Numerische Mathematik manuscript No.
(will be inserted by the editor)

V. Domínguez · I.G. Graham · V.P. Smyshlyaev

A hybrid numerical-asymptotic boundary integral method for high-frequency acoustic scattering

Received: date / Revised: date

Abstract We propose a new robust method for the computation of scattering of high-frequency acoustic plane waves by smooth convex objects in 2D. We formulate this problem by the direct boundary integral method, using the classical combined potential approach. By exploiting the known asymptotics of the solution, we devise particular expansions, valid in various zones of the boundary, which express the solution of the integral equation as a product of explicit oscillatory functions and more slowly varying unknown amplitudes. The amplitudes are approximated by polynomials (of minimum degree d) in each zone using a Galerkin scheme. We prove that the underlying bilinear form is continuous in L_2, with a continuity constant that grows mildly in the wavenumber k. We also show that the bilinear form is uniformly L_2-coercive, independent of k, for all k sufficiently large. (The latter result depends on rather delicate Fourier analysis and is restricted in 2D to circular domains, but it also applies to spheres in higher dimensions.) Using these results and the asymptotic expansion of the solution, we prove superalgebraic convergence of our numerical method as $d \to \infty$ for fixed k. We also prove that, as $k \to \infty$, d has to increase only very modestly to maintain a fixed error bound ($d \sim k^{1/9}$ is a typical behaviour). Numerical experiments show that the method suffers minimal loss of accuracy as $k \to \infty$, for a fixed number of degrees of freedom. Numerical solutions with a relative error of about 10^{-5} are obtained on domains of size $\mathcal{O}(1)$ for k up to 800 using about 60 degrees of freedom.

Mathematics Subject Classification (2000) High frequency · Acoustic scattering · Boundary integral method · Hybrid numerical-asymptotic · wave-number robust

1 Introduction

In this paper we devise, analyse and implement an algorithm for computing the scattering of plane waves in two space dimensions by an arbitrary smooth convex scatterer. Our method is based on the direct boundary integral method and the Galerkin approximation, but instead of using ansatz functions constructed directly

Víctor Domínguez was supported by the Bath Institute for Complex Systems, FEDER/MCYT MTM2004-01905 and Gobierno de Navarra, resolución 18/2005

We would like to thank Simon Chandler-Wilde, Ralf Hiptmair, Vladimir Kamotski and Steve Langdon for useful discussions.

V. Domínguez
Dep. Matemática e Informática, Universidad Pública de Navarra, Campus de Arrosadía s/n, Pamplona 31006, Spain.
E-mail: victor.dominguez@unavarra.es

I.G. Graham
Department of Mathematical Sciences, University of Bath, Bath, BA2 7AY, United Kingdom.
E-mail: I.G.Graham@bath.ac.uk

V.P. Smyshlyaev
Department of Mathematical Sciences, University of Bath, Bath, BA2 7AY, United Kingdom.
E-mail: vps@maths.bath.ac.uk

from (piecewise) polynomials, we use an ansatz which incorporates the rigorously known asymptotic properties of the normal derivative of the total field, obtainable from e.g., [37] – see also [15,17].

If the incident field is a plane wave $\exp(ik\mathbf{x}\cdot\widehat{\mathbf{a}})$, for some unit vector $\widehat{\mathbf{a}}$, and if the scatterer is supposed to be of "sound soft" type, then the scattered field everywhere exterior to the object can be determined by computing the "surface current" $v := \partial_\nu u$, on the boundary Γ of the scatterer, where ∂_ν denotes the normal derivative on Γ, directed outward from the scatterer, and u is the total field (incident plus scattered fields). It is also known that, in the case of a convex scatterer, v can be expressed as

$$v(\boldsymbol{\gamma}(s),k) = kV(s,k)\exp(ik\boldsymbol{\gamma}(s)\cdot\widehat{\mathbf{a}})\ , \tag{1.1}$$

with $\boldsymbol{\gamma}$ denoting arc-length parametrisation on Γ, where $V(\cdot,k)$ oscillates less rapidly than $v(\cdot,k)$, for large k. The asymptotic analysis for $k\to\infty$ also implies that Γ can be decomposed into four zones: the illuminated zone, the shadow zone and the transition or "Fock" domains which lie between these ([22]). In each of these regions, explicit estimates can be given for the rate of growth of $V(s,k)$. The choice of precise boundaries between the transition zones and the other zones turns out to be of particular importance in practice.

Our Galerkin method only approximates the slowly varying amplitudes $V(s,k)$ in each zone, and in computational experiments suffers negligible loss of accuracy for a fixed number of degrees of freedom as $k\to\infty$. This should be compared to the usual boundary element methods, which approximate the complete oscillatory solution v by (piecewise) polynomials. Although these can be very effective for moderate frequencies (e.g. [23]), to preserve accuracy as k grows, it is normally necessary to increase the number of degrees of freedom with at least $\mathcal{O}(k)$, leading to a growth in complexity as k increases. The method in the present paper avoids such a growth of complexity.

The model problem considered here is chosen because it contains enough features of more general scattering problems to be of general interest, but is also simple enough to allow a substantial amount of rigorous analysis. Extensions to more general scattering geometries may have to take account of more complicated asymptotics, such as diffraction from edges or corner points (see [6,8,9] and the references therein) or multiple scattering ([11,12,20]). The idea of this paper is to show that k-robust numerical methods with a complete error analysis are possible, and to prepare the ground for further developments.

We remark that several other authors have considered the introduction of oscillatory ansatz functions into numerical boundary element schemes. Our method is most closely related to those of Abboud et. al. [1], Bruno et. al. [11] and Giladi and Keller [26]. In [1] (which considered a different boundary integral equation with spurious frequencies), an ansatz similar to (1.1) was used globally on Γ, the analogue of V was approximated by a boundary element Galerkin method and an estimate for the consistency error was then stated. This suggested that h must be chosen proportional to $k^{-1/3}$ to preserve accuracy as $k\to\infty$. In [11] the ansatz (1.1) was used directly, the resulting integral equation for $V(\cdot,k)$ was solved by a Nyström scheme, and k-independent convergence was observed experimentally. In the Nyström method, a cube root change of variable was employed in the transition zones (where $V(\cdot,k)$ still oscillates, but more moderately than $v(\gamma(\cdot),k)$) in order to achieve robustness as $k\to\infty$. In the Galerkin scheme of the present paper the oscillation in the transition zones is handled by allowing these zones to shrink with k (according to a delicate formula described below) and using polynomial approximations there which are independent of the polynomials employed in the illuminated part. Neither [1] nor [11] attempt a rigorous error analysis. The paper [26] considers a collocation method also based on an ansatz similar to (1.1) but with a more particular treatment of "creeping waves" just behind the shadow boundary. Numerical results are given showing good robustness properties, but again no error analysis is given. Since [26] considers only single layer potential formulations it also suffers from spurious frequencies and so a full error analysis will not be possible without modifications.

In related work, Langdon and Chandler-Wilde [31] consider the case of scattering from an infinite impedance line, where the asymptotics of the scattered wave can be derived directly from the mathematical model. They were able to then propose an algorithm based on approximation only of slow variables and moreover to prove rigorously its robustness with respect to k, i.e. the constants in the stability and consistency estimates were shown not to blow up if $k\to\infty$. More recent work of Arden, Chandler-Wilde and Langdon [4,32] has extended the results of [31] to convex polygons. An extensive survey of other approaches to the use of oscillatory ansatz functions in high-frequency scattering is given in [31].

In this paper we devise approximations of $v(\cdot,k)$ in (1.1) by using algebraic polynomial approximations of $V(\cdot,k)$ in certain carefully chosen illuminated, shadow and shadow boundary-zones of Γ. That is, we use a non-standard variant of the $p-$version of the boundary element method to approximate $V(\cdot,k)$. The resulting finite dimensional subspaces are then used inside a Galerkin scheme to obtain approximations of v. We undertake a complete error analysis of this type of approximation and prove its stability and superalgebraic convergence for fixed k. Most importantly, by a careful analysis of the k dependence of the constants appearing in the error analysis, we also prove that, to obtain bounded discretisation error as $k\to\infty$, the polynomial degrees need to grow only very mildly with k, with $k^{1/9}$ being typical theoretical growth rate. Under an assumption on the coercivity of the integral operator (which is satisfied by circular scatterers), we prove that the actual Galerkin error enjoys the same property. In practice the error appears to be essentially independent of k, for the range of k which we have tested.

The plan of this paper is as follows. In §2 we review the scattering problem and its reformulation as a boundary integral equation (using the combined potential approach). In §3 we briefly explain the Galerkin scheme for the boundary integral equation and present the necessary ingredients for its error analysis. In §4 we show that the bilinear form underlying the Galerkin method is continuous in L_2, for any suitably smooth $2D$ boundary, and that the continuity constant grows at worst with $\mathcal{O}(k^{1/2})$, as $k \to \infty$. We also show using a more delicate Fourier analysis that, for the case of spherical geometries (in $2D$ or $3D$), the bilinear form is L_2-coercive, with coercivity constant independent of k, as $k \to \infty$ and the $\mathcal{O}(k^{1/2})$ bound on the continuity constant can be improved to $\mathcal{O}(k^{1/3})$. These results apply to both the Brakhage-Werner combined potential operator and the closely related formal adjoint of it (sometimes called the Burton-Miller operator [14]).

The latter operator is the one of chief interest in this paper, since it appears in the integral equation with the physically relevant solution v in (1.1). In §5 we summarise the rigorous asymptotics of v and in particular we obtain estimates for the derivatives of the slowly varying amplitude V which is approximated in the Galerkin scheme. In §6 we use the results from §5 to derive error estimates for our Galerkin scheme, in which the dependence on the wave number k is made explicit.

In §7 we describe numerical experiments which illustrate the convergence of our algorithm. In our experiments the method converges superalgebraically with respect to the polynomial degrees used and does not degrade at all as $k \to \infty$.

2 The Scattering Problem

Suppose that $\Omega \subset \mathbb{R}^2$ is a closed bounded convex domain with a $\mathcal{C}^\infty$ boundary Γ and that $\Omega' := \mathbb{R}^2 \backslash \overline{\Omega}$ is the corresponding exterior domain. We shall consider an acoustic incident plane wave:

$$u^I(\mathbf{x}) := \exp(ik\mathbf{x}\cdot\hat{\mathbf{a}}) \ , \qquad \mathbf{x} \in \mathbb{R}^2 \ ,$$

with direction given by the unit vector $\hat{\mathbf{a}} \in \mathbb{R}^2$. This is scattered by Ω to produce a scattered wave u^S. The total wave $u := u^I + u^S$ is a solution of the Helmholtz equation:

$$\Delta u + k^2 u = 0 \quad \text{in} \ \ \Omega' \ , \tag{2.1}$$

and u^S satisfies the usual radiation condition:

$$\partial_r u^S(\mathbf{x}) - iku^S(\mathbf{x}) = o(|\mathbf{x}|^{-1/2}), \qquad \text{as } |\mathbf{x}| \to \infty \ ,$$

where ∂_r denotes the radial derivative. We shall consider the sound-soft scattering problem, in which case

$$u = 0 \quad \text{on} \quad \Gamma \quad (\text{equivalently} \quad u^S = -u^I \quad \text{on } \Gamma). \tag{2.2}$$

This problem can be formulated as a boundary integral equation in various ways. Because we wish to exploit known asymptotic formulae involving u (or u^S) for large k, we favour here the direct approach, in which the unknown $v(\mathbf{x}) := (\partial_\nu u)(x)$, $\mathbf{x} \in \Gamma$ satisfies the combined potential integral equation (see, e.g., [19]):

$$\frac{1}{2}v + \mathcal{D}_k v - ik\mathcal{S}_k v = \partial_\nu u^I - iku^I \ , \qquad \text{or, more abstractly,} \qquad \mathcal{R}_k v \ = \ f_k \ , \tag{2.3}$$

where $\mathcal{S}_k$ and $\mathcal{D}_k$ are the single layer operator and its normal derivative given, for $\mathbf{x} \in \Gamma$, by:

$$\mathcal{S}_k v(\mathbf{x}) = \int_\Gamma \Phi_k(\mathbf{x}-\mathbf{y})v(\mathbf{y})\mathrm{d}\mathbf{y} \quad \text{and} \quad \mathcal{D}_k v(\mathbf{x}) \ = \ \int_\Gamma \partial_{\nu(\mathbf{x})}\left\{\Phi_k(\mathbf{x}-\mathbf{y})\right\} v(\mathbf{y})\,\mathrm{d}\mathbf{y} \ , \tag{2.4}$$

and $\Phi_k(\mathbf{x}) = (i/4)H_0^{(1)}(k|\mathbf{x}|)$ (with $H_0^{(1)}$ denoting the Hankel function of the first kind of order zero) is the usual fundamental solution of the Helmholtz equation in 2D.

