COMPLEX OSSERMAN ALGEBRAIC CURVATURE TENSORS AND CLIFFORD FAMILIES

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ABSTRACT. We use methods of algebraic topology to study the eigenvalue structure of a complex Osserman algebraic curvature tensor. We classify the algebraic curvature tensors which are both Osserman and complex Osserman in all but a finite number of exceptional dimensions.

1. INTRODUCTION

In recent years, the Osserman problem has played an important role in the understanding of curvature. The real setting has been studied previously; in this paper, we study the complex setting. We introduce the following notational conventions. Let ∇ be the Levi-Civita connection of a Riemannian manifold (M,g) and let R be the associated Riemann curvature tensor:

$$R(x,y,z,w):=g((\nabla_x\nabla_y-\nabla_y\nabla_x-\nabla_{[x,y]})z,w)\,.$$

The Jacobi operator $\mathcal{J}_R$ and the skew-symmetric curvature operator $\mathcal{R}$ are characterized by the identities:

$$g(\mathcal{J}_R(x)y,z)=R(y,x,x,z)\quad\text{and}\quad g(\mathcal{R}(x,y)z,w)=R(x,y,z,w)\,. \tag{1.a}$$

Motivated by the seminal paper of Osserman [13], one says that (M,g) is *Osserman* if the eigenvalues of $\mathcal{J}_R$ are constant on the sphere bundle $S(M,g)$ of unit tangent vectors. Since the local isometries of a local two-point homogeneous manifold act transitively on $S(M,g)$, such manifolds are Osserman. Osserman wondered if the converse was also true, that is, are Osserman manifolds necessarily local two-point-homogeneous spaces. This question has been called the Osserman conjecture by subsequent authors and has been also considered in the pseudo-Riemannian context; in this paper, we will only work in the Riemannian context and refer to [5, 9] for a discussion of the pseudo-Riemannian setting.

1.1. **Algebraic curvature tensors.** It turned out to be convenient to work in a purely algebraic context in studying the Osserman conjecture. Let $\mathfrak{M}:=(V,\langle\cdot,\cdot\rangle,R)$ be a *model*. This means that V is a vector space of dimension n which is equipped with a positive definite inner product $\langle\cdot,\cdot\rangle$ and that $R\in\otimes^4V^*$ is an *algebraic curvature tensor*, i.e. R satisfies the Riemannian curvature tensor identities:

$$\begin{aligned}&R(x,y,z,t)=-R(y,x,z,t)=R(z,t,x,y),\\&R(x,y,z,t)+R(y,z,x,t)+R(z,x,y,t)=0.\end{aligned}$$

One uses Equation (1.a) to define the Jacobi operator and skew-symmetric curvature operator in this setting as well; $\mathfrak{M}$ is said to be *Osserman* if the eigenvalues of $\mathcal{J}_R(\cdot)$ are constant on the sphere $S(V,\langle\cdot,\cdot\rangle)$ of unit vectors in V. Clearly, if (M,g) is a Riemannian manifold, and if $P\in M$, then $\mathfrak{M}_P:=(T_PM,g_P,R_P)$ defines a model. Conversely, every model is geometrically realizable.

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1.2. **The Osserman conjecture.** This conjecture for Riemannian manifolds was established by Chi [3] in dimensions $n \equiv 1 \pmod 2$, $n \equiv 2 \pmod 4$ and $n = 4$. Subsequent work by Nikolayevsky [11, 12] has established the Osserman conjecture in dimensions $n \neq 16$; the case $n = 16$ is still open. Nikolayevsky used a two step approach following the discussion in [6]. He first showed that any Osserman model is given by a Clifford family as specified in Equation (1.e) of Section 1.7 below except in dimension 16. He then used the Second Bianchi Identity to prove the necessary integrability results to show any Osserman manifold of dimension $n \neq 16$ was locally isometric to a rank 1 symmetric space or was flat. Note that the algebraic classification fails if $n = 16$; indeed the curvature tensor of the Cayley plane is Osserman but it is not given by a Clifford family, i.e. it is not expressible in the form given in Equation (1.e).

1.3. **The higher order Jacobi operator.** There are other related questions. One may follow the discussion of Stanilov and Videv [14] to define a *higher order Jacobi operator* as follows. Let $\{e_1, ..., e_p\}$ be an orthonormal basis for a p-plane $\mathcal{P}$. Set

$$\mathcal{J}_R(\mathcal{P}) = \sum_{i=1}^{p} \mathcal{J}_R(e_i)\,;$$

this is independent of the particular orthonormal basis chosen. If $p = 1$, one recovers the ordinary Jacobi operator. Furthermore, if $p = n$, then $\rho := \mathcal{J}_R(V)$ is the *Ricci operator*; thus the higher order Jacobi operator can also be thought of as a generalization of the Ricci operator to lower dimensional subspaces. One says that a model $\mathfrak{M}$ is *p-Osserman* if the eigenvalues of $\mathcal{J}_R(\mathcal{P})$ are constant on the Grassmannian $Gr_p(V)$ of p-planes. The geometry is very rigid in this setting. If $p = 1$ or if $p = n - 1$, then $\mathfrak{M}$ is p-Osserman if and only if $\mathfrak{M}$ is Osserman. Thus these values of p may be excluded from consideration. If $2 \leq p \leq n - 2$, then it is known [7] that $\mathfrak{M}$ is p-Osserman if and only if $\mathfrak{M}$ has constant sectional curvature c, i.e. that $R = cR_0$ where R_0 is given by:

$$R_0(x, y, z, t) := \langle x, t\rangle\langle y, z\rangle - \langle x, z\rangle\langle y, t\rangle\,. \tag{1.b}$$

1.4. **Complex geometry.** In this paper, we will consider a complex analogue of these questions. Let J denote an Hermitian almost complex structure on $(V, \langle\cdot,\cdot\rangle)$; this means that J is an isometry of $(V, \langle\cdot,\cdot\rangle)$ with $J^2 = -\operatorname{id}$. A 2-plane is said to be holomorphic if it is J-invariant and a real linear transformation T of V is said to be complex linear if $TJ = JT$. We let $\mathbb{CP}(V, \langle\cdot,\cdot\rangle, J)$ be the set of all holomorphic 2 planes. If $x \in S(V, \langle\cdot,\cdot\rangle)$ is a unit vector, let $\pi_x := \operatorname{Span}\{x, Jx\}$. The natural map $x \to \pi_x$ defines the Hopf fibration from $S(V, \langle\cdot,\cdot\rangle)$ to $\mathbb{CP}(V, \langle\cdot,\cdot\rangle, J)$. Let

$$\mathcal{J}_R(\pi_x) := \mathcal{J}_R(x) + \mathcal{J}_R(Jx)$$

be the complex Jacobi operator; this is the restriction of the higher order Jacobi operator to the set of complex 2-planes. The following result is well known, for example see [9]. Conditions (2), (3) of the Lemma simply mean that the operator under consideration is complex linear.

Lemma 1.1. *We say that R and J are* compatible *if any of the following equivalent conditions are satisfied:*

1. $R(x, y, z, t) = R(Jx, Jy, Jz, Jt)$ *for all* $x, y, z, t \in V$.
2. $\mathcal{J}_R(\pi_x)J = J\mathcal{J}_R(\pi_x)$ *for all* $x \in S(V, \langle\cdot,\cdot\rangle)$.
3. $\mathcal{R}(x, Jx)J = J\mathcal{R}(x, Jx)$ *for all* $x \in S(V, \langle\cdot,\cdot\rangle)$.

Note that the curvature R and the almost complex structure J of a Kähler manifold are compatible. Thus this is a very natural geometric condition.

1.5. **The complex Osserman condition.** Instead of the 2-Osserman condition where the eigenvalues are constant on the Grassmannian of 2-planes, we consider a natural weaker condition with constant eigenvalues on the space of holomorphic planes, $\mathbb{CP}(V,\langle\cdot,\cdot\rangle,J)$.

Definition 1.2. *Let $\mathcal{V}:=(V,\langle\cdot,\cdot\rangle,J,R)$. We say that $\mathcal{V}$ is a* complex model *if $\langle\cdot,\cdot\rangle$ is a positive definite inner product on V, if J is an Hermitian almost complex structure on $(V,\langle\cdot,\cdot\rangle)$, and if R is an algebraic curvature tensor on $(V,\langle\cdot,\cdot\rangle)$. We say that $\mathcal{V}$ is* complex Osserman *if*

(1) $\mathcal{V}$ is a complex model.
(2) J and R are compatible, i.e. $\mathcal{J}_R(\pi_x)$ is complex linear for all $x\in S(V,\langle\cdot,\cdot\rangle)$.
(3) The eigenvalues of $\mathcal{J}_R(\pi_x)$ are constant on $\mathbb{CP}(V,\langle\cdot,\cdot\rangle,J)$.

We shall also sometimes simply say that R is complex Osserman *in this situation.*

1.6. **The canonical curvature tensor.** In addition to the tensor of constant sectional curvature +1 defined in Equation (1.b), it is useful to consider the tensor

$$R_\Psi(x,y,z,t):=\langle x,\Psi t\rangle\langle y,\Psi z\rangle-\langle x,\Psi z\rangle\langle y,\Psi t\rangle-2\langle x,\Psi y\rangle\langle z,\Psi t\rangle \tag{1.c}$$

where Ψ is a skew-symmetric endomorphism of $(V,\langle\cdot,\cdot\rangle)$. Such tensors play an important role in studying the space of all algebraic curvature tensors. For example, Fiedler [4] has shown that tensors of this form span the space of all algebraic curvature tensors. In this paper, we shall study tensors of this form where the endomorphism in question defines an Hermitian almost complex structure on $(V,\langle\cdot,\cdot\rangle)$. We note for future reference that

$$\mathcal{J}_{R_0}(x)y=y-\langle y,x\rangle\quad\text{and}\quad\mathcal{J}_{R_\Psi}(x)y=3\langle y,\Psi x\rangle\Psi x\,. \tag{1.d}$$

1.7. **Algebraic curvature tensors given by Clifford families.** We say that a set $\mathcal{F}=\{J_1,\dots,J_\kappa\}$ of Hermitian almost complex structures on $(V,\langle\cdot,\cdot\rangle)$ is a *Clifford family of rank* κ if they are subject to the commutation rules

$$J_iJ_j+J_jJ_i=-\delta_{ij}\,\mathrm{id}\,.$$

We say that a model $(V,\langle\cdot,\cdot\rangle,R)$ is *given by a Clifford family $\mathcal{F}$ of rank κ* if there exist constants c_i with $c_i\neq0$ for $1\le i\le\kappa$ so that

$$R=c_0R_0+c_1R_{J_1}+...+c_\kappa R_{J_\kappa}\,. \tag{1.e}$$

We shall also sometimes say that R is given by a Clifford family in this setting. The relations of Equation (1.d) yield that:

$$\mathcal{J}_R(x)y=c_0\{y-\langle y,x\rangle x\}+3c_1\langle y,J_1x\rangle J_1x+...+3c_\kappa\langle y,J_\kappa x\rangle J_\kappa x\,. \tag{1.f}$$

From this it follows immediately that

$$\begin{aligned}\mathcal{J}_R(\pi_x)y&=c_0\{2y-\langle y,x\rangle x-\langle y,Jx\rangle Jx\}\\&\quad+\sum_{i=1}^{\kappa}3c_i\{\langle y,J_ix\rangle J_ix+\langle y,J_iJx\rangle J_iJx\}\,.\end{aligned} \tag{1.g}$$

1.8. **Reparametrizing Clifford families.** Let $A=(A_{ij})\in O(\kappa)$ be an orthogonal matrix. Set

$$\tilde{\mathcal{F}}:=\{\tilde J_i=A_{i1}J_1+\dots+A_{i\kappa}J_\kappa\}\,.$$

This new Clifford family is said to be a *reparametrization* of $\mathcal{F}$; this defines an equivalence relation on the collection of Clifford families.

