A CLASS OF NONCOMMUTATIVE PROJECTIVE SURFACES

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ABSTRACT. Let $A = \bigoplus_{i \geq 0} A_i$ be a connected graded, noetherian k-algebra that is generated in degree one over an algebraically closed field k. Suppose that the graded quotient ring $Q(A)$ has the form $Q(A) = k(X)[t, t^{-1}, \sigma]$, where σ is an automorphism of the integral projective surface X. Then we prove that A can be written as a naïve blowup algebra of a projective surface $\mathbb{X}$ birational to X. This enables one to obtain a deep understanding of the structure of these algebras; for example, generically they are not strongly noetherian and their point modules are not parametrized by a projective scheme. This is despite the fact that the simple objects in qgr-A will always be in (1-1) correspondence with the closed points of the scheme $\mathbb{X}$.

CONTENTS

1. INTRODUCTION

One of the outstanding problems of noncommutative projective geometry is to classify all noncommutative projective surfaces. There are many ways of phrasing this problem; from a ring-theoretic standpoint it means classifying (hopefully in some geometric way) those connected graded rings that should be thought of as corresponding to surfaces. In this paper, we will achieve just such a classification for a large class of noncommutative rings: noetherian connected graded rings that are birationally 2-geometric, as defined below.

1.1. The results.

Let us first give the statement of the main theorem and then explain its significance and applications. Throughout, k will be an algebraically closed base field. A k-algebra A is called *connected graded* (cg) if $A = \bigoplus_{n \geq 0} A_n$, where $A_0 = k$ and $\dim_k A_n < \infty$ for each n. For a noetherian cg domain A, one can invert the nonzero homogeneous elements to obtain the *graded quotient ring* $Q(A) \cong D[t, t^{-1}; \sigma]$, for some automorphism σ of the division ring $D = Q(A)_0$ (see also Section 3). We say that A *is birationally geometric* if $D \cong k(V)$, the field of rational functions on an integral projective scheme V, and σ is induced from an automorphism of V. If in addition V is a surface, we say that A *is birationally* 2-*geometric*.

The aim of the paper is to classify birationally 2-geometric algebras as naïve blowup algebras, in the sense of [KRS]. These are described as follows. Let X be an integral projective scheme, with zero dimensional subscheme $Z = Z_{\mathcal{I}} = \operatorname{Spec} \mathcal{O}_X / \mathcal{I}$, automorphism σ and σ-ample invertible sheaf $\mathcal{L}$, as defined in Section 8. In a manner reminiscent of the Rees ring construction of the blowup of a commutative scheme, we form the *bimodule algebra*

$$\mathcal{R} = \mathcal{R}(X, Z, \mathcal{L}, \sigma) = \mathcal{O}_X \oplus \mathcal{J}_1 \oplus \mathcal{J}_2 \oplus \cdots, \quad \text{where } \mathcal{J}_n = \mathcal{L}_n \otimes_{\mathcal{O}_X} \mathcal{I}_n \text{ for } \mathcal{I}_n = \mathcal{I} \cdot \sigma^* \mathcal{I} \cdots (\sigma^*)^{n-1} \mathcal{I}.$$

As is described in detail in [KRS] and explained briefly here in Section 8, this bimodule algebra has a natural multiplication and the *naïve blowup algebra of X at Z* is then defined to be the algebra of sections

$$R = R(X, Z, \mathcal{L}, \sigma) = \mathrm{H}^0(X, \mathcal{R}) = k \oplus \mathrm{H}^0(X, \mathcal{J}_1) \oplus \mathrm{H}^0(X, \mathcal{J}_2) \oplus \cdots$$

Although implicitly Z is nonempty, this definition makes perfect sense when $Z = \emptyset$, in which case $B(X, \mathcal{L}, \sigma) = R(X, \emptyset, \mathcal{L}, \sigma)$ is just the *twisted homogeneous coordinate ring* that is so important in other aspects of noncommutative projective geometry; see, for example, [AV] or [SV]. For any zero dimensional subscheme $Z \subset X$ there is a natural embedding $R = R(X, Z, \mathcal{L}, \sigma) \hookrightarrow B = B(X, \mathcal{L}, \sigma)$ and the multiplication in R is also that induced from B.

We can now state the main result of this paper.

Main Theorem 1.1. [Theorem 10.1] *Let A be a cg noetherian domain that is generated in degree one and birationally 2-geometric. Then, up to a finite dimensional vector space, either $A \cong B(\mathbb{X}, \mathcal{L}, \sigma)$ or $A \cong R(\mathbb{X}, Z, \mathcal{L}, \sigma)$, for some projective surface $\mathbb{X}$ with automorphism $\sigma \in \operatorname{Aut}(\mathbb{X})$, σ-ample invertible sheaf $\mathcal{L}$ and zero dimensional subscheme Z.*

Noetherian naïve blowup algebras only appear when the algebras are far from commutative; explicitly for each closed point $z \in Z$ the orbit $\langle \sigma \rangle \cdot z$ will be *critically dense* in the sense that the orbit is infinite and all of its infinite subsets are Zariski dense in $\mathbb{X}$.

Combined with the results from [KRS] and [RS], the alternative $A \cong R(\mathbb{X}, Z, \mathcal{L}, \sigma)$ in Theorem 1.1 has remarkable consequences for the structure of the algebra A and its representations. Before giving these results we need some further definitions. The category of noetherian $\mathbb{Z}$-graded right A-modules, with homomorphisms being graded homomorphisms of degree zero, will be denoted by gr-A. Let tors-A denote the full subcategory of gr-A consisting of modules of finite length, with quotient category qgr-A = gr-A/tors-A. (Thinking of qgr-A as the category of coherent sheaves on the imaginary scheme $\operatorname{Proj} A$ can often provide useful intuition.) *A point module for A* is a cyclic graded A-module $M = \bigoplus_{n \geq 0} M_n$ such that $\dim_k M_n = 1$ for all n. *A point module in* qgr-A is defined to be the image in qgr-A of a cyclic graded A-module $M = \bigoplus_{n \geq 0} M_n$, generated in degree zero, such that $\dim_k M_n = 1$ for all $n \gg 0$. For point modules defined over arbitrary base rings and the formalities on parametrization, see Section 2.

Corollary 1.2. *Let A be as in Theorem 1.1 and keep the notation from that result.*

(I) *If $A \cong B(\mathbb{X}, \mathcal{L}, \sigma)$ then:*

    (a) *A is* strongly noetherian; *i.e., $A \otimes_k C$ is noetherian for all commutative noetherian k-algebras C;*

    (b) qgr-$A \simeq \operatorname{coh} \mathbb{X}$, *the category of coherent sheaves on $\mathbb{X}$;*

    (c) *the set of point modules for A (both in* gr-A *and in* qgr-A*) is parametrized by the scheme $\mathbb{X}$;*

    (d) *A has a balanced dualizing complex in the sense of* [Ye].

(II) *Otherwise $A \cong R(\mathbb{X}, Z, \mathcal{L}, \sigma)$, for some nonempty finite subscheme Z and:*

    (a) *A is not strongly noetherian;*

    (b) *neither the point modules in* gr-A *nor those in* qgr-A *are parametrized by a scheme of locally finite type;*

    (c) *the simple objects in* qgr-A *are the point modules and are naturally in (1-1) correspondence with the closed points of $\mathbb{X}$;*

    (d) qgr-A *has finite cohomological dimension;*

    (e) *A does not have a balanced dualizing complex;*

    (f) *$H^1(A) = \operatorname{Ext}^1_{\text{qgr-}A}(A, A)$ is infinite dimensional;*

    (g) *A does not satisfy generic flatness in the sense of* [ASZ].

Proof of Corollary 1.2: This is really just a case of quoting the literature. To be precise, part (Ia) follows from [ASZ, Proposition 4.13], while part (Ib) is [AV, Theorem 1.3] and (Id) is proved in [Ye, Theorem 7.3]. Part (Ic) follows from [RZ, Theorem 1.1, 1.2] and [KRS, Proposition 10.2]. All the results stated in part (II) are contained in [RS, Theorem 1.1 and Remark 1.2]. $\square$

We should emphasize the striking fact that the properties described by part (II) of the corollary are not exceptional: as soon as $(\mathbb{X}, \sigma)$ has at least one critically dense orbit then the theorem and its corollary

imply that, generically, each noetherian cg subalgebra of $k(\mathbb{X})[t, t^{-1}; \sigma]$ has these properties. Despite the curious features displayed in Corollary 1.2(II), the category qgr-A described there is surprisingly similar to the category of coherent sheaves on a projective scheme. This is illustrated by the following result, which also shows that the properties of the naïve blowup qgr-A are actually closer to those of the original scheme $\mathbb{X}$ than they are to those of the classical blowup $\widetilde{\mathbb{X}}$ of $\mathbb{X}$ at Z.

Proposition 1.3. [RS, Theorem 4.10] *Keep the hypotheses of Theorem 1.1. Then the subcategory of (Goldie) torsion objects in* qgr-$R(\mathbb{X}, Z, \mathcal{L}, \sigma)$ *is equivalent to the category of torsion coherent* $\mathcal{O}_{\mathbb{X}}$-modules.

1.2. **An example.** Here is an easy example that illustrates the sorts of algebras that appear in Theorem 1.1 and its corollary (see, also, Example 4.3).

A typical example of a twisted coordinate ring is the "quantum polynomial ring" $S = S_{pq}$ with generators x, y, z and relations $xy - pyx$, $xz - qzx$ and $yz - qp^{-1}zy$ for p and $q \in k^*$. This can also be written as $S \cong B(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(1), \sigma)$, where $\sigma \in \operatorname{Aut}(\mathbb{P}^2)$ is defined by $(\lambda_0 : \lambda_1 : \lambda_2) \mapsto (\lambda_0 : p\lambda_1 : q\lambda_2)$. It follows that $Q(S)_0 = k(yx^{-1}, zx^{-1}) = k(\mathbb{P}^2)$ and so, in this special case, Theorem 1.1 classifies the noetherian cg algebras A with $Q(A) = Q(S)$. To see an explicit example of case (II) of the corollary, assume that the scalars p, q are algebraically independent over the base field, take Z to be the single reduced point $(1 : 1 : 1)$ and consider the ring $R(\mathbb{P}^2, Z, \mathcal{O}(1), \sigma)$; it is easy to show this ring is equal to $k\langle x - y, x - z\rangle \subseteq S$.

It is also easy to find cg subalgebras of $Q(S_{pq})$ that are not noetherian. For example, [KRS, Proposition 4.8] implies that, for this choice of σ, the naïve blowup algebra $R(\mathbb{P}^2, Z', \mathcal{O}(1), \sigma)$ is not noetherian when $Z' = \{(1 : 0 : 0)\}$. For a detailed and more algebraic examination of these rings, see [Ro1].

1.3. **History.** We briefly explain the history behind these results and their wider relevance.

As we remarked earlier, it is useful to think of qgr-A for a noetherian cg ring A as coherent sheaves on the nonexistent scheme $\operatorname{Proj} A$. Extending this analogy further one can think of a noncommutative projective integral curve, respectively surface, as qgr-A for a noetherian cg domain A with Gelfand-Kirillov dimension two, respectively three. For the purposes of this discussion we will also restrict ourselves to algebras generated in degree one (see Section 1.5 for comments about the general case). Under this analogy, noncommutative integral projective curves have been classified in [AS] (they are nothing more than qgr-$B(X, \mathcal{L}, \sigma) \simeq \operatorname{coh} X$ for integral projective curves X). Similarly, noncommutative projective planes and their algebraic analogues, the Artin-Schelter regular rings with the Hilbert series of a polynomial ring in three variables, have been classified in [ATV1, BP]. Such an Artin-Schelter regular ring either equals $B(\mathbb{P}^2, \mathcal{O}(1), \sigma)$ for some σ or has a factor isomorphic to $B(E, \mathcal{L}, \sigma)$ for a projective curve E in which case (E, σ) determines A. These results are surveyed in [SV], and the reader is referred to that paper for more details.

Motivated by these results, in [Ar] Artin posed the problem of classifying all noncommutative integral projective surfaces and this paper was motivated in part by that question. In one sense, however, Theorem 1.1 is orthogonal to his question, since he was interested in finding the noncommutative algebras whose properties are close to those of commutative integral surfaces. For example, he included the existence of a balanced

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dualizing complex as one of his hypotheses! Of course, if A satisfies the hypotheses of Theorem 1.1 and also has a balanced dualizing complex, then Corollary 1.2(e) implies that A is a twisted homogeneous coordinate ring, in which case the category qgr-A of coherent sheaves on the corresponding noncommutative projective scheme is commutative: qgr-$A \simeq \operatorname{coh} \mathbb{X}$.

Further motivation for this paper comes from recent work on strongly noetherian algebras. These algebras are studied in detail in [ASZ, AZ2] and have many pleasant properties, some of which are described in Section 1.4 below. One is therefore interested in understanding when cg algebras satisfy this property and for birationally geometric algebras this is answered by the following result.

Theorem 1.4. (1) [RZ, Theorem 1.2] *Suppose that A is a strongly noetherian, cg domain that is generated in degree one and birationally geometric. Then, up to a finite dimensional vector space, $A \cong B(\mathbb{X}, \mathcal{L}, \sigma)$, for a projective variety $\mathbb{X}$, with automorphism $\sigma \in \operatorname{Aut}(\mathbb{X})$ and σ-ample invertible sheaf $\mathcal{L}$.*

(2) [ASZ, Proposition 4.13] *Conversely, if $\mathcal{L}$ is a σ-ample invertible sheaf on a projective scheme X, then $B(X, \mathcal{L}, \sigma)$ is strongly noetherian.*

Since noetherian cg domains that are *not* strongly noetherian exist in profusion, even among domains of Gelfand-Kirillov dimension three, it is important to understand their structure and Theorem 1.1 gives one step in that direction.

1.4. **The proofs.** We next want to give the strategy behind the proof of the main theorem, but we first outline the proof of Theorem 1.4(1) as the contrast is illuminating. Strongly noetherian rings are known to have a number of very pleasant properties and in particular one has the following result from [AZ2, Theorem E4.3]: *If A is a strongly noetherian k-algebra then the point modules for A are parametrized by a projective scheme $\mathbb{X}$.* The key step in the proof of Theorem 1.4 is that, under the hypotheses of the theorem and up to a finite dimensional vector space as always, *there exists a surjection $\chi : A \twoheadrightarrow B(\mathbb{X}, \mathcal{L}, \sigma)$ for this scheme $\mathbb{X}$ together with a sheaf $\mathcal{L}$ and automorphism σ.* By hypothesis $Q(A)_0 = k(V)$ for some scheme V and the next step is to regard $Q(A)_{\geq 0} = \sum_{n \geq 0} k(V)t^n$ as a "generic point module" with coefficients from $k(V)$ (see Section 3 for a discussion of this type of module). The existence of this generic point module can then be used to show that χ is injective.

Now suppose that A is merely noetherian. As we have seen, the point modules for A are no longer parametrized by a projective scheme and so there is no easy candidate for the scheme $\mathbb{X}$; indeed the construction of $\mathbb{X}$ forms a major part of the proof of the theorem. The idea is as follows. *A truncated point module of length $n + 1$* is a cyclic graded A-module $M = \bigoplus_{i=0}^{n} M_i$ where $\dim_k M_i = 1$ for each i. By [ATV1, Proposition 3.9] these modules are parametrized by a scheme W_n and there are natural maps $\Phi_{n-1}, \Psi_{n-1} : W_n \to W_{n-1}$ given by further truncation: $\Phi_{n-1}(M) = M/M_n$, respectively $\Psi_{n-1}(M) = M_{\geq 1}[1]$, where [1] denotes the shift in gradation. When A is strongly noetherian, it follows from [AZ2, Corollary E4.5] that Φ_n and Ψ_n become isomorphisms for $n \gg 0$. The scheme $\mathbb{X}$ of Theorem 1.4 is then just $\mathbb{X} = W_n$ for such an n, with automorphism $\sigma = \Psi_n \Phi_n^{-1}$. In contrast, when A is only noetherian

these maps will never be isomorphisms. Fortunately there is a reduced and irreducible component Y_n of W_n such that (a) Φ_n and Ψ_n restrict to maps between the Y_n and (b) $k(Y_n) \cong k(V)$ and so there is an analogous generic truncated point module $\bigoplus_{i=0}^n Q(A)_i$ (see Corollary 3.5).

The main step the proof of Theorem 1.1 is to show that, for $n \gg 0$, one can appropriately blow down $\pi : Y_n \to \mathbb{X}$ to give a scheme $\mathbb{X}$ for which the induced action of the birational map $\sigma = \Psi_n \Phi_n^{-1}$ is actually an automorphism. This is the scheme appearing in the theorem. The scheme $\mathbb{X}$ appears in a second way, which also indicates how it is constructed: given a truncated point module M' corresponding to some $y \in Y_n$ and $n \gg 0$, write M' as the image of a (non-unique) point module M and take the image $\overline{M}$ of M in qgr-A. As Corollary 7.11 shows, it is *this* object that is uniquely determined by M' and the map π corresponds to mapping $M' \mapsto \overline{M}$. The proof of these assertions takes up Sections 2—7. Once $\mathbb{X}$ has been constructed, one can use the ideas from [AS, KRS, RZ] to prove the theorem (see Sections 8—10).

1.5. **Questions.** We end the introduction with some comments and questions. First, Theorem 1.4 is actually stronger than stated since it works for any cg noetherian domain A that is *birationally commutative* in the sense that $Q(A)_0$ is a finitely generated field; in other words one does not need to assume that the automorphism σ is induced from that of a scheme. We conjecture that, for surfaces, Theorem 1.1 also holds under this more general hypothesis, but there is one case we are as yet unable to handle. Using work of Diller and Favre [DF] which, essentially, classifies birational maps of surfaces one has the following result.

Theorem 1.5. [Ro3] *Let $Q = K[t, t^{-1}; \sigma]$, where K is a finitely generated field extension of k of transcendence degree two. Then every cg Ore domain A with graded quotient ring Q has the same Gelfand-Kirillov dimension $d \in \{3, 4, 5, \infty\}$. If $d < \infty$, then $d = 3$ or 5 if and only if σ is induced from an automorphism of some projective surface X with $K \cong k(X)$. If $d = \infty$ then A is not noetherian.*

This just leaves the case when $d = 4$. This definitely can occur; for example take the automorphism $u \mapsto uv, v \mapsto v$ of $k(\mathbb{P}^2) = k(u, v)$. We believe, however, that when $d = 4$ the algebra A can never be noetherian, which would solve the conjecture. See Example 11.3 for a typical example in this dimension. Note, also, that the case $d = 5$ is covered by Theorem 1.1 and so noncommutative surfaces are a little more general than algebras of Gelfand-Kirillov dimension 3.

We end with two further questions. First, as is true of Theorem 1.4, is there a version of Theorem 1.1 describing any birationally geometric A? Second, in the present paper we always assume that our algebras are generated in degree one. In the commutative case this is not so much of a restriction since one can always reduce to that case by taking an appropriate Veronese ring. In the noncommutative case, however, it is a more severe restriction, since it excludes the idealizer examples that appear in [Ro2, AS, SZ]. It would be interesting to know whether, as happened for curves in [AS], one can classify all noetherian cg algebras that are birationally 2-geometric in terms of variants of naïve blowup algebras and idealizers.

2. POINT MODULES

The method we use to introduce geometry into the study of noncommutative algebras is the theory of (truncated) point modules and the schemes that parametrize them. In this section we recall the relevant definitions and prove some basic facts about these objects that will be needed in the sequel. These concepts were first introduced by Artin, Tate and Van den Bergh in [ATV1, Section 3], and more details can be found both there and in [RZ, Section 4]. For the most part our geometric notation is standard and, unless we say otherwise, we will use the definitions given in [Ha].

As in the introduction, fix a cg algebra $A = \bigoplus_{n \geq 0} A_n$ with categories gr-A and qgr-A. The category of all graded right A-modules is written Gr-A, but we will not need the corresponding quotient category Qgr-A. Assume for the rest of the section that A is generated by A_1 as a k-algebra. Given a commutative k-algebra R we write $A_R = A \otimes_k R$, regarded as a graded R-algebra by putting R into degree zero. Following [ATV1, Definition 3.8], a *truncated R-point module of length $n + 1$*, for $n \geq 0$, is a graded cyclic right A_R-module $M = \bigoplus_{i=0}^n M_i$ where $M_0 = R$ and each M_i is a locally free R-module of rank 1. An *R-point module* is given by the same definition, except that now $n = \infty$. Under the identification of R with R^{op}, we can and usually will regard (truncated) right R-point modules as (R, A)-bimodules since this will frequently make their module structures more transparent. Note that if M is a (truncated) R-point module, then the identification $M_0 = R$ defines a unique isomorphism $M \cong A_R/I$ for some right ideal I, and so this rigidifies the given (truncated) R-point module. A (truncated) k-point module is just called a (truncated) point module. Write $\mathcal{P}(R)$, respectively $\mathcal{P}_n(R)$, for the set of isomorphism classes of right R-point modules, respectively truncated right R-point modules of length $n + 1$. We write $\mathcal{P} = \mathcal{P}(k)$, $\mathcal{P}_n = \mathcal{P}_n(k)$ and let $\mathcal{P}^\ell(R), \ldots, \mathcal{P}_n^\ell$ denote the left hand analogues of each of these sets.

Recall that Yoneda's Lemma embeds the category **Schemes** of noetherian k-schemes inside the category of functors from the category **Alg** of commutative noetherian k-algebras to the category **Sets** of sets, where a scheme X corresponds to the functor $h_X : R \mapsto \mathrm{Hom}(\mathrm{Spec}\, R, X)$ (see [EH, Section VI]). A functor $h : \mathbf{Alg} \to \mathbf{Sets}$ is said to be *represented* by the scheme X if h and h_X are isomorphic functors.

We will apply this theory to the functors arising from truncated point modules over our algebra A. By [ATV1, Proposition 3.9] the functor $F_n : \mathbf{Alg} \to \mathbf{Sets}$ defined by $R \mapsto \mathcal{P}_n(R)$ is represented by a (commutative) projective scheme W_n called the nth *truncated point scheme* for A. Since it will be used many times, we recall the functorial property of F_n. Thus, let $M \in \mathcal{P}_n(R)$ be a truncated R-point module corresponding to a morphism of schemes $\theta : \mathrm{Spec}\, R \to W_n$ and let $\alpha : R \to S$ be a homomorphism of commutative noetherian k-algebras. The induced map $\mathrm{Spec}\, S \to \mathrm{Spec}\, R$ will be denoted $\alpha^\star$. Using our convention that R-point modules are written as (R, A)-bimodules, $S \otimes_R M$ is naturally a truncated point module over $A_R \otimes_R S \cong A_S$ and the morphism of schemes $\mathrm{Spec}\, S \to W_n$ corresponding to $S \otimes_R M$ is simply $\theta\alpha^\star$. If we need to remember the map α in this construction, we will write $S_\alpha \otimes M$ for $S \otimes_R M$.

Following [ATV1] the schemes W_n can be constructed explicitly. Write $A \cong T(A_1)/J$, where $J = \bigoplus_{n \geq 0} J_n$ is a homogeneous ideal of the tensor algebra $T(A_1)$. Let $\mathbb{P}$ denote the projective space $\mathbb{P}(A_1^*)$ of lines in A_1^* and for each $n \geq 1$ write $\mathbb{P}^{\times n}$ for the product of n copies of $\mathbb{P}$. For $n \geq 0$, we identify T_n with the global sections of the sheaf $\mathcal{O}(1,1,\ldots,1)$ on $\mathbb{P}^{\times n}$. Under this identification, J_n defines a vector subspace $\widetilde{J}_n$ of $\mathrm{H}^0(\mathbb{P}^{\times n}, \mathcal{O}(1,1,\ldots,1))$ and, by the proof of [ATV1, Proposition 3.9], W_n can be identified with the vanishing locus of $\widetilde{J}_n$ in $\mathbb{P}^{\times n}$. We typically write this embedding as $\iota_n : W_n \hookrightarrow \mathbb{P}^{\times n}$. For each n and $1 \leq a \leq b \leq n$ let $\pi_{a,b} : \mathbb{P}^{\times n} \to \mathbb{P}^{\times(b-a+1)}$ denote the projection onto the coordinates between a and b inclusively. By [ATV1, Proposition 3.5], $\pi_{1,n}(W_{n+1}) \subseteq W_n$ and $\pi_{2,n+1}(W_{n+1}) \subseteq W_n$ and we define projective morphisms $\Phi_n : W_{n+1} \to W_n$ and $\Psi_n : W_{n+1} \to W_n$ by restricting $\pi_{1,n}$, respectively $\pi_{2,n+1}$ to W_{n+1}. We call the collection of schemes and morphisms (W_n, Φ_n, Ψ_n) the *point scheme data* for the algebra A, where the embeddings ι_n are to be understood.

The morphisms Φ_n and Ψ_n are closely related to truncations of modules. Let R be a commutative noetherian k-algebra and, for an R-module V, set $V^* = \mathrm{Hom}_R(V, R)$. If $M = \bigoplus M_i \in \mathrm{Gr}\text{-}A$ and $n \in \mathbb{Z}$, then the *shift* $M[n]$ of M is defined by $M[n]_r = M_{n+r}$. Given a truncated R-point module of length $n+2$, say $M = \bigoplus_{i=0}^{n+1} M_i \in \mathcal{P}_{n+1}(R)$, we define the *right truncation* of M to be

$$(2.1) \qquad \Phi'_n(M) \;=\; M/M_{n+1} \;=\; M_0 \oplus \cdots \oplus M_n \;\in\; \mathcal{P}_n(R),$$

and we define the *left truncation* of M to be the right A_R-module

$$(2.2) \qquad \Psi'_n(M) \;=\; M_1^* \otimes_R M[1]_{\geq 0} \;=\; (M_1^* \otimes_R M_1) \oplus \cdots \oplus (M_1^* \otimes_R M_{n+1}) \;\in\; \mathcal{P}_n(R).$$

The canonical identification $M_1^* \otimes_R M_1 = R$ ensures that $\Psi'_n(M)$ really is a truncated R-point module.

By repeating the previous few paragraphs with left modules in place of right modules, we can define the analogous *left point scheme data* $(W_n^\ell, \Phi_n^\ell, \Psi_n^\ell)$. For example, W_n^ℓ is a projective scheme which represents the functor $F_n^\ell : \mathbf{Alg} \to \mathbf{Sets}$ defined by $R \mapsto \mathcal{P}_n^\ell(R)$. As is shown by the next lemma, $(W_n^\ell, \Phi_n^\ell, \Psi_n^\ell)$ is a mirror of (W_n, Φ_n, Ψ_n) and so we will not need to consider it separately.

For the rest of this section, we analyze the morphisms Φ_n and Ψ_n and begin with some elementary properties that will be used frequently.

Lemma 2.3. (1) *For $n \geq 0$, the maps $\mathcal{P}_{n+1}(R) \to \mathcal{P}_n(R)$ given by the rules $M \mapsto \Phi'_n(M)$, and $M \mapsto \Psi'_n(M)$ induce the respective morphisms $\Phi_n : W_{n+1} \to W_n$ and $\Psi_n : W_{n+1} \to W_n$ between the representing schemes.*

(2) *There are isomorphisms $\gamma_n : W_n \xrightarrow{\sim} W_n^\ell$ for $n \geq 0$ satisfying $\gamma_n^{-1} \Phi_n^\ell \gamma_{n+1} = \Psi_n$ and $\gamma_n^{-1} \Psi_n^\ell \gamma_{n+1} = \Phi_n$.*

(3) *For all $n \geq 0$ we have $\Psi_n \Phi_{n+1} = \Phi_n \Psi_{n+1}$.*

Proof. (1) See the proof of [ATV1, Proposition 3.9].

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(2) Fix $n \geq 0$ and suppose that $M = \bigoplus_{i=0}^n M_i \in \mathcal{P}_n(R)$, for some noetherian commutative k-algebra R. Define $M^\vee = \left(\bigoplus_{i=-n}^0 M_{-i}^* \otimes_R M_n \right)[-n]$ as a variant of the Matlis dual and note that $M^\vee$ is a graded left A_R-module, under the convention that A acts from the left and R from the right. By [KRS, Proposition 10.2(1)][1] $M^\vee \in \mathcal{P}_n^\ell(R)$.

Similarly, if $N \in \mathcal{P}_n^\ell(R)$, then $N^\vee = \left(\bigoplus_{i=-n}^0 N_n \otimes_R N_{-i}^* \right)[-n] \in \mathcal{P}_n(R)$. The two operations are inverses, compatible with change of rings, and so the map $M \mapsto M^\vee$ induces an isomorphism $\gamma_n : W_n \xrightarrow{\sim} W_n^\ell$ between the representing schemes.