The operator $\mathcal{R}_k$ in (2.3) is closely related to the classical *Brakhage-Werner* operator [10]:

$$\mathcal{P}_k := \frac{1}{2}I + \mathcal{L}_k - ik\mathcal{S}_k \ , \tag{2.5}$$

where

$$\mathcal{L}_k v(\mathbf{x}) \ = \ \int_\Gamma \partial_{\nu(\mathbf{y})}\left\{\Phi_k(\mathbf{x}-\mathbf{y})\right\} v(\mathbf{y})\,\mathrm{d}\mathbf{y} \ , \tag{2.6}$$

is the double layer potential, and which arises in the indirect approach to scattering problems. We are only interested in this paper in approximating (2.3) (since its solution v has physical meaning and the asymptotic analysis below exploits this), but the estimates in §4 will be valid for both $\mathcal{R}_k$ and $\mathcal{P}_k$.

Suppose

$$\Gamma = \{\boldsymbol{\gamma}(s) : s \in [0, 2\pi]\} \tag{2.7}$$

is a 2π-periodic parametrisation of Γ chosen, for convenience, to be proportional to arc-length parametrisation on Γ. Then (2.3) can be rewritten as an equation on $[0, 2\pi]$ in an obvious way. Using the same symbols to denote transformed functions and operators, we write (2.3) again as

$$\mathcal{R}_k v(s) \; = \; f_k(s) \; , \quad s \in [0, 2\pi] \; . \tag{2.8}$$

For any measurable $\Lambda \subset [0, 2\pi]$, let $(v, w)_{L_2(\Lambda)}$ denote the usual L_2 inner product of complex-valued functions on Λ and let $\| \cdot \|_{L_2(\Lambda)}$ denote the induced norm. When $\Lambda = [0, 2\pi]$ we denote these simply by (v, w) and $\|v\|$ and we write $L_2[0, 2\pi]$ simply as L_2. We also let $L_\infty = L_\infty[0, 2\pi]$, with norm $\| \cdot \|_{L_\infty}$. The variational formulation of (2.8) then is to seek $v \in L_2$ such that

$$a_k(v, w) := (\mathcal{R}_k v, w) \; = \; (f_k, w) \; , \quad \text{for all} \quad w \in L_2 \; . \tag{2.9}$$

Below we shall give estimates for the coercivity and continuity of the sesquilinear form a_k. Although these would hold for any η chosen proportional to k, for convenience we simply set $\eta = k$ here. Other authors have considered different choices of η for different purposes (e.g. [29, 28, 3, 25]).

3 A Non-standard Galerkin Method

When a plane wave is incident on a convex two dimensional object, the two tangency points T_1 and T_2 naturally divide the boundary into an "illuminated zone" (**I**) and a "shadow zone" (**S**), as depicted in Figure 1 .

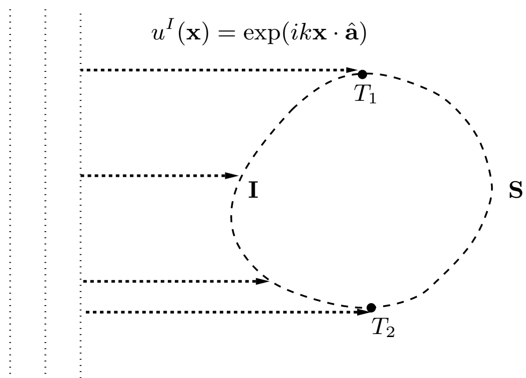

Fig. 1 Physical domain: **I** denotes the illuminated zone, **S** the shadow zone and T_1, T_2 the tangency points.

We choose the parametrization $\boldsymbol{\gamma}$ (see (2.7)) so that the point $\boldsymbol{\gamma}(0)$ lies in the shadow zone, halfway (with respect to arc length) between T_2 and T_1. Let $\boldsymbol{\gamma}(t_i) = T_i$, for $i = 1, 2$. Then we can choose three overlapping closed subintervals $\Lambda_i \subset (0, 2\pi)$, $i = 1, \ldots, 3$ with the properties:

$$t_i \in \Lambda_i \backslash \left(\cup_{j \neq i} \Lambda_j \right) \; , \; i = 1, 2 \; , \quad \text{and} \quad \{\boldsymbol{\gamma}(s) : s \in \Lambda_3\} \subset I \; .$$

(That is, if we define $\Gamma_i := \boldsymbol{\gamma}(\Lambda_i)$, $i = 1, 2, 3$, then the tangency point T_i lies only in Γ_i ($i = 1, 2$) and the illuminated zone contains Γ_3.) We assume further that $0 \in \Lambda_4 := [0, 2\pi] \backslash \left(\cup_{j=1}^3 \Lambda_j \right)$, (i.e. the shadow zone contains $\Gamma_4 := \boldsymbol{\gamma}(\Lambda_4)$).

We introduce a corresponding *partition of unity* $\{\chi_j : j = 1, 2, 3, 4\}$, where, for each j, $\chi_j \in L_\infty$ and

$$\text{supp} \; \chi_j = \Lambda_j \; , \quad 0 \le \chi_j(s) \le 1 \; , \quad \text{and} \quad \left(\sum_{j=1}^{4} \chi_j \right)(s) = 1 \; , \quad s \in [0, 2\pi] \; . \tag{3.1}$$

The Λ_j (and hence χ_j) may depend on k, but we do not reflect this in the notation. The Λ_i may overlap their neighbours in either a single point or an interval with non-empty interior and the χ_j may be either characteristic functions of the Λ_j or a classical smooth partition of unity. (In the numerical experiments in §7 we use a smooth partition of unity and we adjust the sizes of the Λ_i as k increases; this turns out to be important in achieving robustness with respect to k and is used in a key way in the proofs below.)

Underlying our Galerkin method is the assumption that for each $j = 1, 2, 3$, there exist explicitly defined oscillatory functions $e_j(\cdot, k) \in L_\infty$, such that the ($k$- dependent) solution v of the integral equation (2.3) satisfies

$$\chi_j(s)v(s) = k\ \chi_j(s)\ e_j(s,k)\ V_j(s,k)\ , \quad s \in [0, 2\pi]\ , \tag{3.2}$$

with unknown "modulating amplitudes" $V_j(s, k)$ which have to be computed, but which vary more slowly than $v(s, k)$. For our problem this abstract assumption will be made concrete in §6. In the shadow zone Λ_4, v is exponentially small as $k \to \infty$ (see also §6) and so here v can be well-approximated by zero (see §7 for further remarks on this point). Note that the explicit factor k appearing on the right-hand side of (3.2) reflects the fact that since $v = \partial_\nu u$, the amplitude of v will grow with $\mathcal{O}(k)$ as $k \to \infty$. (See Theorem 5.4 for a precise justification of this.)

On the assumption that (3.2) holds, we formulate our Galerkin scheme as follows. For any integer $d \geq 0$, let $\mathbb{P}^d$ denote the algebraic polynomials of degree d or less. For $j = 1, 2, 3$ choose integers $d_j \geq 0$ and set

$$\mathcal{V}_k^j = \mathrm{span}\{k\ \chi_j(s)\ e_j(s,k)\ s^\ell :\ \ell = 0, \dots, d_j\}\ .$$

Then define

$$\mathcal{V}_k^{\mathbf{d}} = \oplus_{j=1}^3 \mathcal{V}_k^j\ , \quad \text{a space of dimension} \quad \#(\mathbf{d}) = d_1 + d_2 + d_3 + 3\ .$$

Note that any function in $\mathcal{V}_k^{\mathbf{d}}$ vanishes in $\Lambda_4 \backslash \{\cup_{j=1}^3 \Lambda_j\}$, i.e. in the shadow zone. Hence, we approximate the solution of our problem by zero in this part of the boundary.

The corresponding Galerkin scheme for (3.2) is then to seek $\tilde{v} \in \mathcal{V}_k^{\mathbf{d}}$ such that

$$a_k(\tilde{v}, \tilde{w}) = (f_k, \tilde{w}), \quad \text{for all} \quad \tilde{w} \in \mathcal{V}_k^{\mathbf{d}}\ . \tag{3.3}$$

The abstract theory of the Galerkin method for coercive problems is standard:

Lemma 3.1 (Céa's Lemma) *Suppose a_k satisfies, for all $v, w \in L_2$, the two assumptions:*

$$\text{continuity} \quad |a_k(v,w)| \ \leq\ B_k\|v\|\|w\|\ , \quad B_k > 0\ , \tag{3.4}$$

$$\text{coercivity} \quad |a_k(v,v)| \ \geq\ \alpha_k\|v\|^2\ , \quad \alpha_k > 0\ . \tag{3.5}$$

Then both the weak form (2.9) *and its Galerkin approximation* (3.3) *have unique solutions* ($v \in L_2$ *and* $\tilde{v} \in \mathcal{V}_k^{\mathbf{d}}$)*. Moreover,*

$$\|v - \tilde{v}\| \ \leq\ \left(\frac{B_k}{\alpha_k}\right)\ \|v - \tilde{w}\|\ , \quad \textit{for all}\ \tilde{w}\ \in\ \mathcal{V}_k^{\mathbf{d}}\ . \tag{3.6}$$

To apply this result to our problem, we use (3.1) and (3.2) to write

$$v(s) \ =\ \sum_{j=1}^4 \chi_j(s)v(s) \ =\ k\sum_{j=1}^3 \chi_j(s)e_j(s,k)V_j(s,k) \ +\ \chi_4(s)v(s)\ ,$$

and write $\tilde{w} \in \mathcal{V}_k^d$ as

$$\tilde{w}(s) = k\sum_{j=1}^3 \chi_j e_j(s,k)p_j^{d_j}(s)\ , \quad \text{for some} \quad p_j^{d_j} \in \mathbb{P}^{d_j}\ , \quad j = 1, 2, 3.$$

From these formulae with (3.6), and recalling (3.1), the following corollary follows easily.

Corollary 3.2

$$\|v - \tilde{v}\| \ \leq\ \left(\frac{B_k}{\alpha_k}\right)\left[k\ \sum_{j=1}^3 \left\{\|e_j(\cdot,k)\|_{L_\infty} \inf_{p \in \mathbb{P}^{d_j}} \|V_j(\cdot,k) - p\|_{L_2(\Lambda_j)}\right\} + \|v\|_{L_2(\Lambda_4)}\right].$$

In our applications the functions $e_j(\cdot,k)$ will be exponentials with pure imaginary argument, having therefore modulus equal to 1. Thus the only issues for the error estimate in Corollary 3.2 will be to estimate the constants B_k and α_k (this is done in §4), together with the errors of the polynomial approximations of $V_j(\cdot,k)$ and the size of v in the shadow zone. (The latter are investigated in §6).

We would like to remark that our purpose in this paper is to establish rigorously that Galerkin methods based on asymptotic information for k large can deliver $k-$robust methods. In this first paper we concentrate on investigating the $k-$ dependence of the constants α_k and B_k and of the error in approximation of $V_j(\cdot,k)$ in the space $\mathcal{V}_k^{\mathbf{d}}$. The numerical analysis of methods for efficiently evaluating the integrals arising in the implementation of the Galerkin method will be a focus of later work. Related work in this direction is in [11, 24].