1.9. **Summary of results.** In this paper we begin the study of complex Osserman manifolds by concentrating on the analysis of complex Osserman models. In Section 2, we give necessary and sufficient conditions so that a model $\mathcal{V}=(V,\langle\cdot,\cdot\rangle,J,R)$ is complex Osserman and we show that R is necessarily Einstein if $\mathcal{V}$ is complex Osserman. We also give a topological result in Theorem 2.4 which controls the eigenvalue structure of $\mathcal{J}_R(\pi_x)$ if $\mathcal{V}$ is complex Osserman. In Section 3, we recall results of Adams on the existence of Clifford families and discuss some reparametrization results. We also present some examples of complex Osserman models and show Theorem 2.4 is sharp.

Work of Nikolayevsky shows that any Osserman model $(V,\langle\cdot,\cdot\rangle,R)$ is given by a Clifford family except in dimension 16. We divide our study into two cases depending on the rank κ of the structure in question.

We study the case $\kappa>3$ in Section 4 and show:

Theorem 1.3. *Let $\mathcal{V}:=(V,\langle\cdot,\cdot\rangle,J,R)$ where $R=c_0R_0+c_1R_{J_1}+...+c_\kappa R_{J_\kappa}$ is given by a Clifford family of rank $\kappa\geq4$ on a vector space V of dimension n. The following assertions hold:*

(1) Let $c_0=0$. If $\kappa=4,5$, assume $n\geq 2^\kappa$ and, if $\kappa\geq6$, assume $n\geq\kappa(\kappa-1)$. Then $\mathcal{V}$ is not complex Osserman.
(2) Let $c_0\neq0$. If $\kappa=4$ assume $n\geq32$, if $\kappa=5,6,7$ assume $n\geq2^\kappa$, if $\kappa\geq8$ assume $n\geq\kappa(\kappa-1)$. Then $\mathcal{V}$ is not complex Osserman.

Note that, as a consequence of Lemma 3.1 below, the hypothesis $n\geq\kappa(\kappa-1)$ in Theorem 1.3 is not a restriction when $\kappa\geq16$. Consequently, there are only a finite number of possibly exceptional dimensions and ranks when $\kappa\geq4$.

Section 5 is devoted to the study of Clifford families of lower rank. Results in this section are summarized in the following theorem:

Theorem 1.4. *Let $\mathcal{V}:=(V,\langle\cdot,\cdot\rangle,J,R)$. Let $\mathcal{F}=\{J_i\}$ be a Clifford family on a vector space V of dimension n. Let $c_i\neq0$ be given for $1\leq i\leq\kappa$ where $\kappa\leq3$.*

(1) Rank $\kappa=0$. Let $R=c_0R_0$. Then $\mathcal{V}$ is complex Osserman.
(2) Rank $\kappa=1$. Let $R=c_0R_0+c_1R_{J_1}$.
  (a) *If $c_0=0$, then $\mathcal{V}$ is complex Osserman if and only if $JJ_1=\pm J_1J$.*
  (b) *If $c_0\neq0$, then $\mathcal{V}$ is complex Osserman if and only if $J=\pm J_1$ or $JJ_1=-J_1J$.*
(3) Rank $\kappa=2$. Let $R=c_0R_0+c_1R_{J_1}+c_2R_{J_2}$. Then $\mathcal{V}$ is complex Osserman if and only if there exists a reparametrization $\{\tilde J_1,\tilde J_2\}$ of $\mathcal{F}$ so that one has $R=c_0R_0+\tilde c_1R_{\tilde J_1}+\tilde c_2R_{\tilde J_2}$ and so that one of the following holds:
  (a) *$c_0=0$, $J\tilde J_1=\tilde J_1J$ and $J\tilde J_2=-\tilde J_2J$.*
  (b) *Either $J=\tilde J_1$ or $J=\tilde J_1\tilde J_2$.*
(4) Rank $\kappa=3$. Let $R=c_0R_0+c_1R_{J_1}+c_2R_{J_2}+c_3R_{J_3}$.
  (a) *Assume $n\geq12$. If $c_0=0$, then $\mathcal{V}$ is complex Osserman if and only if there exists a reparametrization $\{\tilde J_1,\tilde J_2,\tilde J_3\}$ of $\mathcal{F}$ so that one has $R=\tilde c_1R_{\tilde J_1}+\tilde c_2R_{\tilde J_2}+\tilde c_3R_{\tilde J_3}$ and that $J=\tilde J_1$ or $J=\tilde J_2\tilde J_3$.*
  (b) *Assume $n\geq16$. If $c_0\neq0$, then $\mathcal{V}$ is complex Osserman if and only if there exists a reparametrization $\{\tilde J_1,\tilde J_2,\tilde J_3\}$ of $\mathcal{F}$ so that one has $R=c_0R_0+\tilde c_1R_{\tilde J_1}+\tilde c_2R_{\tilde J_2}+\tilde c_3R_{\tilde J_3}$, $J=\tilde J_1$, and $\tilde J_1\tilde J_2\tilde J_3=\mathrm{id}$.*

Remark 1.5. From Theorem 1.4 we obtain the following geometric conclusions:

(1) Let (M,g) be a manifold of constant sectional curvature. Then (M,g) is complex Osserman with respect to any Hermitian almost complex structure J.
(2) Let (M,g,J) be a Kähler manifold which has constant holomorphic sectional curvature. Then (M,g,J) is complex Osserman with respect to J.

(3) Let $(M, g, \{J_1, J_2, J_3\})$ be a quaternionic Kähler manifold which has constant quaternionic sectional curvature, where $\{J_1, J_2, J_3\}$ forms a locally defined quaternionic structure. Then, for any $J \in \mathrm{Span}\{J_1, J_2, J_3\}$, (M, g, J) is complex Osserman.

Note that if (M, g) is Osserman of dimension different from 16, then it is isometric to one of these three examples or is flat [3, 11, 12].

2. Algebraic preliminaries

In this section we present some foundational results. Our first result is the well known observation:

Lemma 2.1. *Let $\mathcal{V}_i := (V, \langle\cdot,\cdot\rangle, R_i)$, with $i = 1, 2$, be models. If $\mathcal{J}_{R_1}(x) = \mathcal{J}_{R_2}(x)$ for all x in V, then $R_1 = R_2$.*

What is perhaps somewhat surprising is that this observation fails for the complex Jacobi operator as we shall see in Theorem 3.6. Let $\mathrm{Spec}\{\mathcal{J}_R(\pi_x)\}$ be the spectrum of $\mathcal{J}_R(\pi_x)$ and let $E_\lambda(\pi_x)$ be the eigenspace associated to the eigenvalue λ of $\mathcal{J}_R(\pi_x)$. Since $\mathcal{J}_R(\pi_x)$ is self-adjoint, $\mathcal{J}_R(\pi_x)$ is diagonalizable with real eigenvalues. Thus we have an orthogonal direct sum decomposition

$$V = \oplus_\lambda E_\lambda(\pi_x)$$

for any $x \in S(V, \langle\cdot,\cdot\rangle)$. The following lemma is an immediate consequence of Lemma 1.1 and provides a criterion for complex Osserman curvature tensors:

Lemma 2.2. *$\mathcal{V} = (V, \langle\cdot,\cdot\rangle, J, R)$ is complex Osserman if and only if*

(1) *$JE_\lambda(\pi_x) = E_\lambda(\pi_x)$ for all $\pi_x \in \mathbb{CP}(V, \langle\cdot,\cdot\rangle, J)$ and $\lambda \in \mathrm{Spec}\{\mathcal{J}_R(\pi_x)\}$.*
(2) *$\mathrm{Spec}\{\mathcal{J}_R(\pi_x)\} = \mathrm{Spec}\{\mathcal{J}_R(\pi_y)\}$ for all $\pi_x, \pi_y \in \mathbb{CP}(V, \langle\cdot,\cdot\rangle, J)$.*

An model $(V, \langle\cdot,\cdot\rangle, R)$ is said to be Einstein if $\rho(\cdot,\cdot) = c\,\langle\cdot,\cdot\rangle$ for a constant c, where by ρ we denote the Ricci tensor. In general, p-Osserman models are Einstein. This result generalizes to become:

Lemma 2.3. *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, J, R)$ be complex Osserman. Then $\mathcal{V}$ is Einstein.*

Proof. Assume that $\mathcal{V}$ is complex Osserman. Let $x \in S(V, \langle\cdot,\cdot\rangle)$. As R is compatible, $R(y, Jx, Jx, z) = R(Jy, x, x, Jz)$ and thus $\mathcal{J}_R(Jx) = -J\mathcal{J}_R(x)J$. Consequently

$$\rho(x, x) = \mathrm{Tr}\{\mathcal{J}_R(x)\} = \mathrm{Tr}\{\mathcal{J}_R(Jx)\} = \tfrac{1}{2}\,\mathrm{Tr}\{\mathcal{J}_R(\pi_x)\} = \tfrac{1}{2}\textstyle\sum_\lambda \lambda \dim\{E_\lambda(\pi_x)\}$$

is independent of $x \in S(V, \langle\cdot,\cdot\rangle)$. This implies $\rho(\cdot,\cdot) = c\,\langle\cdot,\cdot\rangle$. Consequently $\mathcal{V}$ is Einstein. □

Methods of algebraic topology can be used to control the eigenvalue structure of a complex Osserman model. In particular, no more than 3 distinct eigenvalues may occur.