Given $M = \bigoplus_{i=0}^{n+1} M_i \in \mathcal{P}_{n+1}(R)$, then a short calculation shows that

$$((M^\vee)_{\leq n})^\vee \cong \bigoplus_{i=0}^n M_1^* \otimes_R M_{i+1} = \Psi_n'(M) \in \mathcal{P}_n(R),$$

and this isomorphism is compatible with change of rings. Using part (1) and its left-sided analog we conclude that $\gamma_n^{-1} \Phi_n^\ell \gamma_{n+1} = \Psi_n$ as maps from W_{n+1} to W_n. The proof that $\gamma_n^{-1} \Psi_n^\ell \gamma_{n+1} = \Phi_n$ is analogous.

(3) The morphisms $\pi_{1,n}\, \pi_{2,n+2}$ and $\pi_{2,n+1}\, \pi_{1,n+1}$ give the same projection $\mathbb{P}^{\times(n+2)} \to \mathbb{P}^{\times n}$. Restriction to the subscheme $W_n \subseteq \mathbb{P}^n$ gives $\Phi_n \Psi_{n+1} = \Psi_n \Phi_{n+1}$. $\qquad \square$

In order to avoid any possible ambiguity we will clarify our conventions about a morphism of schemes $f : Z \to W$. If we say that f^{-1} is defined at a point $w \in W$, then we do mean that it is defined as a morphism from an open neigbourhood of w to an open neighbourhood of the (necessarily singleton) set $f^{-1}(w)$. As such, f^{-1} automatically defines a local isomorphism at w, meaning that it defines an isomorphism from an open neighbourhood of w to an open neighbourhood of $f^{-1}(w)$. Finally, we adopt the usual convention that an automorphism τ acts on functions by $(\tau(\theta))(z) = \theta(\tau(z))$ for $z \in Z$.

Lemma 2.4. *Let $f : Z \to W$ be a proper morphism of finite type schemes and suppose that $w \in W$ is a closed point with set-theoretic fibre $f^{-1}(w) = \{z\}$ a single point. Then:*

(1) *If the induced map $f_z^\sharp : \mathcal{O}_{W,w} \to \mathcal{O}_{Z,z}$ is an isomorphism, then f^{-1} is a local isomorphism at w.*

(2) *If $f_*(\mathcal{O}_Z) = \mathcal{O}_W$, then f^{-1} is a local isomorphism at w.*

Proof. (1) It is routine to show that, since $f_z^\sharp$ is an isomorphism, there are open neighborhoods $w \in V \subseteq W$ and $z \in U \subseteq Z$ such that f restricts to an isomorphism $U \to V$. Since the fibre $f^{-1}(w) = \{z\}$ is a singleton and f is a closed morphism, we can shrink V to a smaller open set $V' \ni w$ whose inverse image satisfies $f^{-1}(V') \subseteq U$. Then f is injective over V' and f^{-1} is defined at w.

(2) Since $f_*(\mathcal{O}_Z) = \mathcal{O}_W$, the map $f_z^\sharp$ can be identified with the map

$$\widehat{f} : \varinjlim_{w \in V} \mathcal{O}_Z(f^{-1}(V)) \to \varinjlim_{z \in U} \mathcal{O}_Z(U),$$

---

[1] The proof in [KRS] omitted the term $\otimes M_n$ which is need to ensure that $M_1^\vee = R$, but otherwise the proofs are identical.

where U and V run through open sets in their respective schemes. As in part (1), for any such U containing z there is an open set V containing w with $f^{-1}(V) \subseteq U$. It follows that $\widehat{f}$ is an isomorphism and so the result follows from part (1). $\qquad\square$

It follows from [AZ2, Theorem E4.3] together with Lemma 2.3 that the point scheme data (W_n, Φ_n, Ψ_n) stabilizes for strongly noetherian rings, in the sense that Φ_n and Ψ_n are isomorphisms for large n. In contrast, as we remarked in the introduction, this need not be the case for arbitrary noetherian rings. Fortunately, as we show in Proposition 2.5, the point scheme data does at least stabilize locally, and this will form the starting point of many of the results in this paper.

In the next few results, we let χ be a symbol which denotes either Φ or Ψ; thus any property claimed for a map χ_n is supposed to hold for both Φ_n and Ψ_n.

Proposition 2.5. *Assume that A is a noetherian, cg k-algebra generated in degree 1 and pick $n_0 \geq 0$. Let $\{w_n \in W_n : n \geq n_0\}$ be a sequence of (not necessarily closed) points such that $\chi_n(w_{n+1}) = w_n$ for all $n \geq n_0$. Then, for all $n \gg n_0$, the fibre $\chi_n^{-1}(w_n) = \{w_{n+1}\}$ is a singleton and χ_n^{-1} is a local isomorphism at w_n.*

Proof. Assume that $\chi = \Phi$; the proof for $\chi = \Psi$ follows similarly using left modules and Lemma 2.3(2).

Suppose first that the points $\{w_n\}$ are closed points. Then this sequence $\{w_n\}$ corresponds to a right k-point module $M = A/J$; thus J is a right ideal with $\dim_k J_n = \dim_k A_n - 1$ for all $n \geq 0$. Choose m_0 such that J is generated in degrees $\leq m_0$. If possible, pick $m \geq m_0$ such that the fibre $\Phi_m^{-1}(w_m)$ contains (at least) two distinct closed points, say w_{m+1} and y, corresponding to the truncated point modules $M_{\leq(m+1)}$ and N. Write $N = A/I$ for some right ideal I and notice that $I_{\leq m} = J_{\leq m}$, but $I_{m+1} \neq J_{m+1}$. However, as A is generated in degree one and J is generated in degrees $\leq m$, this implies that $I_{m+1} \supseteq I_m A_1 = J_m A_1 = J_{m+1}$. Since $\dim_k A_{m+1}/I_{m+1} = 1 = \dim A_{m+1}/J_{m+1}$, this forces $I_{m+1} = J_{m+1}$, a contradiction.

Thus $\Phi_n^{-1}(w_n) = \{w_{n+1}\}$ for all $n \gg 0$. By [ATV1, Proposition 3.6(ii)] and the last line of the proof of that result, the induced map of noetherian local rings $\widehat{\Phi}_n : \mathcal{O}_{W_n, w_n} \to \mathcal{O}_{W_{n+1}, w_{n+1}}$ is surjective for any such n. For sufficiently large n, the map $\widehat{\Phi}_n$ is therefore an isomorphism. By [Ha, Corollary II.4.8(e)] Φ_n is proper and so Lemma 2.4(1) implies that Φ_n^{-1} is a local isomorphism at w_n.

Now suppose that the points $w_n \in W_n$ are not necessarily closed and write $\overline{w}_n$ for the closure of w_n. By properness, $\Phi_n(\overline{w}_{n+1})$ is closed, and thus equal to $\overline{w}_n$, for any $n \geq n_0$. Thus, beginning with any closed point $y_{n_0} \in \overline{w}_{n_0}$, we can choose a sequence of closed points $y_n \in \overline{w}_n$ such that $\Phi_n(y_{n+1}) = y_n$, for each $n \geq n_0$. For $n \gg 0$ the first part of the proof shows that the fibre $\Phi_n^{-1}(y_n) = \{y_{n+1}\}$ is a singleton and that Φ_n^{-1} is a local isomorphism at y_n. But then Φ_n^{-1} is then also local isomorphism at w_n and so certainly $\Phi_n^{-1}(w_n) = w_{n+1}$ is a singleton for all such n. $\qquad\square$

For $m > n \geq 0$ define morphisms $\Phi_{m,n}$ and $\Psi_{m,n}$ by

(2.6) $\qquad \Phi_{m,n} = \Phi_n \Phi_{n+1} \cdots \Phi_{m-1} : W_m \to W_n \qquad$ and $\qquad \Psi_{m,n} = \Psi_n \Psi_{n+1} \cdots \Psi_{m-1} : W_m \to W_n.$

Corollary 2.7. *Fix $n_0 \geq 0$. For each $n \geq n_0$ let G_n be a finite set of (not necessarily closed) points of W_n such that $\chi_n(G_{n+1}) \subseteq G_n$. Then for $n \gg n_0$, the cardinality $|G_n|$ is constant and χ_n gives a bijection between G_{n+1} and G_n. Moreover, if $g \in G_n$ for any such n, then χ_n^{-1} is a local isomorphism at g.*

Proof. As before, it suffices to prove the result for $\chi = \Phi$. For each $n \geq n_0$, let H_n be the subset of points $w \in G_n$ for which there exists $m > n$ and two distinct points $y, z \in G_m$ such that $\Phi_{m,n}(y) = w = \Phi_{m,n}(z)$. Obviously $\Phi_n(H_{n+1}) \subseteq H_n$, so if $H_{n+1} \neq \emptyset$ for some $n > n_0$, then $H_n \neq \emptyset$. Assume that $H_n \neq \emptyset$ for all $n \geq n_0$. Then we claim that there exists a sequence of points $\{w_n \in H_n : n \geq n_0\}$ such that $\Phi_n(w_{n+1}) = w_n$ for all $n \geq n_0$. To see this, for $w \in H_n$, set $\ell(w) = \sup\{r : w = \Phi_{r+n,n}(y) \text{ for some } y \in H_{n+r}\}$. As H_{n_0} is finite we may pick $w_{n_0} \in H_{n_0}$ with $\ell(w_{n_0})$ maximal. If $\ell(w_{n_0}) = t < \infty$, then $H_j = \emptyset$ for all $j > t + n_0$, contradicting our assumption. So $\ell(w_{n_0}) = \infty$. Now we may construct the sequence w_n inductively: having chosen $w_n \in H_n$ with $\ell(w_n) = \infty$, since H_{n+1} is finite there exists $w_{n+1} \in H_{n+1}$ such that $\Phi_n(w_{n+1}) = w_n$ and $\ell(w_{n+1}) = \infty$.

Thus if $H_n \neq \emptyset$ for all $n \geq n_0$, there is a sequence of points $w_n \in W_n$ such that (i) $\Phi_n(w_{n+1}) = w_n$ for all $n \geq n_0$, and (ii) for infinitely many choices of n the set theoretic fibre $\Phi_n^{-1}(w_n)$ is not a singleton. This contradicts Proposition 2.5. So H_n is empty for $n \gg n_0$ and Φ_n induces an injection from G_{n+1} to G_n for all such n. Since each G_n is finite, this implies that Φ_n must induce a bijection for G_{n+1} to G_n for all $n \gg n_0$. The final statement of the corollary now follows from the final assertion of Proposition 2.5. $\qquad\square$

3. Birationally commutative domains

Fix a noetherian cg k-algebra A that is generated in degree one, with point scheme data (W_n, Φ_n, Ψ_n). In this section we examine how the ideal structure of A is reflected in the varietal structure of the schemes W_n. For the algebras A of interest, this will allow us to isolate subschemes $Y_n \subseteq W_n$ that better reflect the geometry of A.

If A is also a domain then [NV, C.I.1.6 and A.I.4.3] shows that the set $\mathcal{C}$ of nonzero homogeneous elements of A forms an Ore set with corresponding localization $Q(A) = A_{\mathcal{C}} \cong D[t, t^{-1}; \tau]$. Here τ is an automorphism of the division ring D, and $D[t, t^{-1}; \tau]$ is the twisted Laurent polynomial ring defined by $td = d^{\tau}t$ for all $d \in D$. We call $Q(A)$ the *graded quotient ring* of A and D the *function ring* of A. If D is a field, we call A *birationally commutative*, and note that in this case D will be a finitely generated field extension of k; see, for example, Corollary 3.5(1). (One reason for the notation is that, if A were the homogeneous coordinate ring of an integral projective scheme Y, then D would just be the function field $k(Y)$.)

We begin with a useful observation about the point scheme data for factor rings.

Lemma 3.1. *Let A be a noetherian cg k-algebra, generated in degree 1, with a homogeneous ideal I. Let (W_n, Φ_n, Ψ_n) be the point scheme data for A, and let $(\widehat{W}_n, \widehat{\Phi}_n, \widehat{\Psi}_n)$ be the point scheme data for A/I. Then, for all $n \geq 0$, $\widehat{W}_n$ is canonically identified with a closed subscheme of W_n with $\widehat{\Phi}_n = \Phi_n|_{\widehat{W}_n}$ and $\widehat{\Psi}_n = \Psi_n|_{\widehat{W}_n}$, respectively.*

Moreover, as sets, $\Phi_n^{-1}(\widehat{W}_n) = \widehat{W}_{n+1}$ and $\Psi_n^{-1}(\widehat{W}_n) = \widehat{W}_{n+1}$ for all $n \gg 0$.

Proof. We are given a presentation $A = T(A_1)/J$ so that $W_n \subseteq \mathbb{P}^{\times n} = \mathbb{P}(A_1^*)^{\times n}$ is the zero set $\mathcal{V}(\widetilde{J}_n)$ of the multi-linearization $\widetilde{J}_n$ of J_n. Now, $A/I \cong T/K$ for some ideal $K \supseteq J$ and so $\widehat{W}_n$ is defined to be a subscheme of $\mathbb{P}\big((A_1/K_1)^*\big)^{\times n}$. However, we can identify $\mathbb{P}\big((A_1/K_1)^*\big)^{\times n}$ with a closed subscheme of $\mathbb{P}(A_1^*)^{\times n}$ via the embedding $(A_1/K_1)^* \hookrightarrow A_1^*$, after which $\widehat{W}_n$ is the zero set of the larger vector space $\widetilde{K}_n \supseteq \widetilde{J}_n$. Thus $\widehat{W}_n$ is indeed closed in W_n. (If one considers the corresponding point modules, the embedding $\widehat{W}_n \hookrightarrow W_n$ is nothing more than the identification of point modules over A/K with the point modules over A annihilated by K.) Since Φ_n and Ψ_n are defined as restrictions of projection maps $\mathbb{P}^{\times(n+1)} \to \mathbb{P}^{\times n}$, the maps $\widehat{\Phi}_n$ and $\widehat{\Psi}_n$ are just further restrictions of those maps.

Assume that I is generated, as a right A-module, in degrees $\leq n_0$; thus $I_n A_1 = I_{n+1}$ for $n \geq n_0$. For $n \geq n_0$, let $z \in \widehat{W}_n$ and $y \in W_{n+1}$ be closed points with $\Phi_n(y) = z$. Thus z corresponds to a truncated point module A/L where L is a right ideal of A containing I and y corresponds to a truncated point module A/M with $M_{\leq n} = L_{\leq n} \supseteq I_{\leq n}$. As $n \geq n_0$ this implies that $M \supseteq M_{\leq n}A \supseteq I$ and hence that $y \in \widehat{W}_{n+1}$. In other words, $\Phi_n^{-1}(\widehat{W}_n) = \widehat{W}_{n+1}$ for all $n \geq n_0$. The assertion $\Psi_n^{-1}(\widehat{W}_n) = \widehat{W}_{n+1}$ follows by working with left modules and using Lemma 2.3(2). $\square$

If A is a strongly noetherian domain which is birationally commutative, then it is isomorphic in large degree to a twisted homogeneous coordinate ring $B = B(X, \mathcal{L}, \sigma)$ (see Section 1.5 or [RZ, Theorem 1.2]) and there is a natural correspondence between ideals of A and σ-invariant subschemes $Z \subseteq X$ (see [AS, Lemma 4.4]). If A is merely noetherian then the correspondence, which forms the content of the next proposition, is more subtle, basically because the morphisms Ψ_n and Φ_n need not be isomorphisms for $n \gg 0$.

We begin by discussing the point modules defined by function fields of factors of A. Let $B = \bigoplus B_n$ be a birationally commutative cg algebra that is generated in degree one, with graded quotient ring $Q(B) = K[t, t^{-1}; \sigma]$ for some field K and $\sigma \in \mathrm{Aut}(K)$. Since $B_1 \neq 0$ we may assume that $t \in B_1$, as is often convenient in these situations. Regard $\mathcal{G} = K[t; \sigma] = Q(B)_{\geq 0}$ as a right B_K-module, where B acts by right multiplication but K acts from the left. Since $t \in B_1$, $\mathcal{G}$ is cyclic and hence is a K-point module for B. Now consider the shift $\mathcal{G}[1]_{\geq 0}$ of $\mathcal{G}$. It is immediate that the map $\mu : \mathcal{G} \to \mathcal{G}[1]_{\geq 0}$, given by $t^m \ell \mapsto t^{m+1}\ell$ for $\ell \in K$, is an isomorphism of graded right B-modules. However, this is not an isomorphism of B_K-modules since the K-action is twisted: given $a \in K$ and $g = t^m \ell \in \mathcal{G}_{\geq 0}$ for some $\ell \in K$, then the left K-action is given by

$$a \circ \mu(g) = at^{m+1}\ell = t^{m+1}a^{\sigma^{-(m+1)}}\ell = \mu\left(a^{\sigma^{-1}} \circ (t^m \ell)\right) = \mu(a^{\sigma^{-1}} \circ g).$$

There is a second way of writing the module $\mathcal{G}[1]_{\geq 0}$. In the notation from page 7, consider the homomorphism $\sigma : K = R \to K = S$ and the resulting B_S-module $K_\sigma \otimes \mathcal{G} = S \otimes_R \mathcal{G}$. (We have introduced R and S in order to distinguish between the two copies of K.) As right B-modules, $\mathcal{G} \cong S \otimes_R \mathcal{G}$ under the map $\lambda : g \mapsto 1 \otimes g$.

However, the S-module action is again twisted: if $s * r = sr^\sigma$ denotes the action of $r \in R$ on $s \in S$ then

$$a \circ \lambda(g) = a \otimes g = (1 * a^{\sigma^{-1}}) \otimes g = 1 \otimes a^{\sigma^{-1}} g = \lambda(a^{\sigma^{-1}} \circ g) \quad \text{for } a \in K = S \text{ and } g \in \mathcal{G}.$$

Comparing the two displayed equations shows that:

Lemma 3.2. *Keep the notation from the previous paragraph. Then $\mathcal{G}[1]_{\geq 0} \cong K_\sigma \otimes \mathcal{G}$ as A_K-modules.* $\qquad\square$

Proposition 3.3. *Let A be a noetherian cg k-algebra that is generated in degree 1. Then there is a bijection between the sets:*

(i) *Completely prime, homogeneous ideals P of A (excluding $A_{\geq 1}$) such that A/P is birationally commutative.*

(ii) *Sequences of (not necessarily closed) points $\{\eta_n \in W_n : n \gg 0\}$ such that $\Phi_n(\eta_{n+1}) = \Psi_n(\eta_{n+1}) = \eta_n$ for all $n \gg 0$, where two such sequences are identified if they agree for all large n.*

Under this correspondence the function ring $Q(A/P)_0$ in part (i) is isomorphic to the field $k(\eta_n)$ in part (ii).

Proof. **Step 1**. Suppose that P is a completely prime ideal as in (i). Let $\overline{A} = A/P$ and write $Q(\overline{A}) \cong K[t, t^{-1}; \sigma]$, where K is a field and $t \in \overline{A}_1$. The idea is to think of $\mathcal{G} = Q(\overline{A})_{\geq 0}$ as a "generic" point module for $\overline{A}$. More precisely, regard $\mathcal{G}$ as a K-point module for A, as discussed above. For $n \geq 1$, the module $\mathcal{G}_{\leq n} = \bigoplus_{i=0}^n Kt^i$ is a truncated K-point module of length $n+1$ for A and so corresponds to a morphism $\theta_n : \operatorname{Spec} K \to W_n$. By Lemma 2.3(1), $\Phi_n \theta_{n+1} = \theta_n$ for all $n \gg 0$.

Lemma 2.3(1) also implies that the left truncation $\Psi'_n(\mathcal{G}_{\leq n+1}) = \bigoplus_{i=0}^n \mathcal{G}[1]_i$ corresponds to the morphism $\Psi_n \theta_{n+1} : \operatorname{Spec} K \to W_n$ for $n \gg 0$. By Lemma 3.2, $\Psi'_n(\mathcal{G}_{\leq n+1}) \cong K_\sigma \otimes \mathcal{G}_{\leq n}$ and so $\Psi_n \theta_{n+1} = \theta_n \sigma^\star$, where $\sigma^\star : \operatorname{Spec} K \to \operatorname{Spec} K$ is the morphism induced by σ. If $\eta_n = \theta_n(\eta) \in W_n$ denotes the image of the point $\eta = \operatorname{Spec} K$ then $\Psi_n(\eta_{n+1}) = \Psi_n \theta_{n+1}(\eta) = \theta_n \sigma^\star(\eta) = \theta_n(\eta) = \eta_n$. On the other hand, the previous paragraph shows that $\eta_n = \theta_n(\eta) = \Phi_n \theta_{n+1}(\eta) = \Phi_n(\eta_{n+1})$ for all $n \gg 0$. Thus $\{\eta_n\}$ is a sequence of points as in (ii).

Step 2. Conversely, suppose that we are given points $\eta_n \in W_n$ as in (ii), say for $n \geq n_0$. By Proposition 2.5, we can enlarge n_0 if necessary so that, for all $n \geq n_0$, both Ψ_n^{-1} and Φ_n^{-1} are defined and hence local isomorphisms at η_n. Define $L = k(\eta_{n_0})$ to be the residue field at η_{n_0}; this identification also induces a morphism $\rho_{n_0} : \operatorname{Spec} L \to W_{n_0}$ with $\operatorname{Im} \rho_{n_0} = \eta_{n_0}$. For $n \geq n_0$, we can inductively define $\rho_{n+1} = \Phi_n^{-1}(\rho_n)$; thus $\rho_{n+1} : \operatorname{Spec} L \to W_{n+1}$ satisfies $\operatorname{Im} \rho_{n+1} = \eta_{n+1}$, and $\rho_n = \Phi_n(\rho_{n+1})$ for all $n \geq n_0$. This means that there is an L-point module M for A which corresponds to the maps ρ_n.

Since Φ_n^{-1} and Ψ_n^{-1} are defined at η_n for $n \geq n_0$, our hypotheses imply that $\Psi_n \Phi_n^{-1}$ is a local isomorphism at η_n with $\Psi_n \Phi_n^{-1}(\eta_n) = \eta_n$. Then there is a unique scheme automorphism $\widetilde{\tau}_n : \operatorname{Spec} L \to \operatorname{Spec} L$ satisfying $\rho_n \widetilde{\tau}_n = \Psi_n \Phi_n^{-1} \rho_n = \Psi_n \rho_{n+1}$. We claim that the morphisms $\widetilde{\tau}_n$ are all equal to each other. Indeed, for $n \geq n_0$, Lemma 2.3(3) implies that

(3.4) $$\rho_n \widetilde{\tau}_n = \Psi_n \rho_{n+1} = \Psi_n \Phi_{n+1} \rho_{n+2} = \Phi_n \Psi_{n+1} \rho_{n+2} = \Phi_n \rho_{n+1} \widetilde{\tau}_{n+1} = \rho_n \widetilde{\tau}_{n+1}.$$

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Thus $\widetilde{\tau}_n = \widetilde{\tau}_{n+1} = \widetilde{\tau}$, say.

By Lemma 2.3(1), the truncations $N = \Phi'_n(M) = M_{\leq n+1}/M_{n+1}$ and $N' = \Psi'_n(M_{\leq n+1}) = (M_{\leq n+1}[1])_{\geq 0}$ of $M_{\leq n+1}$ correspond to the scheme morphisms $\Phi_n\rho_{n+1} = \rho_n$ and $\Psi_n\rho_{n+1} = \rho_n\widetilde{\tau}$, respectively. By the functoriality of F_n, as discussed on page 7, $N' \cong L_\tau \otimes N$, where $\tau : L \to L$ is the morphism induced from $\widetilde{\tau}$. Thus $M[1]_{\geq 0} \cong L_\tau \otimes M$. Let $P = \operatorname{ann}_A M_s$ for some $0 \leq s$. As the A-module and L-module actions on M commute, P is also the annihilator of $(L_\tau \otimes M)_s = M_{s+1}$. Hence, by induction, $P = \operatorname{ann}_A M$. Note that, as M is a point module, $P \neq A_{\geq 1}$.

We will show that P is a completely prime ideal such that A/P is birationally commutative, as in (i). Suppose first that $a \in A_u$ and $b \in A_v$ are homogeneous elements with $a, b \notin P$. Thus $M_0 a \neq 0$ which, since M_u is a 1-dimensional L-module, implies that $M_0 a = M_u$. Similarly, as $P = \operatorname{ann}_A M_u$, we find that $M_0 ab = M_u b = M_{uv}$. Thus, $ab \notin P$ and P is a completely prime ideal of A. Set $\overline{A} = A/P$ and write $Q = Q(\overline{A}) = Q_0[t, t^{-1}; \gamma]$, for some $\gamma \in \operatorname{Aut}(Q_0)$ and $t \in \overline{A}_1$. Given nonzero homogeneous elements $m \in M_s$, $a \in \overline{A}_u$ and $b \in \overline{A}_v$ with $u \geq v$ then, once again, $M_{s+u} = Lma = M_{s+u-v}b$ and so there exists $n \in M_{s+u-v}$ with $nb = ma$. Then $m \otimes ab^{-1} = n \otimes 1$ in $M \otimes_A Q$, and so $M \cong M \otimes_A Q_{\geq 0}$ already has a right $Q_{\geq 0}$-module structure. As Q_0 is a division ring and its right action on M commutes with that of L, there is an injection of rings $f : Q_0 \hookrightarrow \operatorname{End}_L M_0 = L$. Thus Q_0 is isomorphic to a subfield K of L. In particular, $\overline{A}$ is birationally commutative.

Step 3. We need to show that the operations from Steps 1 and 2 are mutual inverses. To begin, we assume that we have chosen a sequence $\{\eta_n\}$ as in (ii), and performed Step 2 to construct a completely prime ideal P as in (i). Keep all of the notation of that step.

As L is a field, the right action of Q_0 on M_0 coincides with the left action of the field $K \subseteq L$. Thus, for any $0 \neq m \in M_0$ the A_L-module surjection $L \otimes_k Q_{\geq 0} \to M$ given by $\ell \otimes q \mapsto \ell m q$ induces a graded surjective A_L-module homomorphism $\alpha : L \otimes_K Q_{\geq 0} \to M$. Since $Q_{\geq 0} = Q_0[t; \gamma]$, each graded piece both of $L \otimes_K Q_{\geq 0}$ and of M is isomorphic to L. Thus α is an isomorphism.

We now perform Step 1 with the ideal P to construct maps $\theta_n : \operatorname{Spec} K \to W_n$ corresponding to the K-point module $Q_{\geq 0}$. If $f^\star : \operatorname{Spec} L \to \operatorname{Spec} K$ is the morphism associated to the injection of fields $f : K \hookrightarrow L$ then the isomorphism $L_f \otimes Q_{\geq 0} \cong M$ implies that ρ_n factors as $\theta_n f^\star = \rho_n$ for all $n \geq n_0$. Thus $\eta_n = \operatorname{Im}(\theta_n)$ for all $n \geq n_0$ and so performing Step 2 followed by Step 1 gives the identity operation.

This also proves the final assertion of the proposition. Indeed, as the morphism ρ_{n_0} is defined by the identification $k(\eta_{n_0}) = L$ the factorization $\theta_{n_0} f^\star = \rho_{n_0}$ forces f to be the equality $K = L$. Thus Step 2 maps the set of points $\{\eta_n\}$ to a completely prime ideal P such that A/P has function ring $K = L = k(\eta_n)$ for $n \gg 0$.

Step 4. Conversely, suppose we have chosen an ideal P as in (i) and performed Step 1, to obtain a sequence of maps $\theta_n : \operatorname{Spec} K \to W_n$ corresponding to the point module $Q(A/P)_{\geq 0} = K[t; \sigma]$, together with points $\eta_n = \operatorname{Im}(\theta_n) \in W_n$ as in (ii). Then we may perform Step 2 to construct a point module M. As in Step 2, write

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$L = k(\eta_{n_0})$ with associated morphisms $\rho_n : \operatorname{Spec} L \to W_n$ for $n \geq n_0$. In this case θ_{n_0} factors as $\theta_{n_0} = \rho_{n_0}\widetilde{f}$ for a unique morphism of schemes $\widetilde{f} : \operatorname{Spec} K \to \operatorname{Spec} L$, with corresponding ring map $f : L \to K$. Thus $\Phi_{n,n_0}\theta_n = \theta_{n_0} = \rho_{n_0}\widetilde{f} = \Phi_{n,n_0}\rho_n\widetilde{f}$ for all $n \geq n_0$. By the choice of n_0, Φ_{n,n_0}^{-1} is a local isomorphism at η_{n_0} and so $\theta_n = \rho_n\widetilde{f}$ for all $n \geq n_0$. By functoriality this implies that $K_f \otimes M \cong Q(A/P)_{\geq 0}$ as A_K-modules, and hence that $\operatorname{ann}_A M = \operatorname{ann}_A(Q(A/P)_{\geq 0}) = P$. Thus performing Step 1 followed by Step 2 is also the identity operation. $\qquad\square$

One of the most important cases of Proposition 3.3 occurs when P is the zero ideal.