4 Coercivity and Continuity

In this section we prove the continuity and the coercivity of the bilinear form a_k appearing in the variational formulation (2.9). Since all the estimates of this section are equally valid for both the (Burton-Miller) operator $\mathcal{R}_k$ and the (Brakhage-Werner) operator $\mathcal{P}_k$ (see (2.3) or (2.5)), we shall in this section consider the more general sesquilinear form:

$$a_k(v,w) := (\mathcal{A}_k v, w)\ , \quad \text{where} \quad \mathcal{A}_k = \mathcal{R}_k \ \text{ or } \ \mathcal{A}_k = \mathcal{P}_k\ . \tag{4.1}$$

For any boundary Γ, the continuity of a_k can be established by bounding $\|\mathcal{A}_k\|$ and then applying the trivial estimate

$$|a_k(v,w)| \le \|\mathcal{A}_k\|\|v\|\|w\|\ . \tag{4.2}$$

(By the Riesz representation theorem, $\|\mathcal{A}_k\|$ is the the least upper bound on a_k.) We shall use this approach in the case of an arbitrary closed smooth boundary Γ in §4.2. However our proof of coercivity requires Fourier analysis, and so is restricted to circular or spherical domains. Since the Fourier analysis can also be used to give a sharper estimate for the continuity constant (in fact this was first done by K. Giebermann [25]), we give the coercivity proof in the first subsection, and demonstrate the continuity in the following one.

4.1 Fourier analysis

We first consider the $2D$ case; the extension to $3D$ will be discussed in Theorem 4.12. Suppose Γ is the unit circle, with parametrisation $\boldsymbol{\gamma}(\theta) = (\cos\theta, \sin\theta)$. Since in this case $\mathcal{D}_k = \mathcal{L}_k$, we assume throughout this subsection that $\mathcal{A}_k = \mathcal{R}_k = \frac{1}{2} + \mathcal{D}_k - ik\mathcal{S}_k$. Then, following (2.7) and (2.8), we can also identify the operators $\mathcal{S}_k$ and $\mathcal{D}_k$ with operators on spaces of 2π-periodic functions, which will be given the same names. In the standard way, we represent any 2π-periodic L_2 function v as

$$v(\theta) = \frac{1}{2\pi} \sum_{m\in\mathbb{Z}} \widehat{v}(m)\exp(im\theta)\ , \quad \text{where} \quad \widehat{v}(m) := \int_0^{2\pi} v(\theta)\ \exp(-im\theta)\,\mathrm{d}\theta\ \ ,$$

in which case, the L_2-inner product and norm are given by

$$(v,w) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} \widehat{v}(m)\overline{\widehat{w}(m)}\ , \quad \text{and} \quad \|v\|^2 = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} |\widehat{v}(m)|^2\ \ . \tag{4.3}$$

We denote the Bessel functions of the first and second kind of order m by J_m and Y_m and introduce the corresponding Hankel function of the first kind $H_m^{(1)} = J_m + iY_m$. We then have the following Fourier representations of $\mathcal{S}_k$ and $\mathcal{D}_k$.

Lemma 4.1 *For all $v \in L_2$*

$$\text{(i)} \qquad (\mathcal{S}_k v)(\theta) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} \widehat{\sigma}_k(m)\,\widehat{v}(m)\exp(im\theta)$$

$$\text{(ii)} \quad (\tfrac{1}{2}v + \mathcal{D}_k v)(\theta) = (\tfrac{1}{2}v + \mathcal{L}_k v)(\theta) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} \widehat{\delta}_k(m)\widehat{v}(m)\exp(im\theta)\ ,$$

with the symbols given by

$$\widehat{\sigma}_k(m) = \frac{\pi i}{2} J_{|m|}(k) H^{(1)}_{|m|}(k)\ , \qquad \widehat{\delta}_k(m) = \frac{k\pi i}{2} J'_{|m|}(k) H^{(1)}_{|m|}(k)\ , \tag{4.4}$$

where the prime denotes differentiation with respect to k.

Proof. The proof can be easily obtained by adapting a procedure in [28] from the sphere to the circle. Let u be any radiating solution of the Helmholtz equation exterior to Γ and let v be any solution of the Helmholtz equation interior to Γ. Then Green's identities on Γ read:

$$\mathcal{L}_k u - \mathcal{S}_k \partial_\nu u = \frac{1}{2} u \ , \qquad \mathcal{L}_k v - \mathcal{S}_k \partial_\nu v = -\frac{1}{2} v \ . \tag{4.5}$$

Using polar coordinates, and substituting $u(\mathbf{x}) = H^{(1)}_{|m|}(kr)\exp(im\theta)$ and $v(\mathbf{x}) = J_{|m|}(kr)\exp(im\theta)$ into (4.5), yields a system of simultaneous equations which determine the quantities $\mathcal{S}_k[\exp(im\cdot)]$ and $\mathcal{L}_k[\exp(im\cdot)]$. Solving these with the help of the Wronskian formula from [2, (9.1.16)]:

$$\frac{\pi k}{2}(J_m(k)Y_m'(k) - J_m'(k)Y_m(k)) = 1, \tag{4.6}$$

one readily obtains the result.

■

As a result of Lemma 4.1, we obtain the following Fourier representation for the operator $\mathcal{A}_k = \frac{1}{2} + \mathcal{D}_k - ik\mathcal{S}_k = \frac{1}{2} + \mathcal{L}_k - ik\mathcal{S}_k$:

$$\mathcal{A}_k v(\theta) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} \widehat{\rho}_k(m)\widehat{v}(m)\exp(im\theta) \quad \text{with} \quad \widehat{\rho}_k(m) = \frac{\pi k}{2}\ H^{(1)}_{|m|}(k)\ (J_{|m|}(k) + iJ'_{|m|}(k)). \tag{4.7}$$

(Note that the choice ik of coupling parameter has ensured that the contributions to $\mathcal{A}_k$ from the single- and double-layer components are of the same order as $k \to \infty$.) Then

$$a_k(v,w) = (\mathcal{A}_k v, w) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} \widehat{\rho}_k(m)\widehat{v}(m)\overline{\widehat{w}(m)}\ , \tag{4.8}$$

and since $\widehat{\rho}_k(m)$ is even in m, we can obtain the continuity constant B_k and coercivity constant α_k by showing that

$$\sup_{m\in\mathbb{N}\cup\{0\}} |\widehat{\rho}_k(m)| \le B_k \quad \text{and} \quad \inf_{m\in\mathbb{N}\cup\{0\}} \Re(\widehat{\rho}_k(m)) \ge \alpha_k\ , \tag{4.9}$$

where $\Re$ denotes the real part of a complex number.

The first main result in this subsection is that coercivity holds independently of k, as $k \to \infty$:

Theorem 4.2 *There exists $\xi > 0$ such that for all $k \ge \xi$*

$$\inf_{m\in\mathbb{N}\cup\{0\}} \Re(\widehat{\rho}_k(m)) \ge \frac{1}{2}\ .$$

The proof of Theorem 4.2 will be obtained from several lemmas proved below. The first lemma shows three equivalent ways of writing the result of Theorem 4.2. To state it, we need to introduce the moduli of the Hankel function and its derivative (see [2, (9.2.17) and (9.2.18)]) :

$$M_m(k) := |H^{(1)}_m(k)| = \sqrt{J_m^2(k) + Y_m^2(k)}, \qquad N_m(k) := |(H^{(1)}_m(k))'| = \sqrt{(J_m'(k))^2 + (Y_m'(k))^2}. \tag{4.10}$$

Notice that since the real zeros of J_m and Y_m and also those of J_m' and Y_m' interlace ([2, §9.5]), both M_m and N_m are strictly positive functions of k.

Lemma 4.3 *For all $k > 0$ and $m \ge 0$, the following inequalities are equivalent*

(i) $\Re(\widehat{\rho}_k(m)) = \dfrac{\pi k}{2}\Big[J_m^2(k) - J_m'(k)Y_m(k)\Big] \ge \dfrac{1}{2}$;

(ii) $J_m^2(k) - \dfrac{1}{2}(J_m(k)Y_m(k))' \ge 0$;

(iii) $(J_m'(k))^2 + \dfrac{J_m^2(k)}{M_m^2(k)}\left[\dfrac{4}{\pi k} - N_m^2(k)\right] \ge 0.$

Proof. Using the Wronskian formula (4.6), we can write

$$\begin{aligned}\Re(\widehat{\rho}_k(m)) &= \frac{\pi k}{2}\left(J_m^2(k) - J_m'(k)Y_m(k)\right)\\ &= \frac{\pi k}{2}\left(J_m^2(k) - \frac{1}{2}(J_m(k)Y_m(k))'\right) + \frac{1}{2}\frac{\pi k}{2}\left(J_m(k)Y_m'(k) - J_m'(k)Y_m(k)\right)\\ &= \frac{\pi k}{2}\left(J_m^2(k) - \frac{1}{2}(J_m(k)Y_m(k))'\right) + \frac{1}{2}\ ,\end{aligned}$$

and so (i) and (ii) are equivalent. On the other hand, writing

$$J_m^2(k) - \frac{1}{2}(J_m(k)Y_m(k))' = J_m^2(k) - \frac{1}{2}(J_m(k)Y_m'(k) + J_m'(k)Y_m(k))$$

and multiplying the last term by the left-hand side of (4.6), we obtain

$$\begin{aligned}J_m^2(k) - \frac{1}{2}(J_m(k)Y_m(k))' &= J_m^2(k) - \frac{1}{2}\frac{\pi k}{2}\Big[J_m^2(k)(Y_m'(k))^2 - (J_m'(k))^2Y_m^2(k)\Big]\\ &= J_m^2(k) - \frac{\pi k}{4}\Big[J_m^2(k)N_m^2(k) - (J_m'(k))^2M_m^2(k)\Big].\end{aligned}$$

Dividing the last identity by $\pi k M_m^2(k)/4$, we obtain the equivalence of (ii) and (iii). ■

The next result shows inequality (i) for any m provided k is sufficiently large (depending on m).

Proposition 4.4 *For all $m \geq 0$, there exists κ_m depending only on m such that*

$$\Re(\widehat{\rho}_k(m)) \geq \frac{1}{2}\ , \quad \textit{when } k \geq \kappa_m.$$

Proof. Taking into account the asymptotics of the Bessel functions, for fixed m, as $k \to \infty$:

$$\begin{aligned}J_m(k) &= \sqrt{\tfrac{2}{\pi k}}\cos(k - \tfrac{m\pi}{2} - \tfrac{\pi}{4}) + \mathcal{O}(k^{-3/2}),\\ Y_m(k) &= \sqrt{\tfrac{2}{\pi k}}\sin(k - \tfrac{m\pi}{2} - \tfrac{\pi}{4}) + \mathcal{O}(k^{-3/2}),\\ J_m'(k) &= -\sqrt{\tfrac{2}{\pi k}}\sin(k - \tfrac{m\pi}{2} - \tfrac{\pi}{4}) + \mathcal{O}(k^{-3/2}),\\ Y_m'(k) &= \sqrt{\tfrac{2}{\pi k}}\cos(k - \tfrac{m\pi}{2} - \tfrac{\pi}{4}) + \mathcal{O}(k^{-3/2}),\end{aligned}$$

(see [2, (9.2.1), (9.2.2), (9.2.11) and (9.2.12)]) we conclude that

$$\begin{aligned}&J_m^2(k) - \frac{1}{2}(J_m(k)Y_m(k))'\\ &\quad= \frac{2}{\pi k}\Big[\cos^2(k - \tfrac{m\pi}{2} - \tfrac{\pi}{4}) + \tfrac{1}{2}\sin^2(k - \tfrac{m\pi}{2} - \tfrac{\pi}{4}) - \tfrac{1}{2}\cos^2(k - \tfrac{m\pi}{2} - \tfrac{\pi}{4})\Big] + \mathcal{O}(k^{-2})\\ &\quad= \frac{1}{\pi k} + \mathcal{O}(k^{-2}).\end{aligned}$$

The result follows now readily. ■

Notice that, because of the dependence of κ_m on m, we can only use this lemma to deal with a finite set of Fourier coefficients of the operator. For the rest of the Fourier coefficients we need Propositions 4.6 and 4.7 below. First we need a result which is essentially contained in the PhD thesis of K. Giebermann [25] .

