_{E_1} E_1} g(K, K) - \nabla_{\nabla_{E_2} E_2} g(K, K) \\ &= E_1(E_1(g(K, K))) + E_2(E_2(g(K, K))) + 4g(\nabla_{\nabla_L K} K, K) \\ &\quad - 2g(\nabla_{\nabla_{E_1} E_1} K, K) - 2g(\nabla_{\nabla_{E_2} E_2} K, K) \\ &= E_1(E_1(g(K, K))) + E_2(E_2(g(K, K))) - 4g(\nabla_K K, \nabla_L K) \\ &\quad + 2g(\nabla_K K, \nabla_{E_1} E_1) + 2g(\nabla_K K, \nabla_{E_2} E_2) \\ &= E_1(E_1(g(K, K))) + E_2(E_2(g(K, K))) - 8\kappa^2 \\ &\quad + 2g(\nabla_K K, \nabla_{E_1} E_1) + 2g(\nabla_K K, \nabla_{E_2} E_2) \\ &= E_1(E_1(g(K, K))) + E_2(E_2(g(K, K))) - 8\kappa^2 \\ &\quad + 2\kappa g(K, \nabla_{E_1} E_1) + 2\kappa g(K, \nabla_{E_2} E_2). \end{aligned}$$

In the case of $q \in \mathcal{H}_{2,reg}^-$, from Lemma 5.5 we obtain

$$\square g(K, K)(q) = -8\kappa^2 + 8(g(\nabla_{E_2} K, E_1))^2(q). \quad (17)$$

In the case of $q \in \mathcal{H}_{1,reg}^-$, from Lemma 5.6 we obtain

$$\square g(K, K)(q) = -8\kappa^2. \quad (18)$$

Finally, in the case of $q \in (\mathcal{H}_{1,0}^- \cup \mathcal{H}_{2,0}^-) \cap S$, from Lemma 5.7 we obtain

$$\square g(K, K)(q) \geq 0. \quad (19)$$

On the other hand,

$$\begin{aligned}
2g(\nabla K, \nabla K)(q) &= -4g(\nabla_L K, \nabla_K K) + 2g(\nabla_{E_1} K, \nabla_{E_1} K) + 2g(\nabla_{E_2} K, \nabla_{E_2} K) \\
&= -8\kappa^2 + 2(g(\nabla_{E_1} K, E_1))^2 + 2(g(\nabla_{E_1} K, E_2))^2 \\
&\quad - 4g(\nabla_{E_1} K, K)g(\nabla_{E_1} K, L) + 2(g(\nabla_{E_2} K, E_2))^2 \\
&\quad + 2(g(\nabla_{E_2} K, E_1))^2 - 4g(\nabla_{E_2} K, K)g(\nabla_{E_2} K, L) \\
&= -8\kappa^2 + 4(g(\nabla_{E_2} K, E_1))^2.
\end{aligned}$$

We thus have

$$\text{Ric}(K, K) = -2(g(\nabla_{E_2} K, E_1))^2 \quad (20)$$

in $\mathcal{H}_{2,reg}^-$,

$$\text{Ric}(K, K) = 0 \quad (21)$$

in $\mathcal{H}_{1,reg}^-$, and

$$\text{Ric}(K, K) \leq 0$$

in $(\mathcal{H}_{1,0}^- \cup \mathcal{H}_{2,0}^-) \cap \mathcal{S}$. In all cases,

$$\text{Ric}(K, K) \leq 0.$$

for any $q \in \tilde{S} \doteq ((\mathcal{H}_{1,0}^- \cup \mathcal{H}_{2,0}^-) \cap \mathcal{S}) \cup \mathcal{H}_{1,reg}^- \cup \mathcal{H}_{2,reg}^-$. By (9), this suffices to obtain the Proposition. \square

In the Gowdy case, the right hand side of (20) vanishes in view of the twist-free condition for the Killing fields. More on this in Section 6.

Proposition 5.2. *Let (\mathcal{M}, g) be as in the statement of Proposition 5.1, and assume that it satisfies in addition the null convergence condition (4). Then*

$$\text{Ric}(K, K)(q) = 0, \quad (22)$$

on \tilde{S} .