Corollary 3.5. *Let A be a cg birationally commutative noetherian domain which is generated in degree 1 and has graded quotient ring $Q(A) = K[t, t^{-1}; \sigma]$, for some $t \in A_1$ and $\sigma \in \operatorname{Aut}(K)$. Then there exists $n_0 \in \mathbb{N}$ such that:*

(1) *There is a sequence of points $\{\eta_n = \Phi_n(\eta_{n+1}) = \Psi_n(\eta_{n+1}) \in W_n\}$ corresponding to the K-point module $Q(A)_{\geq 0}$ such that $k(\eta_n) \cong K$ for all $n \geq n_0$.*

(2) *If Y_n denotes the closure of η_n in W_n then Y_n is a projective integral scheme such that the morphisms $\phi_n = \Phi_n|_{Y_n}$ and $\psi_n = \Psi_n|_{Y_n}$ are birational (and hence surjective) maps for all $n \geq n_0$.*

Proof. (1) Apply Proposition 3.3 to the ideal $P = 0$. By Step 1 of the proof, the sequence of points that this constructs does indeed correspond to $Q(A)_{\geq 0}$.

2) By Proposition 2.5, Φ_n^{-1} and Ψ_n^{-1} will be defined locally at η_n for all $n \gg 0$ and so ϕ_n and ψ_n will be birational for such n. The rest follows from part (1). $\qquad\square$

Definition 3.6. Keep the hypotheses and notation of Corollary 3.5. The scheme Y_n will be called the *relevant subscheme* of W_n and the data (Y_n, ϕ_n, ψ_n) will be called the *relevant point scheme data* for A. In Proposition 3.8 we will show that Y_n is actually an irreducible component of W_n, after which we will call Y_n the *relevant component of W_n*. We will usually write $\theta_n : \operatorname{Spec} K \to Y_n$ for the morphism defined by $K \cong k(\eta_n)$. In the notation of (2.6), the restrictions of $\Phi_{m,n}$ and $\Psi_{m,n}$ to Y_m will be written $\phi_{m,n}$, respectively $\psi_{m,n}$.

We want to briefly explain why the schemes Y_n are important. So let A be a noetherian cg domain that is generated in degree one and birationally commutative, with $Q(A) = K[t, t^{-1}; \sigma]$. If A is *strongly* noetherian then, as we explained in Section 1.4, A is isomorphic in high degree to a twisted homogeneous coordinate ring $B = B(Y, \mathcal{L}, \sigma)$ where the scheme Y equals W_n for any $n \gg 0$.

Now suppose that A is merely noetherian. The aim of this paper is, of course, to prove that A is a naïve blowup algebra $R(Y, Z_{\mathcal{I}}, \mathcal{L}, \sigma)$, again up to a finite dimensional vector space, for some scheme Y. Since $K = k(Y)$, the scheme Y will necessarily be integral. Unfortunately, Y will typically not equal any W_n or even be birational to some W_n simply because the W_n can have the wrong dimension or have multiple components (see Example 11.1 and Remark 11.2). On the other hand the relevant subscheme Y_n is an irreducible component of W_n for all $n \gg 0$ (see Proposition 3.8, below) and, fortunately, it also contains all

the information we need to reconstruct A. Indeed, the eventual scheme Y will be obtained by blowing down Y_n for any large n.

We will need to consider how the schemes W_n and Y_n are affected by the passage to Veronese rings. Given a noetherian cg domain A generated in degree 1 and $p \geq 1$, the pth-*Veronese ring* $A^{(p)} = \bigoplus_{n=0}^{\infty} A_{np}$ of A is again a cg noetherian domain that is generated in degree one (see [AZ1, Proposition 5.10]) and we write $(W_n^{(p)}, \Phi_n^{(p)}, \Psi_n^{(p)})$ for its point scheme data. If A is also birationally commutative with $Q(A) \cong K[t, t^{-1}; \sigma]$ and relevant subschemes $Y_n \subseteq W_n$, then $A^{(p)}$ will again be birationally commutative (its graded quotient ring is $K[t^p, t^{-p}; \sigma^p]$) and we let $(Y^{(p)}, \phi_n^{(p)}, \psi_n^{(p)})$ denote the relevant point scheme data.

Lemma 3.7. *Let A be a cg birationally commutative noetherian domain that is generated in degree one and pick $p \geq 1$.*

(1) *For all $n \geq 0$ there exist closed embeddings $\rho_n : W_{np} \to W_n^{(p)}$ satisfying $\Phi_{n-1}^{(p)} \rho_n = \rho_{n-1} \Phi_{np,(n-1)p}$ and $\Psi_{n-1}^{(p)} \rho_n = \rho_{n-1} \Psi_{np,(n-1)p}$, in the notation of (2.6).*

(2) *The morphism ρ_n restricts to an isomorphism $Y_{np} \xrightarrow{\sim} Y_n^{(p)}$ for all $n \gg 0$.*

Proof. (1) Let C be any commutative k-algebra. If $M = \bigoplus_{i=0}^{np} M_i$ is a truncated point module of length $np + 1$ over $A_C = A \otimes C$ then $M^{(p)} = \bigoplus_{i=0}^{n} M_{ip}$ is a truncated point module of length $n + 1$ over $A_C^{(p)}$. Since the map $M \mapsto M^{(p)}$ is compatible with change of rings, we obtain a morphism of the representing schemes $\rho_n : W_{np} \to W_n^{(p)}$. The given decompositions of $\Phi_{n-1}^{(p)} \rho_n$ and $\Psi_{n-1}^{(p)} \rho_n$ then follow from Lemma 2.3(1).

It remains to show that ρ_n is a closed embedding. Write $A = T(V)/J$, for $V = A_1$; thus $A^{(p)} \cong T(V^{\otimes p})/\bigoplus_{n=0}^{\infty} J_{np}$. Under these presentations, W_{np} is the zero set $\mathcal{V}(J_{np}) \subseteq \mathbb{P}(V^*)^{\times np}$ while $W_n^{(p)} = \mathcal{V}(J_n^{(p)}) \subseteq \mathbb{P}(V^{* \otimes p})^{\times n}$. The map ρ_n is just the restriction to W_{np} of the Segre embedding $\mathbb{P}(V^*)^{\times np} = (\mathbb{P}(V^*)^{\times p})^{\times n} \longrightarrow \mathbb{P}(V^{* \otimes p})^{\times n}$. As such, ρ_n is a closed embedding.

(2) If $Q(A) = K[t, t^{-1}, \sigma]$ then $Q(A^{(p)}) = K[t^p, t^{-p}, \sigma^p]$. The K-point module $Q(A)_{\geq 0}$ corresponds to a sequence of morphisms $\theta_n : \operatorname{Spec} K \to W_n$ for $n \geq 0$ and, similarly, $Q(A^{(p)})_{\geq 0}$ is a K-point module for $A^{(p)}$ corresponding to morphisms $\theta'_n : \operatorname{Spec} K \to W_n^{(p)}$. Since $Q(A)_{\geq 0}^{(p)} = Q(A^{(p)})_{\geq 0}$, it follows from part (1) that $\rho_n \theta_{np} = \theta'_n$ for each n. Since Y_{np}, respectively $Y_n^{(p)}$, is the closure of the image of θ_{np}, respectively θ'_n, it follows that $\rho_n(Y_{np}) = Y_n^{(p)}$ for each n. As ρ_n is also a closed embedding, it must restrict to an isomorphism $Y_{np} \xrightarrow{\sim} Y_n^{(p)}$. $\square$

To conclude the section, we will show that the relevant subschemes of a birationally commutative ring are irreducible components of the truncated point schemes. This result will not be used elsewhere in the paper.

Proposition 3.8. *Let A be a birationally commutative noetherian cg domain, with point scheme data (W_n, Φ_n, Ψ_n) and relevant subschemes $Y_n \subseteq W_n$. Then Y_n is an irreducible component of W_n for all $n \gg 0$.*

Proof. Keep the notation from Corollary 3.5 and Definition 3.6; in particular, η_n is the generic point of Y_n and Φ_n^{-1} and Ψ_n^{-1} define local isomorphisms at η_n for all $n \geq n_0$. For each such n, let S_n be the set of irreducible components of W_n which contain Y_n.

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Suppose first that $\dim \Phi_n(Z) < \dim Z$ for some $Z \in S_{n+1}$ with $n \geq n_0$. Then for every closed point $z \in \Phi_n(Z)$, and in particular for $z \in \Phi_n(Y_{n+1}) = Y_n$, the fibre $\Phi_n^{-1}(z)$ has dimension ≥ 1. This contradicts the fact that Φ_n^{-1} defines a local isomorphism at η_n and proves that $\dim \Phi_n(Z) = \dim Z$ for all $Z \in S_{n+1}$. Similarly, $\dim \Psi_n(Z) = \dim Z$ for all $Z \in S_{n+1}$. For $n \geq n_0$ let d_n be the maximum dimension of the components in S_n. Since $\Phi_n(Z) \subseteq Z'$ for some $Z' \in S_n$, we have $d_{n+1} \leq d_n$ for $n \geq n_0$. Thus d_n is constant for $n \gg 0$, say $d_n = d$ for $n \geq n_1 \geq n_0$.

For each $n \geq n_1$ let $T_n \subseteq S_n$ consist of those components of dimension d. Let Ω_n be the set of generic points of the components in T_n. Then $\Phi_n(\Omega_{n+1}) \subseteq \Omega_n$ and similarly $\Psi_n(\Omega_{n+1}) \subseteq \Omega_n$ for all $n \geq n_1$. By Corollary 2.7, we may pick $n_2 \geq n_1$ such that for $n \geq n_2$ both Φ_n and Ψ_n give bijections from Ω_{n+1} to Ω_n and both Φ_n^{-1} and Ψ_n^{-1} define local isomorphisms at each point of Ω_n. In particular, $|T_n|$ is a constant value m for $n \geq n_2$. Now $\Phi_n \Psi_{n+1} = \Psi_n \Phi_{n+1}$ by Lemma 2.3(3). Thus, for each $n \geq n_2$, we can label $\Omega_n = \{\mu_{1,n}, \ldots, \mu_{m,n}\}$ in such a way that $\Phi_n(\mu_{i,n+1}) = \mu_{i,n}$ and $\Psi_n(\mu_{i,n+1}) = \mu_{\tau(i),n}$, where τ is a permutation in the symmetric group on $\{1, 2, \ldots, m\}$, independent of the choice of n.

Set $p = |\tau|$ and write $A' = A^{(p)}$, with point scheme data (W_n', Φ_n', Ψ_n') and relevant subschemes $Y_n' \subseteq W_n'$. Let $\rho_n : W_{np} \to W_n'$ be the closed embeddings of Lemma 3.7; thus $Y_n' = \rho_n(Y_{np})$ by that lemma. If η_n' denotes the generic point of each Y_n', then $\Psi_n'(\eta_{n+1}') = \Phi_n'(\eta_{n+1}') = \eta_n'$ for all $n \gg 0$. On the other hand, if $\mu_n' = \rho_n(\mu_{1,np})$, then Lemma 3.7 shows that $\Psi_n'(\mu_{n+1}') = \Phi_n'(\mu_{n+1}') = \mu_n'$ for all $n \gg 0$. Applying Proposition 3.3 to the ring A' shows that the points $\{\eta_n'\}$ correspond to the zero ideal while the $\{\mu_n'\}$ correspond to some (possibly different) completely prime ideal P of A'.

We need to show that $P = 0$. Note that η_n' is in the Zariski closure $Z_n' = \overline{\mu_n'}$ for all $n \geq n_2$. By Proposition 2.5, pick $n_3 \geq n_2$ such that $(\Phi_n')^{-1}$ defines a local isomorphism both at η_n' and at μ_n' for all $n \geq n_3$. Let R be the local ring $R = \mathcal{O}_{Z_{n_3}', \eta_{n_3}'}$ with field of fractions $L \cong k(\mu_{n_3}')$ and residue field $K \cong k(\eta_{n_3}')$. As usual, write $f^\star : \operatorname{Spec} L \to \operatorname{Spec} R$ and $g^\star : \operatorname{Spec} K \to \operatorname{Spec} R$ for the morphisms induced from the corresponding maps $f : R \to L$ and $g : R \to K$. Then we can define morphisms $\alpha_{n_3} : \operatorname{Spec} K \to W_n'$ with image η_n', as well as $\beta_{n_3} : \operatorname{Spec} L \to W_n'$ with image μ_n', and $\gamma_{n_3} : \operatorname{Spec} R \to W_n'$ in such a way that $\alpha_{n_3} = \gamma_{n_3} g^\star$ and $\beta_{n_3} = \gamma_{n_3} f^\star$. Since $(\Phi_{n,n_3}')^{-1}$ is defined as a morphism on $\operatorname{Im}(\gamma_{n_3})$, we can inductively define maps $\gamma_{n+1} = (\Phi_n')^{-1} \gamma_n : \operatorname{Spec} R \to W_{n+1}'$ such that $\Phi_n' \gamma_{n+1} = \gamma_n$ for all $n \geq n_3$. The morphisms $\alpha_n = \gamma_n g^\star$ and $\beta_n = \gamma_n f^\star$ then satisfy $\Phi_n' \alpha_{n+1} = \alpha_n$ and $\Phi_n' \beta_{n+1} = \beta_n$ for all $n \geq n_3$.

The sequences of maps $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ for $n \geq n_3$ correspond to point modules M, N, and Q over the rings K, L, and R, respectively, such that $K_g \otimes_R Q \cong M$ and $L_f \otimes_R Q \cong N$. It follows that $\operatorname{ann}_A N = \operatorname{ann}_A Q \subseteq \operatorname{ann}_A M$. By the correspondence of Proposition 3.3, this means that $P \subseteq (0)$, whence $P = 0$. This forces $\mu_n' = \eta_n'$ for all $n \gg 0$ which, since each ρ_n is a closed embedding, implies that $\mu_{1,np} = \eta_n$ for each such n. Thus Y_n is indeed a component of W_n. $\square$

4. CONTRACTED POINTS

Let A be a noetherian cg domain that is birationally commutative. If the morphisms $\phi_n, \psi_n : Y_{n+1} \to Y_n$ are isomorphisms for $n \gg 0$ then it is not hard to understand the structure of A and, indeed, to prove that A is a twisted homogeneous coordinate ring in large degree (see [RZ, Theorem 4.4, Corollary 4.6]). To understand the general case, it will be useful to examine the the points where the maps ϕ_n and ψ_n are *not* isomorphisms, with particular emphasis on the points where they have infinite fibres, and this is what we do over the next four sections. Eventually, in Theorem 7.1, this information will be used to construct a surjective image of Y_n, for some $n \gg 0$, on which the birational map $\sigma = \psi_n \phi_n^{-1}$ induces an isomorphism.

To achieve this aim we will have to restrict our attention to surfaces and so, from now on, we will assume:

Assumptions 4.1. *A is a cg noetherian domain, generated in degree 1, which is birationally commutative with graded quotient ring $Q = K[t, t^{-1}; \sigma]$, where K is a field with* $\operatorname{tr.deg} K/k = 2$.

The points that interest us are described as follows.

Notation 4.2. Given an algebra A as in (4.1), let (Y_n, ϕ_n, ψ_n) denote the relevant point scheme data from Definition 3.6. By Corollary 3.5, fix an integer n_0 such that $k(Y_n) \cong K$ and the morphisms ϕ_n, ψ_n are birational, for all $n \geq n_0$. For $n \geq n_0$, let S_n^- denote the set of closed points $y \in Y_n$ such that the fibre $\phi_n^{-1}(y)$ has dimension ≥ 1. Similarly, let S_n^+ be the set of points $y \in Y_n$ such that the fibre $\psi_n^{-1}(y)$ has dimension ≥ 1. The points in S_n^- and S_n^+ will be called *contracted points*. As $\dim Y_n = 2$ it follows, for example from [Ha, Exercise II.3.22(d)], that S_n^-, S_n^+ are finite sets.

Example 4.3. In order to understand these concepts, consider the example $A = R(\mathbb{P}^2, Z, \mathcal{O}(1), \sigma)$ from Section 1.2, where Z is the single reduced point $c = (1 : 1 : 1)$ and p, q are algebraically independent over the prime subfield of k. We will not prove the assertions we make about A except to note that, at the level of sets, our description of the closed points of Y_n follows from [Ro1, Theorem 6.6]. To see this, note that, by Corollary 3.5(2), each truncated point module y_n corresponding to a point in Y_n is a homomorphic image of a point module. Those point modules are then described by [Ro1, Theorem 6.6]. The proof that we have the correct description of Y_n as a scheme is more complicated and is omitted.

Write $r_m = \sigma^{1-m}(c) = (1 : p^{1-m} : q^{1-m})$ for $m \in \mathbb{Z}$. If n is sufficiently large then Y_n is the scheme obtained as the blowup $\gamma_n : Y_n \to \mathbb{P}^2$ of $\mathbb{P}^2$ at $Z_n = \{r_1, \ldots, r_n\}$. Write the resulting exceptional curves as $C_i^{(n)} = \gamma_n^{-1}(r_i)$ for $1 \leq i \leq n$ but, by a slight abuse of notation, set $r_m = \gamma^{-1}(r_m)$ for $m \notin [1, n]$. The maps ϕ_n and ψ_n are easily described. The morphism $\phi_n : Y_{n+1} \to Y_n$ is simply obtained by blowing up r_{n+1} to obtain the new exceptional curve $C_{n+1}^{(n+1)}$. On the other hand, $\psi_n : Y_{n+1} \to Y_n$ is obtained by blowing up $r_0 = c$ and then "shifting by σ"; thus $\psi_n^{-1}(r_0) = C_1^{(n)}$ while $\psi_n^{-1}(C_j^{(n)}) = C_{j+1}^{(n+1)}$ for $1 \leq j \leq n$.

We remark that the conventions from the previous paragraph are the ones that are most natural for naïve blowups (see, for example, [KRS]). However, the notation that is most useful for this paper is the opposite—thus $S_n^- = \{r_{n+1}\}$ and $S_n^+ = \{r_0\}$. □

The following result gives the most basic properties of the contracted points.

Lemma 4.4. *Let A satisfy (4.1) and keep the notation of (4.2). Then there exists $n_1 \geq n_0$ such that:*

(1) *For all $p \geq n_0$ one has $\psi_p(S_{p+1}^-) \subseteq S_p^-$ and $\phi_p(S_{p+1}^+) \subseteq S_p^+$.*

(2) *For all $q > p \geq n_1$ the birational map $\psi_{q,p}^{-1}$ is defined at each point $s \in S_p^-$, with $\psi_{q,p}^{-1}(s) \in S_q^-$. Similarly, $\phi_{q,p}^{-1}(t)$ is defined and belongs to S_q^+ for each $t \in S_p^+$.*

(3) *For all $p \geq n_1$, $S_p^- \cap S_p^+ = \emptyset$ and the cardinalities $|S_p^-|$ and $|S_p^+|$ are constant.*

Proof. In parts (1) and (2), it suffices by symmetry to prove the statements for S_p^-.

(1) Recall that we have a closed immersion $\iota_p : W_p \to \mathbb{P}^{\times p}$, and the morphisms $\Phi_p, \Psi_p : W_{p+1} \to W_p$ are defined to be the restrictions of the respective projection maps $\pi_{1,p}, \pi_{2,p+1} : \mathbb{P}^{\times(p+1)} \to \mathbb{P}^{\times p}$. Now let $y \in S_{p+1}^-$, say given by $y = (y_1, \ldots, y_{p+1}) \in \mathbb{P}^{\times(p+1)}$. By definition, y has infinitely many preimages under ϕ_{p+1}, say $z_\alpha = (y_1, \ldots, y_{p+1}, \alpha) \in Y_{p+2} \subseteq \mathbb{P}^{\times(p+2)}$, for some infinite set of $\alpha \in \mathbb{P}$. The points $w_\alpha = \psi_{p+1}(z_\alpha) = (y_2, \ldots, y_{p+1}, \alpha)$ are then distinct elements of Y_{p+1}, each of which satisfies $\phi_p(w_\alpha) = (y_2, y_3, \ldots, y_{p+1}) = \psi_p(y) \in Y_p$. Hence $\psi_p(y) \in S_p^-$.

(2) The sets S_p^- are finite for $p \geq n_0$ and $\Psi_p(S_{p+1}^-) \subseteq S_p^-$ by part (1). By Corollary 2.7 there exists $n_1 \geq n_0$ such that, if $p \geq n_1$ then $|S_p^-|$ is constant and Ψ_p^{-1} is locally an isomorphism on S_p^-. For such p, the restriction ψ_p^{-1} will still be a local isomorphism at S_p^- and so, by induction, the birational map $\psi_{q,p}^{-1} : Y_p \dashrightarrow Y_q$ is defined at S_p^- for all $q > p \geq n_1$.

(3) Repeating the argument of part (2) for the maps Φ_n, we can choose a single n_1 so that the cardinalities $|S_p^-|$ and $|S_p^+|$ remain constant for $p \geq n_1$. Since ψ_p^{-1} is defined on S_p^-, the map ψ_p obviously cannot have a fibre of positive dimension over S_p^-. Thus $S_p^- \cap S_p^+ = \emptyset$. $\square$

By definition, the birational map $\phi_p^{-1} : Y_p \dashrightarrow Y_{p+1}$ is not defined at any point $y \in S_p^-$. Conversely, we would like to say that for $p \gg 0$, the domain of definition of ϕ_p^{-1} is exactly $Y_p \smallsetminus S_p^-$. Although this holds for Example 4.3, in general it is too strong an assertion. However, something weaker is true; as we next show, the points where ϕ_p^{-1} is undefined are either contracted points themselves, or else lie on a curve which contracts at a later stage.

Lemma 4.5. *Let A satisfy (4.1) and keep the notation of (4.2). Fix $n_1 \geq n_0$. Then there exists $n_2 \geq n_1$ with the following properties.*

(1) *For some $m \geq n_2$, let $y \in Y_m$ be a closed point such that the map ϕ_m^{-1} is not defined at y. Then either $y \in S_m^-$, or else there exists $m > p \geq n_1$ and a curve $C \subset Y_m$ such that $y \in C$ and $\phi_{m,p}(C) \in S_p^-$.*

(2) *Suppose that ψ_m^{-1} is not defined at a closed point $y \in Y_m$ for some $m \geq n_2$. Then either $y \in S_m^+$, or else $y \in C \subset Y_m$ for some curve C with $\psi_{m,p}(C) \in S_n^+$ for some $m > p \geq n_1$.*

Proof. As usual, we prove only part (1); the proof of part (2) is analogous.

The main idea of the proof is to study the morphisms $\phi_{m,n}$ through Stein factorizations, as described in [Ha, Corollary III.11.5], and we start with the relevant notation. Choose integers $m > n \geq n_1$ and, for

ease of notation, set $q = n_1$. The *Stein factorization* $\phi_{m,n} = g_{m,n} \circ h_{m,n}$ of the morphism $\phi_{m,n} : Y_m \to Y_n$ has the following properties: $h_{m,n} : Y_m \to Z_{m,n}$ is a projective morphism to a scheme defined by $Z_{m,n} = \mathbf{Spec}(\phi_{m,n})_* \mathcal{O}_{Y_m}$, while the morphism $g_{m,n} : Z_{m,n} \to Y_n$ is finite. We remark that $h_{m,n}$ necessarily has connected fibres and, since $\phi_{m,n}$ is birational, the maps $g_{m,n}$ and $h_{m,n}$ must also be birational.

By Corollary 3.5 $\phi_{m,q} : Y_m \to Y_q$ is a birational surjective morphism for all $m \geq q$ and so we get an ascending chain of coherent sheaves of $\mathcal{O}_{Y_q}$-algebras,

$$(4.6) \qquad \mathcal{O}_{Y_q} \subseteq (\phi_{q+1,q})_* \mathcal{O}_{Y_{q+1}} \subseteq \cdots \subseteq (\phi_{m,q})_* \mathcal{O}_{Y_m} \subseteq \cdots$$

and associated to this a chain of finite morphisms

$$(4.7) \qquad \cdots \xrightarrow{e_m} Z_{m,q} \xrightarrow{e_{m-1}} \cdots \xrightarrow{e_{q+1}} Z_{q+1,q} \xrightarrow{e_q} Z_q = Y_q$$

where $g_{m,q} = e_q \circ e_{q+1} \circ \cdots \circ e_{m-1}$ is the finite morphism half of the Stein factorization of $\phi_{m,q}$. Choose a finite open affine cover $Y_q = \bigcup U_\alpha$ of Y_q. Then looking at the sections of (4.6) over any U_α gives an ascending chain of finite ring extensions, which must eventually stabilize by the finiteness of the integral closure [Ei, Theorem 4.14]. Thus (4.6) must also stabilize and so e_m must be an isomorphism for all $m \gg q$, say for all $m \geq n_2$.

For $m > n_2$ we have a commutative square:

$$
\begin{array}{ccc}
Y_{m+1} & \xrightarrow{\phi_m} & Y_m \\
{\scriptstyle h_{m+1,q}} \downarrow & & \downarrow {\scriptstyle h_{m,q}} \\
Z_{m+1,q} & \xrightarrow{e_m} & Z_{m,q}
\end{array}
$$

By assumption e_m is an isomorphism and so the map $\theta = h_{m,q}\phi_m = e_m h_{m+1,q} : Y_{m+1} \to Z_{m,q}$ has connected fibres, since $h_{m+1,q}$ does. Let $y \in Y_m$ be a closed point where the birational map ϕ_m^{-1} is not defined and consider $z = h_{m,q}(y) \in Z_{m,q}$. If the birational map θ^{-1} were defined at z, then $\phi_m^{-1} = \theta^{-1} h_{m,q}$ would be defined at y, contradicting our assumption. So θ^{-1} is not defined at z.

Suppose now that the set-theoretic fibre $\theta^{-1}(z)$ is a single point z'. By construction, $\theta = e_m h_{m+1,q}$ is projective with $\theta_* \mathcal{O}_{Y_{m+1}} = \mathcal{O}_{Z_{m,q}}$. Thus Lemma 2.4(2) can be applied to show that the birational map θ^{-1} must have been defined at z after all, a contradiction. As θ has connected fibres, we conclude that the fibre $\theta^{-1}(z)$ must contain an irreducible curve C such that $y \in \phi_m(C)$. Since

$$\phi_{m+1,q} = \phi_{m,q}\phi_m = g_{m,q}h_{m,q}\phi_m = g_{m,q}\theta,$$

we see that $\phi_{m+1,q}(C) = g_{m,q}(z)$ is a closed point in Y_q. Now let p be the largest integer with $m \geq p \geq q$ for which $\phi_{m+1,p}(C)$ is a closed point $w \in Y_p$. Then $w \in S_p^-$ and $\phi_{m,p}(y) = w$. If $p = m$, then $w = y \in S_m^-$; otherwise $m > p \geq q$ and $\phi_m(C) = D$ is a curve, containing y, for which $\phi_{m,p}(D) = w \in S_p^-$. $\qquad \square$

Throughout this section A will satisfy Assumptions 4.1 and we keep the notation from Notation 4.2. In particular, $Q = Q(A) = K[t, t^{-1}; \sigma]$ for some $\sigma \in \mathrm{Aut}_k K$. We are interested in the birational map that σ induces on Y_n for $n \gg 0$ and, for simplicity, we continue to write this as $\sigma : Y_n \dashrightarrow Y_n$. As we will see in this section, the birational map σ has a number of special properties.

These properties are fairly technical but it is at least possible to indicate where the results are heading (see Proposition 6.4 and Theorem 7.1 for the formal statements). The basic idea is that we are aiming to prove that Y_n is the blowup $\pi_n : Y_n \to \mathbb{X}_n$ of a scheme $\mathbb{X}_n$ such that (i) σ is induced from an automorphism $\widetilde{\sigma}$ of $\mathbb{X}_n$ and (ii) the points that are blown up all lie on Zariski dense orbits of $\widetilde{\sigma}$. So, consider the set V of closed points $y \in Y_n$ at which some power of σ is undefined or not a local isomorphism. Under a minor extra hypothesis V breaks into a finite union of sets $V = \bigcup V_i$ that are "trying to be" infinite orbits of σ. Formally, within each V_i there are two dense half orbits $V_i^+ = \{y_{j,i} = \sigma^j(t_i) : j \geq 0\}$ and $V_i^- = \{x_{j,i} = \sigma^{-j}(s_i) : j \geq 0\}$. The remaining points in V_i, which essentially correspond to points on the exceptional divisor of π_n, are contained in a finite union of irreducible curves (and possibly some sporadic closed points) C_ℓ all of which are mapped to both s_i and t_i by suitable powers of σ; that is, there exist unique integers $a = a(\ell) > 0$ and $b = b(\ell) < 0$ such that $\sigma^a(C_\ell) = t_i$ and $\sigma^b(C_\ell) = s_i$.