Lemma 4.5 *There exist $\varepsilon > 0$ and $m_0 > 1$ such that for all $m \geq m_0$*

$$(J_mY_m)'(k) \ < \ 0, \qquad \forall k \in (0, m - \varepsilon m^{1/3}].$$

Proof. Define the function f_m by

$$f_m(k) := J_m(k)Y_m(k). \qquad (4.11)$$

Next recall the expressions for the derivatives of the Bessel functions given in [2, (9.1.27)]:

$$\mathcal{C}'_m(k) = -\mathcal{C}_{m+1}(k) + \frac{m}{k}\mathcal{C}_m(k) = \mathcal{C}_{m-1}(k) - \frac{m}{k}\mathcal{C}_m(k), \qquad m \geq 1 \ , \qquad (4.12)$$

where $\mathcal{C}_m$ denotes either J_m or Y_m. Using this, we have

$$\begin{aligned} f'_m(k) &= J'_m(k)Y_m(k) + J_m(k)Y'_m(k) \\ &= \Big(- J_{m+1}(k) + \frac{m}{k}J_m(k)\Big)Y_m(k) + J_m(k)\Big(Y_{m-1}(k) - \frac{m}{k}Y_m(k)\Big) \\ &= J_m(k)Y_{m-1}(k) - J_{m+1}(k)Y_m(k). \end{aligned} \qquad (4.13)$$

Proceeding in a similar way for the second derivative, we obtain

$$\begin{aligned} f''_m(k) &= \Big(J_m(k)Y_{m-1}(k) - J_{m+1}(k)Y_m(k)\Big)' \\ &= J'_m(k)Y_{m-1}(k) + J_m(k)Y'_{m-1}(k) - J'_{m+1}(k)Y_m(k) - J_{m+1}(k)Y'_m(k) \\ &= \Big(J_{m-1}(k) - \frac{m}{k}J_m(k)\Big)Y_{m-1}(k) + J_m(k)\Big(- Y_m(k) + \frac{m-1}{k}Y_{m-1}(k)\Big) \\ &\quad -\Big(J_m(k) - \frac{m+1}{k}J_{m+1}(k)\Big)Y_m(k) - J_{m+1}(k)\Big(- Y_{m+1}(k) + \frac{m}{k}Y_m(k)\Big) \\ &= J_{m-1}(k)Y_{m-1}(k) - \frac{1}{k}J_m(k)Y_{m-1}(k) - 2J_m(k)Y_m(k) \\ &\quad +\frac{1}{k}J_{m+1}(k)Y_m(k) + J_{m+1}(k)Y_{m+1}(k). \end{aligned} \qquad (4.14)$$

Now recall the asymptotic behaviour of the Bessel functions at zero (for fixed $m > 1$):

$$J_m(z) = \frac{1}{m!}\left(\frac{z}{2}\right)^m + \mathcal{O}(z^{m+2}) \qquad Y_m(z) = -\frac{(m-1)!}{\pi}\left(\frac{z}{2}\right)^{-m} + \mathcal{O}(z^{-m+2}) \ , \quad |z| \to 0 \qquad (4.15)$$

(see [2, (9.1.7) and (9.1.9)-(9.1.11)]). Combining (4.15), with (4.11), (4.13) and (4.14) we obtain, for $m > 1$,

$$f_m(0) = -\frac{1}{\pi m}, \quad f'_m(0) = 0, \quad f''_m(0) = -\frac{2}{\pi(m+1)m(m-1)} < 0. \qquad (4.16)$$

It follows immediately that there exists $\eta_m > 0$ (which may depend on m) such that $f'_m(k) < 0$ for $k \in (0, \eta_m]$.

Moreover, since $J_m(k)$ and $Y_m(k)$ have no roots in the interval $(0, m]$ (see [2, (9.5.2)]), it follows that

$$f_m(k) < 0, \qquad k \in [0, m]. \qquad (4.17)$$

Clearly the first positive critical point of f_m must be a local minimum. We will show that there exists $\epsilon > 0$, independent of m, such that this local minimum lies in the interval $(m - \epsilon m^{1/3}, \sqrt{m^2 - 1/4})$, thus proving the result. To proceed, define

$$g_m(k) := k^3 f''_m(k).$$

Then, from [2, (9.1.59)] it follows that g_m satisfies

$$g'_m(k) \ = \ -k(1 + 4k^2 - 4m^2)f'_m(k) - 4k^2 f_m(k). \qquad (4.18)$$

Now, suppose that f_m has two critical points $\eta_1, \eta_2 \in (0, \sqrt{m^2 - 1/4})$. Obviously η_1 is a local minimum and η_2 a local maximum. Then, for $k \in [\eta_1, \eta_2] \subset (0, \sqrt{m^2 - 1/4})$, we have $f'_m(k) \geq 0$ and $1 + 4k^2 - 4m^2 \leq 0$, and combining these with (4.18), we can conclude that

$$g'_m(k) > 0, \qquad \forall k \in [\eta_1, \eta_2].$$

Since $g_m(\eta_1) \geq 0$, this implies $g_m(k) > 0$, for all $k \in [\eta_1, \eta_2]$ and so $f''_m(k) > 0$, for $k \in (\eta_1, \eta_2]$, but this contradicts the fact that η_2 is a local maximum of f_m. Thus f_m has at most a single critical point in $(0, \sqrt{m^2 - 1/4})$.

Now, for any fixed $\varepsilon > 0$, we can use the "transition asymptotics" [2, (9.3.23)-(9.3.30)] to obtain

$$f'_m(m - \varepsilon m^{1/3}) = \frac{2}{m}\varphi(\varepsilon) + \mathcal{O}(m^{-5/3}) \ ,$$

where

$$\varphi(\epsilon) := \mathrm{Ai}(2^{1/3}\varepsilon)\mathrm{Bi}'(2^{1/3}\varepsilon) + \mathrm{Ai}'(2^{1/3}\varepsilon)\mathrm{Bi}(2^{1/3}\varepsilon)$$

and Ai, Bi denote the Airy functions. Then, from [2, (10.4.1)-(10.4.5)] , we have

$$\varphi(0) = 0, \quad \varphi'(0) = \frac{-2^{4/3}}{3^{\frac{1}{6}}\,\Gamma(\frac{1}{3})^2} < 0.$$

Hence there exists $\varepsilon > 0$ such that $\varphi(\varepsilon) < 0$. Thus, by choosing $m_0 \geq 0$ sufficiently large so that $m_0 - \varepsilon m_0^{1/3} \leq \sqrt{m_0^2 - 1/4}$ we have

$$f_m'(m - \varepsilon m^{1/3}) < 0\ , \quad \text{when} \quad m \geq m_0\ .$$

The first positive local extremum of f must then lie to the right of $m - \varepsilon m^{1/3}$ and the result follows. ■

Before continuing with the proof of coercivity, let us recall some more basic properties of the Bessel functions. First

$$J_m(k) > 0, \quad Y_m'(k) > 0, \quad J_m'(k) > 0, \quad \text{and} \quad Y_m(k) < 0\ , \quad \text{when } k \in (0, m)\ . \tag{4.19}$$

These properties can be easily deduced from the bounds on the first positive roots of these functions, namely that all of them are greater than m, and their (limiting) values when $k \to 0$ (see [2, (9.5.2), (9.1.7) and (9.1.9)]).

Moreover, since by definition Y_m is solution of the Bessel differential equation:

$$k^2 Y_m''(k) + k Y_m'(k) + (k^2 - m^2) Y_m(k) = 0,$$

we deduce that

$$Y_m''(k) \ < \ 0, \qquad k \in (0, m). \tag{4.20}$$

Proposition 4.6 *There exists m_1 such that for all $m \geq m_1$*

$$\Re(\widehat{\rho}_k(m)) \geq \frac{1}{2}\ , \quad \text{when } k \in (0, m]\ .$$

Proof. By Lemma 4.3 we just have to show that there exists m_1 such that for all $m \geq m_1$

$$J_m^2(k) - \frac{1}{2}(J_m(k)Y_m(k))' \geq 0\ , \quad k \in (0, m]\ .$$

But then, by Lemma 4.5, we only have to prove it for $m \geq m_0$ and $k \in (m - \varepsilon m^{1/3}, m]$.

Using (4.19), we have

$$\begin{aligned} J_m^2(k) - \frac{1}{2}(J_m(k)Y_m(k))' \ &\geq \ J_m^2(k) - \frac{1}{2}J_m(k)Y_m'(k) \ \geq \ J_m^2(k) - \frac{1}{4}J_m^2(k) - \frac{1}{4}(Y_m'(k))^2 \\ &\geq \ \frac{3}{4}J_m^2(m - \varepsilon m^{1/3}) - \frac{1}{4}(Y_m'(m - \varepsilon m^{1/3}))^2, \end{aligned}$$

where in the last step we have applied (4.19) and (4.20).

Using the asymptotics of the Bessel functions at $m - \varepsilon m^{1/3}$ (cf. [2, (9.3.23) and (9.3.28)]) we deduce that there exists $C > 0$ such that

$$\frac{3}{4}J_m^2(m - \varepsilon m^{1/3}) - \frac{1}{4}(Y_m'(m - \varepsilon m^{1/3}))^2 \geq C m^{-2/3} + \mathcal{O}(m^{-4/3})\ .$$

Therefore, there exists m' such that for all $m \geq m'$ the left hand side of inequality above is strictly positive. The result follows on setting $m_1 = \max\{m_0, m'\}$. ■

Proposition 4.7 *There exists $\kappa' > 0$ such that*

$$\Re(\widehat{\rho}_k(m)) \geq \frac{1}{2} \quad \text{when} \quad \kappa' \leq m \leq k.$$

Proof. Again, from Lemma 4.3, it is sufficient to show that, for some $\kappa' > 0$

$$kN_m^2(k) \leq \frac{4}{\pi}, \qquad \forall k \in [m, \infty) \text{ , when } \quad \kappa' \leq m \leq k.$$

We start by noticing that $M_m(k)$ and $N_m(k)$ are related by cf. [2, (9.2.23)]

$$\frac{\mathrm{d}}{\mathrm{d}k}\Big(k^2 N_m^2(k)\Big) = -(k^2 - m^2)\frac{\mathrm{d}}{\mathrm{d}k}\Big(M_m^2(k)\Big).$$

Therefore, integrating by parts,

$$\begin{aligned} k^2 N_m^2(k) - m^2 N_m^2(m) &= \int_m^k \frac{\mathrm{d}}{\mathrm{d}t}\Big(t^2 N_m^2(t)\Big)\,\mathrm{d}t = -\int_m^k (t^2 - m^2)\frac{\mathrm{d}}{\mathrm{d}t}\Big(M_m^2(t)\Big)\,\mathrm{d}t \\ &= -(k^2 - m^2)M_m^2(k) + 2\int_m^k t M_m^2(t)\,\mathrm{d}t. \end{aligned}$$