Theorem 2.4. *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, J, R)$ be complex Osserman. If $\mathcal{J}_R(\pi_x)$ is not a multiple of the identity (i.e. if $\mathcal{J}_R(\pi_x)$ has at least 2 distinct eigenvalues), then:*

(1) *If $n \equiv 2 \ (mod\ 4)$, there are 2 eigenvalues with multiplicities $(n-2, 2)$.*
(2) *If $n \equiv 0 \ (mod\ 4)$, then one of the following holds:*
   (a) *There are 2 eigenvalues with multiplicities $(n-2, 2)$.*
   (b) *There are 2 eigenvalues with multiplicities $(n-4, 4)$.*
   (c) *There are 3 eigenvalues with multiplicities $(n-4, 2, 2)$.*

Proof. Let $\mathbb{V} := \mathbb{CP}(V, \langle\cdot,\cdot\rangle, J) \times V$ be the trivial bundle over projective space. Lemma 2.2 shows that the eigenspaces

$$E_{\lambda_i}(\pi) := \{v \in V : \mathcal{J}_R(\pi)v = \lambda_i v\}$$

have constant rank and patch together to define smooth vector bundles $E_{\lambda_i}(\pi)$ over $\mathbb{CP}(V, \langle\cdot,\cdot\rangle, J)$ where $\{\lambda_0, ..., \lambda_k\}$ denote the distinct eigenvalues of $\mathcal{J}_R(\pi)$ for

any, and hence for all, $\pi \in \mathbb{CP}(V, \langle\cdot,\cdot\rangle, J)$. This gives the following direct sum decomposition
$$\mathbb{V} = E_{\lambda_0} \oplus \cdots \oplus E_{\lambda_k}.$$
This decomposition is in the category of complex vector bundles since the eigenbundles are invariant under J.

A sub-bundle E of $\mathbb{V}$ is said to be a *geometrically symmetric vector bundle* if for all complex lines σ, τ in $\mathbb{CP}(V, \langle\cdot,\cdot\rangle, J)$, $\tau \subset E(\sigma)$ implies $\sigma \subset E(\tau)$. Let $\lambda_{\min}$ be the minimal eigenvalue of $\mathcal{J}_R(\pi_x)$. We then have the following chain of equivalences for unit vectors x and y:
$$\begin{aligned}
& y \in E_{\lambda_{\min}}(\pi_x) \\
\Leftrightarrow \quad & R(y,x,x,y) + R(y,Jx,Jx,y) = \lambda \\
\Leftrightarrow \quad & R(x,y,y,x) + R(Jx,y,y,Jx) = \lambda \\
\Leftrightarrow \quad & R(x,y,y,x) + R(x,Jy,Jy,x) = \lambda \\
\Leftrightarrow \quad & x \in E_{\lambda_{\min}}(\pi_y).
\end{aligned}$$
This implies that the bundle $E_{\lambda_{\min}}$ is geometrically symmetric. The desired result now follows from results in [8] concerning geometrically symmetric bundles; these results generalize earlier results of Glover et al. [10]. □

We shall show that this result is sharp in Remark 3.5 below by showing that all the possibilities can be realized.

3. Clifford families and associated curvature tensors

A Clifford family $\mathcal{F} = \{J_1, J_2, J_3\}$ of rank 3 is called a *quaternion structure* if $J_1 J_2 = J_3$. Note that V admits a quaternion structure if and only if $\dim\{V\}$ is divisible by 4. One defines the *Adams number* $\nu(n)$ by setting $\nu(1) = 0$, $\nu(2) = 1$, $\nu(4) = 3$, $\nu(8) = 7$, $\nu(16r) = \nu(r) + 8$ and $\nu(m2^s) = \nu(2^s)$ for m odd. One then has the following well known result of Atiyah et al. [2] which is closely related to work of Adams [1] concerning vector fields on spheres:

Lemma 3.1. *There exists a Clifford family of rank κ on V if and only if $\kappa \leq \nu(n)$.*

We now present a useful technical result:

Lemma 3.2. *Let V and W be vector spaces and let $\mathcal{T} = \{T_1, \ldots, T_\kappa\}$ be a family of linear maps $T_i : V \to W$. Assume there is an integer μ so that for any set of constants a_i, not all of which are zero, one has* $\operatorname{Rank}\{a_1T_1 + \cdots + a_\kappa T_\kappa\} \geq \mu$. *Then the following assertions hold:*

(1) If $\mu \geq \kappa$, there exists $x \in V$ so that $\{T_1x, \ldots, T_\kappa x\}$ is a set of linearly independent vectors.
(2) If $\mu \geq 2\kappa$, there exists $x, y \in V$ so that $\{T_1x, \ldots, T_\kappa x, T_1y, \ldots, T_\kappa y\}$ is a set of linearly independent vectors.
(3) Let $T : V \longrightarrow W$ be a linear map so that $Tx \in \operatorname{Span}\{T_1x, \ldots, T_\kappa x\}$ for all $x \in V$. If $\mu \geq 2\kappa$, then $T \in \operatorname{Span}\{T_1, \ldots, T_\kappa\}$.

Proof. In order to prove Assertion (1), suppose $\mu \geq \kappa$. For a given $x \in V$, choose $r(x)$ maximal so that $\{T_1x, \ldots, T_rx\}$ is a linearly independent set of r vectors. Take $x \in V$ so that $r(x)$ is maximal. If $r(x) = \kappa$, then clearly Assertion (1) holds. Suppose $r(x) < \kappa$. We argue for a contradiction. Choose $(a_1, \ldots, a_r)$ so that $a_1T_1x + \cdots + a_rT_rx + T_{r+1}x = 0$ and let
$$S := a_1T_1 + \cdots + a_rT_r + T_{r+1}.$$
As $\operatorname{Rank}\{S\} \geq \mu \geq \kappa$, there is $y \in V$ so that $\{T_1x, \ldots, T_rx, Sy\}$ is a set of $r+1$ linearly independent vectors. Hence, by continuity, there exists $\epsilon > 0$ such that

$\{T_1(x+\epsilon y),\ldots,T_r(x+\epsilon y),Sy\}$ is a set of $r+1$ linearly independent vectors. Consequently $\{T_1(x+\epsilon y),\ldots,T_r(x+\epsilon y),T_{r+1}(x+\epsilon y)\}$ also is a set of $r+1$ linearly independent vectors. Therefore $r(x+\epsilon y)\geq r+1$ which contradicts the choice of x. This contradiction establishes Assertion (1).

Now suppose that $\mu\geq 2\kappa$. By Assertion (1) we may choose $x\in V$ so that $\{T_1x,\ldots,T_\kappa x\}$ is a linearly independent set of κ vectors. Consider the vector space $W_0:=\operatorname{Span}\{T_1x,\ldots,T_\kappa x\}$ and let $\pi:W\longrightarrow W/W_0$ be the natural projection. We apply Assertion (1) to the linear maps $\bar T_i:=\pi T_i:V\longrightarrow W/W_0$ with $\bar\mu=\mu-\kappa\geq\kappa$ to complete the proof of Assertion (2).

We complete the proof by establishing Assertion (3). By assumption, for every $z\in V$, there exist coefficients $a_i(z)$ so that $Tz=a_1(z)T_1z+\ldots+a_\kappa(z)T_\kappa z$. To show that $T\in\operatorname{Span}\{T_1,\ldots,T_\kappa\}$, we must show that the coefficients can be chosen to be independent of z.

By Assertion (2), there are vectors $x,y\in S(V,\langle\cdot,\cdot\rangle)$ so $\{T_1x,...,T_\kappa x,T_1y,...,T_\kappa y\}$ is a collection of 2κ linearly independent vectors. Then, by continuity, this remains true on some open neighborhoods $\mathcal{O}_x$ and $\mathcal{O}_y$ of x and y, respectively. Let $z\in\mathcal{O}_x$ and let $t\in\mathcal{O}_y$. We may then express:

$$\begin{aligned}T(z+t) &= \sum_{i=1}^{\kappa}a_i(z+t)T_i(z+t)=\sum_{i=1}^{\kappa}a_i(z+t)(T_iz+T_it)\\ &= Tz+Tt=\sum_{i=1}^{\kappa}\{a_i(z)T_iz+a_i(t)T_it\}\,.\end{aligned}$$

Since the vectors $\{T_1z,...,T_\kappa z,T_1t,...,T_\kappa z\}$ are linearly independent, this implies $a_i(z)=a_i(z+t)=a_i(t)$ for $z\in\mathcal{O}_x$ and $t\in\mathcal{O}_y$. Thus, for $a_i:=a_i(t)$,

$$Tz=\sum_{i=1}^{\kappa}a_iT_iz\quad\text{for all}\quad z\in\mathcal{O}_x\,.$$

This polynomial identity holds on a non-empty open set and thus holds on all V. This establishes Assertion (3). □

We specialize this result for Clifford families.

Corollary 3.3. Let $\mathcal{F}:=\{J_1,\ldots,J_\kappa\}$ be a Clifford family of rank κ on a vector space of dimension n.

(1) Suppose that $n\geq\kappa$. Then there exists x in V so that the set $\{J_ix\}_{1\leq i\leq\kappa}$ consists of κ linearly independent vectors.

(2) Suppose that $n\geq 2\kappa$. Then there exist x and y in V so that the set $\{J_ix,J_iy\}_{1\leq i\leq\kappa}$ consists of 2κ linearly independent vectors. Furthermore, if $Tx\in\operatorname{Span}_{1\leq i\leq\kappa}\{J_ix\}$ for all x in V, then $T\in\operatorname{Span}_{1\leq i\leq\kappa}\{J_i\}$.

(3) Suppose that $n\geq\kappa(\kappa-1)$. Then there exists x in V so that the set $\{J_jJ_kx\}_{1\leq j<k\leq\kappa}$ consists of $\frac{1}{2}\kappa(\kappa-1)$ linearly independent vectors.

(4) Suppose that $n\geq 2\kappa(\kappa-1)$. Then there exist x and y in V so that the set $\{J_jJ_kx,J_jJ_ky\}_{1\leq j<k\leq\kappa}$ consists of $\kappa(\kappa-1)$ linearly independent vectors. Furthermore, if $Tx\in\operatorname{Span}_{1\leq j<k\leq\kappa}\{J_jJ_kx\}$ for all x in V, then $T\in\operatorname{Span}_{1\leq j<k\leq\kappa}\{J_jJ_k\}$.

Proof. One verifies that $(a_1J_1+\ldots+a_\kappa J_\kappa)^2=-(a_1^2+\ldots+a_\kappa^2)\operatorname{id}$ and thus one has that $\operatorname{Rank}(a_1J_1+\ldots+a_\kappa J_\kappa)=n$ if any coefficient is non-zero. Assertions (1) and (2) now follow from Lemma 3.2. If not all the coefficients vanish, one shows similarly that:

$$\operatorname{Rank}\left(\sum_{j=1}^{\kappa-1}\sum_{k=j+1}^{\kappa}a_{jk}J_jJ_k\right)\geq\frac{n}{2}\,.$$

The remaining assertions of the Lemma now follow. □

We now describe some general properties of models given by Clifford families. We adopt the notation of Equations (1.b) and (1.c).

Lemma 3.4.

(1) Suppose that J is an Hermitian almost complex structure on $(V,\langle\cdot,\cdot\rangle)$. Then $\mathcal{V}:=(V,\langle\cdot,\cdot\rangle,J,c_0R_0+c_1R_J)$ is complex Osserman.

(2) Suppose that $\{J_1,J_2,J_3\}$ is an Hermitian quaternion structure on $(V,\langle\cdot,\cdot\rangle)$. Then $\mathcal{V}:=(V,\langle\cdot,\cdot\rangle,J_1,c_0R_0+c_1R_{J_1}+c_2R_{J_2}+c_3R_{J_3})$ is complex Osserman.