Proof. This follows from Proposition 5.1 by a simple continuity argument. Extend K in a neighborhood of q to be a C^2 null vector. Were $\text{Ric}(K, K)(q) < 0$, this would have to hold for a point $p \in \mathcal{M}$, and this contradicts (4). \square

The above Proposition clearly applies to (\mathcal{M}, g) satisfying Assumptions 1 and 2 of Theorem 4.1, since collisionless matter spacetimes satisfy (4).

6 Aside: Killing horizons

Recall that we call a C^1 hypersurface \mathcal{H} a *Killing horizon* if its normal bundle is spanned by a null vector field which is Killing on \mathcal{H} .

For a C^2 Killing horizon with K a null Killing vector field spanning the normal bundle, Proposition 6.15 of Heusler [15] implies $\text{Ric}(K, K) = 0$.³ Thus,

³The regularity assumptions are not made precise in this proposition, but C^2 is certainly sufficient.

since $\mathcal{H}_{1,reg}^-$ is a C^3 Killing horizon, the result (21) could alternatively have been deduced from this proposition, whose proof in any case is essentially the computation we have done above.

In fact we have

Proposition 6.1. *Let (\mathcal{M}, g) be as in the statement of Proposition 5.1, and assume that it is in fact Gowdy symmetric. Then $\mathcal{H}_{1,reg}^- \cup \mathcal{H}_2^-$ is a C^3 Killing horizon.*

Proof. For $\mathcal{H}_{1,reg}^-$, there is nothing to say, in view of the previous. It suffices thus to show that \mathcal{H}_2^- is a C^3 Killing horizon.

The arguments below are also inspired by a computation in [15]. The result for $\mathcal{H}_{2,reg}^-$ could alternatively be deduced from the results of [4]. Because \mathcal{H}_2^- is *a priori* only Lipschitz, it is not clear that the results of [15, 4] can be applied directly, and we thus give here a self-contained argument.

Without loss of generality, assume $X(p) = K(p)$ is null for some $p \in \mathcal{H}_2^-$, and consider the vector field \hat{K} defined by

$$\hat{K} = X - g(X, Y)(g(Y, Y))^{-1}Y. \quad (23)$$

This is clearly null in a neighborhood of p on \mathcal{H}_2^- , and is orthogonal to Y in a neighborhood in \mathcal{M} of p . Let E_2 be a C^2 section of the C^2 distribution orthogonal to that spanned by X and Y near p . We will show that

$$E_2(g(X, Y)(g(Y, Y))^{-1}) = 0. \quad (24)$$

We compute

$$\begin{aligned} E_2(g(X, Y)(g(Y, Y))^{-1}) &= g(Y, Y)^{-2}(2g(Y, Y)g(\nabla_{E_2}X, Y) \\ &\quad - 2g(\nabla_{E_2}Y, Y)g(X, Y)) \\ &= g(Y, Y)^{-2}(-2g(Y, Y)g(\nabla_YX, E_2) \\ &\quad + 2g(\nabla_YY, E_2)g(X, Y)) \\ &= g(Y, Y)^{-2}(-2g(Y, Y)g(\nabla_XY, E_2) \\ &\quad + 2g(\nabla_YY, E_2)g(X, Y)) \\ &= -2g(Y, Y)^{-1}(g(\nabla_XY, E_2) \\ &\quad - g(X, Y)g(Y, Y)^{-1}g(\nabla_YY, E_2)) \\ &= -2g(Y, Y)^{-1}g(\nabla_{X-g(X, Y)g(Y, Y)^{-1}Y}Y, E_2) \\ &= 0. \end{aligned}$$

In the above, we have used $[X, Y] = 0$, the Killing property and the twist free property of the Killing fields.

Clearly, an identity analogous to (24) holds when E_2 is replaced by X or Y .