Using (cf. [42, §13.74])

$$\frac{2}{\pi k} \leq M_m^2(k) \leq \frac{2}{\pi\sqrt{k^2 - m^2}}, \qquad \forall k \geq m \ ,$$

and $m/k \leq 1$, we obtain

$$\begin{aligned} kN_m^2(k) &\leq \frac{m^2 N_m^2(m)}{k} - \frac{4}{\pi}\left(\frac{k^2 - m^2}{2k^2}\right) + \frac{4}{\pi k}\int_m^k \frac{t}{\sqrt{t^2 - m^2}}\,\mathrm{d}t \\ &\leq \frac{(m^{2/3} N_m(m))^2}{k^{1/3}} + \frac{4}{\pi}\varphi(m/k), \end{aligned}$$

with $\varphi(\theta) = -1/2 + \theta^2/2 + \sqrt{1 - \theta^2}$, for $\theta \in (0, 1]$. Since $\varphi(\theta) < \varphi(0) = 1/2$, for $\theta \in (0, 1]$ and $|m^{2/3} N_m(m)| \leq C$ with C independent of m (see [2, (9.3.33) and (9.3.34)]), we conclude that

$$kN_m^2(k) \leq \frac{2}{\pi} + Ck^{-1/3}.$$

Taking κ' large enough concludes the proof. ■

Gathering the above lemmas together we now have the proof of the first main result.
Proof of Theorem 4.2. Let

$$\widehat{\kappa} := \max\{\kappa', m_1\},$$

where κ', m_1 are as in propositions 4.6 and 4.7. Then

$$\Re(\widehat{\rho}_k(m)) \geq \frac{1}{2}$$

for all $m, k \geq \widehat{\kappa}$. Taking ℓ to be the largest integer smaller than $\widehat{\kappa}$ and

$$\xi := \max\{\widehat{\kappa}, \kappa_0, \ldots, \kappa_\ell\} \ ,$$

with κ_ℓ as in Proposition 4.4, and recalling that $\widehat{\rho}_k(-m) = \widehat{\rho}_k(m)$, the result follows readily. ■

Remark 4.8 *To gain more insight into the result in Theorem 4.2, let us recall that it is shown in* [38, §3.4.37] *that the operator $\Re\{-i\mathcal{S}_k\} = \Im\{\mathcal{S}_k\}$ is actually positive semidefinite for any sufficiently smooth Γ. Thus the coercivity result proved above is a consequence of a very delicate balance between the Fourier coefficients of $\mathcal{D}_k$ and $\mathcal{S}_k$, in such a way that when a specific coefficient of $1/2 + \mathcal{D}_k$ is small that corresponding to $\mathcal{S}_k$ is sufficiently large to compensate it and vice versa.*

The analysis developed above to prove coercivity also provides a proof of continuity on the circle. Because of (4.2), it is sufficient to bound $\|\mathcal{A}_k\|$, and in turn because of (4.7), this is achieved by bounding the terms

$$|J_m^2(k)|, \quad |J_m(k)J_m'(k)|, \quad |J_m(k)Y_m(k)|, \quad \text{and} \quad |J_m'(k)Y_m(k)|$$

uniformly in $m \in \mathbb{Z}$ and for all k large enough. This we now do.

Lemma 4.9 *There exists $C > 0$ independent of m such that*

$$|J_m(k)| + |J_m'(k)| \leq Ck^{-1/3}, \qquad \forall k \in (0, \infty).$$

Proof. In [30] it is proved that

$$|J_m(k)| \leq Ck^{-1/3}.$$

The bound for the derivative is a simple consequence of (4.12). ■

Lemma 4.10 *For all $\kappa > 0$ there exists a constant C independent of m such that for all $k > \kappa$*

$$|J_m(k)Y_m(k)| + |J_m'(k)Y_m(k)| \leq Ck^{-2/3}.$$

Proof. We remark that $J_m(k)Y_m(k)$ and $J_m'(k)Y_m(k)$ when seen as functions of k and m are continuous for $k > 0$, and for all m fixed. Moreover, fixed $m, \kappa_0 > 0$, they are bounded for $k \geq \kappa_0$. Hence, we just have to study these products for all m large enough.

Taking ϵ and m_0 as in Lemma 4.5 we conclude that for all $m \geq m_0$

$$|J_m(k)Y_m(k)| \leq |J_m(m - \varepsilon m^{1/3})Y_m(m - \varepsilon m^{1/3})| \leq \frac{1}{2}M_m^2(m - \varepsilon m^{1/3}), \qquad k \in [0, m - \varepsilon m^{1/3}]$$

(recall that $J_m(k)Y_m(k) < 0$ for all $k \in (0, m]$). Hence, using, [2, (9.3.23)-(9.3.24))] , we get

$$M_m^2(m - \varepsilon m^{1/3}) \leq Cm^{-2/3} \leq C'k^{-2/3} \ , \qquad k \in [0, m - \varepsilon m^{1/3}] \ .$$

For $k \in [m - \varepsilon m^{1/3}, \infty)$ we use [42, §13.74] and [2, (9.3.23)-(9.3.24))] to obtain

$$\begin{aligned} |J_m(k)Y_m(k)| &\leq \frac{1}{2k}kM_m^2(k) \leq \frac{1}{2k}(m - \varepsilon m^{1/3})M_m^2(m - \varepsilon m^{1/3}) \\ &\leq C'k^{-1}(m - \varepsilon m^{1/3})m^{-2/3} \leq C'k^{-1}m^{1/3} \leq C'k^{-2/3} \ . \end{aligned}$$

This finishes the first part of the proof.

On the other hand, by (4.12) and (4.19)

$$0 \leq J_m'(k), J_m(k) \leq J_{m-1}(k), \qquad \forall k \in [0, m-1].$$

Applying the identity (see [2, (9.1.16)])

$$J_m(k)Y_{m-1}(k) + J_{m-1}(k)Y_m(k) = \frac{2}{\pi k} \ ,$$

we conclude that

$$|J_m'(k)Y_m(k)| \leq |J_{m-1}(k)Y_m(k)| \leq \frac{2}{\pi k} + |J_m(k)Y_{m-1}(k)| \leq \frac{2}{\pi k} + |J_{m-1}(k)Y_{m-1}(k)| \ ,$$

and the bound is proved for $k \in (0, m-1]$.

For $k \geq m - 1$, we proceed as follows

$$|J_m'(k)Y_m(k)| \leq \frac{1}{2}\Big[|J_m'(k)|^2 + Y_m^2(k)\Big] \leq \frac{1}{2}\Big[|J_m'(k)|^2 + M_m^2(k)\Big].$$

By Lemma 4.9 and bounding $M_m^2(k)$ as before, we prove the result. ■

Using the last two lemmas we can now bound the continuity constant for the bilinear form a_k in the case of a circular boundary. This result can also be found in [25].

Theorem 4.11 *Let Γ be the unit circle, and $k_0 > 0$. Then, there exists $C > 0$ independent of $k \geq k_0$ such that*

$$\|\mathcal{A}_k\| \leq Ck^{1/3}.$$

We prove in the next theorem that the coercivity and a similar continuity estimate hold also for the sphere. The proof is a simple adaptation of the tools developed to cover the $2D$ case. In $3D$ the operators $\mathcal{S}_k$, $\mathcal{L}_k$ and $\mathcal{D}_k$ are defined analogously to (2.4) and (2.6), but with $\Phi_k(\mathbf{x}) = \frac{1}{4\pi}\frac{\exp(ik|x|)}{|x|}$. Again $\mathcal{R}_k = \mathcal{P}_k$ and so $\mathcal{A}_k = \frac{1}{2} + \mathcal{D}_k - ik\mathcal{S}_k$.

Theorem 4.12 *Consider the operator $\mathcal{A}_k$ on the sphere in $\mathbb{R}^3$. Then there exist $C_2, c_2 > 0$ such that, for all sufficiently large k,*

$$\|\mathcal{A}_k\| \le C_2 k^{1/3}\,, \qquad \Re(\mathcal{A}_k v, v) \ge c_2 \|v\|^2.$$

Proof. Let $\{Y_m^\ell\}$ ($m = 0, 1, \ldots, \ell = -m, -m+1, \ldots, m$) be the spherical harmonics (see [38, §2.4], [19, §2.3]). Then it is well known that $\{Y_m^\ell\}$ is an orthonormal basis of $L_2(\mathbb{S}_2)$ ($\mathbb{S}_2$ denotes the unit sphere in what follows). We introduce the Fourier coefficients defined in the usual way by

$$\widehat{f}(m,\ell) = \int_{\mathbb{S}_2} f(\mathbf{x}) Y_m^\ell(\mathbf{x}) \mathrm{d}\mathbf{x}.$$

Then

$$\mathcal{S}_k v = \sum_{m=0}^{\infty} \widehat{\sigma}_k(m) \sum_{\ell=-m}^{m} \widehat{v}(m,\ell) Y_m^\ell$$
$$\frac{1}{2} v + \mathcal{D}_k v = \sum_{m=0}^{\infty} \widehat{\delta}_k(m) \sum_{\ell=-m}^{m} \widehat{v}(m,\ell) Y_m^\ell$$

with the symbols of the operators given by (compare with (4.4))

$$\widehat{\sigma}_k(m) = ikj_m(k) h_m^{(1)}(k), \qquad \widehat{\delta}_k(m) = ik^2 j_m'(k) h_m^{(1)}(k)\ . \tag{4.21}$$

Here j_m and y_m are the spherical Bessel functions ([2, §10.1])

$$j_m(k) := \sqrt{\frac{\pi}{2k}} J_{m+1/2}(k)\,, \quad y_m(k) := \sqrt{\frac{\pi}{2k}} Y_{m+1/2}(k). \tag{4.22}$$

and $h_m^{(1)} = j_m + iy_m$. The formulae (4.21) can be found for example in [28,25] – see also [13].

Proceeding as before, we conclude that the continuity and coercivity are equivalent to proving

$$\max_{m\in\mathbb{N}\cup\{0\}} k^2 \left|(j_m^2(k) - j_m'(k) y_m(k)) \ + \ ij_m(k)(j_m'(k) + y_m(k))\right| \le C_2 k^{1/3}, \tag{4.23}$$

$$\alpha_k := \min_{m\in\mathbb{N}\cup\{0\}} k^2 (j_m^2(k) - j_m'(k) y_m(k)) \ge c_2 > 0\ . \tag{4.24}$$

The continuity estimate (4.23) is a consequence of lemmas 4.9 and 4.10. For the coercivity estimate (4.24), observe that from (4.22), it is equivalent to prove that, for all $m = 0, 1, \ldots,$ and for all sufficiently large k,

$$\frac{\pi k}{2}\left[J_{m+1/2}^2(k) - J_{m+1/2}'(k) Y_{m+1/2}(k)\right] + \frac{\pi}{4} J_{m+1/2}(k) Y_{m+1/2}(k) \ge c_2 > 0\ .$$

By Theorem 4.2 (see also Lemma 4.3 item (i)) we know that there exists $\xi > 0$ such that, for all $k \ge \xi$,

$$\frac{\pi k}{2}\left[J_{m+1/2}^2(k) - J_{m+1/2}'(k) Y_{m+1/2}(k)\right] \ge \frac{1}{2}.$$

On the other hand, by Lemma 4.10, there exists C independent of k such that

$$|J_{m+1/2}(k) Y_{m+1/2}(k)| \le C k^{-2/3}.$$

Collecting both inequalities, we conclude that $\alpha_k \ge 1/2 - Ck^{-2/3}$ and taking k large enough, the result follows readily. ∎

The extension of the proof of coercivity to an arbitrary boundary Γ remains as an open question at the moment and certainly deserves further investigation.

In the next subsection we shall prove an estimate of $\|\mathcal{A}_k\|$ when Γ is an arbitrary, sufficiently smooth closed curve. For this we must abandon Fourier analysis and adopt more flexible methods.

4.2 General boundaries

Our main result is Theorem 4.14, for which we need the following technical lemma. Recall the function M_m defined in (4.10).