The results mentioned in the previous paragraph only hold for sufficiently large values of n and it is useful to more formally quantify "sufficiently large" as follows:

Definition 5.1. Define n_0 by Corollary 3.5. Then fix some $n_1 \geq n_0$ such that the conclusions of Lemma 4.4 hold. Then take this n_1 in Lemma 4.5 and choose some $n_2 \geq n_1$ satisfying the conclusions of that lemma. Finally, we call any $n > n_2$ a *stable value of* n.

Notation 5.2. Fix once and for all a stable value of n, and set $Y = Y_n$. Recall from Corollary 3.5 that for each $m \geq n$, the map $\theta_m : \mathrm{Spec}\, K \to W_m$ corresponding to the truncated K-point $\bigoplus_{i=0}^m Q_i$ induces an isomorphism $k(\eta_m) \to K$, where η_m is the generic point of the relevant component Y_m. We identify $k(Y) = K$ through the map θ_n and, as before, write $\sigma : Y \dashrightarrow Y$ for the birational map induced by the automorphism $\sigma \in \mathrm{Aut}(K)$. Set $S^+ = S_n^+$ and $S^- = S_n^-$ for the contracted points, as defined in Notation 4.2.

To simplify notation, we often write $\phi_m = \phi$, for any $m \geq n_0$, where the subscript will be clear from the domain or range of ϕ. Similar comments apply to ψ and so, for example, for any $m \geq n_0$ the birational map $\psi_{m+s,m}^{-1}$ is now written ψ^{-s}. The next result, which shows how to interpret $\sigma^m : Y \dashrightarrow Y$ in terms of the morphisms ϕ and ψ, will be used frequently and usually without further comment.

Lemma 5.3. (1) *As birational maps* $Y_p \dashrightarrow Y_{p-m-\ell}$ *one has* $\phi^\ell \psi^m = \psi^m \phi^\ell$ *for any* $\ell, m \in \mathbb{Z}$ *for which all the maps involved are birational (that is, as long as p, $p - m$, $p - \ell$ and $p - m - \ell$ are $\geq n_0$).*

(2) *For any $m \geq 0$, we have $\sigma^m = \psi^m \phi^{-m}$ and $\sigma^{-m} = \phi^m \psi^{-m}$ as birational maps $Y \dashrightarrow Y$.*

Proof. (1) When $\ell, m \geq 0$ this follows from Lemma 2.3(3). When ℓ or m is negative, multiply through by the appropriate (birational) inverse map.

(2) Recall from step 1 of Proposition 3.3 that $\theta_n = \Phi_n \theta_{n+1}$ and $\theta_n \sigma^\star = \Psi_n \theta_{n+1}$, where $\theta_m : \operatorname{Spec} K \to W_m$ is defined by Notation 5.2 and $\sigma^\star : \operatorname{Spec} K \to \operatorname{Spec} K$ is the map of schemes associated to $\sigma \in \operatorname{Aut}(K)$. Restricting to Y we find that $\psi \phi^{-1} \theta_n = \psi \theta_{n+1} = \theta_n \sigma^\star$, and it follows that $\psi \phi^{-1}$ is precisely the birational map $\sigma : Y \dashrightarrow Y$ induced by $\sigma \in \operatorname{Aut}(K)$. That $\sigma^{-1} = \phi \psi^{-1}$ follows immediately, and the results for all $m \geq 1$ follow by induction using part (1). $\qquad\square$

The rest of this section is devoted to proving successively stronger properties of S^+ and S^-, and of certain special extremal subsets thereof, for our fixed stable value of n. The properties of (S^+, σ) and (S^-, σ^{-1}) are analogous and so we will first consider (S^+, σ), referring the reader to Remark 5.9 for (S^-, σ^{-1}). We begin by studying points $\mu \in Y$ which are fixed by some power σ^n and show that they do not interfere with S^+.

Lemma 5.4. *Let $\mu \in Y$ be a point such that σ^p is defined at μ and $\sigma^p(\mu) = \mu$ for some $p > 0$.*

(1) *If μ is a closed point of Y then $\mu \notin S^+$.*

(2) *If μ is the generic point of an irreducible curve C, then $C \cap S^+ = \emptyset$.*

Proof. The proof is by contradiction, so we assume that there is a closed point $z \in S^+$ where either $z = \mu$ in case (1), or $z \in C$ in case (2). By Lemma 4.4(2), ϕ^{-s} is defined at z for any $s \geq 1$ and then $\phi^{-s}(z) \in S^+_{s+n}$. In either case the birational map ϕ^{-s} is also defined at $\mu_n = \mu$ for $s \geq 1$, so we can define $\mu_r = \phi^{n-r}(\mu) = \phi^{-1}(\mu_{r-1}) \in Y_r$ for all $r n$.

We claim that $\psi^p(\mu_{r+p}) = \mu_r$ for all $r \geq n$. When $r = n$ this is just the observation that $\psi^p(\mu_{n+p}) = \psi^p \phi^{-p}(\mu) = \sigma^p(\mu) = \mu$. We then prove it for all $r \geq n$ by induction: assuming the result is true for some $r \geq n$, then

$$\psi^p(\mu_{r+1+p}) = \psi^p \phi^{-1}(\mu_{r+p}) = \phi^{-1} \psi^p(\mu_{r+p}) = \phi^{-1}(\mu_r) = \mu_{r+1}$$

and so the result holds for $r + 1$.

Consider the Veronese ring $A' = A^{(p)}$ of A with point scheme data (W'_m, Φ'_m, Ψ'_m) and relevant point scheme data (Y'_m, ϕ'_m, ψ'_m). By Lemma 3.7(2), there is an isomorphism $\rho_p : Y_{pm} \to Y'_m$ for each $m \gg 0$ and we write $\mu'_m = \rho_m(\mu_{mp})$ for such m. It follows from the previous paragraph and Lemma 3.7(1) that $\psi'(\mu'_{m+1}) = \mu'_m = \phi'(\mu'_{m+1})$ for all such m. By Proposition 3.3, there is then a completely prime ideal P of A' corresponding to this sequence of elements such that $\widehat{A} = A'/P$ is birationally commutative, with function ring $L \cong k(\mu'_m)$ for $m \gg 0$.

Now consider the point scheme data $(\widehat{W}_m, \widehat{\Phi}_m, \widehat{\Psi}_m)$ for $\widehat{A}$. By Lemma 3.1, $\widehat{W}_m$ is a closed subscheme of W'_m for each m, and by construction $\mu'_m \in \widehat{W}_m$ for $m \gg 0$. By assumption L is a field of transcendence degree ≤ 1 over k, so [AS, Theorem 0.1(ii)] implies that the Gelfand-Kirillov dimension of $\widehat{A}$ satisfies $\operatorname{GKdim} \widehat{A} \leq 2$. By [ASZ, Theorem 4.24] $\widehat{A}$ is strongly noetherian and hence, by [AZ2, Theorem E4.4] and Lemma 2.3(2),

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the maps $\widehat{\Phi}_m$ and $\widehat{\Psi}_m$ are isomorphisms for all $m \gg 0$. Lemma 3.1 therefore implies that the fibre $(\Psi')^{-1}(y)$ is a singleton for all $y \in \widehat{W}_m$ and all $m \gg 0$.

Recall that $z = \mu$ in case (1), respectively $z \in C$ in case (2), and set $z_{pm} = \phi^{-pm+n}(z) \in Y_{pm}$ for all $m \gg 0$, so z_{pm} is in the closure of μ_{pm}. Then $z'_m = \rho_m(z_{pm})$ is in the closure of μ'_m, whence $z'_m \in \widehat{W}_m$. By the argument above, the fibre $(\Psi')^{-1}(z'_m)$ is a singleton for all $m \gg 0$. On the other hand, since $z_{pm} \in S^+_{pm}$, we can find an irreducible curve $D_{pm+p} \subset Y_{pm+m}$ such that $\psi^p(D_{pm+p}) = z_{pm}$. Then Lemma 3.7 implies that $\rho_{m+1}(D_{pm+p})$ is a curve in W'_{m+1} that contracts to z'_m under Ψ'; a contradiction. $\qquad\square$

Next, we reinterpret Lemmas 4.4 and 4.5 to see how the contracted points S^+ are related to the points where σ^{-1} is undefined.

Lemma 5.5. (1) *Let $y \in S^+ \subseteq Y$. Then σ^{-1} is not defined at y, but σ^d is defined at y for all $d \geq 0$.*

(2) *Conversely, let $y \in Y$ be a closed point where either ψ^{-1} or σ^{-1} is not defined. Then there exists $e \geq 0$ such that $\sigma^e(y)$ is defined and belongs to S^+.*

Proof. (1) Since $y \in S^+$ there exists, by definition, an irreducible curve $D \subset Y_{n+1}$ such that $\psi(D) = y$. Suppose that $\phi(D) = z$ is also a closed point; thus $z \in S^-$. By Lemma 4.4(1),

$$w = \phi(y) = \phi\psi(D) = \psi\phi(D) = \psi(z) \in S^+_{n-1} \cap S^-_{n-1},$$

contradicting Lemma 4.4(3). This contradiction implies that $\phi(D) = C$ is a curve, and $w = \psi(C) = \phi\psi(D) = \phi(y)$ is a closed point in S^+_{n-1}. By Lemma 4.4(2), ϕ^{-1} is defined at w and so $\sigma = \phi^{-1}\psi$ is defined at every point of C. Since σ contracts C to y, σ^{-1} is not defined at y. On the other hand, by Lemma 4.4(2) again, $\phi^{-d}(y)$ is defined for all $d \geq 0$ and so $\sigma^d(y) = \psi^d\phi^{-d}(y)$ is defined for all $d \geq 0$.

(2) As $\sigma^{-1} = \phi\psi^{-1}$, if σ^{-1} is not defined at y, then the birational map ψ^{-1} cannot be defined at y. Thus we need only consider the case that ψ^{-1} is not defined at y. Since n is a stable value, Lemma 4.5(2) implies that either $y \in S^+$ or $y \in C \subset Y$ for some irreducible curve C with the property that $z = \psi_{n,p}(C) \in S^+_p$ for some $n > p \geq n_1$. In the former case, we can take $e = 0$; in the latter case, Lemma 4.4 implies that $\phi^{-1}_{n,p}(z)$ is defined and in S^+, so taking $e = n - p$ we have $\sigma^e(y) = \psi^e\phi^{-e}(y) = \phi^{-e}\psi^e(y) = \phi^{-e}(z) \in S^+$. $\qquad\square$

It can happen that some power σ^m of σ sends a point $y \in S^+$ to another point in S^+ but, as we show next, this cannot happen for $m \gg 0$. The transition between these two possibilities will be fundamental and so we define the *extremal elements* of S^+ to be

(5.6) $$S^+_\infty = \{y \in S^+ : \forall \, d \geq 1, \;\; \sigma^d(y) \notin S^+\}.$$

This set will turn out to be the set of distinguished elements $S^+_\infty = \{t_i\}$ mentioned in the introduction to this section. The dual set $S^-_\infty = \{s_j\}$ will be defined after Remark 5.9.

Lemma 5.7. *If $y \in S^+$ then $\sigma^e(y) \in S^+_\infty$ for a unique integer e, necessarily with $e \geq 0$.*

Proof. Let $y_0 = y \in S^+$. By Lemma 5.5(1), $\sigma^d(y_0)$ is defined for all $d \geq 0$. If $y_0 \notin S_\infty^+$, then there exists $d_0 > 0$ such that $\sigma^{d_0}(y_0) = y_1 \in S^+$. Similarly, $\sigma^d(y_1)$ is defined for all $d \geq 0$ and so either $y_1 \in S_\infty^+$ or there is $d_1 > 0$ with $y_2 = \sigma^{d_1}(y_1) = y_2 \in S^+$. If this process continues forever, we will produce an infinite sequence of points $y_0, y_1, y_2, \ldots$ in the finite set S^+ and so $y_i = y_j$ for some $j > i$. But then $\sigma^d(y_i) = y_j = y_i$ for $d = d_i + d_{i+1} + \cdots + d_{j-1} > 0$, contradicting Lemma 5.4(1). Thus the process is finite and $\sigma^e(y_0) \in S_\infty^+$ for some $e \geq 0$.

If $\sigma^{e_1}(y_0) = w \in S_\infty^+$ and $\sigma^{e_2}(y_0) = z \in S_\infty^+$ for some (possibly negative) integers $e_2 > e_1$, then Lemma 5.5(1) implies that $\sigma^{e_2-e_1}(w) = z$, contradicting the definition of S_∞^+. So e is unique. $\qquad\square$

We next study the special properties of the points S_∞^+. The intuitive idea is that these points mark the boundary of bad behavior of σ, in the sense that for $y \in S_\infty^+$ the half-orbit $\{\sigma^d(y) : d \geq 0\}$ behaves just like the half-orbit of an automorphism.

Proposition 5.8. (1) *Let* $y \in S_\infty^+$. *Then* σ^e *is defined and a local isomorphism at* $\sigma^d(y)$ *for all* $e, d \geq 0$.

(2) *Let* $y \in S^+$. *The set of points* $\{\sigma^d(y) : d \geq 0\}$ *is Zariski dense in* Y, *and each of these points is nonsingular.*

Proof. (1) By Lemma 5.5(1), $y_d = \sigma^d(y)$ is defined for each $d \geq 0$. If σ^{-1} is undefined at y_d for some $d \geq 1$ then Lemma 5.5(2) implies that $\sigma^e(y_d) = y_{d+e} \in S^+$ for some $e \geq 0$, contradicting the definition of S_∞^+. Thus σ^{-1} is defined at each y_d with $d \geq 1$ and so, by induction, σ^{-d} is defined at y_d for $d \geq 1$. Thus $\sigma^{-d}(y_d) = y$ and σ^d is a local isomorphism at y. In addition, $\sigma = \sigma^{d+1}\sigma^{-d}$ is defined at y_d for all $d \geq 0$, with $\sigma(y_d) = y_{d+1}$. Finally, σ^e is a local isomorphism at every point y_d with $d \geq 0$, since the inverse map σ^{-e} is defined at y_{d+e}.

(2) Given $y \in S_\infty^+$, the half orbit $P = \{y_d = \sigma^d(y) : d \geq 0\}$ is defined by part (1). Suppose first that $y_i = y_j$ for some $j > i \geq 0$. By part (1), σ^{-i} is defined at both y_i and y_j, and so $y = \sigma^{-i}(y_i) = \sigma^{-i}(y_j) = y_{j-i}$. Then $\sigma^{j-i}(y) = y_{j-i} = y$, contradicting Lemma 5.4(1). So $y_i \neq y_j$ for $j \neq i$ and P is infinite.

Suppose that the Zariski closure $\overline{P}$ of P is a closed subset of Y of dimension 1. Then $\overline{P}$ is a finite union $\overline{P} = C_1 \cup \cdots \cup C_e \cup D$ of irreducible curves C_i together with a (possibly empty) finite set of closed points D. Each C_i contains infinitely many of the y_j, and since both σ and σ^{-1} are defined at every closed point in $P \smallsetminus \{y_0\}$, both σ and σ^{-1} are defined at the generic point μ_i of C_i for each i. In particular, σ and σ^{-1} permute the set $\{\mu_1, \ldots, \mu_e\}$. Since D is finite, if there exists $y_d \in D$, then $\sigma^p(y_d) \in C_j$ for some j and some $p > 0$, which forces $y_d \in \sigma^{-p}(C_j) \subseteq \bigcup C_i$, a contradiction. Thus $D = \emptyset$ and $y \in C_\ell$ for some ℓ. But $\sigma^{e!}(\mu_\ell) = \mu_\ell$ and so Lemma 5.4(2) implies that $C_\ell \cap S^+ = \emptyset$, a contradiction. The only remaining possibility is that P is dense in Y.

Finally, since the singular locus of Y is a proper closed subset and P is dense, some point of P is nonsingular. Since σ is a local isomorphism at each y_d with $d \geq 0$, the local rings $\mathcal{O}_{Y,y_d}$ are isomorphic for all $d \geq 0$ and hence every point y_d is nonsingular. $\qquad\square$

Remark 5.9. Recall from Lemma 2.3(2) that working with left modules in place of right modules interchanges the rôles of ϕ and ψ, and hence it also interchanges the rôles of σ and σ^{-1}. This therefore provides analogues for (S^-, σ^{-1}) of the results just proved for (S^+, σ); formally, by combining Lemma 2.3(2) with the left-hand versions of Lemmas 5.5, 5.7 and Proposition 5.8, one obtains the following results:

(1) Let $y \in Y$ be a closed point where either σ or ϕ^{-1} is undefined. Then $\sigma^e(y)$ is defined and in S^- for some $e \le 0$.

By analogy with the definition of S^+ in (5.6), we let $S_\infty^- = \{y \in S^- : \forall\, n \ge 1,\ \sigma^{-n}(y) \notin S^-\}$ denote the *extremal elements* of S^-.

(2) If $y \in S^-$ then σ is not defined at y. However $\sigma^e(y)$ is defined for all $e \le 0$ and $\sigma^e(y) \in S_\infty^-$ for a unique integer e, necessarily with $e \le 0$.

(3) Let $y \in S_\infty^-$. Then σ^e is defined and a local isomorphism at $\sigma^d(y)$, for all $d, e \le 0$. The points $\sigma^d(y)$ are nonsingular for $d \le 0$.

6. Birationally geometric algebras

We continue to suppose that A satisfies Assumptions 4.1 and let $Y = Y_n$ be the relevant component of the truncated point scheme, for some fixed stable value of n. If we want to show that A is a naïve blowup algebra, then we first have to contract Y to a second variety $\mathbb{X}$ where σ becomes a (biregular) automorphism. This is not possible under the present assumptions on Y. However, an obviously necessary condition for the existence of $\mathbb{X}$ is for σ to be represented by an automorphism of *some* projective model Z of $K = k(Y)$. If we make this extra assumption then $\mathbb{X}$ does exist (see Theorem 7.1) and in this section we set the scene by examining the more elementary consequences of this hypothesis.

We begin with the formal definition.

Definition 6.1. Let B be a cg algebra which is a birationally commutative domain, with graded quotient ring $Q(B) \cong K[t, t^{-1}; \sigma]$. Then B is called *birationally geometric* if there is a projective integral scheme Z with $k(Z) = K$ and an automorphism $\tau \in \operatorname{Aut}(Z)$ inducing σ. If moreover Z is a surface then we say that B is *birationally 2-geometric.*

If B is a cg domain with $Q(B) \cong K[t, t^{-1}; \sigma]$ where K is a finitely generated field extension with $\operatorname{tr.deg} K/k = 1$, then B is automatically birationally geometric—take Z to be the unique nonsingular model of K. The surface case is more subtle, but by the following lemma one can at least assume in the definition that the surface is nonsingular.

Lemma 6.2. *Let B be birationally 2-geometric, with $Q(B) = K[t, t^{-1}; \sigma]$. Then there exists a nonsingular surface Z with $k(Z) = K$ and $\tau \in \operatorname{Aut}(Z)$ inducing σ.*

Proof. By hypothesis, there is some projective surface Z' with $K = k(Z')$ and $\tau' \in \text{Aut}(Z')$ inducing σ. Now take a resolution of singularities $\pi : Z \to Z'$ following, say, [Li, Remark B, p. 155]) and check that τ' lifts to an automorphism τ of Z. $\qquad\square$

Remark 6.3. Keep the hypotheses from the lemma. If K is not the function field of a ruled surface then there exists a unique minimal nonsingular model Z of K and so σ is necessarily induced by an automorphism of Z (see [DF, Proposition 7.5]). On the other hand, if K is the function field of a ruled surface, then there do exist automorphisms $\sigma : K \to K$ such that σ is not induced by an automorphism of any projective model Z of K. For example, if $K = k(u, v)$ is purely transcendental, then the automorphism σ defined by $\sigma(u) = u$ and $\sigma(v) = uv$ is one such example. Further examples are constructed in [DF, Remark 7.3].

For the rest of this section, we will assume that A is a birationally 2-geometric cg domain that is generated in degree 1, with graded quotient ring $Q = K[t, t^{-1}; \sigma]$. We continue to use the terminology from Notation 5.2; in particular we fix $Y = Y_n$ for some stable value of n and note that $k(Y) = K$ with induced birational map $\sigma : Y \dashrightarrow Y$. In Section 5, we proved various special properties for the contracted points S_∞^+, and by symmetry, for the points in S_∞^-. The main application of the birationally 2-geometric hypothesis is to show that these two sets are closely interrelated.

Proposition 6.4. *Let A be a birationally 2-geometric cg domain that is generated in degree 1 and assume that $S_\infty^- \neq \emptyset$; say $S_\infty^- = \{s_1, \ldots, s_N\}$. Then S_∞^+ may be uniquely written as $S_\infty^+ = \{t_1, \ldots, t_N\}$ so that, for each i, there exists a unique $d_i \geq 2$ with $\sigma^{d_i}(s_i) = t_i$ and $\sigma^{-d_i}(t_i) = s_i$. Moreover, σ^{d_i} is a local isomorphism at s_i.*

Remark 6.5. Note that, by the proposition and Remark 5.9, $S_\infty^- \neq \emptyset$ if and only if $S_\infty^+ \neq \emptyset$. Moreover, by Remark 5.9 and Lemmas 5.5(1,2) and 5.7, $S_\infty^+ = \emptyset$ if and only if σ is an automorphism of Y.

Proof. Fix $y \in S_\infty^+$. By Lemma 5.8, the set $P = \{y_m = \sigma^m(y) : m \geq 0\}$ is defined. Moreover, σ is defined and a local isomorphism at each point of P, and P is a dense set in Y consisting of nonsingular points. In particular P is infinite and so $y_i \neq y_j$ for $i \neq j$. Let $g : \widetilde{Y} \to Y$ be the normalization of Y and note that g^{-1} defines an isomorphism locally at each nonsingular point of Y, in particular at each point of P. Write $\widetilde{P} = g^{-1}(P) = \{\widetilde{y}_m = g^{-1}(y_m)\}$ and $\widetilde{\sigma} = g^{-1}\sigma g : \widetilde{Y} \dashrightarrow \widetilde{Y}$ for the induced birational map. Clearly $\widetilde{P}$ is dense in $\widetilde{Y}$, and $\widetilde{y}_i \neq \widetilde{y}_j$ for $i \neq j$. By Lemma 6.2, there exists a nonsingular model Z of K and a birational map $f : Z \dashrightarrow \widetilde{Y}$ such that $\tau = f^{-1}\widetilde{\sigma}f$ is an automorphism of Z.

Since f is a birational map between normal surfaces, there are at most finitely many points where f fails to be defined [Ha, Lemma V.5.1], say $G \subseteq Z$, and finitely many points, say $H \subseteq \widetilde{Y}$, where f^{-1} is undefined. Since H can contain only finitely many of the $\widetilde{y}_i$, there exists $m_0 \geq 0$ such that f^{-1} is defined at $\widetilde{y}_m$ for $m \geq m_0$. Set $z_m = f^{-1}(\widetilde{y}_m)$ for $m \geq m_0$ and note that $\{z_m : m \geq m_0\}$ is dense in Z. We compute that

$$\tau(z_m) = \tau f^{-1}(\widetilde{y}_m) = f^{-1}g^{-1}\sigma g(\widetilde{y}_m) = f^{-1}g^{-1}\sigma(y_m) = f^{-1}g^{-1}(y_{m+1}) = z_{m+1} \qquad \text{for all } m \geq m_0.$$

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Thus $\{z_m : m \geq m_0\}$ extends to a full dense τ-orbit $\{z_m = \tau^{m-m_0}(z_{m_0}) : m \in \mathbb{Z}\}$ and again, $z_i \neq z_j$ for $i \neq j$.

Recall that $y = y_0 \in S_\infty^+$, and so σ^{-1} is not defined at y by Lemma 5.5(1). On the other hand, since G is finite we must have $z_m \notin G$ for all $m \ll 0$. Hence

$$\sigma^m(y) = \sigma^{m-m_0}(y_{m_0}) = gf\tau^{m-m_0}f^{-1}g^{-1}(y_{m_0}) = gf\tau^{m-m_0}(z_{m_0}) = gf(z_m)$$

is defined for all $m \ll 0$. Let $a < 0$ be the largest negative integer for which $\sigma^a(y)$ is defined, so $a \leq -2$. Since σ^{a+1} is not defined at y, the map σ is not defined at $\sigma^a(y)$. By Remark 5.9(1) there exists $b \leq 0$ such that $\sigma^b(\sigma^a(y)) = \sigma^{a+b}(y)$ is defined and belongs to S^-, and then by Remark 5.9(2) there exists an integer $c \leq 0$ with $\sigma^c\sigma^{a+b}(y) = \sigma^{a+b+c}(y) \in S_\infty^-$. Set $e = a+b+c \leq -2$. If e' is a second integer with $\sigma^{e'}(y) \in S_\infty^-$, say with $e' > e$, then Remark 5.9(3) implies that $\sigma^{e-e'}$ is defined at $\sigma^{e'}(y)$, with $\sigma^{e-e'}\sigma^{e'}(y) = \sigma^e(y)$, contradicting the definition of S_∞^-. So e is unique.

A symmetric argument shows that for any $s_i \in S_\infty^-$ there exists a unique integer d_i, necessarily with $d_i \geq 2$, such that $\sigma^{d_i}(s_i) \in S_\infty^+$. Define $t_i = \sigma^{d_i}(s_i)$; applying the first part of the proof to t_i, we obtain e_i such that $z = \sigma^{e_i+d_i}(s_i) \in S_\infty^-$. The definition of S_∞^- ensures that $e_i + d_i \geq 0$. If $e_i + d_i > 0$, then Remark 5.9(3) implies that $\sigma^{-e_i-d_i}$ is defined at z and, of course, $\sigma^{-e_i-d_i}(z) = s_i$, again contradicting the definition of S_∞^-. Thus $e_i = -d_i$ and $z = s_i$. Finally, beginning with any $t \in S_\infty^+$, there exists $e \leq -2$ with $\sigma^e(t) = s_i$ for some i. Since $\sigma^{d_i}(s_i) = t_i$ a similar argument forces $e = -d_i$ and $t = t_i$. This defines the bijection between S_∞^+ and S_∞^- and it is immediate that σ^{d_i} is a local isomorphism at s_i since the inverse map σ^{-d_i} is defined at t_i. $\qquad\square$

Notation 6.6. This notation will only apply when A is a birationally 2-geometric cg domain for which σ is not an automorphism of Y; equivalently, by Remark 6.5, when $S_\infty^- \neq \emptyset$. In this case we fix the notation from Proposition 6.4 and write

$$\Delta = \max\{d_i : 1 \leq i \leq N\}.$$

For $1 \leq i \leq N$ and $j \in \mathbb{Z}$, define

$$C_{ij} = \{\text{closed points } z \in Y \text{ such that } \sigma^{-j} \text{ is defined at } z \text{ with } \sigma^{-j}(z) = s_i\}$$

and write

$$D = \bigcup_{\substack{-\Delta < j < -\Delta+d_i \\ 1 \leq i \leq N}} C_{ij} \quad \text{and} \quad E = \bigcup_{\substack{0 < j < d_i \\ 1 \leq i \leq N}} C_{ij}.$$

When C_{ij} happens to be a singleton we write $C_{ij} = \{c_{ij}\}$.

Example 6.7. As an illustration of these concepts, consider Example 4.3 for stable n. Recall that, in that example, the $S^\pm$ are singletons with $S_\infty^- = S^- = \{r_{n+1}\}$ and $S_\infty^+ = S^+ = \{r_0\}$; thus $N = 1$ with $s_1 = c_{10} = r_{n+1}$ and $t_1 = c_{1,n+1} = r_0$. The sets $C_{11}, \ldots, C_{1n}$ are the exceptional divisors $C_j^{(n)}$ from (4.3), except that the ordering is reversed: $C_{11} = C_n^{(n)}, \ldots, C_{1n} = C_1^{(n)}$. All the other C_{uv} are singletons. Finally $\Delta = d_1 = n+1$, and σ^Δ is defined except at the points $\{c_{1,-n}, c_{1,-n+1} \ldots c_{1,-1}\} = \{r_{2n+1}, \ldots, r_{n+2}\}$.