Thus, let W be any vector field in the span of E_2 , X and Y . Let $p_i \rightarrow p$ be a sequence where $p_i \in \mathcal{M}$, the original spacetime. It is clear that the integral curves of W through p_i cannot meet \mathcal{H}_2^- in a small neighborhood of p for small

time. For suppose γ_i was such a curve. There would be a first point of γ_i meeting \mathcal{H}_2^- , i.e., there would be a segment $\gamma_i([t_0, t_1])$ with $\gamma_i([t_0, t_1]) \subset \mathcal{M}$, $\gamma_i(t_0) = p_i$, and $\gamma_i(t_1) \in \mathcal{H}_2^-$. The above computation shows that $W(g(X, Y)/g(Y, Y)) = 0$, identically on $\gamma([t_0, t_1])$. Moreover, this remains true if X , and Y are replaced by any other Killing vector fields $\tilde{Y} \neq \tilde{X}$, $g(\tilde{Y}, \tilde{Y}) > 0$. Choose \tilde{X} such that \tilde{X} is null at the point $\gamma(t_1) \in \mathcal{H}_2^-$. We have $g(\tilde{X}, \tilde{Y})(\gamma(t_1)) = 0$, and thus $g(\tilde{X}, \tilde{Y}) = 0$ identically along $\gamma([t_0, t_1])$. But at p_i , \tilde{X} and \tilde{Y} span a spacelike two-plane, thus there is a unique direction on this plane orthogonal to \tilde{X} at p_i . Applying the above with a Killing field $\tilde{Y} \neq \tilde{X}$ such that \tilde{X} and \tilde{Y} are *not* orthogonal at p_i , we obtain a contradiction.

Since integral curves of W through p are limits of integral curves of W through p_i , it follows that these curves must remain on the boundary of \mathcal{M} , i.e. on \mathcal{H}_2^- . One thus easily sees that \mathcal{H}_2^- is locally an integral manifold of the C^2 distribution spanned by X , Y , and E_2 , and thus C^3 .

The above now shows that $g(X, Y) = 0$ along \mathcal{H}_2^- for any choice of Y , and thus X is null in the connected component of p . We have thus obtained that \mathcal{H}_2^- is a C^3 Killing horizon, as desired. \square

It would be nice to obtain that \mathcal{H}_1^- is C^3 . This would show that there is a dense open subset of \mathcal{H}^- which is a C^3 Killing horizon. See also [10].

Finally, we also note the following

Proposition 6.2. *Let (\mathcal{M}, g) be as in the statement of Proposition 5.1, and suppose that \mathcal{M} is vacuum, i.e. $\text{Ric} = 0$ identically. Then $\mathcal{H}_{1,reg}^- \cup \mathcal{H}_{2,reg}^-$ is a C^3 Killing horizon.*

Proof. For $\mathcal{H}_{1,reg}^-$ there is nothing to show. For $\mathcal{H}_{2,reg}^-$ in the Gowdy case, there is nothing to show in view of the previous proposition.

Let us assume thus that the spacetime is not Gowdy, and let $p \in \mathcal{H}_{2,reg}^-$. Let K be a Killing vector field such that $K(p)$ is null. Formulas (22) and (20) together imply that $g(\nabla_{E_2} K, E_1) = 0$. This implies that the twist quantity of K vanishes. In the vacuum case, this implies that this quantity vanishes on all of \mathcal{M} . Since there is a *unique* Killing vector field whose twist constant vanishes (in view of the assumption that the spacetime \mathcal{M} is *not* Gowdy) it follows that the null Killing vector field at any other $q \in \mathcal{H}_{2,reg}^-$ must be $K(q)$. Thus, $\mathcal{H}_{2,reg}^-$ is a Killing horizon. \square

7 The contradiction

For the following proposition, the reader should again compare with [14]. The considerations in the present paper are again easier, in view of the fact that the Killing fields are globally defined and do not vanish in the original spacetime.

Proposition 7.1. *Let (\mathcal{M}, g) be as in the statement of Theorem 4.1. Then for a dense subset of the dense subset \tilde{S} of Proposition 5.1,*

$$\text{Ric}(K, K) > 0. \tag{25}$$

Proof. We will show that given any $p \in S$, where S is as in the proof of Proposition 5.1, and any open neighborhood $\mathcal{V} \subset \mathcal{H}^-$ of p in \mathcal{H}^- , there exists a $q \in \mathcal{V} \cap \tilde{S}$ satisfying $\text{Ric}(K, K) > 0$.