Lemma 4.13 *For each $m = 0, 1, \ldots$, there exists $C_m > 0$ independent of k such that*

$$\int_0^{2\pi} M_m(2k|\sin t/2|)\,|\sin t/2|^m\,\mathrm{d}t \;\leq\; C_m k^{-1/2}\;, \quad \text{for } k \quad \text{sufficiently large.}$$

Proof. The integral may be written

$$\int_0^{2\pi} M_m(2k|\sin t/2|)\,|\sin t/2|^m\;\mathrm{d}t = 4\int_0^{\pi/2} M_m(2k\sin t)\,\sin^m t\;\mathrm{d}t.$$

Since (see [2, (9.2.4)]),

$$M_m(z) = \sqrt{\frac{2}{\pi z}} + \mathcal{O}(z^{-1}), \quad \text{as } z \to \infty. \tag{4.25}$$

we conclude

$$\int_{\pi/6}^{\pi/2} M_m(2k\sin t)\,\sin^m t\,dt \;\leq\; C_m k^{-1/2}\;,$$

with $C_m > 0$ independent of k. Moreover, with the change of variable $x = 2k\sin t$, we obtain

$$\begin{aligned}\int_0^{\pi/6} M_m(2k\sin t)\,\sin^m t &= (2k)^{-1-m}\int_0^k M_m(x)\frac{x^m}{\sqrt{1-(x/2k)^2}}\,\mathrm{d}x\\ &\leq \frac{2}{(2k)^{m+1}\sqrt{3}}\int_0^k M_m(x)x^m\,\mathrm{d}x \;=:\; I_m(k).\end{aligned}$$

To bound $I_m(k)$, note that, for any fixed $L > 0$, and k sufficiently large, we can write:

$$I_m(k) = \frac{2}{(2k)^{m+1}\sqrt{3}}\left\{\int_0^L M_m(x)x^m\mathrm{d}x + \int_L^k M_m(x)x^m\mathrm{d}x\right\}. \tag{4.26}$$

We notice that $M_m(x)x^m$ is locally integrable in $[0, \infty)$ (cf. [2, (9.1.7)-(9.1.9)]). Hence, using again (4.25), we conclude

$$I_m(k) \;\leq\; C_m k^{-m-1}\left[1 + \int_L^k x^{m-1/2}\mathrm{d}x\;\right], \quad \text{as} \quad k \to \infty\;,$$

from which the result follows readily. ∎

Now we can prove the required continuity estimate for general boundaries.

Theorem 4.14 *Let Γ be a $\mathcal{C}^\infty$ closed curve. Then there exists $C > 0$ independent of k such that*

$$\|\mathcal{A}_k\| \leq C k^{1/2}\;,$$

where $\mathcal{A}_k = \mathcal{R}_k$ or $\mathcal{P}_k$.

Proof. We shall give the proof for $\mathcal{A}_k = \mathcal{R}_k$. The other case is analogous. Let $\boldsymbol{\gamma}$ be the parameterisation in (2.7). Then, with the notational convention in (2.8), the transformed operator $\mathcal{R}_k$ has the form:

$$(\mathcal{R}_k v)(s) = \frac{1}{2}v(s) + k\int_0^{2\pi}\{D_k(s,t)\; - i\;S_k(s,t)\}\,v(t)\,\mathrm{d}t\;, \tag{4.27}$$

where

$$\begin{aligned} S_k(s,t) &:= \frac{i}{4}H_0^{(1)}(k|\boldsymbol{\gamma}(s)-\boldsymbol{\gamma}(t)|)|\boldsymbol{\gamma}'(t)|\;, \quad \text{and}\\ D_k(s,t) &:= -\frac{i}{4}\frac{(\boldsymbol{\gamma}(s)-\boldsymbol{\gamma}(t))\cdot\nu(s)}{|\boldsymbol{\gamma}(s)-\boldsymbol{\gamma}(t)|}H_1^{(1)}(k|\boldsymbol{\gamma}(s)-\boldsymbol{\gamma}(t)|)|\boldsymbol{\gamma}'(t)|\;.\end{aligned}$$

The result is obtained by estimating the L_2 norm of each of the integral operators appearing on the right-hand side of (4.27). Taking the second operator first, we write

$$\int_0^{2\pi} \left| \int_0^{2\pi} S_k(s,t)v(t)\,\mathrm{d}t \right|^2 \mathrm{d}s = \frac{1}{16}\int_0^{2\pi} \left| \int_0^{2\pi} W_k(s,t)H_0^{(1)}(2k|\sin((s-t)/2)|)v(t)\,\mathrm{d}t \right|^2 \mathrm{d}s$$

with

$$W_k(s,t) := \frac{S_k(s,t)}{(i/4)H_0^{(1)}(2k|\sin((s-t)/2)|)} \ .$$

Here we have used a "singularity division" trick: the denominator in $W_k(s,t)$ is the (convolution kernel) which results from evaluating the numerator on the unit circle $\boldsymbol{\gamma}(s) = (\cos s, \sin s)$. The singularity as $s \to t$ on the top and bottom of W_k are the same, and the asymptotic behaviour as $k \to \infty$, for $s \neq t$ are also the same. A little analysis shows that $|W_k(s,t)|$ can be bounded independent of $k \geq 1$, and $s, t \in [0, 2\pi]^2$. Hence

$$\begin{aligned}\int_0^{2\pi} \left| \int_0^{2\pi} S_k(s,t)v(t)\,\mathrm{d}t \right|^2 &\leq C \int_0^{2\pi} \left[\int_0^{2\pi} M_0(2k|\sin((s-t)/2)|)\,|v(t)|\,\mathrm{d}t \right]^2 \mathrm{d}s \\ &\leq C \left[\int_0^{2\pi} M_0(2k|\sin(t/2)|)\,\mathrm{d}t \right]^2 \|v\|^2 \ ,\end{aligned}$$

with C independent of k, where we have used the standard convolution estimate:
$\|f * g\| \leq \|f\|_{L^1(0,2\pi)}\|g\|$. The required estimate for the second integral in (4.27) is now a consequence of Lemma 4.13 with $m = 0$.

The first integral on the right-hand side of (4.27) is estimated in an analogous way, but instead we perform singularity subtraction using the function
$(-i/4)H_1^{(1)}(2k|\sin(s-t)/2|)|2\sin(s-t)/2|$, and apply Lemma 4.13 with $m = 1$. ■

5 Asymptotics of the Normal Derivative of the Total Field

In this section we discuss the high frequency asymptotic behaviour of the function V appearing in the decomposition (1.1), i.e.

$$v(s,k) \ := \ v(\boldsymbol{\gamma}(s),k) \ = \ kV(s,k)\exp(ik\boldsymbol{\gamma}(s)\cdot\widehat{\mathbf{a}}) \ , \quad s \in [0, 2\pi] \ , \tag{5.1}$$

where v is the solution of (2.8). Throughout we shall require that Γ is a simple convex $\mathcal{C}^\infty$ contour with non-vanishing curvature.

The first (nonrigorous) results on this topic were by Fock (e.g. [22]). This was followed by Ludwig [35] (applying the notion of "uniform" asymptotics) and subsequently by the rigorous analysis of Buslyaev [15,17] – see also [5,7].

We begin by quoting, in Theorem 5.1 below, a well-known result from Melrose and Taylor [37]. This is then used in Corollary 5.3 to obtain a decomposition of V into a finite sum of explicit $k-$dependent terms, plus a controllable remainder. In turn this is used to prove Theorem 5.4, which gives bounds on the derivatives of V (valid as $k \to \infty$), suitable for use in the subsequent numerical analysis.

The following result is [37, Theorem 9.27] restricted to the two-dimensional case. (The analysis in [37] is valid for all dimensions.) Note that here the formula in [37, Theorem 9.27] has been scaled by k^{-1}, because k is an explicit factor in (5.1). Theorem 5.1 can essentially be found elsewhere in the literature. For example "Fundamental Theorem" in [17, §1.9] is stated for the case of Neumann boundary conditions and a point source incident wave, although the general method used there can also be directly extended to the Dirichlet boundary conditions and plane wave incidence which are relevant here [16].

Theorem 5.1 *There exists $\Delta > 0$ such that $V(s,k)$ has the asymptotic expansion:*

$$V(s,k) \sim \sum_{\ell,m\geq 0} k^{-1/3-2\ell/3-m} b_{\ell,m}(s)\Psi^{(\ell)}(k^{1/3}Z(s)) \ , \tag{5.2}$$

valid for $s \in I_\Delta := (t_1 - \Delta, t_1 + \Delta) \cup (t_2 - \Delta, t_2 + \Delta)$, where $\boldsymbol{\gamma}(t_1)$ and $\boldsymbol{\gamma}(t_2)$ are the tangency points as described in §3. The functions $b_{\ell,m}$, Ψ and Z have the following properties.

(i) *$b_{\ell,m}$ are $\mathcal{C}^\infty$ complex-valued functions on I_Δ.*
(ii) *Z is a $\mathcal{C}^\infty$ real-valued function on I_Δ, with simple zeros at t_1 and t_2, which is positive-valued on $(t_1,t_2)\cap I_\Delta$ and negative-valued on $(t_2 - 2\pi, t_1) \cap I_\Delta$.*

(iii) $\Psi : \mathbb{C} \to \mathbb{C}$ *is an entire function specified by*

$$\Psi(\tau) := \exp(-i\tau^3/3) \int_c \frac{\exp(-iz\tau)}{\mathrm{Ai}\,(e^{2\pi i/3} z)}\, \mathrm{d}z \ , \qquad (5.3)$$

where Ai *is the Airy function ([2] section 10.4) and c the is contour depicted in Figure 2 (cf.* [37, (9.8)], *where θ is any sufficiently small positive angle, thus ensuring the absolute convergence of* (5.3), *cf.* [5, p.362]*). Note that $A_+(s)$ in [37, (9.8)] is exactly* $\mathrm{Ai}(e^{2\pi i/3}s)$*. Although this is not explicitly stated in [37], it is implicit there - for example compare [37, (3.19)] with [2, (9.3.37)]. Note also that [37] works in the context of the distributional theory of Fourier analysis, thus treating Fourier integrals in the weakest possible sense. Thus, although the domain of integration is not specified in [37, (9.8)], it is implicitly the real line. Deforming the integration path into the complex plane in the way we have described is a standard procedure which allows us to obtain an absolutely convergent integral with the same value.*
As a consequence the asymptotics of $\Psi(\tau)$ for large $|\tau|$ are known. In particular, (see [37, Lemma 9.9]*):*

$$\Psi(\tau) = a_0\tau + a_1\tau^{-2} + a_2\tau^{-5} + \ldots + a_n\tau^{1-3n} + \mathcal{O}(\tau^{1-3(n+1)}) \ , \quad \text{as } \tau \to \infty \ , \quad \text{where } a_0 \neq 0 \ , \qquad (5.4)$$

and this expansion remains valid for all derivatives of Ψ by formally differentiating each term on the right hand side, including the error term. Moreover, there exists $\beta > 0$ and $c_0 \neq 0$ such that for any $n \in \mathbb{N} \cup \{0\}$

$$D_\tau^n \Psi(\tau) = c_0 \ D_\tau^n\{\exp(-i\tau^3/3 - i\tau\alpha_1)\} \ (1 + \mathcal{O}(\exp(-|\tau|\beta))) \ , \quad \text{as } \tau \to -\infty \ , \qquad (5.5)$$

where $\alpha_1 = \exp(-2\pi i/3)\nu_1$ and $\nu_1 < 0$ is the right-most root of Ai*. (Recall that the roots of* $\mathrm{Ai}(z)$ *are all real and negative cf.* [2] *section 10.4). Hence, when $\tau \to -\infty$ the function Ψ, as well as its derivatives decrease exponentially but in a very oscillating way.*

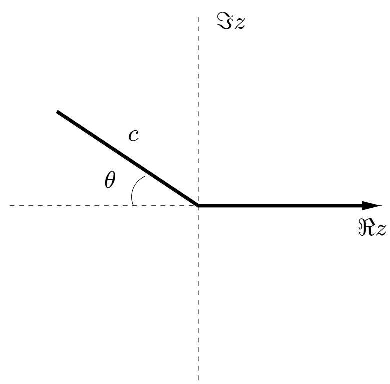

Fig. 2 The contour c in the complex plane

Remark 5.2 *We would like to make several clarifying remarks about Theorem 5.1.*

The function Ψ defined in (5.3) *is often called "Fock's integral" and can be found in the works of Fock e.g. [22, §7, 12]. Its properties were studied in detail in e.g. [15, §3]. (Note that Fock's 1965 book [22] is in fact a collection of English translations of his earlier papers written in Russian, in particular Chapter 7 (1948) and Chapter 12 (1949).)*