(3) Let $\mathcal{F}:=\{J_1,\dots,J_\kappa\}$ be a Clifford family and let $\tilde{\mathcal{F}}:=\{\tilde J_1,\dots,\tilde J_\kappa\}$ be a reparametrization of $\mathcal{F}$. Then $R_{J_1}+\dots+R_{J_\kappa}=R_{\tilde J_1}+\dots+R_{\tilde J_\kappa}$.

Proof. Let $\mathcal{V}$ be as in Assertion (1). We use Equation (1.g) to see that:

$$\mathcal{J}_R(\pi_x)y=\begin{cases}(c_0+3c_1)y & \text{if } y\in\operatorname{Span}\{x,Jx\},\\ 2c_0 & \text{if } y\perp\operatorname{Span}\{x,Jx\}.\end{cases}$$

Hence J and $\mathcal{J}_R(\pi_x)$ commute and the eigenvalues are constant. Thus $\mathcal{V}$ is complex Osserman by Lemma 2.2; the proof of Assertion (2) is similar and follows from a calculation in this instance that:

$$\mathcal{J}_{R_J}(\pi_x)y=\begin{cases}(c_0+3c_1)y & \text{if } y\in\operatorname{Span}\{x,J_1x\},\\ (2c_0+3c_2+3c_3)y & \text{if } y\in\operatorname{Span}\{J_2x,J_3x\},\\ 2c_0 & \text{if } y\perp\operatorname{Span}\{x,J_1x,J_2x,J_3x\}.\end{cases}$$

We complete the proof by verifying that Assertion (3) holds. If $x\in S(V)$, then the vectors $\{J_1x,...,J_\kappa x\}$ form an orthonormal set. Let $\sigma_{\mathcal{F}}x$ be orthogonal projection on the subspace

$$S_1^{\mathcal{F}}(x):=\operatorname{Span}\{J_1x,...,J_\kappa x\}\,.$$

We then have $\sum_i\langle x,J_ix\rangle J_ix=\sigma_{\mathcal{F}}x$. Let $R=R_{J_1}+...+R_{J_\kappa}$. By Equation (1.f), $\mathcal{J}_R(x)=3\sigma_{\mathcal{F}}(x)$. If $\tilde{\mathcal{F}}$ is a reparametrization of $\mathcal{F}$, then $S_1^{\mathcal{F}}(x)=S_1^{\tilde{\mathcal{F}}}(x)$. Consequently $\mathcal{J}_R(x)=\mathcal{J}_{\tilde R}(x)$ so by Lemma 2.1, $R=\tilde R$. □

Remark 3.5. Theorem 2.4 places restrictions on the possible eigenvalue multiplicities of the complex Jacobi operator defined by a complex Osserman model. We may use Lemma 3.4 to show that in fact all these possibilities occur. Suppose first that the dimension n of V is even. Let J be an Hermitian almost complex structure on $(V,\langle\cdot,\cdot\rangle)$.

(1) If $R=3R_0+R_J$, then $J_R(\pi_x)=6\,\mathrm{id}$.

(2) If $R=R_0+R_J$, then the eigenvalues of $J_R(\pi_x)$ are $(2,4)$ and the eigenvalue multiplicities are $(n-2,2)$.

If n is divisible by 4, there are additional eigenvalue multiplicities which can be realized. Let $\{J_1,J_2,J_3\}$ be a quaternion structure on $(V,\langle\cdot,\cdot\rangle)$ and let $J=J_1$.

(1) If $R=3R_0+3R_{J_1}+R_{J_2}+R_{J_3}$, then $\mathcal{J}_R(\pi_x)$ are $(6,12)$ and the eigenvalue multiplicities are $(n-4,4)$.

(2) If $R=R_0+R_{J_1}+R_{J_2}+R_{J_3}$, then the eigenvalues of $\mathcal{J}_R(\pi_x)$ are $(2,4,8)$ and the eigenvalue multiplicities are $(n-4,2,2)$.

Lemma 2.1 shows that the Jacobi operator determines the full curvature tensor, i.e. that if $\mathcal{J}_R(x)=0$ for all x in V, then $R=0$. Similarly, the higher order Jacobi operator determines the full curvature operator. To see that this is true, one may argue as follows. Let $2\le p\le n-1$. Assume that $\mathcal{J}_R(\sigma)=0$ for every p-plane σ. Let x and y be unit vectors in V. Choose additional unit vectors $\{e_2,...,e_p\}$ so that $\{x,e_2,...,e_p\}$ is an orthonormal basis for a p-plane σ_x and so that $\{y,e_2,...,e_p\}$ is an orthonormal basis for a p-plane σ_y. Then

$$\mathcal{J}_R(x)y=(\mathcal{J}_R(x)-\mathcal{J}_R(y))y=(\mathcal{J}_R(\sigma_x)-\mathcal{J}_R(\sigma_y))y=0\,.$$

This shows that $\mathcal{J}_R=0$ and hence $R=0$ by Lemma 2.1.

However an analogous property does not hold for the complex Jacobi operator. This is, perhaps, to be expected on dimensional grounds. The domain of the usual Jacobi operator is V which is n-dimensional. The domain of the higher order Jacobi operator is the dimension of the p-dimensional Grassmannian which has dimension greater than n for $2 \leq p \leq n-2$. However, the domain of the complex Jacobi operator is $\mathbb{CP}(V,\langle\cdot,\cdot\rangle,J)$ which is $n-2$ dimensional. One has the following result:

Theorem 3.6. *Let V be a vector space of dimension n. Assume n is divisible by 4 and that n is at least 8. Then there exists a model $\mathcal{V}=(V,\langle\cdot,\cdot\rangle,J,R)$ which is complex Osserman, which is not Osserman, which is not given by a Clifford family, and which has $\mathcal{J}_R(\pi_x)=0$ for all x.*

Proof. Since the dimension of V is divisible by 4, we can find a quaternion structure $\{K_1,K_2,K_3\}$ on V. Since $n\geq 8$, we may take a non-trivial decomposition of V as a quaternion module in the form $V=V_+\oplus V_-$. Define a new Clifford family on V which is not a quaternion structure by setting $J_1:=K_1$, $J_2:=K_2$, and $J_3:=\mp J_1J_2$ on $V_\pm$. We then have $J_1J_2J_3x=\pm x$ for $x\in V_\pm$. Define

$$R:=R_{J_2}-R_{J_1J_2}-R_{J_3}+R_{J_1J_3}\,.$$

Let $x_\pm\in S(V_\pm)$. Equation (1.d) yields that:

$$\mathcal{J}_R(x_+)y=\begin{cases}6y & \text{if} \quad y\in\operatorname{Span}\{J_2x_+\}=\operatorname{Span}\{J_1J_3x_+\},\\ -6y & \text{if} \quad y\in\operatorname{Span}\{J_3x_+\}=\operatorname{Span}\{J_1J_2x_+\}\,.\end{cases}$$

On the other hand, if we take $x_0=(x_++x_-)/\sqrt{2}$, then

$$\mathcal{J}_R(x_0)y=\begin{cases}3y & \text{if} \quad y\in\operatorname{Span}\{J_2x_0,J_1J_3x_0\}=\operatorname{Span}\{J_2x_+,J_2x_-\},\\ -3y & \text{if} \quad y\in\operatorname{Span}\{J_1J_2x_0,J_3x_0\}=\operatorname{Span}\{J_3x_+,J_3x_-\}\,.\end{cases}$$

This shows that $\mathcal{V}$ is not Osserman. As any model given by a Clifford family is necessarily Osserman, $\mathcal{V}$ is not given by a Clifford family. On the other hand, the complex Jacobi operator with respect to $J=J_1$ is given by

$$\begin{aligned}\mathcal{J}_R(\pi_x)y &= 3\langle y,J_2x\rangle J_2x+3\langle y,J_2J_1x\rangle J_2J_1x-3\langle y,J_1J_2x\rangle J_1J_2x\\ &- 3\langle y,J_1J_2J_1x\rangle J_1J_2J_1x-3\langle y,J_3x\rangle J_3x-3\langle y,J_3J_1x\rangle J_3J_1x\\ &+ 3\langle y,J_1J_3x\rangle J_1J_3x+3\langle y,J_1J_3J_1x\rangle J_1J_3J_1x\\ &= 0\,.\end{aligned}$$

This shows $\mathcal{J}_R(\pi_x)=0$ for all x as desired. Thus $\mathcal{V}$ is complex Osserman. □

4. Curvature and higher order Clifford families

In this section, we establish Theorem 1.3 by studying models with

$$R=c_0R_0+c_1R_{J_1}+...+c_\kappa R_{J_\kappa}$$

where $\{J_1,...,J_\kappa\}$ is a Clifford family of rank $\kappa\geq 4$ on $(V,\langle\cdot,\cdot\rangle)$. We remark that the work of [3, 11, 12] shows tensors of this kind do not arise in the geometric context. In Section 4.1 we study the case $c_0=0$ and in Section 4.2 we study the case $c_0\neq 0$. We shall always assume that the constants c_1, ..., c_κ are non-zero.

4.1. Curvature given by a Clifford family with $c_0=0$.

Throughout this section we shall assume that

$$R=c_1R_{J_1}+...+c_\kappa R_{J_\kappa}$$

where c_1, ..., c_k are non-zero constants and where $\{J_1,...,J_\kappa\}$ is a Clifford family of rank κ on a vector space V of dimension n. Let $\mathcal{V}:=(V,\langle\cdot,\cdot\rangle,J,R)$. We suppose that $\mathcal{V}$ is complex Osserman. We first show that this implies that J has the form $J=\sum_{i<j}c_{ij}J_iJ_j$. We then derive a contradiction by studying the eigenvalue structure and by studying the coefficients c_{ij}. The eigenvalue multiplicity estimates of Theorem 2.4 will play a crucial role in our analysis.

We shall have to impose certain conditions on n; these conditions are automatic for κ large. We begin with a technical result:

Lemma 4.1. *Let* $\mathcal{V} := (V, \langle\cdot,\cdot\rangle, J, R = c_1 R_{J_1} + \cdots + c_\kappa R_{J_\kappa})$ *be a complex Osserman model where* $\{J_1, ..., J_\kappa\}$ *is a Clifford family of rank* κ *on a vector space of dimension* n. *Assume that* $\kappa \geq 4$ *and that* $n \geq 2\kappa + 5$. *If* $x \in S(V, \langle\cdot,\cdot\rangle)$, *then*

(1) $\operatorname{Rank}\{\mathcal{J}_R(\pi_x)\} \leq 4$.

(2) $Jx \in \operatorname{Span}_{i\leq 4, i\neq j}\{J_i J_j x\}$.

Proof. Equation (1.g) shows $\operatorname{Rank}\{\mathcal{J}_R(\pi_x)\} \leq 2\kappa$. Consequently 0 is an eigenvalue of multiplicity at least $n - 2\kappa \geq 5$. Theorem 2.4 then shows that 0 is an eigenvalue of multiplicity at least $n - 4$. Consequently, as desired, $\operatorname{Rank}\{\mathcal{J}_R(\pi_x)\} \leq 4$.