The following basic properties of the sets C_{ij} will prove useful.

Lemma 6.8. *Keep the assumptions and notation from Notation 6.6.*

    (1) *If $y \notin C_{ij}$ for all i, j, then σ^m is defined and a local isomorphism at y for all $m \in \mathbb{Z}$.*

    (2) *If $y \in C_{ij}$ for some i and j, then σ^d is defined at y for all $d \leq -j$ and $d \geq -j + d_i$.*

    (3) *If $j \notin [1, d_i - 1]$, then $C_{ij} = \{c_{ij}\}$ is a singleton, with $c_{ij} = \sigma^j(s_i)$ a nonsingular point. The map σ is defined and a local isomorphism at c_{ij} for $j < 0$ and $j \geq d_i$.*

    (4) *The sets C_{ij} are pairwise disjoint closed subsets of Y.*

    (5) *Suppose that $y \in C_{ij}$ for some i and j. If σ^d is defined at y for some $d \in \mathbb{Z}$, then $\sigma^d(y) \in C_{i,j+d}$.*

Proof. (1) Suppose that $y \in Y$ is a point where some σ^m is either not defined or is defined but fails to be a local isomorphism. In the former case, set $z = y$; in the latter case, set $z = \sigma^m(y)$ and note that σ^{-m} is not defined at z. In either case, some nonzero power of σ is undefined at z, so by Lemma 5.5(2), Lemma 5.7 and Remark 5.9(1,2), we have either $\sigma^a(z) \in S_\infty^+$ or $\sigma^a(z) \in S_\infty^-$ for some $a \in \mathbb{Z}$. Applying Proposition 6.4, if necessary, we conclude that $\sigma^b(z) = \sigma^c(y) \in S_\infty^-$ for some $b, c \in \mathbb{Z}$. Thus $y \in C_{ij}$ for some i, j.

(2) When $d \leq -j$, Remark 5.9(3) implies that $\sigma^d(y) = \sigma^{d+j}\sigma^{-j}(y) = \sigma^{d+j}(s_i)$ is defined. On the other hand, Proposition 6.4 implies that $\sigma^{d_i}(s_i) = t_i$ and so, for $d \geq -j + d_i$, Proposition 5.8(1) implies that $\sigma^d(y) = \sigma^{d+j}(s_i) = \sigma^{d-(d_i-j)}(t_i)$ is defined.

(3) If $j \notin [1, d_i - 1]$, then part (2) implies that $\sigma^j(s_i)$ is defined. Hence, for any $z \in C_{ij}$ one has $z = \sigma^j \sigma^{-j}(z) = \sigma^j(s_i)$. Thus $C_{ij} = \{\sigma^j(s_i)\}$ is a singleton. For $j \geq d_i$, Proposition 5.8 implies that c_{ij} is nonsingular and σ is a local isomorphism at c_{ij}. Similarly, when $j < 0$, Remark 5.9(3) says that c_{ij} is nonsingular and that σ^{-1} is a local isomorphism at $c_{i,j+1}$. Thus σ is a local isomorphism at c_{ij}.

(4,5) Suppose that $y \in C_{ij} \cap C_{i'j'}$, where $j \geq j'$. Then $\sigma^{-j}(y) = s_i$ and $\sigma^{-j'}(y) = s_{i'}$. Since $s_i \in S_\infty^-$, Remark 5.9(3) implies that $\sigma^{j'-j}(s_{i'})$ is defined and thus equal to s_i. By the definition of S_∞^- this forces $j = j'$ and $s_i = s_{i'}$, whence $i = i'$.

Before completing the proof of (4), we prove (5). Suppose that σ^d is defined at $y \in C_{ij}$ but that $\sigma^d(y) \notin C_{uv}$ for any u, v. Then part (1) implies that $\sigma^{-d-j}\sigma^d(y) = \sigma^{-j}(y)$ is defined. But this element must equal s_i and so $\sigma^d(y) \in C_{i,j+d}$. This contradiction implies that $\sigma^d(y) \in C_{uv}$ to begin with. But this implies that $\sigma^{-v+d}(y) = s_u$ which, by the previous paragraph, forces $u = i$ and $v = d + j$.

We return to the proof of (4) and suppose that y is in the closure of some C_{ij}. By parts (1) and (2), and for any $m \gg 0$, σ^m is defined in a neighbourhood of y. However, if $z \in C_{ij}$, then parts (2) (3) and (5) imply that $\sigma^m(z) = c_{i,j+m}$ for such an integer m. Therefore, by continuity, $\sigma^m(y) = c_{i,j+m}$. But then $\sigma^{-j}(y) = \sigma^{-j-m}(c_{i,j+m}) = s_i$ and so $y \in C_{ij}$ after all. Thus C_{ij} is closed. $\square$

The significance of the computations in this section is that we tightly prescribe the closed points where σ^Δ is either undefined or not a local isomorphism.

Corollary 6.9. *Keep the assumptions and notation from Notation 6.6.*

(1) *The points where σ^Δ is undefined are all contained in the finite set D.*

(2) *The set of points in $U = Y \smallsetminus D$ where σ^Δ is not a local isomorphism is contained in the closed (but possibly infinite) set E.*

Proof. (1) Suppose that σ^Δ is undefined at the point y. By Lemma 6.8(1), $y \in C_{ij}$ for some i, j and then Lemma 6.8(2) implies that $-\Delta < j < -\Delta + d_i$. Thus $y \in D$. Given i, j with $-\Delta < j < -\Delta + d_i$, then $j \leq 0$ and so the set C_{ij} is a singleton by Lemma 6.8(3). As there are finitely many such pairs $\{i, j\}$, the set D is finite.

(2) If $y \in Y \smallsetminus D$ is a point where σ^Δ is not a local isomorphism, then Lemma 6.8(1) again shows that $y \in C_{ij}$ for some i, j. Then $\sigma^{-\Delta}$ cannot be defined at $z = \sigma^\Delta(y)$. By Lemma 6.8(2) this forces $0 < j < d_i$ and $y \in E$. $\qquad\square$

7. Constructing the automorphism

We continue to assume that A is a birationally 2-geometric cg domain, generated in degree 1, with graded quotient ring $Q = K[t, t^{-1}; \sigma]$ and we maintain the notation from Proposition 6.4 and Notation 6.6. We are now ready to show that the stable scheme Y can be contracted to a second projective scheme $\mathbb{X}$ such that the birational map σ descends to an automorphism on $\mathbb{X}$. The idea behind the result is rather easy: If $y \notin \bigcup_{i,j} C_{ij}$ then Lemma 6.8(1) implies that σ^m is defined at y for all $m \in \mathbb{Z}$. On the other hand, Lemma 6.8(2,3) implies that σ^m is defined on all of C_{ij} and $\sigma^m(C_{ij}) = \{c_{i,j+m}\}$ is a singleton provided that $m \notin [-j, -j + d_i]$. Thus, by patching U and $\sigma^m(U')$ for appropriate open subsets U and U' of Y and some $m \gg 0$, we may hope to construct a new scheme Y' for which the images of the C_{ij} are singletons $\{c'_{ij}\}$. At this point it is easy to define an action of σ on Y' by setting $\sigma(c'_{ij}) = c'_{i,j+1}$. That this idea works forms the content of the next theorem.

Theorem 7.1. *Let A be a birationally 2-geometric cg domain that is generated in degree 1 and write $Y = Y_n$ for some stable value of n with its induced birational action of σ. Then there exists a surjective birational morphism $\pi : Y \to \mathbb{X}$, where $\mathbb{X} = \mathbb{X}_n$ is a projective surface, such that:*

(1) *$\widetilde{\sigma} = \pi \sigma \pi^{-1}$ is an automorphism of $\mathbb{X}$, and*

(2) *$\pi^{-1} : \mathbb{X} \dashrightarrow Y$ is defined except at a finite set P of nonsingular closed points of $\mathbb{X}$, each of which lies on an dense $\widetilde{\sigma}$-orbit.*

Remark 7.2. In the special case of Example 4.3, $\mathbb{X} \cong \mathbb{P}^2$ and the morphism $\pi : Y \to \mathbb{X}$ is simply the map γ_n from that example that blows up $\mathbb{P}^2$ at the n points $\{r_1, \ldots, r_n\}$.

Proof. If $S_\infty^- = \emptyset$ then σ is an automorphism by Remark 6.5 and so we can simply take $\mathbb{X} = Y$ and let π be the identity. We may therefore assume that $S_\infty^- \neq \emptyset$ and hence, by Remark 6.5 again, that $S_\infty^+ \neq \emptyset$.

Define the integer Δ and the sets C_{ij}, D, and E as in Notation 6.6. Set $U_1 = Y \smallsetminus D$ and notice that, by construction, $E \subset U_1$. By Corollary 6.9(1,2) we know that σ^Δ is defined at every point of U_1, while σ^Δ

is defined and a local isomorphism on the open subset $U_1 \smallsetminus E$ of Y. By the definition of the C_{ij}, we also know that $U_2 = \sigma^\Delta(U_1)$ is equal to $Y \smallsetminus E$, and so, in particular, U_2 is open in Y with $Y = U_1 \cup U_2$. By Lemma 6.8(2), $\sigma^\Delta(E) = \bigcup\{c_{ij} : 1 \le i \le N, \Delta < j < \Delta + d_i\}$ is a finite set of points.

We now want to construct the surface $\mathbb{X}$. Formally, we will do this by glueing two open sets, but the construction is probably clearer if we first indicate what is happening at the level of elements. So, set $H = Y \smallsetminus \{C_{ij} : 1 \le i \le N, j \in \mathbb{Z}\}$, let $\widetilde{H}$ be an isomorphic copy of H and for each i, j as above define an abstract point $\widetilde{c}_{ij}$. Then the new scheme $\mathbb{X}$ will simply be $\widetilde{H} \cup \{\widetilde{c}_{ij} : 1 \le i \le N, j \in \mathbb{Z}\}$, with the projection $\pi : Y \to \mathbb{X}$ defined by the given isomorphism $H \to \widetilde{H}$ together with $\pi(C_{ij}) = \widetilde{c}_{ij}$.

To make this formal, and especially to define the topology on $\mathbb{X}$, recall that Y is covered by the open sets U_1 and U_2. Let $V_1 = U_2$ and $V_2 = U_2$ be two new copies of U_2, and set $V_{12} = U_2 \smallsetminus \sigma^\Delta(E)$ and $V_{21} = U_1 \smallsetminus E$; thus V_{12} is an open subset of V_1 and V_{21} is open in V_2. The morphism $\sigma^\Delta|_{U_1}$ restricts to an isomorphism $f : V_{21} \to V_{12}$. Now think of V_1 and V_2 as new quasi-projective schemes and forget their embeddings in Y. Then we can use the isomorphism f to glue V_2 and V_1 along their open subschemes V_{21} and V_{12} to obtain a new scheme $\mathbb{X}$. Define a morphism $\pi : Y \to \mathbb{X}$ by defining $\pi|_{U_1} = \sigma^\Delta : U_1 \to U_2 = V_1$ and letting $\pi|_{U_2} : U_2 \to U_2 = V_2$ be the identity map. It is immediate that these definitions agree on $U_1 \cap U_2 = V_{21}$, and so π is indeed a well-defined morphism. It is obviously surjective and birational. By construction, π is a local isomorphism except possibly at points in the set $E = Y \smallsetminus U_2$, and so the morphism π^{-1} is defined except possibly at the finite set of points $P = \pi(E) \subset \mathbb{X}$.

As was noted earlier, if $C_{ij} \subseteq E$, then Lemma 6.8(3,5) implies that $\widetilde{c}_{ij} = \pi(C_{ij}) \cong \sigma^\Delta(C_{ij})$ is indeed a closed point of $\mathbb{X}$, while for $C_{ij} \not\subseteq E$ it follows from Lemma 6.8(3,4) that both C_{ij} and $\widetilde{c}_{ij} = \pi(C_{ij})$ consist of a single closed point. Since $H = Y \smallsetminus \{C_{ij} : 1 \le i \le N, j \in \mathbb{Z}\} \subset U_1 \cap U_2$, this justifies the comments of the paragraph before last. We next want prove that the birational map $\widetilde{\sigma} = \pi \sigma \pi^{-1} : \mathbb{X} \dashrightarrow \mathbb{X}$ is actually an automorphism and once again this is particularly easy to describe at the level of sets: $\widetilde{\sigma}(\widetilde{c}_{ij}) = \widetilde{c}_{i,j+1}$, while the action of $\widetilde{\sigma}$ on $\widetilde{H} = \pi(H)$ is induced from the fact that σ is defined on H.

Formally, let $E' = E \cup \{s_i : 1 \le i \le N\}$ and $U_2' = Y \smallsetminus E' \subseteq U_2$. Also, write $P' = \pi(Z') = P \cup \pi(\{s_i\})$ and $V_2' = \pi(U_2') = \mathbb{X} \smallsetminus P'$. By Lemma 6.8(1,3), $\sigma : U_2' \to U_2$ is defined and therefore an open immersion. Since $\pi|_{U_2} : U_2 \to V_2$ is an isomorphism, this implies that $\widetilde{\sigma} = \pi \sigma \pi^{-1} : V_2' \to V_2$ is also an immersion. Next, set $U_1' = U_1 \smallsetminus \bigcup_i C_{i,-\Delta}$ and $V_1' = \pi(U_1') \subseteq V_1$. Recall that V_1 was defined to be a copy of U_2 and write this identification as $g : V_1 \to U_2$. By the definition of $\pi|_{U_1}$, we see that $g\pi = \sigma^\Delta$ as morphisms from U_1 to U_2. Since g maps V_1' isomorphically onto U_2', it follows that $\widetilde{\sigma}' = g^{-1}\sigma g$ is defined as a morphism from V_1' to V_1 and is even an immersion. As birational maps, $g^{-1}\sigma g = \pi \sigma^{-\Delta} \sigma \sigma^\Delta \pi^{-1} = \pi \sigma \pi^{-1}$. Thus $\widetilde{\sigma}$ and $\widetilde{\sigma}'$ represent the same birational map $\mathbb{X} \dashrightarrow \mathbb{X}$, and so they patch together to give a morphism $\widetilde{\sigma} : V_1' \cup V_2' \to V_1 \cup V_2$.

From the definitions of D and E in Notation 6.6 it is clear that $Y = U_1' \cup U_2'$. Applying π, this gives $V_i' \cup V_2' = \mathbb{X}$. Thus $\widetilde{\sigma} : \mathbb{X} \to \mathbb{X}$ is a globally defined birational morphism which, by the previous paragraph, is a local isomorphism at every point. Thus the inverse birational map is defined at every point of $\mathbb{X}$ and

so $\tilde{\sigma}$ must be an isomorphism. It is also clear that, at the level of points, $\tilde{\sigma}$ has the form described in the paragraph before last; in particular $\tilde{\sigma}(\tilde{c}_{ij}) = \tilde{c}_{i,j+1}$ for each i, j.

It remains to prove that $\mathbb{X}$ is projective, which follows from the next, more general proposition. $\qquad\square$

Lemma 7.3. *Let W be an integral scheme of finite type. Let $\mathcal{M}_1, \mathcal{M}_2$ be invertible sheaves on W with $M_i \subseteq H^0(X, \mathcal{M}_i)$ such that each M_i generates $\mathcal{M}_i$. Suppose that the map $W \to \mathbb{P}(M_1^*)$ is an immersion. If M denotes the image of $M_1 \otimes M_2$ in $H^0(X, \mathcal{M}_1 \otimes \mathcal{M}_2)$, then the map $W \to \mathbb{P}(M^*)$ is an immersion.* $\qquad\square$
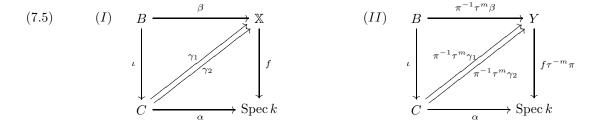
Proposition 7.4. *Let Y be a projective k-scheme and $\mathbb{X}$ be an integral k-scheme of finite type with a birational surjective morphism $\pi : Y \to \mathbb{X}$. Assume that $\mathbb{X}$ has an automorphism τ and a finite set P of nonsingular closed points such that*

  (i) *π^{-1} is defined on $U = \mathbb{X} \smallsetminus P$, and*

  (ii) *every point of P lies on an infinite orbit of τ.*

Then $\mathbb{X}$ is a projective scheme.

Proof. We begin by proving that $\mathbb{X}$ is separated and proper, for which we use the valuative criteria from [Ha, Section II.4]. By [Ha, Exercise II.4.11(c)], we need only consider discrete valuation rings in the criteria. So, fix a discrete valuation ring R with field of fractions L and write $\iota : B = \operatorname{Spec} L \hookrightarrow C = \operatorname{Spec} R$ for the induced morphism. Let c denote the closed point of C and write b for the generic point of C (and B). We also have the structure map $f : \mathbb{X} \to \operatorname{Spec} k$ and assume that there exist morphisms $\alpha : C \to \operatorname{Spec} k$, $\beta : B \to \mathbb{X}$ such that $\alpha\iota = f\beta$.
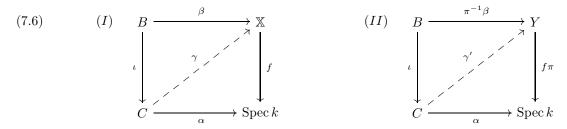
We first prove that $\mathbb{X}$ is separated, so assume that there exist two morphisms $\gamma_i : C \to \mathbb{X}$ both of which make (7.5)(I) commute.

(7.5) $\qquad$ (I)

Write $y_1 = \gamma_1(c)$ and $y_2 = \gamma_2(c)$. By assumption (ii) we choose $m \gg 0$ such that $\tau^m(y_1)$ and $\tau^m(y_2)$ lie in $U = \mathbb{X} \smallsetminus P$; this forces $\tau^m\beta(b) \in U$ as well. Then $\pi^{-1}\tau^m\gamma_i : C \to Y$ is a well-defined morphism for $i = 1, 2$, and similarly $\pi^{-1}\tau^m\beta : B \to Y$ is well-defined. Moreover, the diagram (7.5)(II) commutes. Since Y is separated, [Ha, Theorem II.4.3] implies that $\pi^{-1}\tau^n\gamma_1 = \pi^{-1}\tau^n\gamma_2$ and hence $\gamma_1 = \gamma_2$. By [Ha, Theorem II.4.3], again, this implies that $\mathbb{X}$ is separated.

We now prove properness. By hypothesis, there exists $m \gg 0$ such that $\mathbb{X}$ is covered by the open sets $U \cong \pi^{-1}(U)$ and $\tau^m(U)$, so $\mathbb{X}$ is noetherian. By the Valuative Criterion for Properness [Ha, Theorem II.4.7]

it therefore suffices to find a morphism $\gamma : C \to \mathbb{X}$ making (7.6)(I) commute.

(7.6)   (I)

$$
\begin{array}{ccc}
B & \xrightarrow{\ \beta\ } & \mathbb{X} \\
{\scriptstyle\iota}\downarrow & {\scriptstyle\gamma}\ \nearrow & \downarrow{\scriptstyle f} \\
C & \xrightarrow[\ \alpha\ ]{} & \operatorname{Spec} k
\end{array}
$$

(II)

$$
\begin{array}{ccc}
B & \xrightarrow{\ \pi^{-1}\beta\ } & Y \\
{\scriptstyle\iota}\downarrow & {\scriptstyle\gamma'}\ \nearrow & \downarrow{\scriptstyle f\pi} \\
C & \xrightarrow[\ \alpha\ ]{} & \operatorname{Spec} k
\end{array}
$$

First, if $x = \beta(b)$ is a closed point of $\mathbb{X}$ then clearly putting $\gamma(c) = \gamma(b) = y$ gives a well-defined map γ making (7.6)(I) commute. Alternatively, if x is not a closed point, then $x \notin P$ and so π^{-1} is defined at x. As Y is proper, there then exists $\gamma' : C \to Y$ making the diagram (7.6)(II) commute. Thus $\gamma = \pi\gamma'$ makes (7.6)(I) commute and $\mathbb{X}$ is proper.

Before proving that $\mathbb{X}$ is projective, we prove the following easy facts. The constant sheaf of rational functions on Y and on $\mathbb{X}$ will be written $\mathcal{K}$.

Sublemma 7.7. *Keep the notation from the proposition. Suppose that $\mathcal{M} \subset \mathcal{K}$ is an invertible sheaf on Y and set $\mathcal{F} = \pi_*\mathcal{M} \subset \mathcal{K}$. Then:*

(1) *The reflexive hull $\mathcal{L} = \mathcal{F}^{**} = \mathcal{H}om_{\mathcal{O}_{\mathbb{X}}}(\mathcal{H}om_{\mathcal{O}_{\mathbb{X}}}(\mathcal{F}, \mathcal{O}_{\mathbb{X}}), \mathcal{O}_{\mathbb{X}})$ of $\mathcal{F}$ is invertible.*

(2) *$\mathcal{L}$ is the unique reflexive (or invertible) sheaf on $\mathbb{X}$ satisfying $\mathcal{L}|_U = \mathcal{F}|_U$.*

(3) *$\mathcal{L}^{\otimes n}$ is the reflexive hull of $\mathcal{F}^n$, where the multiplication takes place inside $\mathcal{K}$.*

Proof. (1) Since $\mathcal{F}|_U = \mathcal{M}|_{\pi^{-1}(U)}$ under the identification of $\mathcal{O}_U$ with $\mathcal{O}_{\pi^{-1}(U)}$, the sheaf $\mathcal{L}$ is locally free on U. On the other hand, if $p \in P$, then $\mathcal{O}_{\mathbb{X},p}$ is regular and hence factorial by [Ei, Theorem 19.19] and so any reflexive ideal of $\mathcal{O}_{\mathbb{X},p}$ is cyclic. Thus $\mathcal{L}$ is locally free at each point of $\mathbb{X}$.

(2) We need only prove this locally at a closed point $y \in P$. If $\mathcal{Q}$ is a second reflexive sheaf with $\mathcal{Q}|_U = \mathcal{F}|_U$, then certainly $(\mathcal{Q}_y)_{\mathfrak{p}} = (\mathcal{F}_y)_{\mathfrak{p}} = (\mathcal{L}_y)_{\mathfrak{p}}$ for all height one prime ideals $\mathfrak{p}$ of the regular local ring $\mathcal{O}_{\mathbb{X},y}$. But $\mathcal{Q}_y = \bigcap\{(\mathcal{Q}_y)_{\mathfrak{p}} : \mathfrak{p} \text{ height } 1\}$. Thus $\mathcal{Q}_y = \mathcal{L}_y$ and hence $\mathcal{Q} = \mathcal{L}$.

(3) Since $\mathcal{L}^{\otimes n}|_U = \mathcal{F}^n|_U = (\pi_*(\mathcal{M}^{\otimes n}))|_U$, this follows from part (2) applied to $\mathcal{M}^{\otimes n}$. $\qquad\square$

We return to the proof of the proposition and prove that $\mathbb{X}$ is projective. Let $\mathcal{M} \subset \mathcal{K}$ be any very ample invertible sheaf on Y and set $\mathcal{F} = \pi_*\mathcal{M}$ with reflexive hull $\mathcal{L} = \mathcal{F}^{**}$. By the sublemma $\mathcal{L}$ is locally free. Since $\mathcal{M}$ is generated by its sections $M = \mathrm{H}^0(Y, \mathcal{M})$ and $M = \mathrm{H}^0(\mathbb{X}, \mathcal{F}) \subseteq \mathrm{H}^0(\mathbb{X}, \mathcal{L})$, the sections $\mathrm{H}^0(\mathbb{X}, \mathcal{L})$ do at least generate the sheaf $\mathcal{L}|_U = \mathcal{F}|_U \cong \mathcal{M}|_{\pi^{-1}(U)}$ as an $\mathcal{O}_U$-module. Since $P = \mathbb{X} \smallsetminus U$ is finite and $\mathbb{X}$ is proper, it therefore follows from [Fu, Corollary 1.14] that, for any $n \gg 0$, the sheaf $\mathcal{L}^{\otimes n}$ is generated by its global sections $V_1 = \mathrm{H}^0(\mathbb{X}, \mathcal{L}^{\otimes n})$. For $r \in \mathbb{Z}$, write $\mathcal{N}_r = (\tau^{-r})^*(\mathcal{L}^{\otimes n})$.

Let $\theta : \mathbb{X} \to \mathbb{P}(V_1^*) = \mathbb{P}^d$ denote the map determined by V_1 and write $V_2 = \mathrm{H}^0(Y, \mathcal{M}^{\otimes n})$. Then Sublemma 7.7(3) implies that $V_2 = \mathrm{H}^0(\mathbb{X}, \pi_*\mathcal{M}^{\otimes n}) \subseteq \mathrm{H}^0(Y, \mathcal{L}^{\otimes n}) = V_1$. Since $\mathcal{M}^{\otimes n}$ is very ample, V_2 defines a closed immersion of Y into $\mathbb{P}(V_2^*)$ and hence an immersion $\theta' : \pi^{-1}(U) \hookrightarrow \mathbb{P}(V_2^*)$. Since θ' factors through θ, this ensures that θ defines an immersion of U into $\mathbb{P}(V_1^*)$.

By hypothesis (ii) of the proposition, $\mathbb{X} = U \cup \tau^m(U)$ for $m \gg 0$ and, for some such m, we consider the map $\alpha : \mathbb{X} \to \mathbb{P}(V_3^*)$ determined by $V_3 = \mathrm{H}^0(\mathbb{X}, \mathcal{N}_0 \otimes \mathcal{N}_m)$. Since the $\mathcal{N}_r$ are generated by their sections, we can apply Lemma 7.3 with $W = U$ to show that the restriction of α to U is an immersion. For exactly the same reasons the map $\theta_1 : \mathbb{X} \to \mathbb{P}^d$ determined by $\mathrm{H}^0(\mathbb{X}, \mathcal{N}_m)$ immerses $\tau^m(U)$ into $\mathbb{P}^d$ and so the restriction of α to $\tau^m(U)$ is also an immersion.

Since α is defined on all of $\mathbb{X} = U \cup \tau^m(U)$, we can define C to be the closure in $\mathbb{P}(V_3^*)$ of $\alpha(\mathbb{X})$. Since $\mathbb{X}$ is proper and irreducible, $C = \alpha(\mathbb{X})$ and obviously the corresponding map $\beta : \mathbb{X} \to C$ is birational. The conclusion of the previous paragraph implies that β is a local isomorphism at every point of $\mathbb{X}$. Therefore, for any point $c \in C$ we can define an inverse map to β in a neighbourhood of c and so the inverse birational map β^{-1} is defined at every point of C. Hence $\beta : \mathbb{X} \to C$ is an isomorphism. In other words, $\alpha : \mathbb{X} \to \mathbb{P}(V_3^*)$ is a closed immersion and $\mathbb{X}$ is projective. $\qquad\square$

Remark 7.8. For future reference we note that the last part of the proof of Proposition 7.4 proves the following fact: Suppose that $\mathcal{M}$ is a very ample invertible sheaf on Y and let $\mathcal{B} = (\pi_* \mathcal{M})^{**}$. Then, for all $q \gg 0$, the sheaf $\mathcal{B} \otimes (\tau^q)^* \mathcal{B}$ is an ample invertible sheaf on $\mathbb{X}$.

Although the construction of the scheme $\mathbb{X}_n$ in Theorem 7.1 depends upon n, this is unimportant as there is a natural isomorphism $\mathbb{X}_n \cong \mathbb{X}_{n+1}$ for any stable value of n. Before proving this we need a technical lemma.

Lemma 7.9. *Let $y \in Y = Y_n$, for a fixed stable value of n. Then there exists $m_0 \geq 0$ such that, for all $m \geq m_0$ and $r \geq 0$, σ^m is defined at y and the birational map ϕ^{-r} is defined at $\sigma^m(y)$.*

Proof. Let $w \in Y$ and suppose that ϕ^{-r} is not defined at w for some $r \geq 1$. If we choose the minimal such r then ϕ^{-1} is undefined at $\widetilde{w} = \phi^{-r+1}(w) \in Y_{n+r-1}$. Since $n + r - 1$ is again a stable value, Remark 5.9(1) implies that $\sigma^e(\widetilde{w}) \in S_{n+r-1}^-$ for some $e \leq 0$. Thus $\sigma^{e+r-1} = \psi^{r-1}\sigma^e\phi^{-r+1}$ is defined at w and, by Lemma 4.4(1), $\sigma^{e+r-1}(w) \in S^-$. Using Notation 6.6 and Remark 5.9(2), it follows that $w \in C_{i,j}$ for some i, j.