The asymptotics in (5.5) *are not given explicitly in [37] but may be deduced easily by applying the theory of residues to the contour integral in* (5.3) *- see, e.g. [5, p.393], [15, Lemma 8]. The function $Z(s)$ can be found explicitly in the two-dimensional case, see e.g. [15, (45.1)].*

The precise meaning of the asymptotic expansion in (5.2) *can be made clear using the symbol classes of Hörmander (see, e.g. [27, p.236, Definition 7.8.1], [37, p. 249]). In the context of our problem, a function $p = p(s,k)$ (where $s \in [0,2\pi]$ and $k \in (0,\infty)$), is said to lie in the class $S^\mu_{\rho,\delta}$ (for all (s,k) in some subdomain of $[0,2\pi] \times (0,\infty)$), if*

$$|D_k^\alpha D_s^n p(s,k)| \ \leq \ C_{\alpha,n}(1+k)^{\mu-\rho\alpha+\delta n} \ , \quad \alpha, \ n \in \mathbb{N} \cup \{0\} \ .$$

Using this formalism, (5.2) *can be written more explicitly as follows: Choose any* $\mu < 0$. *Then, for all* $L, M \in \mathbb{N} \cup \{0\}$ *sufficiently large (depending on* μ*), the remainder*

$$R_{L,M} := V(s,k) - \sum_{\ell,m=0}^{L,M} k^{-1/3-2\ell/3-m} b_{\ell,m}(s) \Psi^{(\ell)}(k^{1/3}Z(s)) \ ,$$

satisfies

$$R_{L,M} \in S^{\mu}_{2/3,1/3} \quad \textit{for } (s,k) \in I_{\Delta} \times (0,\infty) \tag{5.6}$$

(i.e. $\rho = 2/3$, $\delta = 1/3$ *in this case). Note that although the symbol class* $S^{\mu}_{2/3,1/3}$ *is explicit in [37, (1.17)], it remains implicit in much of the rest of [37]. The above interpretation of* (5.2) *is equivalent to saying that* (5.2) *is a conventional asymptotic expansion and remains so under term-by term differentiation on both sides with respect to both* s *and* k. *This standard use of the* $\sim$ *symbol is prevalent throughout the rigorous asymptotics and microlocal analysis literature and it always has the same meaning.*

In view of our explicit definition, the functions $b_{\ell,m}$, Ψ *and* Z *are all necessarily smooth (*$\mathcal{C}^\infty$*) functions of their arguments.*

It is also clear that the function Z *can be extended (nonuniquely) to a* $\mathcal{C}^\infty$ 2π*-periodic function which is positive-valued on* (t_1, t_2) *and negative-valued on* $(t_2 - 2\pi, t_1)$ *and from now on we assume this extension has been made.*

Corollary 5.3 *With the same notation as Theorem 5.1, the functions* $b_{\ell,m}$ *can be extended to* 2π*-periodic* $\mathcal{C}^\infty$ *functions such that, for all* $L, M \in \mathbb{N} \cup \{0\}$, *the decomposition*

$$V(s,k) = \Big[\sum_{\ell,m=0}^{L,M} k^{-1/3-2\ell/3-m} b_{\ell,m}(s) \Psi^{(\ell)}(k^{1/3}Z(s)) \Big] + R_{L,M}(s,k) \ . \tag{5.7}$$

holds for all $s \in [0, 2\pi]$, *with remainder term satisfying, for all* $n \in \mathbb{N} \cup \{0\}$,

$$|D^n_s R_{L,M}(s,k)| \le C_{L,M,n}(1+k)^{\mu+n/3} \ , \quad \textit{where} \quad \mu := -\min\left\{ \frac{2}{3}(L+1),\ (M+1) \right\} \ . \tag{5.8}$$

and $C_{L.M,n}$ *is independent of* k.

Proof. Our first step is to derive the result for $s \in I_\Delta$ from Theorem 5.1. To do this, simply choose any $L, M \in \mathbb{N} \cup \{0\}$ and μ as in (5.8) and, for convenience, set

$$a_{\ell,m}(s,k) = k^{-1/3-2\ell/3-m} b_{\ell,m}(s) \Psi^{(\ell)}(k^{1/3}Z(s)).$$

Then, by Remark 5.2, there exist integers $L' \ge L$ and $M' \ge M$ such that $R_{L',M'} \in S^{\mu}_{2/3,1/3}$ and (by (5.7) written in two different ways),

$$R_{L,M}(s,k) = \left\{ \sum_{\ell=L+1}^{L'} \sum_{m=0}^{M'} + \sum_{\ell=0}^{L} \sum_{m=M+1}^{M'} \right\} a_{\ell,m}(s,k) \ + \ R_{L',M'}(s,k) \ . \tag{5.9}$$

Now, using the expansions (5.4), (5.5) and the fact that $\Psi \in \mathcal{C}^\infty(\mathbb{R})$, we obtain, for all $\tau \in \mathbb{R}$, the estimates

$$|\Psi(\tau)| \ \le \ C_0(1+|\tau|) \ , \tag{5.10}$$
$$|\Psi'(\tau)| \ \le \ C_1 \ , \tag{5.11}$$
$$|\Psi^{(\ell)}(\tau)| \ \le \ C_\ell(1+|\tau|)^{-2-\ell}, \quad \text{for} \quad \ell \ge 2 \ , \tag{5.12}$$

with C_ℓ independent of k. Using (5.10) - (5.12), a direct calculation shows that $a_{0,m}(s,k) \in S^{-m}_{2/3,1/3}$ and $a_{\ell,m}(s,k) \in S^{-1/3-2\ell/3-m}_{2/3,1/3}$, when $\ell \ge 1$. Hence for the range of ℓ and m in the double summations in (5.9), we have $a_{\ell,m} \in S^{\mu}_{2/3,1/3}$, with μ as chosen above. Combining this and the stated property of L', M' with (5.9) shows that the first term on the right-hand side of (5.9) is in $S^{\mu}_{2/3,1/3}$ for $(s,k) \in I_\Delta \times (0,\infty)$. Since $R_{L',M'}$ has the same property, the required result follows.

To complete the proof we shall extend the result to all $s \in [0, 2\pi]$, with appropriate 2π-periodic $\mathcal{C}^\infty$ complex-valued functions $b_{\ell,m}$. A way to establish this is, for example, by employing known asymptotic results in the illuminated zone $(t_1 + \Delta/2, t_2 - \Delta/2)$ ("geometrical optics") and in the shadow zone $(t_2 - 2\pi + \Delta/2, t_1 - \Delta/2)$ ("extra-polynomial decay" – i.e. decay faster that any inverse power of k).

Consider first the illuminated zone. The required asymptotics can be found (for example) in [37, (1.15)], stated there without proof or a precise reference as a commonly well-known fact as $V(s,k) \sim \sum_{j\geq 0} k^{-j} d_j(s)$, with $d_j \in \mathcal{C}^\infty(t_1 + \Delta/2,\, t_2 - \Delta/2)$. Interpreting this as in Corollary 5.3, this may be written:

$$V(s,k) \;=\; \sum_{j=0}^{N} k^{-j} d_j(s) \;+\; r_N(s,k), \tag{5.13}$$

where, for all $n \geq 0$,

$$|D_s^n r_N(s,k)| \;\leq\; c_{N,n}(1+k)^{-N-1}\;, \qquad s \in (t_1 + \Delta/2, t_2 - \Delta/2), \tag{5.14}$$

with $c_{N,n}$ independent of k (but dependent on Δ). (The first term in (5.13) is the well-known geometric optics approximation $d_0(s) = 2i\boldsymbol{\nu}(s).\hat{\mathbf{a}}$, where $\boldsymbol{\nu}$ is the outward unit normal to the scatterer Γ.) For additional statements and proofs, see e.g. [36, Thm II], [18], [17, Thm 8] and [45]. (The reference [17] discusses the Neumann boundary condition and point source incidence; the extension to Dirichlet conditions is straightforward while plane wave incidence can be handled by viewing it as a limit of a sequence of increasingly remote and appropriately magnified point sources, as in e.g., [6, §A.2].)

We argue next that (5.13) implies the validity of (5.7) on all of (t_1, t_2), with $b_{\ell,m}$ replaced by a suitable $\mathcal{C}^\infty$ extension. To see this, first suppose that such an extension does exist and denote it $\tilde{b}_{\ell,m}$. Then, for $s \in (t_1 + \Delta/2,\, t_2 - \Delta/2)$, replacing $\Psi^{(\ell)}(k^{1/3}Z(s))$ in (5.7) by its asymptotics as $k \to \infty$ (using (5.4)), and performing a straightforward re-arrangement, we see that (5.7) adopts the form (5.13) for sufficiently large L and M, provided the $\tilde{b}_{\ell,m}$ satisfy the equations (with a_n as in (5.4)):

$$\begin{aligned}
d_0(s) &= a_0 \tilde{b}_{0,0}(s) Z(s),\\
d_1(s) &= a_0\left(\tilde{b}_{0,1}(s)Z(s) + \tilde{b}_{1,0}\right) + a_1 \tilde{b}_{0,0}(s)(Z(s))^{-2},\\
d_j(s) &= a_0\left(\tilde{b}_{0,j}(s)Z(s) + \tilde{b}_{1,j-1}\right) + \\
&\qquad \sum_{\substack{\ell+m+n=j\\ n\geq 1}} \left[\prod_{p=0}^{\ell-1}(1-3n-p)\right] a_n \tilde{b}_{\ell,m}(s)(Z(s))^{1-3n-\ell}, \quad j \geq 2\;,
\end{aligned} \tag{5.15}$$

(with the convention that $\prod_{p=0}^{-1} x_p := 1$). Since $a_0 \neq 0$ and $Z(s) \neq 0$ for $s \in (t_1 + \Delta/2,\, t_2 - \Delta/2)$, it is easy to see that one can always (non-uniquely) select $\tilde{b}_{\ell,m}$ so that the equations (5.15) hold for this range of s. Inserting the chosen $\tilde{b}_{\ell,m}$ into (5.7) and using this to define $R_{L,M}(s,k)$ for $s \in (t_1 + \Delta, t_2 - \Delta)$, we then see that (5.8) holds there.

Finally we join the illuminated and transition zones by "matching" $\tilde{b}_{\ell,m}$ and $b_{\ell,m}$ in the overlapped regions $(t_1 + \Delta/2, t_1 + \Delta)$ and $(t_2 - \Delta, t_2 - \Delta/2)$ in a standard way, i.e. re-defining $b_{\ell,m}(s)$ on $[t_1 - \Delta, t_2 + \Delta]$ as $\chi(s)\tilde{b}_{\ell,m}(s) + (1-\chi(s))b_{\ell,m}(s)$, where $\chi \in \mathcal{C}^\infty$ is a cut-off function: $\chi(s) \equiv 1$ for $s \in [t_1 + \Delta, t_2 - \Delta]$, $\chi(s) \equiv 0$ for $s \leq t_1 + \Delta/2$ and $s \geq t_2 - \Delta/2$.

The extension into the (deep) shadow zone $(t_2 - 2\pi + \Delta, t_1 - \Delta)$ is performed in a similar way by employing results on extra-polynomial decay of the total wave field, see e.g. [36, §2], [17, §1.9]. These results imply an asymptotic expansion of the form (5.13)-(5.14) with all the coefficients $d_j(s)$ being identically zero, for $s \in (t_2 - 2\pi + \Delta/2, t_1 - \Delta/2)$. Combining this with (5.5) implies in fact that any $\mathcal{C}^\infty$ continuation of $b_{\ell,m}$ into the shadow zone would suffice. Notice in passing that although this argument is sufficient for the present proof, it can be further sharpened to prove exponential, rather than extra-polynomial, decay in the deep shadow. This is important later in the paper (see Theorem 6.5).