The vectors $\{J_1 x, ..., J_\kappa x\}$ form an orthonormal set for $x \in S(V, \langle\cdot,\cdot\rangle)$. Let $\alpha_i(x) := \langle J_i x, J_1 J x\rangle$ be the Fourier coefficients of $J_1 J x$. Let

$$\begin{aligned} U(x) &:= \operatorname{Span}\{J_1 x, \ldots, J_\kappa x, J_1 J x\},\\ V(x) &:= \operatorname{Span}\{J_2 J x, \ldots, J_\kappa J x\},\\ W(x) &:= U(x) + V(x). \end{aligned}$$

Note that $\operatorname{Range}\{\mathcal{J}_R(\pi_x)\} \subset W(x)$. If $\dim\{U(x)\} \leq \kappa$, then $J_1 J x \in \operatorname{Span}_i\{J_i x\}$. Since $J_1 J x \perp J_1 x$, we have that $Jx \in \operatorname{Span}_{i>1}\{J_1 J_i x\}$ and Assertion (2) follows.

Suppose on the other hand that $\dim\{U(x)\} = \kappa + 1$ or equivalently that

$$\alpha_1^2 + \cdots + \alpha_\kappa^2 < 1\,. \tag{4.a}$$

Let ρ be the projection on $W(x)/V(x)$. Then

$$\begin{aligned} \rho\mathcal{J}_R(\pi_x) J_i x &= \rho\{3c_i J_i x + 3c_1\alpha_i J_1 J x\},\\ \rho\mathcal{J}_R(\pi_x) J_1 J x &= \rho\{3c_1 J J_1 x + 3c_1\alpha_1 J_1 x + ... + 3c_\kappa\alpha_\kappa J_\kappa x\}. \end{aligned}$$

Hence $\rho\mathcal{J}_R(\pi_x) = \rho M$ on $U(x)$, where

$$M := 3\begin{pmatrix} c_1 & 0 & \ldots & 0 & c_1\alpha_1\\ 0 & c_2 & \ldots & 0 & c_2\alpha_2\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & \ldots & c_\kappa & c_\kappa\alpha_\kappa\\ c_1\alpha_1 & c_1\alpha_2 & \ldots & c_1\alpha_\kappa & c_1 \end{pmatrix}$$

We compute $\det(M) = 3^{\kappa+1} c_1^2 c_2 \ldots c_\kappa (1 - \alpha_1^2 - \cdots - \alpha_\kappa^2)$. Thus by Equation (4.a), $\det(M) \neq 0$ so M is invertible. Consequently,

$$\dim\{\rho U(x)\} = \dim\{\rho M U(x)\} = \dim\{\rho\mathcal{J}_R(\pi_x) U(x)\} \leq \operatorname{Rank}\{\mathcal{J}_R(\pi_x)\} \leq 4.$$

The short exact sequence

$$0 \to V(x) \to W(x) \to W(x)/V(x) = \rho U(x) \to 0$$

shows that $\dim\{W(x)\} = \dim\{V(x)\} + \dim\{\rho U(x)\} \leq (\kappa - 1) + 4$. Therefore

$$\begin{aligned} \dim\{\operatorname{Span}_{i\leq 4}\{J_i x\} \cap \operatorname{Span}_i\{J_i J x\}\} &= 4 - (\dim\{W(x)\} - \kappa)\\ &\geq 4 + \kappa - (\kappa + 3) > 0\,. \end{aligned}$$

Hence, there exist non-zero constants a_i and b_j so that

$$a_1 J_1 x + a_2 J_2 x + a_3 J_3 x + a_4 J_4 x = b_1 J_1 J x + \cdots + b_\kappa J_\kappa J x.$$

We multiply by $b_1 J_1 + ... + b_\kappa J_\kappa$ to invert this relation and conclude thereby that $Jx \in \operatorname{Span}\{x, \{J_i J_j x\}_{i\leq 4, i\neq j}\}$. Since $Jx \perp x$, we may conclude as desired that $Jx \in \operatorname{Span}_{i\leq 4, i\neq j}\{J_i J_j x\}$. □

We continue our study by reducing to the cases $\kappa = 4$ and $\kappa = 5$:

Lemma 4.2. *Let $\mathcal{V} := (V, \langle\cdot,\cdot\rangle, J, R = c_1 R_{J_1} + \cdots + c_\kappa R_{J_\kappa})$ be a complex Osserman model where $\{J_1, ..., J_\kappa\}$ is a Clifford family of rank κ on a vector space of dimension n. Assume that $\mathcal{V}$ is complex Osserman, that $n \geq \kappa(\kappa-1)$, and that $\kappa \geq 4$. Then $\kappa \leq 5$.*

Proof. Suppose $\kappa \geq 6$. By Corollary 3.3 we know that there exists $x \in V$ such that $\{J_i J_j x\}_{i<j}$ is a linearly independent set of $\frac{1}{2}\kappa(\kappa-1)$ vectors. By Lemma 4.1,

$$Jx = \sum_{1 \leq i \leq 6, i<j} a_{ij}(x) J_i J_j x.$$

Moreover, the sum may be restricted to $i \leq 4$ and, since the coefficients a_{ij} are uniquely determined, we get $a_{56}(x) = 0$. By permuting the role of the indices we may conclude that all the coefficients vanish. As this is not possible, $\mathcal{V}$ can not be a complex Osserman model. □

The analysis of the cases $\kappa = 4$ and $\kappa = 5$ to complete the proof of Theorem 1.3 (1) is a bit technical. We shall outline the proof but omit details in the interests of brevity. We assume $\dim(V) \geq 16$ throughout.

Lemma 4.3. *Let $\mathcal{V} := (V, \langle\cdot,\cdot\rangle, J, R = c_1 R_{J_1} + \cdots + c_\kappa R_{J_\kappa})$ where $\{J_1, ..., J_\kappa\}$ is a Clifford family of rank κ on a vector space of dimension n. Assume that $n \geq 2^\kappa$ and that $\kappa = 4, 5$. Then:*

1. *Suppose that $\mathcal{V}$ is complex Osserman. Then there exists a reparametrization $\tilde{\mathcal{F}} = \{\tilde{J}_1, ..., \tilde{J}_\kappa\}$ of the family $\mathcal{F} = \{J_1, ..., J_\kappa\}$ so that $J = \tilde{J}_1 \tilde{J}_2$ and so that $R = \tilde{c}_1 R_{\tilde{J}_1} + \cdots + \tilde{c}_\kappa R_{\tilde{J}_\kappa}$.*
2. *If $\kappa = 5$, then $\mathcal{V}$ is not complex Osserman.*
3. *If $\kappa = 4$, then $\mathcal{V}$ is not complex Osserman.*

Proof. Since $\kappa = 4$ or $\kappa = 5$ we have $2\kappa + 5 < 16 \leq n$. Thus Lemma 4.1 implies $Jx \in \mathrm{Span}_{i \neq j}\{J_i J_j x\}$ for all $x \in S(V, \langle\cdot,\cdot\rangle)$. One can show there exists $x, y \in V$ so $\{J_j J_k x, J_j J_k y\}_{j<k}$ is an orthonormal set of $\kappa(\kappa-1)$ linearly independent vectors. Thus the argument used to establish Lemma 3.2 proves that

$$J = \sum_{i=1}^{\kappa-1} \sum_{j=i+1}^{\kappa} a_{ij} J_i J_j\,.$$

One can now show that there exists a suitable reparametrization; as the argument is straightforward, if a bit lengthy, we shall omit the details.

Suppose that $\kappa = 5$. By Assertion (1), we may suppose that $J = J_1 J_2$. As noted above, there exists $x \in S(V, \langle\cdot,\cdot\rangle)$ such that $\{J_i J_j x\}_{i<j}$ is an orthonormal set and, thus, $\{J_1 x, J_2 x, J_3 x, J_4 x, J_5 x, J_1 J_2 J_3 x, J_1 J_2 J_4 x, J_1 J_2 J_5 x\}$ is also an orthonormal set. Therefore

$$\mathcal{J}_R(\pi_x) y = \begin{cases} 3(c_1 + c_2) y & \text{if } y \in \mathrm{Span}\{J_1 x, J_2 x\}, \\ 3 c_3 y & \text{if } y \in \mathrm{Span}\{J_3 x, J_1 J_2 J_3 x\}, \\ 3 c_4 y & \text{if } y \in \mathrm{Span}\{J_4 x, J_1 J_2 J_4 x\}, \\ 3 c_5 y & \text{if } y \in \mathrm{Span}\{J_5 x, J_1 J_2 J_5 x\}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mathrm{Rank}\{\mathcal{J}_R(\pi_x) y\} > 4$. Hence, by Lemma 2.4, R is not complex Osserman. Assertion (2) now follows.

Finally suppose $\kappa = 4$. Again, we may suppose $J = J_1 J_2$. Since $(J_1 J_2 J_3)^2 = \mathrm{id}$, there exists $x \in S(V, \langle\cdot,\cdot\rangle)$ such that $J_1 J_2 J_3 x = \pm x$, and hence

$$\{x, J_1 x, J_2 x, J_3 x, J_4 x, J_1 J_2 J_4 x\}$$

is an orthonormal set. Note that

$$\mathcal{J}_R(\pi_x)y=\left\{\begin{array}{ll}3c_3y & \text{if } y\in\operatorname{Span}\{x,J_3x\},\\ 3(c_1+c_2)y & \text{if } y\in\operatorname{Span}\{J_1x,J_2x\},\\ 3c_4y & \text{if } y\in\operatorname{Span}\{J_4x,J_1J_2J_4x\},\\ 0 & \text{if } y\perp\operatorname{Span}\{x,J_3x,J_1x,J_2x,J_4x,J_1J_2J_4x\}.\end{array}\right.$$

Now, since $(J_1J_2J_3J_4)^2=\operatorname{id}$, there exists $y\in S(V,\langle\cdot,\cdot\rangle)$ such that $J_1J_2J_3J_4y=\pm y$ and

$$\mathcal{J}_R(\pi_x)y=\left\{\begin{array}{ll}3(c_1+c_2)y & \text{if } y\in\operatorname{Span}\{J_1x,J_2x\},\\ 3(c_3+c_4)y & \text{if } y\in\operatorname{Span}\{J_3x,J_4x\},\\ 0 & \text{if } y\perp\operatorname{Span}\{J_1x,J_2x,J_3x,J_4x\}.\end{array}\right.$$

Since the eigenvalues are different, R is not complex Osserman. □

4.2. **Curvature given by a Clifford family with $c_0\neq 0$.** This section is devoted to the proof of Assertion (2) of Theorem 1.3. Although there is some parallelism between cases $c_0=0$ and $c_0\neq 0$, the approach we follow now is slightly different. However, in the interests of brevity, we will refer to arguments in Section 4.1 whenever possible. We begin by studying a reduced complex Jacobi operator where the effect of c_0 has been normalized.