Return to the proof of the lemma and suppose that $y \notin C_{ij}$ for any i, j. Then Lemma 6.8(1) implies that $y_m = \sigma^m(y)$ is defined for all m. If ϕ^{-r} is undefined at $w = y_m$ for some $m \geq 0$ and $r \geq 0$, then the previous paragraph shows that $y_m \in C_{ij}$ for some i, j. This forces $y \in C_{i,j-m}$, a contradiction. Thus in this case we may take $m_0 = 0$.

Otherwise, $y \in C_{ij}$ some i, j, and so $\sigma^e(y) = t_i \in S_\infty^+$ for $e = d_i - j$. Set $z = t_i$ and notice that, by Lemma 4.4(2), ϕ^{-r} is defined at z for all $r \geq 0$. For $d \geq 0$, the map σ^d is defined and a local isomorphism at z by Proposition 5.8. Hence $\phi^{-r} = \psi^d\phi^{-r-d}\sigma^{-d}$ is also defined at $z_d = \sigma^d(z)$ for all $r \geq 0$. Finally, $\sigma^{e+d}(y) = z_d$ for all $d \geq 0$ and so the conclusion of the lemma holds with $m_0 = \max(0, e)$. $\qquad\square$

Lemma 7.10. *The scheme $\mathbb{X}_n$ is independent of the stable value of n in the sense that, for any such n, there exists an isomorphism $\alpha_n : \mathbb{X}_{n+1} \to \mathbb{X}_n$ such that $\pi_n \phi_n = \alpha_n \pi_{n+1}$.*

Proof. Let $y \in \mathbb{X}_n$. By Theorem 7.1(2), π_n^{-1} is defined at $\widetilde{\sigma}_n^e(y)$ for all $e \gg 0$, say for all $e \geq e_0$. Set $z = \widetilde{\sigma}_n^{e_0}(y)$. Then by Lemma 7.9, we may choose $e_1 \geq 0$ such that ϕ_n^{-1} is defined at $\widetilde{z}_e = \sigma_n^e \pi_n^{-1}(z) = \pi_n^{-1} \widetilde{\sigma}_n^e(z)$, for all $e \geq e_1$. So, for all w in some neighborhood of y we may define

$$\alpha_{n,m}^{-1}(w) = \widetilde{\sigma}_{n+1}^{-m} \circ \pi_{n+1} \circ \phi_n^{-1} \circ \pi_n^{-1} \circ \widetilde{\sigma}_n^m(w), \qquad \text{for} \quad m \geq m_0 = e_0 + e_1.$$

By Lemma 5.3, we have the equality $\sigma_{n+1} \phi_n^{-1} = \phi_n^{-1} \sigma_n$ of birational maps. Also, $\widetilde{\sigma}_n \pi_n = \pi_n \sigma_n$ as birational maps by the definition of $\widetilde{\sigma}_n$. Thus $\alpha_{n,m}^{-1} = \widetilde{\sigma}_{n+1}^{-m} \pi_{n+1} \phi_n^{-1} \pi_n^{-1} \widetilde{\sigma}_n^m = \pi_{n+1} \phi_n^{-1} \pi_n^{-1}$ as birational maps, independent of the choice of $m \geq m_0$. Thus the birational map $\alpha_n^{-1} = \pi_{n+1} \phi_n^{-1} \pi_n^{-1}$ is defined in a neighborhood of y which, since y was arbitrary, implies that α_n^{-1} is globally defined. A similar (but easier) argument shows that the birational map $\alpha_n = \pi_n \phi_n \pi_{n+1}^{-1}$ is globally defined, and so α_n is an isomorphism. $\square$

We end the section by giving a more algebraic interpretation of the scheme $\mathbb{X}$; the set of closed points in $\mathbb{X}$ is canonically isomorphic to a set $\widetilde{P}_{\mathrm{qgr}}$ of k-point modules in qgr-A. This is both suggestive and curious. It is curious since, as is shown by [KRS, Theorem 0.1], $\widetilde{P}_{\mathrm{qgr}}$ is definitely *not* represented by a scheme of finite type when A is a naïve blowup algebra. However, it does suggest that there should be some category of modules over A which is represented by $\mathbb{X}$.

Formally, for $m \geq n$, a closed point $y \in Y_m$ can be identified with a truncated k-point module M_y of length $m + 1$. Mimicking the definition in Section 2, let $\widetilde{P}_m$ denote the set of isomorphism classes of all such truncated point modules and let $\widetilde{P} = \varprojlim \widetilde{P}_m$ denote the set of isomorphism classes of point modules corresponding these truncated modules. Finally, let $\widetilde{P}_{\mathrm{qgr}}$ denote the image of $\widetilde{P}$ in qgr-A. Although we will not prove it here, the set $\widetilde{P}_{\mathrm{qgr}}$ is the image in qgr of the set of all point modules from gr-A and hence also the set of all point modules in qgr-A in the sense of the introduction.

Corollary 7.11. *Pick a stable value of n and set $\mathbb{X} = \mathbb{X}_n$. There is a bijective correspondence ρ from the set of closed points in the scheme $\mathbb{X}$ to the set of objects in $\widetilde{P}_{\mathrm{qgr}}$ with the following property: Given $\widetilde{y} \in \mathbb{X}$, write $\widetilde{y} = \pi(y)$ for some $y \in Y$, say corresponding to the truncated point module $M = M_y \in \widetilde{P}_n$. Then $\rho(\widetilde{y})$ is the image $\overline{N}$ in qgr-A of any point module $N \in \widetilde{P}$ satisfying $M \cong N_{\leq n}$.*

Proof. Let $y \in Y = Y_n$, for a fixed stable value of n and recall the definitions from Notation 6.6 and Theorem 7.1. We first claim that the map $\pi' : \widetilde{P}_n \to \widetilde{P}_{\mathrm{qgr}}$ given by $y \mapsto \overline{N}$ is well-defined.

To prove this, pick $m \geq 0$ by Lemma 7.9 so that the map ϕ^{-r} is defined at $\sigma^m(y)$ for all $r \geq 0$. We now translate this into module theory. Let $M_y \in \widetilde{P}_n$ be the truncated point module corresponding to the point $y \in Y_n$. By Corollary 3.5(2), the maps ϕ_d are surjective for $d \geq n$, and so there is a sequence of points $y_d \in Y_d$ with $y_n = y$ and $\phi(y_{d+1}) = y_d$ for all $d \geq 0$. Corresponding to this sequence is a point module N such that $N_{\leq n} \cong M_y$. The truncation shift $L = N[m]_{\geq 0}$ is also a truncated point module but now, by (2.1) and (2.2), $L_{\leq n}$ corresponds to the point $\sigma^m(y) \in Y$. Since ϕ^{-r} is defined at $\sigma^m(y)$ for all $r \geq 0$, L must be the unique point module L' with $L'_{\leq n} \cong L_{\leq n}$. In other words, $N[m]_{\geq 0}$ and hence $N_{\geq m}$ are uniquely determined by M_y and y. Since N and $N_{\geq m}$ have the same image in $\widetilde{P}_{\mathrm{qgr}}$, this says that π' is well defined.

In order to show that the correspondence ρ is well-defined and bijective, it suffices to prove the following: given closed points $x_1 \neq x_2 \in Y$ then $\pi(x_1) = \pi(x_2)$ if and only if $\pi'(x_1) = \pi'(x_2)$. By the construction of π in Theorem 7.1, $\pi(x_1) = \pi(x_2)$ if and only if $x_1, x_2 \in C_{ij}$ for some i, j. But in this case, Lemma 6.8(2,3,5) implies that $\sigma^m(x_1) = \sigma^m(x_2) = c_{i,j+m}$ for $m \gg 0$. Now suppose that $N(i)$ are point modules with $N(i)_{\leq n} = M_{x_i}$ for $i = 1, 2$. Then, just as in the previous paragraph, it follows from (2.1) and (2.2) that $L(i) = N(i)[m]_{\geq 0}$ is a truncated point module corresponding to $\sigma^m(x_i)$, which is independent of i. In other words, $N(1)$ and $N(2)$ have isomorphic tails, $\overline{N(1)} \cong \overline{N(2)}$, and ρ is well-defined. Conversely, if $\pi'(x_1) = \pi'(x_2)$, then it follows from the argument of the previous paragraph that $\sigma^m(x_1) = \sigma^m(x_2)$ for all $m \gg 0$. Thus $x_1 = x_2 \in C_{ij}$ for some i, j and so $\pi(x_1) = \pi(x_2)$. $\qquad\square$

8. Naïve blowing up

We now have the data required for the statement of the Main Theorem 1.1 and in the next three sections we show how to use this to describe the relevant cg domains as naïve blowup algebras. In this section we give the definitions and some of the basic properties of these algebras. Despite their name, naïve blowups have properties that are very different from the classical case—just look at Corollary 1.2. They have been studied in detail in [KRS]; but only for the case where one is blowing up a single closed point, whereas for applications in this paper we need to be able to naïvely blow up a zero dimensional subscheme of the given scheme and this complicates several of the results from [KRS]. A detailed examination of these more general rings is given in the companion paper [RS], to which the reader is referred for the details behind the assertions given here.

The following assumptions will be in force throughout the section.

Assumptions 8.1. *Fix an integral projective scheme X of dimension $d \geq 2$ with an automorphism τ and field of rational functions $K = k(X)$. The corresponding sheaf of rational functions will be written $\mathcal{K}$. Let $\mathcal{L} \subset \mathcal{K}$ be an invertible sheaf and let $\mathcal{I}$ be an ideal sheaf such that the associated scheme $Z_{\mathcal{I}} = \mathcal{V}(\mathcal{I})$ is zero dimensional. We allow $\mathcal{I} = \mathcal{O}_X$ but, to avoid trivialities, we assume that if $s \in Z_{\mathcal{I}}$ then s lies on an infinite τ-orbit in X.*

Given an $\mathcal{O}_X$-module $\mathcal{M}$, write $\tau^*(\mathcal{M}) = \mathcal{M}^\tau$. If $\mathcal{M} \subseteq \mathcal{K}$, set $\mathcal{M}_0 = \mathcal{O}_X$ and $\mathcal{M}_n = \mathcal{M}\mathcal{M}^\tau \cdots \mathcal{M}^{\tau^{n-1}}$ for $n > 0$, under the multiplication induced from $\mathcal{K}$. Of course, $\mathcal{M}_n \cong \mathcal{M} \otimes \mathcal{M}^\tau \otimes \cdots \otimes \mathcal{M}^{\tau^{n-1}}$ when $\mathcal{M}$ is invertible but we will also need use the construction for $\mathcal{M} = \mathcal{L}\mathcal{I}$ where $\mathcal{M}\mathcal{M}^\tau$ is typically not isomorphic to $\mathcal{M} \otimes \mathcal{M}^\tau$.

Given this data, and regarding $\mathcal{K}[t, t^{-1}; \tau]$ as a sheaf of $\mathcal{O}_X$-algebras in the natural way, we can form the *bimodule algebra*

$$(8.2) \qquad \mathcal{R}(X, Z_{\mathcal{I}}, \mathcal{L}, \tau) = \bigoplus_{n \geq 0} \mathcal{R}_n t^n \subseteq \mathcal{K}[t, t^{-1}; \tau], \qquad \text{where} \quad \mathcal{R}_n = \mathcal{L}_n \mathcal{I}_n \cong \mathcal{L}_n \otimes \mathcal{I}_n \quad \text{for } n \geq 0.$$

The definition of $\mathcal{R} = \mathcal{R}(X, Z_{\mathcal{I}}, \mathcal{L}, \tau)$ in [KRS] or [RS] is slightly different—although equivalent—since it is given in terms of $\mathcal{O}_X$-bimodules. One advantage of the present definition is that the multiplication in $\mathcal{R}$ is induced from that in $\mathcal{K}[t, t^{-1}; \tau]$ and is given by the natural map

$$(8.3) \qquad \mathcal{R}_m t^m \otimes \mathcal{R}_n t^n \longrightarrow \mathcal{R}_m t^m \mathcal{R}_n t^n = \mathcal{R}_m \mathcal{R}_n^{\tau^m} t^{m+n} = \mathcal{R}_{m+n} t^{m+n}.$$

We remark that, as is noted in [AV, p. 252], $\mathcal{R}(X, Z_{\mathcal{I}}, \mathcal{L}, \tau)$ is not a sheaf of algebras in the usual sense, since the presence of σ means one has to play a game of musical chairs with the open sets. The *naïve blowup algebra of X at $Z_{\mathcal{I}}$* is then defined to be the ring of global sections

$$(8.4) \quad R(X, Z_{\mathcal{I}}, \mathcal{L}, \tau) \;=\; \mathrm{H}^0(X, \mathcal{R}) = \bigoplus_{n \geq 0} R_n = \bigoplus_{n \geq 0} \overline{R}_n t^n \qquad \text{where} \quad \overline{R}_n = \mathrm{H}^0(X, \mathcal{I}_n \otimes \mathcal{L}_n) \quad \text{for } n \geq 0.$$

As a matter of convention we write a graded subspace of R or $Q(R)$ as $I = \bigoplus I_n = \bigoplus \overline{I}_n t^n$, for some $\overline{I}_n \subset k(X)$, but we do not use the overline for subsets of $\mathcal{R}$.

Although we have implicitly assumed that $\mathcal{I} \neq \mathcal{O}_X$ in these definitions, they make perfect sense when $\mathcal{I} = \mathcal{O}_X$. In that case one obtains the bimodule algebra $\mathcal{B}(X, \mathcal{L}, \sigma) = \mathcal{R}(X, \emptyset, \mathcal{L}, \tau)$ whose ring of global sections is the familiar *twisted homogeneous coordinate ring* $B(X, \mathcal{L}, \tau) = R(X, \emptyset, \mathcal{L}, \tau) = \bigoplus_{n \geq 0} \mathrm{H}^0(X, \mathcal{L}_n) t^n$ from [AV]. We note in passing that $R(X, Z_{\mathcal{I}}, \mathcal{L}, \tau) \subseteq B(X, \mathcal{L}, \tau)$ for any $Z_{\mathcal{I}}$.

By definition, a *graded right $\mathcal{R}$-module* is a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M} = \bigoplus_{i \in \mathbb{Z}} \mathcal{M}_n$ together with a right $\mathcal{O}_X$-module map $\mu : \mathcal{M} \otimes \mathcal{R} \to \mathcal{M}$ satisfying the usual axioms. The module $\mathcal{M}$ is called *coherent* (as an $\mathcal{R}$-module) if each $\mathcal{M}_n$ is a coherent $\mathcal{O}_X$-module, with $\mathcal{M}_n = 0$ for $n \ll 0$, such that the natural map $\mu_n : \mathcal{M}_n \otimes \mathcal{R}_1^{\tau^n} \to \mathcal{M}_{n+1}$ is surjective for $n \gg 0$. (By [RS, Lemma 2.11], this is equivalent to the somewhat more general definition from [Vd]. The superscript of τ^n that appears here does not appear in that lemma since the two papers use slightly different notation; in [RS] $\mathcal{R}_1$ is a bimodule into which one absorbs the action of τ.) The bimodule algebra $\mathcal{R}$ is *(right) noetherian* if all (right) ideals of $\mathcal{R}$ are coherent.

The usual definitions for graded modules over cg algebras also apply to bimodule algebras. In particular, the graded right $\mathcal{R}$-modules form an abelian category Gr-$\mathcal{R}$, with homomorphisms graded of degree zero. The subcategory of coherent modules is denoted gr-$\mathcal{R}$ and is abelian when $\mathcal{R}$ is noetherian [KRS, Proposition 2.10]. Let tors-$\mathcal{R}$ denote the full subcategory of gr-$\mathcal{R}$ consisting of coherent $\mathcal{O}_X$-modules, and write qgr-$\mathcal{R}$ for the quotient category gr-$\mathcal{R}$/tors-$\mathcal{R}$. The category qgr-$\mathcal{R}$ is called *the naïve blowup* of X at $Z_{\mathcal{I}}$ (see, also, Theorem 8.5).

Naïve blowup algebras only behave well if the sequence $\{\mathcal{R}_n = \mathcal{L}_n \mathcal{I}_n\}$ is ample in the following sense. Assume that $\mathcal{M} \subset \mathcal{K}$. Then the sequence $\{\mathcal{M}_n : n \geq 0\}$ is *ample*, equivalently $\mathcal{M}$ is *τ-ample*, if for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ the sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M}_n$ is generated by its global sections and $\mathrm{H}^i(X, \mathcal{F} \otimes \mathcal{M}_n) = 0$ for all $i > 0$ and all $n \gg 0$. If $\mathcal{M}$ is an invertible sheaf, then τ-ampleness is a subtle but well-understood concept (see [Ke]), but much less is known when $\mathcal{M}$ is not invertible.

A basic method for relating $\mathcal{R}$ to R is given by the following result; see [KRS, Theorem 2.12] or, for a more general version, [Vd, Theorem 5.2]:

Theorem 8.5. (Van den Bergh) *Keep Assumptions 8.1 and let* $\mathcal{R} = \mathcal{R}(X, Z_{\mathcal{I}}, \mathcal{L}, \tau)$. *Assume that* $\mathcal{R}$ *is noetherian and that* $\{\mathcal{R}_n : n \in \mathbb{N}\}$ *is a ample sequence of* $\mathcal{O}_X$-*modules. Then the naïve blowup algebra* $R = R(X, Z_{\mathcal{I}}, \mathcal{L}, \tau)$ *is noetherian, and there is an equivalence of categories* qgr-$\mathcal{R} \to$ qgr-R *given by* $\mathrm{H}^0(X, -)$. $\quad\square$

In order to use this theorem to pass from R to $\mathcal{R}$ we need to know when $\mathcal{R}$ is noetherian and when $\{\mathcal{R}_n\}$ is an ample sequence. The answer is provided by the next two results.

Theorem 8.6. *Keep Assumptions 8.1 and assume that* $\mathcal{L}$ *is* τ-*ample and that each point* $z \in Z_{\mathcal{I}}$ *lies on a Zariski dense* τ-*orbit. Then the sequences* $\{\mathcal{R}_n = \mathcal{L}_n \otimes \mathcal{I}_n\}$ *and* $\{\mathcal{R}_n^{\tau^{-n}}\}$ *are ample.*

Proof. The ampleness of $\{\mathcal{R}_n\}$ is proved in [RS, Theorem 3.1]. In order to prove the second claim write $\mathcal{R}_n^{\tau^{-n}} = \mathcal{S} \otimes \mathcal{S}^{\alpha} \otimes \cdots \otimes \mathcal{S}^{\alpha^{n-1}}$ for $\mathcal{S} = \mathcal{L}^{\tau^{-1}} \otimes \mathcal{I}^{\tau^{-1}}$ and $\alpha = \tau^{-1}$. By [Ke, Corollary 5.1], $\mathcal{L}^{\alpha}$ is α-ample and clearly the other hypotheses of the theorem also hold for α and $\mathcal{S}$. So [RS, Theorem 3.1] also shows that $\mathcal{S}$ is α-ample; equivalently, that $\{\mathcal{R}_n^{\tau^{-n}}\}$ is ample. $\quad\square$

The noetherian property for $\mathcal{R}$ and R requires stronger hypotheses. For $z \in X$, the τ-orbit $\{\tau^i(z) : i \in \mathbb{Z}\}$ is called *critically dense* if it is infinite and each infinite subset of this orbit is (Zariski) dense in X. We say that $Z_{\mathcal{I}}$ is *saturating* if either $Z_{\mathcal{I}} = \emptyset$ or, for each point $z \in Z_{\mathcal{I}}$, the τ-orbit $\{\tau^i(z) : i \in \mathbb{Z}\}$ is critically dense.

Proposition 8.7. *Keep Assumptions 8.1. Then*

(1) *The bimodule algebra* $\mathcal{R} = \mathcal{R}(X, Z_{\mathcal{I}}, \mathcal{L}, \tau)$ *is noetherian if and only if* $Z_{\mathcal{I}}$ *is a saturating subset of* X.

(2) *Assume that* $\mathcal{L}$ *is* τ-*ample and that either* $Z_{\mathcal{I}} = \emptyset$ *or that each point* $z \in Z_{\mathcal{I}}$ *lies on a Zariski dense* τ-*orbit. Then* $R(X, Z_{\mathcal{I}}, \mathcal{L}, \tau)$ *is noetherian if and only if* $Z_{\mathcal{I}}$ *is saturating.*

Proof. (1) See [RS, Proposition 2.12].

(2) If $Z_{\mathcal{I}}$ is saturating then this follows from part (1) combined with Theorems 8.6 and 8.5. If $Z_{\mathcal{I}}$ is not saturating then it follows from [RS, Proposition 3.16]. $\quad\square$

9. Surjectivity in large degree

The aim of this section is to show that if a cg algebra is "close to" being a naïve blowup algebra, then it actually equals one (see Theorem 9.2 for the precise statement). This is a generalization of [AS, Theorem 4.1], which proved the analogous result for twisted homogeneous coordinate rings, and the present proof closely follows the argument from [AS].

The following assumptions and notation will remain in force through to the end of the proof of Theorem 9.2.

Assumptions 9.1. *Fix an integral projective scheme* X *of dimension* ≥ 2 *with sheaf of rational functions* $\mathcal{K}$, *an automorphism* τ *and a* τ-*ample invertible sheaf* $\mathcal{L} \subset \mathcal{K}$. *Let* $\mathcal{I} \subseteq \mathcal{O}_X$ *be an ideal sheaf defining a zero dimensional subscheme* $Z_{\mathcal{I}}$ *of* X *and write* $\mathcal{R}_n = \mathcal{L}_n \mathcal{I}_n \cong \mathcal{L}_n \otimes \mathcal{I}_n$ *for each* $n \geq 0$. *Let* P_1 *be a finite set of closed points of* X, *with* $\mathrm{Supp}\, Z_{\mathcal{I}} \subseteq P_1$, *and assume that each* $x \in P_1$ *lies on a (Zariski) dense* τ-*orbit of* X.

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For $n \geq 1$ write $P_n = \bigcup_{j=0}^{n-1} \tau^{-j}(P_1)$; thus, $P_n \supseteq \bigcup_{j=0}^{n-1} \tau^{-j}(\text{Supp } Z_\mathcal{I}) = \text{Supp } \mathcal{O}_X/\mathcal{I}_n$. We continue to use the notation from the previous section; in particular a naïve blowup algebra will be written in the form $R = R(X, Z_\mathcal{I}, \mathcal{L}, \tau) = \bigoplus \overline{R}_n t^n \subseteq K[t, t^{-1}; \tau]$, with $\overline{R}_n = \mathrm{H}^0(X, \mathcal{R}_n)$, as described in (8.4).

We can now state the first form of the surjectivity result. For rings generated in degree one, this will be further refined in Corollary 9.13.

Theorem 9.2. *Keep the data from Assumptions 9.1. Let $T = \bigoplus_{n \geq 0} \overline{T}_n t^n$ be a cg noetherian subring of the naïve blowup algebra $R = R(X, Z_\mathcal{I}, \mathcal{L}, \tau)$ such that*

(1) *for $n \gg 0$, $\overline{T}_n$ generates the sheaf $\mathcal{R}_n$, and*

(2) *for $n \gg 0$, the birational map $\theta_n : X \dashrightarrow \mathbb{P}(\overline{T}_n^*)$ defined by the sections $\overline{T}_n \subseteq \overline{R}_n \subseteq \mathrm{H}^0(X, \mathcal{L}_n)$ restricts to give an immersion $(X \smallsetminus P_n) \hookrightarrow \mathbb{P}(\overline{T}_n^*)$.*

Then $T = R$ up to a finite dimensional vector space and $Z_\mathcal{I}$ is saturating.

Note that, by (1), the sections $\overline{T}_n$ do generate $\mathcal{L}_n(X \smallsetminus P_n)$ and so part (2) makes sense. The proof of Theorem 9.2 will be obtained through a series of lemmas and so the assumptions and notation from the theorem will remain in force throughout that proof. The first lemma is a standard application of τ-ample sheaves.

Lemma 9.3. *Let $V \subseteq \mathrm{H}^0(X, \mathcal{F})$ be a vector space that generates the coherent sheaf $\mathcal{F}$ and pick $r \in \mathbb{Z}$. Then:*

(1) *For $p \gg 0$, the natural homomorphism $\alpha : V \otimes_k \mathrm{H}^0(X, \mathcal{R}_p^{\tau^r}) \to \mathrm{H}^0(X, \mathcal{F} \otimes \mathcal{R}_p^{\tau^r})$ is surjective.*

(2) *For $p \gg 0$, the natural homomorphism $\beta : V^{\tau^p} \otimes_k \mathrm{H}^0(X, \mathcal{R}_p) \to \mathrm{H}^0(X, \mathcal{F}^{\tau^p} \otimes \mathcal{R}_p)$ is surjective.*

Proof. (1) Set $V' = V^{\tau^{-r}} \subseteq \mathrm{H}^0(X, \mathcal{F}^{\tau^{-r}})$. Clearly α is surjective if and only if $\alpha' : V' \otimes \mathrm{H}^0(X, \mathcal{R}_p) \to \mathrm{H}^0(X, \mathcal{F}^{\tau^{-r}} \otimes \mathcal{R}_p)$ is surjective. Thus we may assume that $r = 0$.

There is a short exact sequence $0 \to \mathcal{H} \to V \otimes_k \mathcal{O}_X \to \mathcal{F} \to 0$ for the appropriate sheaf $\mathcal{H}$. Tensoring this sequence with $\mathcal{R}_p$ gives an exact sequence

$$0 \longrightarrow \mathcal{G} \xrightarrow{\alpha} \mathcal{H} \otimes \mathcal{R}_p \longrightarrow V \otimes_k \mathcal{R}_p \longrightarrow \mathcal{F} \otimes \mathcal{R}_p \longrightarrow 0.$$

The sheaf $\mathcal{G}$ is supported at the finite set of points P_p where $\mathcal{R}_p$ need not be locally free, and so $\mathrm{H}^j(X, \mathcal{G}) = 0$ for $j > 0$. Therefore, if $\mathcal{A}_p = \text{Coker}(\alpha)$, then $\mathrm{H}^j(X, \mathcal{A}_p) = \mathrm{H}^j(X, \mathcal{H} \otimes \mathcal{R}_p)$ for all $j > 0$ and all $p \geq 0$. As $\{\mathcal{R}_p\}$ is ample, this implies that $\mathrm{H}^j(X, \mathcal{A}_p) = 0$, for all $j > 0$ and $p \gg 0$. Thus, taking global sections of the last displayed equation gives the required surjection $V \otimes_k \mathrm{H}^0(X, \mathcal{R}_p) \to \mathrm{H}^0(X, \mathcal{F} \otimes \mathcal{R}_p)$.

(2) By applying τ^{-p} to the given equation, it suffices to prove surjectivity of the map $V \otimes \mathrm{H}^0(X, \mathcal{R}_p^{\tau^{-p}}) \to \mathrm{H}^0(X, \mathcal{F} \otimes \mathcal{R}_p^{\tau^{-p}})$. Now use the proof of part (1), combined with the second assertion from Theorem 8.6. $\quad\square$

Lemma 9.4. *The natural map $\mathrm{H}^0(X, \mathcal{R}_m \otimes \mathcal{R}_n^{\tau^m}) \to \mathrm{H}^0(X, \mathcal{R}_{n+m})$ is a surjection for any $m, n \geq 0$.*

Proof. Since $\mathcal{R}_{m+n} = \mathcal{R}_m \mathcal{R}_n^{\tau^m}$, the natural surjective map from the tensor product to the product fits into a short exact sequence $0 \to \mathcal{H} \to \mathcal{R}_m \otimes \mathcal{R}_n^{\tau^m} \to \mathcal{R}_{m+n} \to 0$ for some sheaf $\mathcal{H}$. Since $\mathcal{R}_m$ is locally free except

at a finite set of points, $\mathcal{H}$ is supported on a finite set of points and thus $\mathrm{H}^1(X, \mathcal{H}) = 0$. Taking global sections of the short exact sequence gives the required surjection. $\qquad\square$

Lemma 9.5. *R is finitely generated as a left and right T-module. Moreover, R is noetherian, and $Z_\mathcal{I}$ is a saturating subscheme of X.*

Proof. Fix $n \gg 0$ for which $\overline{T}_n$ generates $\mathcal{R}_n$. Then Lemma 9.3(1), with $\mathcal{F} = \mathcal{R}_n$ and $V = \overline{T}_n \subseteq \mathrm{H}^0(X, \mathcal{R}_n)$, together with Lemma 9.4, implies that the multiplication map

$$\overline{T}_n \otimes \overline{R}_p^{\tau^n} = \overline{T}_n \otimes \mathrm{H}^0(X, \mathcal{R}_p^{\tau^n}) \longrightarrow \mathrm{H}^0(X, \mathcal{R}_n \otimes \mathcal{R}_p^{\tau^n}) \longrightarrow \mathrm{H}^0(X, \mathcal{R}_{n+p}) = \overline{R}_{n+p}$$

is surjective for all $p \gg 0$; say for all $p \geq p_0$. Therefore, R is finitely generated by $R_0 \oplus \cdots \oplus R_{p_0+n-1}$ as a left T-module. An analogous argument, using Lemma 9.3(2), shows that R_T is finitely generated.