Let $K(p), L(p), E_1(p), E_2(p)$ be null frame at p such that X and Y lie in the plane spanned by $K(p)$ and $E_1(p)$. Let $X(p) = aK(p) + bE_1(p)$, $Y(p) = cK(p) + dE_1(p)$. We have that $\max(|a|, |c|) > 0$. Consider the geodesic through p with

$$\dot{\gamma}(p) = \frac{1}{4}\delta(\max(|a|, |c|))^{-1}L + 2\delta^{-1}(\max(|a|, |c|)K).$$

We clearly have $g(\dot{\gamma}, \dot{\gamma}) = -1$, while

$$|g(\dot{\gamma}, X)(p)| = \left| \frac{1}{4}\delta g(L, K)a(\max(|a|, |c|))^{-1} \right| \leq \delta/2, \quad (26)$$

$$|g(\dot{\gamma}, Y)(p)| = \left| \frac{1}{4}\delta g(L, K)c(\max(|a|, |c|))^{-1} \right| \leq \delta/2. \quad (27)$$

Since γ is a geodesic and X , and Y are Killing we have $\dot{\gamma}g(\dot{\gamma}, X) = 0$, $\dot{\gamma}g(\dot{\gamma}, Y) = 0$ and thus the inequalities (26), (27) hold throughout γ in \mathcal{M} .

Now, γ must intersect the Cauchy hypersurface at some time T . By continuity of geodesic flow, for every neighborhood \mathcal{V} in \mathcal{H}^- of p there exists a neighborhood \mathcal{U}_0 of $\gamma'(T) \in P$ in the topology of $P \cap \pi^{-1}(\Sigma)$ such that geodesics with initial condition on \mathcal{U} intersect \mathcal{H}^- transversally in \mathcal{V} . So select \mathcal{U}_0 so that this is the case for our chosen \mathcal{V} , and then, in view of Assumption 2 and the continuity of f , select \mathcal{U}_1 such that $f > \epsilon > 0$ on an open $\mathcal{U}_1 \subset \mathcal{U}_0$. By the properties of the geodesic flow, the projection on \mathcal{H}^- of the set of all geodesics with initial condition in \mathcal{U}_1 contains a nonempty open set \mathcal{V}_1 .

Let $q \in \tilde{S} \cap \mathcal{V}_1 \neq \emptyset$ and let V be a null generator for \mathcal{H}^- at q . By Proposition 5.1, $\text{Ric}(V, V) = 0$. On the other hand, since f can easily be seen to extend continuously to $P \cap \pi^{-1}(\mathcal{H}^-)$, we have that $f > 0$ at some point of $P \cap \pi^{-1}(q)$, and thus in an open set. In particular, the integral defined by (3) is strictly positive at q when contracted twice with the null vector V .

Extend V arbitrarily as a C^2 null vector field in a neighborhood of q . In the original spacetime \mathcal{M} , the right hand side of (3) is equal to $\text{Ric}(V, V)$ in view of (1). Taking a sequence of points $q_i \rightarrow q$, with $q_i \in \mathcal{M}$, then $\text{Ric}(V, V)(q_i) \rightarrow \text{Ric}(V, V)(q)$ by the fact that $\tilde{\mathcal{M}}$ is assumed C^2 . By Fatou's lemma, the right hand side of (3) contracted twice with V at q is less than or equal to the limit of its value at q_i when contracted twice with V . The former we have just shown to be strictly positive, while the latter equals $\text{Ric}(V, V)(q)$. Thus $\text{Ric}(V, V)(q) > 0$, as desired. \square

The proof of Theorem 4.1 follows immediately from Propositions 5.2 and 7.1.

8 Comments

The techniques of this paper are tied heavily to continuity of the curvature tensor. The paper thus does not address weaker conditions of inextendibility,

for instance, inextendibility as a C^0 metric, which may be more correct from a physical point of view. See [7, 11, 12].

It is clear that Assumption 2 of Theorem 4.1 can be weakened to the assumption that the equations (1)–(3) are satisfied with (3) replaced by

$$T_{\alpha\beta}(x) = \tilde{T}_{\alpha\beta}(x) + \int_{\pi^{-1}(x)} p_{\alpha} p_{\beta} f$$

where $\tilde{T}_{\alpha\beta}(x)$ satisfies $\tilde{T}_{\alpha\beta} V^{\alpha} V^{\beta} \geq 0$ for all null V^{α} . Thus strong cosmic censorship can be shown in T^2 symmetry when the Einstein-Vlasov system is extended to include other matter fields, *provided that Assumption 1 can also be shown for the maximal development of suitable data.*

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