Lemma 4.4. *Let $\mathcal{V}:=(V,\langle\cdot,\cdot\rangle,J,R=c_0R_0+c_1R_{J_1}+\dots+c_\kappa R_{J_\kappa})$ be a complex Osserman model where $\{J_1,...,J_\kappa\}$ is a Clifford family of rank κ on a vector space of dimension n. Assume that $\kappa\geq 4$. If $4\leq\kappa\leq 7$, assume that $n\geq 2^\kappa$. If $\kappa\geq 8$, assume that $n\geq\kappa(\kappa-1)$. Let $\tilde{\mathcal{J}}_R(\pi_x)=\mathcal{J}_R(\pi_x)-2c_0\operatorname{id}$. Then:*

1. *(1)* $\operatorname{Rank}\{\tilde{\mathcal{J}}_R(\pi_x)\}\leq 4$.
2. *(2)* $Jx\in\operatorname{Span}\{J_ix,J_jJ_kx\}_{i,j<k}$ *for all* $x\in V$.
3. *(3) If* $\kappa\geq 6$*, then* $Jx\in\operatorname{Span}\{J_iJ_jx\}_{i\leq 6}$ *for all* $x\in V$.
4. *(4)* $\kappa\leq 5$.

Proof. We use Equation (1.g) to see that:

$$\tilde{\mathcal{J}}_R(\pi_x)y=-c_0\langle y,x\rangle x-c_0\langle y,Jx\rangle Jx+3\sum c_i(\langle y,J_ix\rangle J_ix+\langle y,J_iJx\rangle J_iJx)\,.$$

Consequently $\operatorname{Rank}\{\tilde{\mathcal{J}}_R(\pi_x)\}\leq 2\kappa+2$ and 0 is an eigenvalue with multiplicity at least $n-2\kappa-2$. Since $n-2\kappa-2>4$ and as we have simply shifted the spectrum, Theorem 2.4 may be used to derive Assertion (1).

To prove Assertion (2), we compute that:

$$\begin{aligned}&\tilde{\mathcal{J}}_R(\pi_x)x=-c_0x+\textstyle\sum_i 3c_i\langle x,J_iJx\rangle J_iJx,\\ \text{(4.b)}\qquad&\tilde{\mathcal{J}}_R(\pi_x)Jx=-c_0Jx+\textstyle\sum_i 3c_i\langle Jx,J_ix\rangle J_ix,\\ &\tilde{\mathcal{J}}_R(\pi_x)J_ix=-c_0\langle J_ix,Jx\rangle Jx+3c_iJ_ix+\textstyle\sum_j 3c_j\langle J_ix,J_jJx\rangle J_jJx\,.\end{aligned}$$

Define:

$$\begin{aligned}M&:=\operatorname{diag}(-c_0,3c_1,...,3c_\kappa),\\ U(x)&:=\operatorname{Span}\{x,J_1x,...,J_\kappa x\},\\ V(x)&:=\operatorname{Span}\{Jx,J_1Jx,...,J_\kappa Jx\},\\ W(x)&:=U(x)+V(x)\,.\end{aligned}$$

Let ρ denote projection on $W(x)/V(x)$. We then have that $\rho\tilde{\mathcal{J}}(\pi_x)=\rho M$ on $U(x)$. As M is invertible, the following inequalities hold:

$$\begin{aligned}&\dim\{\rho U(x)\}=\dim\{\rho\tilde{\mathcal{J}}_R(\pi_x)U(x)\}\leq 4,\\ &\dim\{W(x)\}\leq 4+\kappa+1,\\ &\dim\{U(x)\cap V(x)\}\geq\kappa+1+\kappa+1-\kappa-5=\kappa-3>0\,.\end{aligned}$$

Therefore, there exists a non-trivial relationship

$$(a_0+a_1J_1+...+a_\kappa J_\kappa)Jx=(b_0+b_1J_1+...+b_\kappa J_\kappa)x\,.$$

We invert this relationship by multiplying by $(a_0 - a_1 J_1 - ... - a_\kappa J_\kappa)$. Since $Jx \perp x$, we may conclude that $Jx \in \operatorname{Span}\{J_i x, J_j J_k x\}$ and establish Assertion (2).

If $\kappa \geq 6$, then we can derive a stronger result. We estimate that:

$$\begin{aligned} &\dim\left\{\{\operatorname{Span}\{J_1 x, ..., J_6 x\} \cap \operatorname{Span}\{J_1 Jx, ..., J_\kappa Jx\}\right\} \\ &\geq 6 + \kappa - \dim(W) \geq 6 + \kappa - \kappa - 5 > 0\,. \end{aligned}$$

Assertion (3) now follows using a similar argument to that used to establish Assertion (2).

To establish Assertion (4), we assume to the contrary that $\kappa \geq 6$ and argue for a contradiction. By Assertion (3), we have that $Jx \in \operatorname{Span}\{J_i J_j x\}_{i \leq 6, j \neq i}$. The argument used to establish Lemma 4.2 shows that $\kappa \leq 7$. Thus we have that $\kappa = 6$ or $\kappa = 7$. Since $n \geq 2\kappa(\kappa - 1)$, Corollary 3.3 and Assertion (3) show that $J \in \operatorname{Span}\{J_i J_j\}$. One may show there exists $x \in V$ such that $x \perp J_i J_j J_k x$ for any i, j, k and such that $J_1 J_2 x \perp \operatorname{Span}\{J_i J_j x\}_{(i,j) \neq (1,2)}$. Thus, since $Jx \perp J_i x$ for this specific x, Equation (4.b) yields:

$$\begin{aligned} &\tilde{\mathcal{J}}_R(\pi_x)x = -c_0 x, \qquad \tilde{\mathcal{J}}_R(\pi_x)Jx = -c_0 Jx, \\ &\tilde{\mathcal{J}}_R(\pi_x)J_i x = 3c_i J_i x + \sum_{j=1}^{\kappa} 3c_j \langle J_i x, J_j Jx\rangle J_j Jx\,. \end{aligned}$$

Hence the subspace $\operatorname{Span}\{x, Jx\}$ is invariant under $\tilde{\mathcal{J}}(\pi_x)$. We clear the previous notation. By applying the argument used to prove Assertion (2) to the sets

$$\begin{aligned} U(x) &:= \operatorname{Span}\{J_1 x, ..., J_\kappa x\}, \\ V(x) &:= \operatorname{Span}\{J_1 Jx, ..., J_\kappa Jx\}, \\ W(x) &:= U(x) + V(x), \end{aligned}$$

we obtain $Jx = \sum_{i \leq 3, i<j} a_{ij} J_i J_j x$. Thus in particular $a_{45} = 0$. Since the coefficients a_{ij} were universal and independent of x, we can permute the indices to see that $a_{ij} = 0$ for all $i < j$, which is impossible. □

It remains to show that a Clifford family of rank $\kappa = 4$ or $\kappa = 5$ can not give a complex Osserman model. As in the case $c_0 = 0$ these ranks are treated independently. However, the present situation is a bit more difficult. We present sketch of proofs describing the main ideas involved; full details are available from the authors upon request but are omitted here in the interests of brevity.

Lemma 4.5. *Let $\mathcal{V} := (V, \langle\cdot,\cdot\rangle, J, R = c_0 R_0 + c_1 R_{J_1} + \cdots + c_\kappa R_{J_\kappa})$ be a complex Osserman model where $\{J_1, ..., J_\kappa\}$ is a Clifford family of rank $\kappa = 4$ or $\kappa = 5$ on a vector space of dimension $n \geq 32$.*

(1) If $\kappa = 5$, then $\mathcal{V}$ is not complex Osserman.
(2) If $\kappa = 4$, then $\mathcal{V}$ is not complex Osserman.

Proof. Suppose that $\kappa = 5$, that $n \geq 32$, and that $\mathcal{V}$ is complex Osserman. We argue for a contradiction. Using similar techniques to those which were used to prove Lemma 4.4, one shows that $J \notin \operatorname{Span}\{J_i J_j\}_{i \neq j}$. Consider the set

$$C := \{x \in V : Jx \in \operatorname{Span}\{J_i x\}\}\,.$$

One shows that C is a closed nowhere dense set. So, working in the complementary set C^c and using similar arguments to those which were used to prove Lemma 4.1 applied to the sets

$$\begin{aligned} U(x) &:= \operatorname{Span}\{J_1 x, ..., J_5 x, Jx\}, \\ V(x) &:= \operatorname{Span}\{J_1 Jx, ..., J_5 Jx\}, \\ W(x) &:= U(x) + V(x), \end{aligned}$$

one shows that $Jx \in \operatorname{Span}\{J_iJ_jx\}_{i\neq j}$ and, therefore, $J \in \operatorname{Span}\{J_iJ_j\}_{i\neq j}$, which is false. This proves Assertion (1).

Suppose that $\kappa = 4$. By Lemma 4.4 we know that $Jx \in \operatorname{Span}\{J_ix, J_jJ_kx\}_{j<k}$ for all $x \in V$. Since $n \geq 32$, one can show that there exists $x, y \in S(V, \langle\cdot,\cdot\rangle)$ so that $\{J_ix, J_{jk}x, J_iy, J_{jk}y\}_{j<k}$ is an orthonormal set. The argument given to establish Lemma 3.2 (3) then shows there exist constants a_i and a_{jk} so that

$$J = \sum_{i=1}^{4} a_iJ_i + \sum_{j<k} a_{jk}J_jJ_k\,.$$

The compatibility between J and R shows that the constants a_i vanish so

$$J = \sum_{i<j} a_{ij}J_iJ_j\,.$$

In this situation one may reparametrize the Clifford family so $J = \tilde{J}_1\tilde{J}_2$. A straightforward calculation now shows $\operatorname{Rank}\{\tilde{\mathcal{J}}_{\pi_x}\} \geq 6$, which contradicts Theorem 2.4. □

5. Classification for Clifford families of lower rank

In this section we prove Theorem 1.4 by studying complex Osserman models which are given by Clifford families of rank κ for $0 \leq \kappa \leq 3$. Section 5.1 deals with the case $\kappa = 0$, Section 5.2 deals with $\kappa = 1$, and Section 5.3 deals with $\kappa = 2$. We shall omit much of the analysis when discussing the case $\kappa = 3$ in Section 5.4 in the interests of brevity as it is similar to the other cases; again, details are available upon request from the authors. Throughout Section 5, we suppose that $R = c_0R_0 + c_1R_{J_1} + ... + c_\kappa R_{J_\kappa}$.

5.1. Clifford families of rank 0.

Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, J, R = c_0R_0)$. Then we have

$$\mathcal{J}_R(\pi_x)y = c_0(2y - \langle y, x\rangle x - \langle y, Jx\rangle Jx).$$

Hence $J\mathcal{J}_R(\pi_x) = \mathcal{J}_R(\pi_x)J$ and the eigenvalues are $(c_0, 2c_0)$ with multiplicities $(2, n-2)$ for any $x \in S(V, \langle\cdot,\cdot\rangle)$. Consequently, R is complex Osserman.