As T is noetherian, this certainly implies that R is noetherian. By Assumption 9.1 the hypotheses of Proposition 8.7(2) are therefore satisfied and, by that result, $Z_\mathcal{I}$ is saturating. $\qquad\square$

Lemma 9.6. *It suffices to prove Theorem 9.2 for some pair of Veronese rings $T^{(d)} \subseteq R^{(d)}$.*

Proof. The Veronese ring $R^{(d)} \cong R(X, Z_{\mathcal{I}_d}, \mathcal{L}_d, \tau^d)$ is again a naïve blowup algebra with data satisfying Assumptions 9.1 and so the hypotheses of Theorem 9.2 do carry over to the Veronese rings $T^{(d)} \subseteq R^{(d)}$.

Assume that the theorem holds for $T^{(d)} \subseteq R^{(d)}$; thus $T_s^{(d)} = R_s^{(d)}$ for all $s \gg 0$, and hence $\overline{T}_{sd} = \overline{R}_{sd}$ for all such s. By hypothesis, we can find n_0 such that $\overline{T}_n$ generates $\mathcal{R}_n$ for all $n \geq n_0$. Pick n satisfying $n_0 \leq n \leq n_0 + d$. Then Lemma 9.3(1), with $p = sd$, $r = n$ and $V = \overline{T}_n \subseteq \mathrm{H}^0(X, \mathcal{R}_n)$, together with Lemma 9.4, implies that the multiplication map

$$\overline{T}_n \otimes \overline{T}_{sd}^{\tau^n} = \overline{T}_n \otimes \overline{R}_{sd}^{\tau^n} \longrightarrow \overline{R}_{n+sd}$$

is surjective for $s \gg 0$, say for all $s \geq s_0$. Since there are only finitely many choices of n, we may assume that s_0 is chosen independently of the choice of n. Hence, $\overline{T}_u = \overline{R}_u$ for all $u \geq n_0 + s_0 d$. $\qquad\square$

The key step in the proof of Theorem 9.2 is to relate ideals of R to τ-invariant ideal sheaves on X. This is given by the next result.

Proposition 9.7. *Let $M = \bigoplus_{n \geq 0} \overline{M}_n t^n$ be a two-sided ideal of the naïve blowup algebra R. Then there exists a τ-invariant sheaf of ideals $\mathcal{J} \subseteq \mathcal{O}_X$ such that $\mathrm{H}^0(X, \mathcal{J}\mathcal{R}_n) = \overline{M}_n$ for all $n \gg 0$.*

Proof. Let $\mathcal{R} = \mathcal{R}(X, Z_\mathcal{I}, \mathcal{L}, \tau) = \bigoplus_{n \geq 0} \mathcal{R}_n t^n$ be the bimodule algebra associated to R, in the sense of (8.2). By Theorem 8.6 the sequence $\{\mathcal{R}_n\}$ is ample and by Proposition 8.7 $\mathcal{R}$ is noetherian, so Theorem 8.5 applies. By that result, there exists $n_0 \geq 0$ and a coherent submodule $\mathcal{G} = \bigoplus_{n \geq 0} \mathcal{G}_n t^n \subseteq \mathcal{R}$ such that $\overline{M}_n = \mathrm{H}^0(X, \mathcal{G}_n)$ for each $n \geq n_0$. By the definition of coherent $\mathcal{R}$-modules in Section 8 and increasing n_0 if necessary, we may assume that $\mathcal{G}_{n+1} = \mathcal{G}_n \mathcal{R}_1^{\tau^n}$ for all $n \geq n_0$. Since $\mathcal{R}_m = \mathcal{R}_1 \cdots \mathcal{R}_1^{\tau^{m-1}}$, induction implies

that $\mathcal{G}_{n+m} = \mathcal{G}_n \mathcal{R}_m^{\tau^n}$ for $m > 0$ and $n \geq n_0$. Equivalently, if $\mathcal{H}_n = \mathcal{G}_n \mathcal{L}_n^{-1} \cong \mathcal{G}_n \otimes \mathcal{L}_n^{-1}$, then $\mathcal{H}_n \subseteq \mathcal{O}_X$ is a sheaf of ideals satisfying

$$(9.8) \qquad \mathcal{H}_{n+m} = \mathcal{H}_n \mathcal{I}_m^{\tau^n} \subseteq \mathcal{H}_n \subseteq \mathcal{H}_{n_0} \qquad \text{for each } n \geq n_0 \text{ and } m \geq 0.$$

As $\mathcal{G}_{m+n_0} = \mathcal{G}_{n_0} \mathcal{R}_m^{\tau^{n_0}}$ for all $m \geq 0$, the ampleness of $\{\mathcal{R}_m\}$ implies that $\mathcal{G}_n$ is generated by its sections $\overline{M}_n$ for all $n \gg 0$.

Now consider the fact that M is also a left ideal of R. The hypotheses on R and $\mathcal{R}$ are left-right symmetric and so the equivalence of categories, Theorem 8.5, also applies to left ideals. Thus there also exists a coherent left submodule $\mathcal{G}' = \bigoplus_{m \geq 0} \mathcal{G}'_m t^m \subseteq \mathcal{R}$ such that $\mathrm{H}^0(Y, \mathcal{G}'_n) = \overline{M}_n$ for all $n \gg 0$. By ampleness, again, $\mathcal{G}'_n$ is generated by its sections $\overline{M}_n$ for all $n \gg 0$. Thus $\mathcal{G}'_n = \mathcal{G}_n$ for all $n \gg 0$, which we may assume happens for all $n \geq n_0$. As $\mathcal{G}'$ is a left submodule of $\mathcal{R}$ this implies that $\mathcal{R}_m \mathcal{G}_n^{\tau^m} = \mathcal{R}_m (\mathcal{G}'_n)^{\tau^m} \subseteq \mathcal{G}'_{n+m} = \mathcal{G}_{n+m}$ for all $m \geq 0$ and $n \geq n_0$. Equivalently

$$(9.9) \qquad \mathcal{I}_m \mathcal{H}_n^{\tau^m} \subseteq \mathcal{H}_{n+m} \qquad \text{for each } n \geq n_0 \text{ and } m \geq 0.$$

Set $P_\infty = \{\tau^i(P_1) : i \in \mathbb{Z}\}$; thus $P_\infty \supseteq \mathrm{Supp}\, \mathcal{O}_X / \mathcal{I}_r$ for all $r \geq 0$. If $\mathcal{N} \subseteq \mathcal{O}_X$ is a sheaf of ideals, write $\mathcal{S}(\mathcal{N})$ for the largest sheaf of ideals with the property that $\mathrm{Supp}\, \mathcal{S}(\mathcal{N})/\mathcal{N} \subseteq P_\infty$. Clearly $\mathcal{S}(\mathcal{I}\mathcal{N}) = \mathcal{S}(\mathcal{N})$ and so it follows from (9.8) and (9.9) that

$$\mathcal{S}(\mathcal{H}_{n_0})^\tau = \mathcal{S}(\mathcal{H}_{n_0}^\tau) = \mathcal{S}(\mathcal{I}\mathcal{H}_{n_0}^\tau) \subseteq \mathcal{S}(H_{n_0+1}) \subseteq \mathcal{S}(\mathcal{H}_{n_0}).$$

Since X is noetherian, this forces $\mathcal{S}(\mathcal{H}_{n_0})^\tau = \mathcal{S}(\mathcal{H}_{n_0})$. Thus $\mathcal{J} = \mathcal{S}(\mathcal{H}_{n_0})$ is a τ-invariant sheaf of ideals and we will show that it satisfies the conclusions of the proposition.

We therefore need to show that $\mathcal{G}_n = \mathcal{J}\mathcal{R}_n$ in high degree. Tensoring this statement by $\mathcal{L}_n^{-1}$, we need to show that $\mathcal{H}_n = \mathcal{J}\mathcal{I}_n$ for $n \gg 0$. Since X is noetherian, there exists an ideal sheaf $\mathcal{A}$ such that $\mathcal{H}_{n_0} \supseteq \mathcal{J}\mathcal{I}_{n_0}\mathcal{A}$, and such that $\mathcal{O}_X/\mathcal{A}$ is supported at a finite number of points of P_∞. For any $m \geq 0$, (9.9) gives

$$\mathcal{H}_{n_0+m} \supseteq \mathcal{I}_m \mathcal{H}_{n_0}^{\tau^m} \supseteq \mathcal{I}_m \mathcal{I}_{n_0}^{\tau^m} \mathcal{A}^{\tau^m} \mathcal{J}^{\tau^m} = \mathcal{A}^{\tau^m} \mathcal{I}_{n_0+m} \mathcal{J}.$$

On the other hand, (9.8) implies that

$$\mathcal{H}_{n_0+m} = \mathcal{H}_{n_0} \mathcal{I}_m^{\tau^{n_0}} \supseteq \mathcal{I}_m^{\tau^{n_0}} \mathcal{I}_{n_0} \mathcal{A}\mathcal{J} = \mathcal{A}\mathcal{I}_{n_0+m}\mathcal{J}.$$

Since $\mathcal{O}_X/\mathcal{A}$ is supported at a finite number of points, each lying on an infinite τ-orbit, $\mathcal{A}^{\tau^m} + \mathcal{A} = \mathcal{O}_X$ for $m \gg 0$. Combining the last two displayed equations therefore shows that $\mathcal{H}_{n_0+m} \supseteq \mathcal{J}\mathcal{I}_{n_0+m}$, for all $m \gg 0$.

As $\mathcal{J}$ is τ-invariant, $\mathrm{Supp}\, \mathcal{O}_X/\mathcal{J}$ is a proper τ-invariant closed subspace of X. Since each $x \in P_\infty$ lies on a dense τ-orbit, this forces $W = \mathrm{Supp}(\mathcal{O}_X/\mathcal{J}) \cap \mathrm{Supp}(\mathcal{O}_X/\mathcal{I}_n)$ to be empty. But $\mathcal{J} \cap \mathcal{I}_n / \mathcal{J}\mathcal{I}_n$ is supported on W. Thus $\mathcal{J} \cap \mathcal{I}_n = \mathcal{J}\mathcal{I}_n$ for all n and so the previous paragraph implies that $\mathcal{H}_n \supseteq \mathcal{J} \cap \mathcal{I}_n$ for all $n \gg n_0$. On the other hand, $\mathcal{H}_n \subseteq \mathcal{H}_{n_0} \subseteq \mathcal{S}(\mathcal{H}_{n_0}) = \mathcal{J}$ by (9.8) and $\mathcal{H}_n \subseteq \mathcal{I}_n$ simply because $\mathcal{G}_n \subseteq \mathcal{R}_n = \mathcal{L}_n \mathcal{I}_n$. Thus $\mathcal{H}_n \subseteq \mathcal{J} \cap \mathcal{I}_n = \mathcal{J}\mathcal{I}_n$ and hence $\mathcal{H}_n = \mathcal{J}\mathcal{I}_n$ for all $n \gg n_0$. Equivalently, $\mathcal{G}_n = \mathcal{J}\mathcal{R}_n$ and $\overline{M}_n = \mathrm{H}^0(X, \mathcal{J}\mathcal{R}_n)$, for all $n \gg 0$. $\qquad\square$

Remark 9.10. The proof of Proposition 9.7 also proves the following result: Let $\mathcal{G} = \bigoplus \mathcal{G}_n t^n$ be an ideal of the bimodule algebra $\mathcal{R}$. Then there exists a τ-invariant sheaf of ideals $\mathcal{J} \subseteq \mathcal{O}_X$ such that $\mathcal{G}_n = \mathcal{J}\mathcal{R}_n$ for all $n \gg 0$.

Proof of Theorem 9.2. First, by Lemma 9.6 we may replace R and T by Veronese rings and thereby assume that conditions (1) and (2) of the theorem actually hold for all $n \geq 1$. By Assumptions 9.1, R and hence T are contained in $k(X)[t, t^{-1}; \tau]$ and by condition (1), $T_1 \neq 0$; say $0 \neq t_1 = at \in T_1$ for some $a \in k(X)$. Therefore, $k(X)[t, t^{-1}; \tau] = k(X)[t_1, t_1^{-1}; \tau_1]$ and one even has $\tau = \tau_1$ since they differ by conjugation by the element $a \in k(X)$. Since T is noetherian, it has a graded Goldie quotient ring which, by universality, will have the form $Q(T) = F[t_1, t_1^{-1}; \tau] \subseteq Q(R) \subseteq k(X)[t_1, t_1^{-1}; \tau]$ for some field $F \subseteq k(X)$ invariant under τ.

We claim that $F = k(X)$. To see this, set $V = X \smallsetminus P_1$ and note that, by conditions (1) and (2), $\overline{T}_1$ generates $\mathcal{L}|_V$ and induces an immersion $\chi : V \hookrightarrow \mathbb{P}(\overline{T}_1^*)$. Fix a basis $\{s_j : 0 \leq j \leq r\}$ of $\overline{T}_1$, write $V_0 = \{x \in V : (s_0)_x \notin \mathfrak{m}_x \mathcal{L}_x\}$ and let $U_0 = \{x_0 \neq 0\}$ be the standard open affine subset of $\mathbb{P}(\overline{T}_1^*)$ with $\mathcal{O}(U_0) = k[x_1 x_0^{-1}, \ldots, x_r x_0^{-1}]$. Then $\chi(V_0)$ is open in its closure inside U_0. As in the proof of [Ha, Theorem II.7.1], the morphism $\chi|_{V_0}$ is then given by the map of rings $\alpha : k[x_1 x_0^{-1}, \ldots, x_r x_0^{-1}] \to \Gamma(V_0, \mathcal{O}_{V_0})$ defined by $x_j x_0^{-1} \mapsto s_j s_0^{-1}$, and it follows that $\Gamma(V_0, \mathcal{O}_{V_0})$ is a localization of the image of α. On the other hand, as X is an integral scheme, it has the same field of rational functions as both V and V_0 and so $k(X)$ is generated as a field by the $s_j s_0^{-1} \in F$. Thus $k(X) = F$.

It follows from the previous paragraph that $Q(T) = Q(R) = k(X)[t_1, t_1^{-1}; \tau]$ and so R/T is Goldie torsion as a right (or left) T-module. By Lemma 9.5, R/T is therefore a finitely generated, torsion right T-module and so the left annihilator $J_\ell = \ell\text{-ann}_T(R/T)$ is a non-zero ideal of T. Similarly, $J_r = r\text{-ann}_T(R/T)$ and $M = J_r J_\ell$ are non-zero ideals of T. But M is also an ideal of R. So, if $M = \bigoplus M_n$ then Proposition 9.7 produces a non-zero τ-invariant ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$ such that

$$(9.11) \qquad \mathrm{H}^0(X, \mathcal{J}\mathcal{R}_n)t^n = M_n \qquad \text{for all} \quad n \gg 0.$$

We next consider the factor bimodule algebra $\mathcal{R}/\mathcal{J}\mathcal{R} = \bigoplus_{n \geq 0} \mathcal{R}_n t^n / \mathcal{J}\mathcal{R} t^n$. Let $W \subseteq X$ be the τ-invariant subscheme defined by $\mathcal{J}$. As every point of $\mathrm{Supp}\, Z_{\mathcal{I}_n} \subseteq P_n$ lies on a dense τ-orbit, and W is a proper τ-invariant subscheme of X, it follows that $Z_{\mathcal{I}_n} \cap W = \emptyset$ and hence that $\mathcal{R}_n/\mathcal{J}\mathcal{R}_n = \mathcal{R}_n \otimes_{\mathcal{O}_X} \mathcal{O}_W = \mathcal{L}_n \otimes_{\mathcal{O}_X} \mathcal{O}_W$ for all $n \geq 0$. In other words $\mathcal{R}/\mathcal{J}\mathcal{R} \cong \bigoplus_{n \geq 0}(\mathcal{L}_n|_W)\overline{t}^{\,n}$, where $\overline{t}$ is now a formal symbol denoting the image of t. If we write $\overline{\mathcal{L}} = \mathcal{L}|_W$, then $\mathcal{L}_n|_W \cong \overline{\mathcal{L}} \otimes_{\mathcal{O}_W} \overline{\mathcal{L}}^\tau \otimes \cdots \otimes \overline{\mathcal{L}}^{\tau^{n-1}} = \overline{\mathcal{L}}_n$ and so $\mathcal{R}/\mathcal{J}\mathcal{R}$ is just the bimodule algebra $\mathcal{B}(W, \mathcal{L}|_W, \tau|_W)$, as defined in Section 8. Taking global sections we obtain

$$\mathrm{H}^0(X, \mathcal{R}/\mathcal{J}\mathcal{R}) \;=\; \bigoplus_{n \geq 0} \mathrm{H}^0(X, \mathcal{L}_n \otimes_{\mathcal{O}_X} \mathcal{O}_W)\overline{t}^{\,n} \;=\; \bigoplus_{n \geq 0} \mathrm{H}^0(W, \mathcal{L}_n|_W)\overline{t}^{\,n} \;=\; B(W, \mathcal{L}|_W, \tau|_W),$$

where $B = B(W, \mathcal{L}|_W, \tau|_W)$ is the ordinary twisted homogeneous coordinate ring.

Now take global sections of the short exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{J}\mathcal{R} \to \mathcal{R} \to \mathcal{R}/\mathcal{J}\mathcal{R} \to 0$ to obtain the exact sequence of finitely generated right R-modules $0 \to N \to R \to B$, where $N = \mathrm{H}^0(X, \mathcal{J}\mathcal{R})$.

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Set

$$(9.12) \qquad T' = \frac{T+N}{N} = \bigoplus_{n \geq 0} \overline{T}'_n \overline{t}^n \subseteq R' = \frac{R}{N} \subseteq B = B(W, \mathcal{L}|_W, \tau|_W).$$

We claim that the hypotheses of the theorem pass to the pair of rings $T' \subseteq B$ or, more precisely, that $T' \subseteq B$ satisfy the analogous hypotheses of [AS, Theorem 4.1]. To see this, observe first that $\overline{T}'_n$ is the restriction to W of the global sections contained in $\overline{T}_n \subseteq \overline{R}_n = \mathrm{H}^0(X, \mathcal{R}_n)$. Therefore, by condition (1) of the theorem, $\overline{T}'_n$ generates the sheaf $\mathcal{R}_n|_W = \mathcal{L}_n|_W = \overline{\mathcal{L}}_n$ for all $n \gg 0$. Moreover, the rational map $\theta'_n : W \dashrightarrow \mathbb{P}(\overline{T}'^*_n)$ is the restriction of the rational map $\theta_n : X \dashrightarrow \mathbb{P}(\overline{T}^*_n)$. Once again, as W is τ-invariant, $W \cap P_n = \emptyset$ for all n, and so, by condition (2), θ'_n is an everywhere-defined immersion. Since W is proper, its image is closed and so θ'_n is a closed immersion. Since $\mathcal{L}$ is τ-ample, it follows from [Ke, Proposition 2.3(2) and Corollary 5.1] that $\mathcal{L}|_W$ is both $\tau|_W$-ample and $(\tau|_W)^{-1}$-ample as an invertible sheaf over W. Moreover, by [Ke, Theorem 1.2(3)], B has finite Gelfand-Kirillov dimension. The hypotheses of [AS, Theorem 4.1] are therefore satisfied and, by that result, B/T' is finite dimensional. But (9.11) implies that $N_n = \mathrm{H}^0(X, \mathcal{J}\mathcal{R})_n \subseteq T_n$ for all $n \gg 0$ and so (9.12) implies that R/T is also finite-dimensional.

Finally, $Z_\mathcal{I}$ is saturating by Lemma 9.5. $\qquad\qquad\square$

In applications of Theorem 9.2 it is inconvenient to have to check conditions (1) and (2) of that result for infinitely many choices of n. Fortunately, for rings generated in degree one, just one choice will do. In this final result we do not assume that Assumptions 9.1 are satisfied.

Corollary 9.13. *Let X be an integral projective scheme, with $\dim X \geq 2$, sheaf of rational functions $\mathcal{K}$ and with a τ-ample invertible sheaf $\mathcal{L} \subset \mathcal{K}$ for some $\tau \in \mathrm{Aut}(X)$. Let $T = \bigoplus \overline{T}_m t^m$ be a noetherian cg subalgebra of $B(X, \mathcal{L}, \tau) = \bigoplus \mathrm{H}^0(X, \mathcal{L}_m) t^m \subseteq k(X)[t, t^{-1}; \tau]$ that is generated in degree 1. Let P be a finite set of closed points in X such that each $x \in P$ lies on a dense τ-orbit. Finally, suppose that there exists $n \geq 1$ such that:*

(i) *$\overline{T}_n$ generates the sheaf $\mathcal{L}_n$, except, perhaps, at points of P.*

(ii) *The birational map $X \dashrightarrow \mathbb{P}(\overline{T}^*_n)$ defined by $\overline{T}_n$ restricts to give an immersion $(X \smallsetminus P) \hookrightarrow \mathbb{P}(\overline{T}^*_n)$.*

Set $\mathcal{R}_1 = \overline{T}_1 \mathcal{O}_X \subseteq \mathcal{L}$ and let $\mathcal{I} = \mathcal{R}_1 \mathcal{L}^{-1}$. Then $Z_\mathcal{I}$ is a saturating zero-dimensional subscheme of X with $\mathrm{Supp}\, Z_\mathcal{I} \subseteq P$. Moreover $T \subseteq R = R(X, Z_\mathcal{I}, \mathcal{L}, \tau)$ with $\dim_k T/R < \infty$.

Proof. By definition, $\overline{T}_1$ generates the sheaf $\mathcal{R}_1 = \mathcal{I}\mathcal{L}$. Since T is generated in degree 1, the space $\overline{T}_m = \overline{T}_1 \overline{T}_1^\tau \cdots \overline{T}_1^{\tau^{m-1}}$ generates $\mathcal{R}_m = \mathcal{R}_1 \mathcal{R}_1^\tau \ldots \mathcal{R}_1^{\tau^{m-1}} = \mathcal{I}_m \mathcal{L}_m$ for all $m \geq 1$. Thus we have the following inclusion of subrings of $k(X)[t, t^{-1}; \tau]$:

$$(9.14) \qquad T = \bigoplus_{m \geq 0} \overline{T}_m t^m \subseteq \bigoplus_{m \geq 0} \mathrm{H}^0(X, \mathcal{R}_m) t^m = R(X, Z_\mathcal{I}, \mathcal{L}, \tau).$$

It is convenient to replace the integer n by 1 in the statement of the corollary, which we will accomplish by passing to Veronese rings. Now the hypotheses of the corollary clearly pass across to the rings $T^{(n)} \subseteq B(X, \mathcal{L}_n, \tau^n)$ (in which case take $n = 1$ in the statement of the result). Assume for the moment that the

corollary holds for this pair of rings; thus $\operatorname{Supp} Z_{\mathcal{I}_n} \subseteq P$ and $T^{(n)}$ is equal in large degree to $R(X, Z_{\mathcal{I}_n}, \mathcal{L}_n, \tau^n)$ or, in other words, $\overline{T}_{jn} = \mathrm{H}^0(X, \mathcal{R}_{jn})$ for all $j \gg 0$. This ensures that Assumptions 9.1 also hold for these Veronese rings, and we can apply Lemma 9.3 with $V = \overline{T}_a \subseteq \mathrm{H}^0(X, \mathcal{F})$ where $\mathcal{F} = \mathcal{R}_a$ and $r = a$ for any $1 \leq a \leq n$. Together with Lemma 9.4, this shows that the multiplication map

$$\overline{T}_a \otimes \overline{T}_{jn}^{\tau^a} = \overline{T}_a \otimes \mathrm{H}^0(X, \mathcal{R}_{jn}^{\tau^a}) \longrightarrow \mathrm{H}^0(X, \mathcal{R}_a \otimes \mathcal{R}_{jn}^{\tau^a}) \longrightarrow \mathrm{H}^0(X, \mathcal{R}_{jn+a})$$

is surjective for all $j \gg 0$. Thus $\overline{T}_m = \mathrm{H}^0(X, \mathcal{R}_m)$ for $m \gg 0$. Moreover, since we are assuming that $Z_{\mathcal{I}_n}$ is saturating and supported along P, clearly the same holds for $Z_{\mathcal{I}}$. Therefore, if the corollary holds for $T^{(n)} \subseteq B(X, \mathcal{L}_n, \tau^n)$, it also holds for $T \subseteq B(X, \mathcal{L}, \tau)$.

Thus we have reduced to the case $n = 1$ and we aim to show that the hypotheses (1) and (2) of Theorem 9.2 are satisfied for all $m \geq 1$. We have already seen that $\overline{T}_m$ generates $\mathcal{R}_m$ for all $m \geq 1$, so hypothesis (1) of the theorem holds. Set $P_1 = P$ and, for $m \geq 1$, define $P_m = \bigcup_{i=0}^{m-1} \tau^{-i}(P_1) \supseteq \operatorname{Supp} \mathcal{O}_X/\mathcal{I}_m$. Note that $\overline{T}_1$ generates $\mathcal{L}$ along $X \smallsetminus P_1$. For any $j \in \mathbb{Z}$, condition (ii) implies that the space $\overline{T}(j) = \overline{T}_1^{\tau^j}$ defines an immersion $X \smallsetminus P_n \hookrightarrow X \smallsetminus \tau^{-j}(P_1) \hookrightarrow \mathbb{P}(\overline{T}(j)^*)$, and so an easy induction using Lemma 7.3 shows that $\overline{T}_m$ defines an immersion $\left(X \smallsetminus P_m\right) \hookrightarrow \mathbb{P}(\overline{T}_m^*)$ for all $m \geq 1$. Thus hypothesis (2), and hence all the hypotheses, of Theorem 9.2 are verified and the corollary follows. $\qquad\square$

10. The main theorem

We can now put everything together to prove the main theorem of the paper, which is an elaboration of Theorem 1.1.

Theorem 10.1. *Let $A = \bigoplus_{i \geq 0} A_i$ be a cg noetherian domain that is generated in degree one and birationally 2-geometric. Then*

(1) *Up to a finite dimensional vector space, A is isomorphic to a naïve blowup algebra $R(\mathbb{X}, Z_{\mathcal{I}}, \mathcal{L}, \widetilde{\sigma})$, where $Z_{\mathcal{I}}$ is either empty or a saturating, zero dimensional subscheme of $\mathbb{X}$.*

(2) *The data $\{\mathbb{X}, Z_{\mathcal{I}}, \mathcal{L}, \widetilde{\sigma}\}$ can be made more precise as follows:*

    (i) *For any stable value of n, the scheme $\mathbb{X} = \mathbb{X}_n$ is the blow down $\pi_n(Y_n)$ of the relevant component Y_n of the truncated point scheme, as constructed by Theorem 7.1.*

    (ii) *The automorphism $\widetilde{\sigma} = \pi_n \sigma \pi_n^{-1}$ of $\mathbb{X}$ is induced from the birational map $\sigma_n = \psi_n \phi_n^{-1}$ of Y_n.*

    (iii) *If one writes $A = \bigoplus \overline{A}_n t^n \subset Q(A) = k(\mathbb{X})[t, t^{-1}, \sigma]$ for some $0 \neq t \in A_1$, then $\mathcal{L} = \left(\overline{A}_1 \mathcal{O}_{\mathbb{X}}\right)^{**}$ with $\overline{A}_1 \mathcal{O}_{\mathbb{X}} = \mathcal{I}\mathcal{L}$.*

Before proving the theorem we need some subsidiary results and we begin by setting up the notation that will remain in force throughout the section. Fix a stable value of n, let Y_n denote the relevant component of W_n and drop the subscript from $Y = Y_n$, etc. Let $\pi : Y \to \mathbb{X}$ be the birational surjective morphism defined by Theorem 7.1. Recall from that result that $\widetilde{\sigma} = \pi \sigma \pi^{-1}$ is a biregular automorphism of $\mathbb{X}$, and that the set of points where π^{-1} is not defined is a finite set of nonsingular points P, each lying on a dense $\widetilde{\sigma}$-orbit. Set

$V = \mathbb{X} \smallsetminus P$ and $U = \pi^{-1}(V) \subseteq Y$; thus $U \cong V$. As in the statement of the theorem, fix $t \in A_1 \smallsetminus \{0\}$ and write $A_i = \overline{A}_i t^i$ for $i \geq 0$; thus each $\overline{A}_i \subset K = Q(A)_0$ and $A = \bigoplus \overline{A}_n t^n \subset Q(A) = K[t, t^{-1}; \sigma]$. We recall that $K \cong k(Y) = k(\mathbb{X})$ by Corollary 3.5 and we always write the corresponding sheaf of rational functions (on Y or $\mathbb{X}$) as $\mathcal{K}$.

We begin with some elementary observations about invertible sheaves.

Lemma 10.2. *Let $\mathcal{A} = \sum_{i=1}^{a} s_i \mathcal{O}_Y \subset \mathcal{K}$ be a sheaf generated by sections $s_j \in \mathrm{H}^0(Y, \mathcal{A})$ and set $\mathcal{B} = \pi_* \mathcal{A} \subset \mathcal{K}$ with reflexive hull $\mathcal{P} = \mathcal{B}^{**}$. If $\mathcal{A}(u) = \sum_{i=1}^{a} s_i^{\sigma^{u-1}} \mathcal{O}_Y$ is invertible for $1 \leq u \leq v$, then $\mathcal{B}(u) = \pi_* \mathcal{A}(u)$ has reflexive hull $\mathcal{P}(u) = \mathcal{P}^{\widetilde{\sigma}^{u-1}}$ for $1 \leq u \leq v$. Moreover, $\mathcal{P}(1) \cdots \mathcal{P}(v)$ is the reflexive hull of $\pi_* \big(\mathcal{A}(1) \cdots \mathcal{A}(v) \big)$.*

Proof. As $\widetilde{\sigma}$ and σ induce the same automorphism of K, we have $s_i^{\widetilde{\sigma}^v} = s_i^{\sigma^v}$ for all i, v. Let $\tau = \widetilde{\sigma}^{u-1}$ for some $1 \leq u \leq v$. Since $\mathcal{P}|_V = \sum s_i \mathcal{O}_V$ we have $\mathcal{P}^\tau|_{\tau^{-1}(V)} = \sum s_i^\tau \mathcal{O}_{\tau^{-1}(V)}$. Set $V' = V \cap \tau^{-1}(V)$ and note that $\mathbb{X} \smallsetminus V'$ consists of a finite set of closed nonsingular points of Y. Since

$$\mathcal{B}(u)|_{V'} = \pi_* \big(\mathcal{A}(u)|_{\pi^{-1}(V')} \big) = \sum_{i=1}^{a} s_i^\tau \mathcal{O}_{V'} = \mathcal{P}^\tau|_{V'},$$

Sublemma 7.7(2) implies that $\mathcal{P}^\tau = \mathcal{B}(u)^{**}$. The proof of the last assertion is similar. $\square$

Lemma 10.3. (1) *The scheme $\mathbb{X}$ has a $\widetilde{\sigma}$-ample invertible sheaf.*

    (2) *Let $\mathcal{P}$ be an invertible sheaf on $\mathbb{X}$, and write $\mathcal{P}_\ell = \mathcal{P} \otimes \mathcal{P}^{\widetilde{\sigma}} \otimes \cdots \otimes \mathcal{P}^{\widetilde{\sigma}^{\ell-1}}$ for $\ell \geq 1$. If $\mathcal{P}_n \otimes \mathcal{P}_n^{\widetilde{\sigma}^q}$ is ample for some $n \geq 1$ and all $q \gg 0$ then $\mathcal{P}$ is $\widetilde{\sigma}$-ample.*

Proof. (1) As A is noetherian, [SeZ, Theorem 0.1] implies that A has sub-exponential growth. Since $Q(A) = K[t, t^{-1}; \sigma]$, [RZ, Proposition 3.5 and Lemma 2.3(2)] together with [Ke, Theorem 1.2(2)] imply that $\mathbb{X}$ has a $\widetilde{\sigma}$-ample invertible sheaf.

    (2) By part (1) and [Ke, Theorem 1.3], $\widetilde{\sigma}$ is quasi-unipotent in the sense of [Ke]. By hypothesis, $\mathcal{F} = \mathcal{P}_n \otimes \mathcal{P}_n^{\widetilde{\sigma}^{mn}}$ is ample for some $m \geq 1$. Hence, so is $\mathcal{G} = \mathcal{F} \otimes \mathcal{F}^{\widetilde{\sigma}^n} \otimes \cdots \otimes \mathcal{F}^{\widetilde{\sigma}^{nm-n}}$. But expanding terms, one finds that $\mathcal{G} = \mathcal{P}_{2nm}$. By [Ke, Theorem 1.3], $\mathcal{P}$ is therefore $\widetilde{\sigma}$-ample. $\square$

Proof of Theorem 10.1. We continue to use the notation set up immediately after the statement of the theorem. The aim of the proof is to apply Corollary 9.13 and this will be easy to do once we understand the relationship between sheaves of $\mathcal{O}_Y$-modules generated by the $\overline{A}_j$ and those induced from the embedding of Y into $\mathbb{P}(A_1^*)$. Thus to begin, we want to be very precise about the maps of schemes involved, after which we will translate the resulting identifications into ones involving sheaves.

    Fix a basis $\{x_j = a_j t : 0 \leq j \leq r\}$ of A_1, where each $a_j \in \overline{A}_1 \subset K$. Write $\mathbb{P} = \mathbb{P}(A_1^*)$ and let $\iota : Y \hookrightarrow W \hookrightarrow \mathbb{P}^{\times n}$ denote the restriction to Y of the natural inclusion described in Section 2. For $1 \leq m \leq n$, let $\rho_m : \mathbb{P}^{\times n} \to \mathbb{P}$ be the projection onto the mth copy of $\mathbb{P}$. As usual, we regard $Q(A)_{\geq 0}$ as a "generic" right K-point module, where K acts by left multiplication. By Corollary 3.5, the truncated K-point module $\bigoplus_{i=0}^{n} Q(A)_i$ corresponds to a map $\theta : \eta = \operatorname{Spec} K \to Y$ and hence, for $1 \leq m \leq n$, induces

a map

$$(10.4) \qquad \chi_m : \eta \xrightarrow{\theta} Y \xrightarrow{\iota} \mathbb{P}^{\times n} \xrightarrow{\rho_m} \mathbb{P}.$$

As in Notation 5.2, we identify $K = k(Y)$ through θ.

We need the precise form of θ, so we recall its construction from [ATV1]. Write $\underline{a} = (a_0 : a_1 : \cdots : a_r)$, regarded as a K-valued point of $\mathbb{P}$. The decomposition $\bigoplus_{i=0}^{n} Q(A)_i = \bigoplus_{i=0}^{n} Kt^i$ fixes the generator t^i of the free K-module $Q(A)_i$ and the right action of A_1 is then given by the formula $t^i x_j = a_j^{\sigma^i} t^{i+1}$ for all $i, j \geq 0$. Thus, by the proof of [ATV1, Proposition 3.9] (see [ATV1, Equations 3.10 and 3.11] in particular), the map $\iota\theta$ associates to η the K-valued point $(\underline{a}, \underline{a}^{\sigma}, \ldots, \underline{a}^{\sigma^{n-1}})$ of $\mathbb{P}^{\times n}$. Equivalently, for $1 \leq m \leq n$, χ_m associates η to the K-valued point $\underline{a}^{\sigma^{m-1}}$ of $\mathbb{P}$. Alternatively, regard the $\{x_j\}$ as coordinate functions on $\mathbb{P}$ and set $U_j = \{p \in \mathbb{P} : x_j(p) \neq 0\}$ for $0 \leq j \leq r$. Then, for $0 \leq j \leq r$ and $1 \leq \ell \leq n$, the map of rings $\mathcal{O}(U_j) = k[x_1 x_j^{-1}, \ldots, x_r x_j^{-1}] \to K$ induced by χ_ℓ is simply defined by $x_i x_j^{-1} \mapsto s_i(\ell) s_j(\ell)^{-1}$ for $s_m(\ell) = a_m^{\sigma^{\ell-1}}$.

We now use this information to construct invertible sheaves on Y, which we do by following the proof of [Ha, Theorem II.7.1]. Fix $1 \leq m \leq n$ and let $P(m) = K$, regarded as a locally free sheaf on $\eta = \operatorname{Spec} K$ generated by the sections $\{s_i(m) : 0 \leq i \leq r\}$. By the proof of [Ha, Theorem II.7.1(b)], this data defines a morphism $\chi'_m : \eta \to \mathbb{P}$ given by $x_i x_j^{-1} \mapsto s_i(m) s_j(m)^{-1}$ as a homomorphism $\mathcal{O}(U_j) \to K$; in other words it defines the morphism χ_m. Conversely, by [Ha, Theorem II.7.1(a)], we can identify $P(m) = \chi_m^* \mathcal{O}_{\mathbb{P}}(1)$ via the explicit formula $s_i(m) = \chi_m^*(x_i)$ for $0 \leq i \leq r$.

Now apply [Ha, Theorem II.7.1(a)] to the map $\widetilde{\chi}_m = \rho_m \iota : Y \to \mathbb{P}$ for some $1 \leq m \leq n$. Since $\mathcal{O}_Y$ is embedded into $\mathcal{K}$ via θ, we have $\widetilde{\chi}_m^*(x_i) = \chi_m^*(x_i) = s_i(m)$ for each i. Therefore, [Ha, Theorem II.7.1(a)] shows that the elements $\{s_i(m) : 0 \leq i \leq r\}$ generate a locally free sheaf of $\mathcal{O}_Y$-modules $\mathcal{N}(m) \cong \widetilde{\chi}_m^*(\mathcal{O}_{\mathbb{P}}(1)) = \iota^* \rho_m^*(\mathcal{O}_{\mathbb{P}}(1))$.

To summarize, as $\{a_i^{\sigma^{m-1}} : 0 \leq i \leq r\}$ is a k-basis of $\overline{A}_1^{\sigma^{m-1}}$, we have shown:

$$(10.5) \qquad \overline{A}_1^{\sigma^{m-1}} \subseteq \operatorname{H}^0(Y, \mathcal{N}(m)) \subseteq K \qquad \text{and} \qquad \mathcal{N}(m) = \overline{A}_1^{\sigma^{m-1}} \mathcal{O}_Y \subseteq \mathcal{K} \qquad \text{for } 1 \leq m \leq n.$$

Now consider the invertible sheaf of $\mathbb{P}^{\times n}$-modules $\mathcal{O}(1, \ldots, 1) = \rho_1^* \mathcal{O}_{\mathbb{P}}(1) \otimes \rho_2^* \mathcal{O}_{\mathbb{P}}(1) \otimes \cdots \otimes \rho_n^* \mathcal{O}_{\mathbb{P}}(1)$, where the tensor products are over $\mathcal{O}_{\mathbb{P}^{\times n}}$. Since pull-backs commute with tensor products, and the tensor product of the $\mathcal{N}(m)$ is isomorphic to their product inside $\mathcal{K}$, it follows that $\mathcal{M} = \mathcal{N}(1)\mathcal{N}(2) \cdots \mathcal{N}(n)$ satisfies

$$\mathcal{M} \cong \iota^* \rho_1^* \mathcal{O}_{\mathbb{P}}(1) \otimes \cdots \otimes \iota^* \rho_n^* \mathcal{O}_{\mathbb{P}}(1) = \iota^* \mathcal{O}(1, \ldots, 1).$$

Therefore, by (10.5),

$$(10.6) \quad \overline{A}_n = \overline{A}_1 \overline{A}_1^{\sigma} \cdots \overline{A}_1^{\sigma^{n-1}} \subseteq \operatorname{H}^0(Y, \mathcal{N}(1)) \cdots \operatorname{H}^0(Y, \mathcal{N}(n)) \subseteq \operatorname{H}^0(Y, \mathcal{M}), \qquad \text{where} \qquad \mathcal{M} = \overline{A}_n \mathcal{O}_Y.$$

Since $H^0\big(\mathbb{P}^{\times n}, \mathcal{O}(1, \ldots, 1)\big) = H^0\big(\mathbb{P}^{\times n}, \rho_1^*\mathcal{O}_{\mathbb{P}}(1)\big) \otimes_k \cdots \otimes_k H^0\big(\mathbb{P}^{\times n}, \rho_n^*\mathcal{O}_{\mathbb{P}}(1)\big)$, it follows that $\overline{A}_n$ is just the image of the natural map

$$\iota^* : H^0(\mathbb{P}^{\times n}, \mathcal{O}(1, 1, \ldots, 1)) \to H^0(Y, \iota^*(\mathcal{O}(1, 1, \ldots, 1)) = H^0(Y, \mathcal{M}).$$

On the other hand, $\mathcal{O}(1, 1, \ldots, 1)$ is a very ample invertible sheaf over $\mathbb{P}^{\times n}$ and so its set of global sections $N = H^0\big(\mathbb{P}^{\times n}, \mathcal{O}(1, \ldots, 1)\big)$ defines a closed immersion $\xi : \mathbb{P}^{\times n} \hookrightarrow \mathbb{P}(N^*)$. By restriction, ξ defines a closed immersion of Y into projective space; to be precise, $\overline{N} = \iota^*(N)$ defines a closed immersion of Y into $\mathbb{P}(\overline{N}^*) \hookrightarrow \mathbb{P}(N^*)$. Comparing this with the conclusion of the previous paragraph we deduce:

(10.7) $\mathcal{M}$ is invertible and the sections $\overline{A}_n \subseteq H^0(Y, \mathcal{M})$ define a closed immersion $Y \hookrightarrow \mathbb{P}(\overline{A}_n^*)$.

We now push everything forward to $\mathbb{X} = \mathbb{X}_n$ via the map $\pi : Y \to \mathbb{X}$ constructed in Theorem 7.1. Recall that $V = \mathbb{X} \smallsetminus P$ and that $\pi^{-1} : V \to U = \pi^{-1}(V)$ is an isomorphism. For each m, write $\mathcal{F}(m) = \pi_*\mathcal{N}(m)$, with reflexive hull $\mathcal{L}(m) = \mathcal{F}(m)^{**}$. By Sublemma 7.7(1), each $\mathcal{L}(m)$ is an invertible sheaf of $\mathcal{O}_{\mathbb{X}}$-modules contained in $\mathcal{K}$. Moreover, Lemma 10.2 and (10.5) imply that $\mathcal{L}(m) = \mathcal{L}(1)^{\tilde{\sigma}^{m-1}}$ for $1 \leq m \leq n$ and that $\mathcal{L}(1)\mathcal{L}(2) \cdots \mathcal{L}(n) = (\pi_*\mathcal{M})^{**}$. In other words, $(\pi_*\mathcal{M})^{**} = \mathcal{L}_n = \mathcal{L}\mathcal{L}^{\tilde{\sigma}} \cdots \mathcal{L}^{\tilde{\sigma}^{n-1}}$ for $\mathcal{L} = \mathcal{L}(1) \subset \mathcal{K}$.

By (10.7), $\mathcal{M}$ is a very ample invertible sheaf on Y while $\mathcal{L}_n = (\pi_*\mathcal{M})^{**}$ by definition. Thus, Remark 7.8 implies that $\mathcal{L}_n \otimes (\mathcal{L}_n)^{\tilde{\sigma}^q}$ is ample for all $q \gg 0$. Hence, by Lemma 10.3(2) and the previous paragraph, $\mathcal{L}$ is a $\tilde{\sigma}$-ample invertible sheaf.

We are now ready to check the hypotheses of Corollary 9.13. By its construction in Theorem 7.1, $\tilde{\sigma}$ induces the automorphism σ of K. Since $\mathcal{L} \subset \mathcal{K}$ this implies that $B(\mathbb{X}, \mathcal{L}, \tilde{\sigma}) = \bigoplus_{n \geq 0} H^0(\mathbb{X}, \mathcal{L}_n)t^n \subset K[t; \sigma]$. As A is generated in degree one by $A_1 = \overline{A}_1 t \subseteq H^0(\mathbb{X}, \mathcal{L})t$, it follows that A is a subring of $B(\mathbb{X}, \mathcal{L}, \tilde{\sigma})$, and hence $\overline{A}_n t^n \subseteq H^0(\mathbb{X}, \mathcal{L}_n)t^n$. Since $\mathcal{L}$ is a $\tilde{\sigma}$-ample invertible sheaf by the previous paragraph, and P consists of a finite set of points lying on dense $\tilde{\sigma}$-orbits by Theorem 7.1, the hypotheses from the first three sentences of Corollary 9.13 are satisfied in the present setup. By (10.6), $\mathcal{L}_n|_V = \pi_*\mathcal{M}|_V = \mathcal{M}|_U$ is generated by $\overline{A}_n$ and so condition (1) of the corollary holds, while condition (2) of the corollary follows from (10.7). We can therefore apply the corollary to find a saturating zero-dimensional subscheme $Z_{\mathcal{I}} \subset \mathbb{X}$ such that $A = R(\mathbb{X}, Z_{\mathcal{I}}, \mathcal{L}, \tilde{\sigma})$, up to a finite dimensional vector space. This proves part (1) of the theorem.

Finally, parts (2.i) and (2.ii) of the theorem both follow from Theorem 7.1 together with our construction of $\mathbb{X}$. Corollary 9.13 also ensures that $\mathcal{I}$ is defined by $\mathcal{L}\mathcal{I} = \overline{A}_1\mathcal{O}_{\mathbb{X}}$, while (10.5) shows that $\mathcal{L} = (\pi_*\mathcal{M})^{**} = (\overline{A}_1\mathcal{O}_{\mathbb{X}})^{**}$. Thus part (2.iii) of the theorem holds. $\qquad\square$

As an application of the theorem, we get the following criterion for a noetherian cg domain to be a twisted homogeneous coordinate ring. Up to changes in phraseology (the set S_∞^+ does not appear in [RZ]) this result is also a special case of [RZ, Theorem 4.4].

Corollary 10.8. *Let A be a noetherian cg domain that is generated in degree one and birationally 2-geometric. Suppose that the set of extremal elements S_∞^+, as defined in (5.6), is empty. Then, up to a finite dimensional vector space, $A \cong B(\mathbb{X}, \mathcal{L}, \sigma)$, where $\mathbb{X} = Y_n$ for any stable value of n.*

Proof. The first paragraph of the proof of Theorem 7.1 shows that $\mathbb{X} = Y = Y_n$. Therefore, the set P constructed in the proof of Theorem 10.1 is also empty and so Corollary 9.13—or indeed [AS, Theorem 4.1]— shows that $A \cong R(\mathbb{X}, \emptyset, \mathcal{L}, \sigma) = B(Y, \mathcal{L}, \sigma)$ up to a finite dimensional vector space. $\qquad\square$

11. EXAMPLES

In this section, we give two examples which illustrate the earlier results in the paper. The first example shows that the truncated point schemes W_n and the relevant components Y_n are, in general, distinct; indeed the dimension can differ by arbitrarily large amounts. Surprisingly, this happens when one takes the naïve blowup algebra at a high power of a sheaf of maximal ideals.

Example 11.1. Let X be an integral projective scheme with $\dim X \geq 2$ and $\sigma \in \operatorname{Aut} X$ for which there exists a very ample and σ-ample invertible sheaf $\mathcal{L}$ and a point $c \in X$ such that the orbit $\langle \sigma \rangle \cdot c$ is critically dense. Write $\mathcal{I} = \mathcal{I}_c$ for the ideal sheaf of the point c. Then, for any integer $t \geq 1$ there exist $p, q \in \mathbb{N}$ with the following properties:

  (1) The ring $R = R(X, Z_{\mathcal{I}^p}, \mathcal{L}^q, \sigma)$ is noetherian and generated in degree one.
  (2) For any $n \geq 1$ the truncated point scheme W_n for R satisfies $\dim W_n \geq t$.
  (3) For any $n \geq 1$ there exists a point $w \in W_n$ such that the fibre $\phi_n^{-1}(w)$ has dimension $\geq t$.

Remarks 11.2. (1) Note that, if X is a surface in this example and n is a stable value then $\dim Y_n = 2$, by Corollary 3.5. Therefore, if $t > 2$, then $W_n \neq Y_n$ for all $n \gg 0$.

(2) An explicit example can be obtained by using the data from Section 1.2; thus, modulo a slight change of notation, take $X = \mathbb{P}^2$ with $\mathcal{L} = \mathcal{O}(1)$, $c = (1 : 1 : 1)$ and σ defined by $(\lambda_0 : \lambda_1 : \lambda_2) \mapsto (\lambda_0 : \alpha\lambda_1 : \beta\lambda_2)$, where $\alpha, \beta \in k^*$ are algebraically independent over the prime subfield of k.

(3) This result is in striking contrast to the commutative case, where blowing up at a maximal ideal $\mathcal{I}$ and at some power $\mathcal{I}^p$ of $\mathcal{O}_X$ gives the same scheme [Ha, Exercise II.7.11].

Proof. In the proof, tensor products will be taken over $\mathcal{O}_X$. As $\dim X \geq 2$ we can pick p such that $\operatorname{length} \mathcal{I}^p / \mathcal{I}^{p+1} \geq t + 1$. Set $\mathcal{J} = \mathcal{I}^p$. We want to pick q so that there are no complications arising from cohomology. Explicitly, by [RS, Proposition 3.20 and Corollary 3.21] we can choose $q \geq 1$ such that:

  (i) the ring $R(X, Z_{\mathcal{J}}, \mathcal{L}^q, \sigma)$ is generated in degree one, and
  (ii) $\operatorname{H}^1(X, \mathcal{I}^{\sigma^u} \otimes \mathcal{J}_a \otimes \mathcal{L}_a^q) = 0$, for all $u \in \mathbb{Z}$ and $a \geq 1$.

For these values of p and q set $R = R(X, Z_{\mathcal{J}}, \mathcal{L}^q, \sigma)$ and $\mathcal{R} = \mathcal{R}(X, Z_{\mathcal{J}}, \mathcal{L}^q, \sigma)$. The ring R is noetherian by Proposition 8.7(2) and so (1) holds. Let $\mathcal{H} = \mathcal{I}^{\sigma^n}$ for some $n \geq 1$. The point of the proof will be to find a t-dimensional family of truncated point modules of length $n + 2$ among the quotients of the R-module $M = \operatorname{H}^0(X, \mathcal{R}/\mathcal{H}\mathcal{R})$.

Taking cohomology of the exact sequence $0 \to \mathcal{H}\mathcal{R} \to \mathcal{R} \to \mathcal{R}/\mathcal{H}\mathcal{R} \to 0$ gives the exact sequence of R-modules

$$0 \longrightarrow \mathrm{H}^0(X, \mathcal{H}\mathcal{R}) \longrightarrow R \xrightarrow{\chi} M \longrightarrow \bigoplus_{a=0}^{\infty} \mathrm{H}^1(X, \mathcal{H}\mathcal{J}_a \otimes \mathcal{L}_a^q).$$

Suppose first that $a \geq 1$; thus $\mathrm{H}^1(X, \mathcal{H} \otimes \mathcal{J}_a \otimes \mathcal{L}_a^q) = 0$, by the choice of q. Since the kernel $\mathcal{D}$ of the natural surjection $\mathcal{H} \otimes \mathcal{J}_a \otimes \mathcal{L}_a^q \twoheadrightarrow \mathcal{H}\mathcal{J}_a \otimes \mathcal{L}_a^q$ is supported at the point $\sigma^{-n}(c)$, certainly $\mathrm{H}^2(X, \mathcal{D}) = 0$. Combining these observations shows that $\mathrm{H}^1(X, \mathcal{H}\mathcal{J}_a \otimes \mathcal{L}_a^q) = 0$ as well and so $\chi_a : R_a \to M_a$ is surjective for all $a \geq 1$. When $a = 0$, $R_0 = M_0 = k$ and $\mathrm{H}^0(X, \mathcal{H}\mathcal{R}_0) = \mathrm{H}^0(X, \mathcal{H}) = 0$. Thus χ_0 is an isomorphism, χ is surjective and M is cyclic.

For any $r \geq 0$,

$$M_r = \mathrm{H}^0\left(X, \frac{\mathcal{J}_r \otimes \mathcal{L}_r^q}{\mathcal{H}\mathcal{J}_r \otimes \mathcal{L}_r^q}\right) \cong \mathrm{H}^0\left(X, \frac{\mathcal{J}_r}{\mathcal{H}\mathcal{J}_r}\right).$$

If $r \leq n$ then $\mathcal{H} = \mathcal{I}^{\sigma^n}$ and $\mathcal{J}_r = \mathcal{J} \cdots \mathcal{J}^{\sigma^{r-1}}$ are comaximal and so $M_r \cong \mathrm{H}^0(X, \mathcal{O}_X/\mathcal{H}) \cong k$. However,

$$M_{n+1} \cong \mathrm{H}^0\left(X, \frac{\mathcal{J} \cdots \mathcal{J}^{\sigma^n}}{\mathcal{H}\mathcal{J} \cdots \mathcal{J}^{\sigma^n}}\right) \cong \mathrm{H}^0\left(X, \frac{\mathcal{H}^p}{\mathcal{H}^{p+1}}\right).$$

This has dimension $T \geq t + 1$ by the choice of p. For each $(T-1)$-dimensional subspace $V \subset M_{n+1}$, let $M(V) = M/(VR + M_{n+2}R) \cong R/Q(V)$, say. By the above analysis, this is a cyclic R-module with $\dim_k M(V)_j = 1$ for $0 \leq j \leq n + 1$; in other words $M(V)$ is a truncated point module of length $n + 2$. Now, truncated point modules R/Q and R/Q' are isomorphic if and only if $Q = Q'$. Since $Q(V_1) \neq Q(V_2)$ if $V_1 \neq V_2$, we have therefore constructed a family of truncated point modules of length $n + 2$ parametrized by $\mathbb{P}(M_{n+1}^*)$. Thus $\dim W_{n+1} \geq t$ for all $n \geq 0$, proving (2). Finally, each of the modules $M(V)$ has the same truncation $M/M_{n+1}R$ and so, by Lemma 2.3(1), the fibre of ϕ_n over the point $w \in W_n$ corresponding to $M/M_{n+1}R$ also has dimension at least t. $\qquad \square$

In order to illustrate the questions raised in Section 1.5, we end by giving an example of a (non-noetherian) cg domain A that is birationally commutative but not birationally 2-geometric. As mentioned there, we conjecture that examples of this type are always non-noetherian.

Example 11.3. Let $K = k(u, v)$ be a rational function field over k, and define $\sigma : K \to K$ by $u \mapsto u$, $v \mapsto uv$. Let $Q = K[z, z^{-1}; \sigma]$ and $A = k\langle uz, vz, z\rangle \subset Q$. Then A is a non-noetherian domain that is birationally commutative but not birationally 2-geometric.

Proof. It is a routine exercise, which we leave to the reader, to show that $\mathrm{GKdim}\, A = 4$ (mimic the proof of [MR, Example 8.2.16]). Thus A is an Ore domain by [MR, Corollary 8.1.21] and since $u, v \in A_1 A_1^{-1}$ it is clear that Q is the graded ring of fractions of A.

In order to prove that A is not noetherian, write $f_n = (vz)z^n = vz^{n+1}$ for $n \geq 0$ and let $I^{(n)} = \sum_{i=0}^n A f_i$. Note that any element $a \in A_r$ has the form $a = qz^r$, for some $q \in k[u, v]$. Hence, if $r > 0$ then $af_{n-r} = qz^r vz^{n-r+1} = qu^r vz^{n+1} \in k[u, v] u f_n$. Thus, $f_n \notin I^{(n-1)}$ for $n \geq 1$ and $I^{(0)} \subsetneq I^{(1)} \subsetneq \ldots$ gives a proper ascending chain of left ideals of A.

Clearly A is birationally commutative. There are several ways to see that A is not birationally 2-geometric. First of all, [DF, Remark 7.3] shows that there is no projective model Y of $K = k(u, v)$ with an automorphism inducing the given $\sigma \in \operatorname{Aut}(K)$, so A is not birationally 2-geometric by definition. Alternatively if A were birationally 2-geometric, then [RZ, Theorem 1.6] would imply that $\operatorname{GKdim}(A) = \operatorname{GKdim}(B)$ for some twisted homogeneous coordinate ring $B(Y, \mathcal{L}, \sigma)$ with $Q(B) = K[t, t^{-1}; \sigma]$, where Y would necessarily be a surface and $\mathcal{L}$ would be σ-ample. But, by [AV, Theorem 1.7], no such twisted homogeneous coordinate ring has GK-dimension 4. $\qquad\square$

Acknowledgments

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