5.2. Clifford families of rank 1.

We have that:

Lemma 5.1. *Let J and J_1 be Hermitian almost complex structures on $(V, \langle\cdot,\cdot\rangle)$.*

(1) Let $\mathcal{V} := (V, \langle\cdot,\cdot\rangle, J, R = c_1R_{J_1})$ where $c_1 \neq 0$. The following assertions are equivalent:
   - (a) *R and J are compatible.*
   - (b) *$JJ_1 = J_1J$ or $JJ_1 = -J_1J$.*
   - (c) *$\mathcal{V}$ is complex Osserman.*

(2) Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, J, R = c_0R_0 + c_1R_{J_1})$ where $c_0c_1 \neq 0$. Then $\mathcal{V}$ is complex Osserman if and only if $J = \pm J_1$ or $JJ_1 = -J_1J$.

Proof. Suppose that $\mathcal{V} := (V, \langle\cdot,\cdot\rangle, J, R = c_1R_{J_1})$ and that J and R are compatible. By Equation (1.g),

$$\mathcal{J}_R(\pi_x)y = 3\langle y, J_1x\rangle J_1x + 3\langle y, J_1Jx\rangle J_1Jx\,.$$

Hence $\operatorname{Range}\{\mathcal{J}_R(\pi_x)\} = \operatorname{Span}\{J_1x, J_1Jx\}$ and, since J and R are compatible, we have $J(\operatorname{Span}\{J_1x, J_1Jx\}) \subset \operatorname{Span}\{J_1x, J_1Jx\}$. Since $JJ_1x \perp J_1x$, necessarily $JJ_1x = \epsilon_xJ_1Jx$, where $\epsilon_x = \pm 1$. By continuity, since $S(V, \langle\cdot,\cdot\rangle)$ is connected, ϵ_x is constant. Then $JJ_1 = J_1J$ or $JJ_1 = -J_1J$. If this condition holds, then it is easily verified that $\mathcal{V}$ is complex Osserman. Finally, if $\mathcal{V}$ is complex Osserman, then necessarily R and J are compatible. Assertion (1) now follows.

Next suppose that $\mathcal{V} := (V, \langle\cdot,\cdot\rangle, J, R = c_0R_0 + c_1R_{J_1})$ is a complex Osserman model where $c_0 \neq 0$ and $c_1 \neq 0$. Since R and J are compatible and since R_0 and J are compatible, R_{J_1} and J are compatible as well. Thus by Assertion (1), $JJ_1 = J_1J$ or $JJ_1 = -J_1J$. We now show that $JJ_1 = J_1J$ implies $J = \pm J_1$. We suppose to the contrary that $J \neq \pm J_1$ and argue for a contradiction. Because $(JJ_1)^2 = \text{id}$, we can use JJ_1 to define a $\mathbb{Z}_2$ grading on V by decomposing $V = V_+ \oplus V_-$ where $J = \pm J_1$ on $V_\pm$.

Let $x_\pm \in S(V_\pm)$ and let $x_0 = (x_+ + x_-)/\sqrt{2}$. Then one has that:

$$\mathcal{J}_R(\pi_{x_+})y = \begin{cases} (c_0 + 3c_1)y & \text{if } y \in \text{Span}\{x_+, Jx_+\}, \\ 2c_0 y & \text{if } y \perp \text{Span}\{x_+, Jx_+\}, \end{cases}$$
$$\mathcal{J}_R(\pi_{x_0})y = \begin{cases} c_0 y & \text{if } y \in \text{Span}\{x_+, Jx_0\}, \\ (2c_0 + 3c_1)y & \text{if } y \in \text{Span}\{J_1 x_0, J_1 J x_0\}, \\ 2c_0 y & \text{if } y \perp \text{Span}\{x_0, Jx_0, J_1 x_0, JJ_1 x_0\}. \end{cases}$$

This shows that the eigenvalues of $\mathcal{J}_R(\pi_{x_+})$ are $(c_0 + 3c_1, 2c_0)$ with multiplicities $(2, n-2)$ (if $3c_1 = c_0$ then $2c_0$ has multiplicity n). Furthermore, the eigenvalues of $\mathcal{J}_R(\pi_{x_0})$ are $(c_0, 2c_0 + 3c_1, 2c_0)$ with multiplicities $(2, 2, n-4)$. So the eigenvalues are different in both cases. This contradiction shows that if $JJ_1 = J_1 J$, then $J = \pm J_1$.

Conversely, if $JJ_1 = -J_1 J$ or if $J = \pm J_1$, then a straightforward calculation shows $\mathcal{V}$ is complex Osserman. □

5.3. **Clifford families of rank** 2. We first suppose that $c_0 = 0$.

Lemma 5.2. *Let J be an Hermitian almost complex structure and let $\{J_1, J_2\}$ be a Clifford family on $(V, \langle \cdot, \cdot \rangle)$. Let $\mathcal{V} := (V, \langle \cdot, \cdot \rangle, J, R = c_1 R_{J_1} + c_2 R_{J_2})$ be complex Osserman. If x is a unit vector, set $\alpha(x) := \langle J_1 J_2 x, Jx \rangle$. Then:*

(1) $\alpha(x)$ is constant on $S(V, \langle \cdot, \cdot \rangle)$.

(2) Either $\alpha = 0$, or $\alpha = 1$, or $\alpha = -1$.

(3) Suppose that $\alpha = \pm 1$. Then $J = \pm J_1 J_2$ and $\text{Rank}\{\mathcal{J}_R(\pi_x)\} = 2$.

(4) Suppose that $\alpha = 0$. Then $\text{Rank}\{\mathcal{J}_R(\pi_x)\} = 4$. *Furthermore:*

(a) *if $c_1 \neq c_2$ then $JJ_1 = J_1 J$ and $JJ_2 = -J_2 J$ or $JJ_1 = -J_1 J$ and $JJ_2 = J_2 J$.*

(b) *if $c_1 = c_2$ then there exists a reparametrization $\{\tilde{J}_1, \tilde{J}_2\}$ of $\{J_1, J_2\}$ so that $R = c_1 R_{\tilde{J}_1} + c_2 R_{\tilde{J}_2}$, $J\tilde{J}_1 = \tilde{J}_1 J$ and $J\tilde{J}_2 = -\tilde{J}_2 J$.*

Proof. Since $\mathcal{V}$ is complex Osserman, Equation (1.g) shows that

$$\text{Range}\{\mathcal{J}_R(\pi_x)\} \subset \text{Span}\{J_1 x, J_1 Jx, J_2 x, J_2 Jx\}.$$

Consequently,

$$\begin{aligned}
\mathcal{J}_R(\pi_x) J_1 x &= 3c_1 J_1 x + 3\alpha(x) c_2 J_2 Jx, \\
\mathcal{J}_R(\pi_x) J_2 Jx &= 3\alpha(x) c_1 J_1 x + 3c_2 J_2 Jx, \\
\mathcal{J}_R(\pi_x) J_1 Jx &= 3c_1 J_1 Jx - 3\alpha(x) c_2 J_2 x, \\
\mathcal{J}_R(\pi_x) J_2 x &= -3\alpha(x) c_1 J_1 Jx + 3c_2 J_2 x.
\end{aligned}$$

Thus $V_1(x) := \text{Span}\{J_1 x, J_2 Jx\}$ and $V_2(x) := \text{Span}\{J_2 x, J_1 Jx\}$ are $\mathcal{J}_R(\pi_x)$ invariant subspaces. Note that $J(V_1(x)) = V_2(x)$, that $V_1(x) \perp V_2(x)$, and that

$$\text{Range}(\mathcal{J}_R(\pi_x)) = V_1(x) \oplus V_2(x).$$

If $\alpha(\bar{x}) = \pm 1$ for some $\bar{x} \in S(V, \langle \cdot, \cdot \rangle)$, then $\text{Rank}\{\mathcal{J}_R(\pi_{\bar{x}})\} = 2$. Since R is complex Osserman, $\mathcal{J}_R(\pi_x)$ has constant rank. In such a case we get $\alpha(x) = \pm 1$ for all $x \in S(V, \langle \cdot, \cdot \rangle)$. On the other hand if $\alpha(x) \neq \pm 1$, then

$$\mathcal{J}_R(\pi_x)|_{V_1(x)} = \begin{pmatrix} 3c_1 & 3\alpha(x) c_1 \\ 3\alpha(x) c_2 & 3c_2 \end{pmatrix}, \quad \text{and}$$
$$\mathcal{J}_R(\pi_x)|_{V_2(x)} = \begin{pmatrix} 3c_1 & -3\alpha(x) c_1 \\ -3\alpha(x) c_2 & 3c_2 \end{pmatrix}.$$

Consequently, $\det\{\mathcal{J}_R(\pi_x)|_{V_1(x)+V_2(x)}\} = (9c_1 c_2(1 - \alpha(x)^2))^2$. Since the eigenvalues of $\mathcal{J}_R(\cdot)$ are constant, the determinant of $\mathcal{J}_R(\cdot)$ is constant and consequently $\alpha(x)$ does not depend on x. This establishes Assertion (1). The proof of Assertion (2) is a bit technical and is omitted in the interests of brevity. It relies on the fact that J preserves the eigenspaces of $\mathcal{J}_R(\pi_x)$; details are available from the authors.

The possible values of $\operatorname{Rank}\{\mathcal{J}_R(\pi_x)\}$ are 2 and 4, which correspond to $\alpha = \pm 1$ or $\alpha \neq \pm 1$, respectively. If $\alpha = \pm 1$, then $J = \pm J_1 J_2$ since Jx and $J_1 J_2 x$ are unit vectors. Assertion (3) now follows.

On the other hand, if $\alpha = 0$ then, by polarizing the identity $\langle J_1 J_2 x, Jx\rangle = 0$, we see that $\langle J_1 J_2 x, Jy\rangle + \langle J_1 J_2 y, Jx\rangle = 0$ and consequently $J_1 J_2 J + J J_1 J_2 = 0$. Furthermore, $\{J_1 x, J_1 Jx, J_2 x, J_2 Jx\}$ is an orthonormal set for any $x \in S(V, \langle\cdot,\cdot\rangle)$.

Suppose that $c_1 \neq c_2$ and that $\mathcal{J}_R$ has three different eigenvalues $(0, 3c_1, 3c_2)$. As J preserves the eigenspaces of $\mathcal{J}_R(\pi_x)$, J preserves the spaces $\operatorname{Span}\{J_1 x, J_1 Jx\}$ and $\operatorname{Span}\{J_2 x, J_2 Jx\}$. Consequently, $JJ_1 = \pm J_1 J$ and $JJ_2 = \pm J_2 J$. Since one has that $JJ_1J_2 + J_1J_2J = 0$, the only possibilities are $JJ_1 = J_1 J$ and $JJ_2 = -J_2 J$ or $JJ_1 = -J_1 J$ and $JJ_2 = J_2 J$.

Suppose that $c_1 = c_2$. In such a case there are only two distinct eigenvalues for $\mathcal{J}_R(\pi_x)$ and $\operatorname{Range}\{\mathcal{J}_R(\pi_x)\} = \operatorname{Span}\{J_1x, J_2x, J_1Jx, J_2Jx\}$ is a 4-dimensional eigenspace. Since J preserves this eigenspace and $J_1 Jx \perp J_1 x, J_2 x$ we have

$$JJ_1x = \langle JJ_1x, J_1Jx\rangle J_1Jx + \langle JJ_1x, J_2Jx\rangle J_2Jx\,.$$

Set $\Theta_1 = JJ_1$ and $\Theta_2 = JJ_2$, then $\langle \Theta_1^2 x, x\rangle^2 + \langle \Theta_2\Theta_1 x, x\rangle^2 = 1$. Also note that

$$\begin{aligned}
&\Theta_1\Theta_1^* = JJ_1J_1J = \mathrm{id}, \Theta_2\Theta_2^* = JJ_2J_2J = \mathrm{id},\\
&\Theta_1\Theta_2^* + \Theta_2\Theta_1^* = JJ_1J_2J + JJ_2J_1J = 0,\\
&\Theta_1\Theta_2 = JJ_1JJ_2 = JJ_1JJ_1J_2J_1 = -JJ_1J_1J_2JJ_1 = \Theta_2\Theta_1\,.
\end{aligned}$$

Consequently, Θ_1 and Θ_2 are commuting orthogonal maps. Let

$$V = V_+ \oplus V_- \oplus V_1 \oplus \cdots \oplus V_k$$

be a skew-diagonalization of Θ_1, such that $\Theta_1 = \pm\,\mathrm{id}$ on $V_\pm$ and Θ_1 is a rotation through an angle θ_i, $0 < \theta_i < \pi$, on V_i. After some technical fuss, one may show that there is a reparametrization $\{\tilde J_1, \tilde J_2\}$ such that the previous decomposition is reduced to $V = V_+ \oplus V_-$ and hence $J\tilde J_1 = \tilde J_1 J$. Also, since $JJ_1J_2 = -J_1J_2J$ as noted above, $J\tilde J_2 = -\tilde J_2 J$. □

We complete the proof of Theorem 1.4 (3) by studying models with $c_0 \neq 0$. First we establish the following consequence of the compatibility between J and R for a Clifford family of rank at most 3.

Lemma 5.3. *Let $R = c_1R_{J_1} + c_2R_{J_2} + c_3R_{J_3}$ be an algebraic curvature tensor given by a Clifford family of rank 3. Suppose R is compatible with an Hermitian almost complex structure J. If $Jx = (a_1J_1 + a_2J_2 + a_3J_3)x$ for all $x \in V$, then $(c_i - c_j)a_ia_j = 0$ for $i \neq j$.*

Proof. Compute

$$\begin{aligned}
JR(x,Jx)x &= c_0x - 3c_1a_1JJ_1x - 3c_2a_2JJ_2x - 3c_3a_3JJ_3x,\\
R(x,Jx)Jx &= c_0x - 3c_1a_1J_1Jx - 3c_2a_2J_2Jx - 3c_3a_3J_3Jx.
\end{aligned}$$

Now, since R and J are compatible, $JR(x,Jx)x = R(x,Jx)Jx$ so

$$(c_1 - c_2)a_1a_2J_1J_2x + (c_1 - c_3)a_1a_3J_1J_3x + (c_2 - c_3)a_2a_3J_2J_3x = 0.$$

Since $\{J_1J_2x, J_1J_3x, J_2J_3x\}$ is an orthogonal set, the desired equalities follow. □

Lemma 5.4. *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, J, R = c_0R_0 + c_1R_{J_1} + c_2R_{J_2})$ be complex Osserman. If $\dim\{V\} \geq 12$, then there exists a reparametrization $\{\tilde J_1, \tilde J_2\}$ of $\{J_1, J_2\}$ such that $R = c_0R_0 + \tilde c_1R_{\tilde J_1} + \tilde c_2R_{\tilde J_2}$ and either $J = \tilde J_1$ or $J = \tilde J_1\tilde J_2$.*

Proof. Let $\tilde{\mathcal{J}}_R(\pi_x) = \mathcal{J}_R(\pi_x) - 2c_0 \operatorname{id}$ be the reduced complex Jacobi operator. As $\mathcal{J}_R(\pi_x)$ is complex Osserman, $\tilde{\mathcal{J}}_R(\pi_x)$ has rank at most 4. Let $\alpha(x) := \langle J_1 J_2 x, Jx\rangle$, $\alpha_1(x) := \langle J_1 x, Jx\rangle$ and $\alpha_2(x) := \langle J_2 x, Jx\rangle$. Then

$$\begin{aligned}
\tilde{\mathcal{J}}_R(\pi_x)x &= -c_0 x - 3c_1\alpha_1(x)J_1Jx - 3c_2\alpha_2(x)J_2Jx,\\
\tilde{\mathcal{J}}_R(\pi_x)Jx &= -c_0 Jx + 3c_1\alpha_1(x)J_1x + 3c_2\alpha_2(x)J_2x,\\
\tilde{\mathcal{J}}_R(\pi_x)J_1x &= -c_0\alpha_1(x)Jx + 3c_1J_1x + 3c_2\alpha(x)J_2Jx,\\
\tilde{\mathcal{J}}_R(\pi_x)J_2x &= -c_0\alpha_2(x)Jx - 3c_1\alpha(x)J_1Jx + 3c_2J_2x,\\
\tilde{\mathcal{J}}_R(\pi_x)J_1Jx &= c_0\alpha_1(x)x + 3c_1J_1Jx - 3c_2\alpha(x)J_2x,\\
\tilde{\mathcal{J}}_R(\pi_x)J_2Jx &= c_0\alpha_2(x)x + 3c_1\alpha(x)J_1x + 3c_2J_2Jx.
\end{aligned}$$

Consider the subspace $W(x) := \operatorname{Span}\{x, J_1x, J_2x, Jx, J_1Jx, J_2Jx\}$ and notice that $\operatorname{Range}\{\tilde{\mathcal{J}}_R(\pi_x)\} \subset W(x)$. We wish to show that dim $W(x) < 6$. On the contrary, suppose $\dim\{W(x)\} = 6$. From the previous calculations we get the matrix associated to $\tilde{\mathcal{J}}_R(\pi_x)_{|W(x)}$ and compute:

$$\det(\tilde{\mathcal{J}}_R(\pi_x)_{|W(x)}) = 3^4c_0^2c_1^2c_2^2(-1+\alpha(x)^2+\alpha_1(x)^2+\alpha_2(x)^2)^2\,.$$

Since $\dim\{V\} \geq 12$ we apply Theorem 2.4 to get $\det(\tilde{\mathcal{J}}_R(\pi_x)_{|W(x)}) = 0$ and hence $\alpha^2+\alpha_1^2+\alpha_2^2 = 1$. Since $\alpha(x)$, $\alpha_1(x)$ and $\alpha_2(x)$ are the Fourier coefficients of Jx with respect to $\{J_1J_2x, J_1x, J_2x\}$, we get $Jx = \alpha(x)J_1J_2x + \alpha_1(x)J_1x + \alpha_2(x)J_2x$ which contradicts the assumption that $\dim\{W(x)\} = 6$.

Hence $\dim\{W(x)\} \leq 5$ and $\operatorname{Span}\{x, J_1x, J_2x\} \cap \operatorname{Span}\{Jx, J_1Jx, J_2Jx\}$ is non-trivial. Moreover, there exists a unit vector $(\rho_0, \rho_1, \rho_2) \in \mathbb{R}^3$ such that

$$(\rho_0 + \rho_1J_1 + \rho_2J_2)Jx \in \operatorname{Span}\{x, J_1x, J_2x\}\,.$$

Let $\{J_1, J_2, J_1J_2\}$ give V a quaternion structure $\mathbb{H}$. As $Jx \in \mathbb{H}x$,

$$Jx = a_1(x)J_1x + a_2(x)J_2x + a_3(x)J_3x.$$

The following argument shows that $a_i(\cdot)$ are constant functions in $S(V, \langle\cdot,\cdot\rangle)$. Let $x, y \in S(V, \langle\cdot,\cdot\rangle)$. Since $\dim\{\mathbb{H}x + \mathbb{H}y\} \leq 8$, there exists $z \in S(V, \langle\cdot,\cdot\rangle)$ such that $z \perp \mathbb{H}x, \mathbb{H}y$. Then $\mathbb{H}x \perp \mathbb{H}z$ and for $w := \frac{1}{\sqrt{2}}(x+z)$ we have:

$$J(w) = \frac{1}{\sqrt{2}}\sum_i a_i(w)J_i(x+z) = \frac{1}{\sqrt{2}}\sum_i (a_i(x)J_i(x) + a_i(z)J_i(z)).$$

which implies that $a_i(x) = a_i(w) = a_i(z)$. Similarly, $a_i(y) = a_i(z)$.

Therefore $J = a_1J_1 + a_2J_2 + a_3J_1J_2$. By Lemma 5.3 with $c_3 = 0$, we have:

$$(c_1 - c_2)a_1a_2 = c_2a_2a_3 = c_1a_1a_3 = 0\,.$$

Then either $J = \pm J_3$ or $J = a_1J_1 + a_2J_2$ and we may reparametrize $\{J_1, J_2\}$ by $\{\tilde{J}_1, \tilde{J}_2\}$ so that $J = \tilde{J}_1$. □

5.4. **Clifford families of rank 3.** Let $\{J_1, J_2, J_3\}$ be a Clifford family on V. The dual structure, which is always a quaternion structure, is given by

$$\{J_1^* := J_2J_3,\ J_2^* := J_3J_1,\ J_3^* := J_1J_2\}\,.$$

We use this structure to establish Assertion (4) of Theorem 1.4; in the interest of brevity we shall simply outline the proof rather than giving full details. Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, J, R = c_0R_0 + c_1R_{J_1} + c_2R_{J_2} + c_3R_{J_3})$ be complex Osserman, where c_0 may be 0. Then one has the following:

(1) If $J = a_1J_1 + a_2J_2 + a_3J_3$, then there exists a reparametrization $\{\tilde{J}_1, \tilde{J}_2, \tilde{J}_3\}$ so that $R = c_0R_0 + \tilde{c}_1R_{\tilde{J}_1} + \tilde{c}_2R_{\tilde{J}_2} + \tilde{c}_3R_{\tilde{J}_3}$ and $J = \tilde{J}_1$.

(2) Suppose $J_1J_2 \neq J_3$.
  (a) Then $J \neq J_1$. Furthermore, if $c_0 \neq 0$, then $J \neq J_2J_3$.

(b) Suppose that $Jx \in \text{Span}\{J_1x, J_2x, J_3x, J_1^*x, J_2^*x, J_3^*x\}$ for some element $x \in V$ with $x = (x_+ + x_-)/\sqrt{2}$ where $J_1J_2x_\pm = J_3x_\pm$. Then $c_0 = 0$, and there is a reparametrization $\{\tilde{J}_1, \tilde{J}_2, \tilde{J}_3\}$ such that one has $R = \tilde{c}_1R_{\tilde{J}_1} + \tilde{c}_2R_{\tilde{J}_2} + \tilde{c}_3R_{\tilde{J}_3}$ and $J = \tilde{J}_2\tilde{J}_3$.

The classification in Theorem 1.4 (4) follows from these observations and from a careful analysis of the rank of the matrix associated to $\mathcal{J}_R(\pi_x)$. The technique is similar to that developed in Lemma 5.4.

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