■

We make use of this theorem by deriving the following estimates for the derivatives of $V(s,k)$ with respect to s, which will be directly useful in our numerical analysis.

Theorem 5.4 *For all $n \in \mathbb{N} \cup \{0\}$ there exist constants $C_n > 0$ independent of k and s such that for all k sufficiently large,*

$$|D_s^n V(s,k)| \leq \begin{cases} C_n, & n = 0, 1,\\ C_n\left[1 + \displaystyle\sum_{j=2}^{n} k^{(j-1)/3}(1 + k^{1/3}|\omega(s)|)^{-j-2}\right], & n \geq 2, \end{cases} \tag{5.16}$$

where $\omega(s) := (s - t_1)(t_2 - s)$.

Proof. Throughout this proof C_n denotes a generic constant independent of s and k but possibly depending on n, whose value may change from line to line. Note first that, by the properties of Z (see Theorem 5.1 and 5.2), we have

$$Z(s) = h(s)\omega(s) \ , \tag{5.17}$$

where h is a smooth positive real function bounded away from zero on $s \in [0, 2\pi]$.

Choosing any $n \in \mathbb{N} \cup \{0\}$ we can select L, M so that $-\mu \geq n/3$, where μ is defined in (5.8). Then apply Corollary 5.3 to obtain

$$V(s,k) = A_{L,M}(s,k) + R_{L,M}(s,k),$$

where

$$A_{L,M}(s,k) := k^{-1/3} \sum_{\ell=0}^{L} k^{-2\ell/3} B_{\ell,M}(s) \Psi^{(\ell)}(k^{1/3} Z(s)), \qquad B_{\ell,M} := \sum_{m=0}^{M} k^{-m} b_{\ell,m}(s) \ .$$

and the derivatives of $R_{L,M}$ are bounded as in (5.8). Since $\mu + n/3 \leq 0$, it follows that

$$|D_s^n R_{L,M}(s,k)| \ \leq \ C_n \ , \quad \text{for all } k \ .$$

By the Leibnitz rule, since all derivatives of $B_{\ell,M}$ are bounded independently of k, we obtain

$$|D_s^n A_{L,M}(s,k)| \leq k^{-1/3} \sum_{\ell=0}^{L} k^{-2\ell/3} \left| D_s^n \left[B_{\ell,M}(s) \Psi^{(\ell)}(k^{1/3} Z(s)) \right] \right| \tag{5.18}$$

$$\leq C_n k^{-1/3} \sum_{j=0}^{n} \sum_{\ell=0}^{L} k^{(j-2\ell)/3} \left| \Psi^{(\ell+j)}(k^{1/3} Z(s)) \right|. \tag{5.19}$$

We split the sum on the right-hand side of (5.19) into three components: for $j = 0$, $j = 1$ and $j \geq 2$. For the third component, we use (5.12) and (5.17) to obtain

$$k^{-1/3} \sum_{j=2}^{n} \sum_{\ell=0}^{L} k^{(j-2\ell)/3} |\Psi^{(\ell+j)}(k^{1/3} Z(s))| \leq C_n k^{-1/3} \sum_{j=2}^{n} \sum_{\ell=0}^{L} k^{(j-2\ell)/3} (1 + k^{1/3} |\omega(s)|)^{-2-j-\ell}$$

$$\leq C_n \sum_{j=2}^{n} k^{(j-1)/3} (1 + k^{1/3} |\omega(s)|)^{-j-2} \left[\sum_{\ell=0}^{L} k^{-2\ell/3} (1 + k^{1/3} |\omega(s)|)^{-\ell} \right].$$

The result for $j = 0, 1$ follows analogously, using (5.10), (5.11). ■

The next result is now a simple consequence of Theorem 5.4.

Corollary 5.5 *For all $n \geq 1$ there exists C_n independent of k and s such that, for k sufficiently large,*

$$|D_s^n V(s,k)| \leq C_n (1+k)^{(n-1)/3} \ , \quad s \in [0, 2\pi] \ .$$

We note that this estimate holds globally, including therefore near the tangency points $s = t_1$ and $s = t_2$. However this estimate is not sharp in the illuminated zone, well away from the tangency points, where the fact that $\omega(s)$ is bounded away from zero allows the negative power of $(1 + k^{1/3}|\omega(s)|)$ in the right-hand side of (5.16) to become active. The different estimates in the illuminated and shadow zones will be exploited in the next section.

6 Error Estimates

In this section we study the approximation of the solution v of (2.8) by the non-standard Galerkin procedure outlined in §3. In this case, via (5.1), the expansion (3.2) holds for $j = 1, 2, 3$ with

$$e_j(s,k) = \exp(ik\boldsymbol{\gamma}(s) \cdot \widehat{\mathbf{a}}) \ , \quad V_j(s) = \begin{cases} V(s), & s \in \Lambda_j \ , \\ 0, & \text{otherwise} \ . \end{cases} \tag{6.1}$$

For good approximation of V_j, we need to choose the domains Λ_j carefully, and possibly depending on k. For parameters $\varepsilon, \delta \in (0, 1/3]$, $c_1 > 0$, $c_2 > 0$ and for sufficiently large $k > 0$, we define

– the transition zones:

$$\Lambda_1 := [t_1 - c_2 k^{-1/3+\delta}, t_1 + c_1 k^{-1/3+\varepsilon}], \quad \Lambda_2 := [t_2 - c_1 k^{-1/3+\varepsilon}, t_2 + c_2 k^{-1/3+\delta}]; \tag{6.2}$$

– the illuminated zone:

$$\Lambda_3 := [t_1 + c_1 k^{-1/3+\varepsilon}, t_2 - c_1 k^{-1/3+\varepsilon}] \ ; \tag{6.3}$$

– and the shadow zone:

$$\Lambda_4 := [t_2 - 2\pi + c_2 k^{-1/3+\delta}, t_1 - c_2 k^{-1/3+\delta}] \ . \tag{6.4}$$

The zones Λ_j touch only at their endpoints.

We shall see that the choice of the constants c_1, c_2 does not affect the asymptotic behaviour of the error in approximation of each V_j as $k \to \infty$, but tuning c_1, c_2 turns out in practice to be useful, if we seek methods which perform well across a range of wavenumbers k. The zones Λ_j depend on ε, δ, c_1 and c_2 but we do not reflect this in the notation.

Recall that $\mathbb{P}^d$ denotes the univariate polynomials of degree $\leq d$. Given an interval $I = (a,b)$ and $n \in \mathbb{N}\cup\{0\}$, we introduce the (semi)norms (for suitable f):

$$|f|_{n,I} := \left[\int_a^b |f^{(n)}(s)|^2 (s-a)^n (b-s)^n \, \mathrm{d}s \right]^{1/2} .$$

Then it is well-known (cf. [40, Cor. 3.12]) that there exists $C_n > 0$ such that, for all nonnegative integers n with $n \leq d+1$,

$$\inf_{p \in \mathbb{P}^d(x)} \|f - p\|_{L_2(I)} \leq C_n d^{-n} |f|_{n,I}. \tag{6.5}$$

(Again throughout this section, C_n denotes a generic $n-$dependent and $k-$ independent constant.) We shall use this to obtain bounds for polynomial approximation of V_j in zone Λ_j, for $j = 1,2,3$. Our first lemma concerns the illuminated zone Λ_3.

Proposition 6.1 *For $n \geq 2$ and for sufficiently large k, there exists C_n independent of k and ε such that*

$$|V(\cdot,k)|_{n,\Lambda_3} \leq C_n \left[1 + k^{-(1+3\varepsilon)/2} k^{(1-3\varepsilon)n/6} \right].$$

Proof. Let us define

$$a_{k,\varepsilon} := t_1 + c_1 k^{-1/3+\varepsilon}, \quad b_{k,\varepsilon} := t_2 - c_1 k^{-1/3+\varepsilon}.$$

By Theorem 5.4

$$|V(\cdot,k)|_{n,\Lambda_3} \leq C_n \left[1 + k^{-1/3} \max_{j=2,\ldots,n} k^{j/3} \left\{ \int_{a_{k,\varepsilon}}^{b_{k,\varepsilon}} \frac{(s-a_{k,\varepsilon})^n (b_{k,\varepsilon}-s)^n}{(1+k^{1/3}|\omega(s)|)^{2j+4}} \mathrm{d}s \right\}^{1/2} \right]. \tag{6.6}$$

To estimate this, we assume without loss of generality $0 < t_1 < \pi < t_2 < 2\pi$ and write

$$k^{2j/3} \int_{a_{k,\varepsilon}}^{b_{k,\varepsilon}} \frac{(s-a_{k,\varepsilon})^n (b_{k,\varepsilon}-s)^n}{(1+k^{1/3}|\omega(s)|)^{2j+4}} \mathrm{d}s =$$

$$k^{2j/3} \int_{a_{k,\varepsilon}}^{\pi} \frac{(s-a_{k,\varepsilon})^n (b_{k,\varepsilon}-s)^n}{(1+k^{1/3}|\omega(s)|)^{2j+4}} \mathrm{d}s \ + \ k^{2j/3} \int_{\pi}^{b_{k,\varepsilon}} \frac{(s-a_{k,\varepsilon})^n (b_{k,\varepsilon}-s)^n}{(1+k^{1/3}|\omega(s)|)^{2j+4}} \mathrm{d}s \ . \tag{6.7}$$

Consider the first term on the right hand side. Since $\omega(s) \geq (t_2 - \pi)(s - t_1) > 0$ when $s \in [a_{k,\varepsilon}, \pi]$, we have

$$\begin{aligned} k^{2j/3} \int_{a_{k,\varepsilon}}^{\pi} \frac{(s-a_{k,\varepsilon})^n (b_{k,\varepsilon}-s)^n}{(1+k^{1/3}|\omega(s)|)^{2j+4}} \mathrm{d}s &\leq C_n k^{2j/3} \int_{a_{k,\varepsilon}}^{\pi} \frac{(s-a_{k,\varepsilon})^n}{k^{(2j+4)/3}(s-t_1)^{2j+4}} \, \mathrm{d}s \\ &= C_n k^{-4/3} \int_{a_{k,\varepsilon}}^{\pi} (s-t_1)^{-2j-4} (s-a_{k,\varepsilon})^n \, \mathrm{d}s \\ &\leq C_n k^{-4/3} \int_{a_{k,\varepsilon}}^{\pi} (s-t_1)^{n-2j-4} \, \mathrm{d}s \ . \end{aligned}$$

It is easy to see that, for all $j = 0, \ldots, n$, the right-hand side can be bounded (up to a constant factor) by the case of $j = n$. Hence, for $j = 0, \ldots, n$,

$$k^{2j/3} \int_{a_{k,\varepsilon}}^{\pi} \frac{(s-a_{k,\varepsilon})^n (s-b_{k,\varepsilon})^n}{(1+k^{1/3}|\omega(s)|)^{2j+4}} \mathrm{d}s \ \leq \ C_n \left(1 + k^{-(1/3+3\varepsilon)} k^{(1-3\varepsilon)n/3} \right) .$$

An analogous estimate holds for the second integral in the right-hand side of (6.7). Combining these in (6.6) yields the result. ∎

Remark 6.2 *The preceding result can be extended to cover different "growths" of the interval in terms of k. For instance, if*

$$a_{k,\varepsilon} = t_1 + c_1 \log k\, k^{-1/3} \qquad b_{k,\varepsilon} = t_2 - c_1 \log k\, k^{-1/3}$$

we can prove

$$|V(\cdot,k)|_{n,\Lambda_3} \le C_n[1 + k^{-1/2+n/6}(\log(k))^{-(n+3)/2}] \ .$$

Now we can give an estimate for the error in approximating V_3 by polynomials.

Theorem 6.3 *For all fixed n, there exists $C_n > 0$ such that, for all $d \ge n-1$ and k sufficiently large,*

$$\inf_{p\in\mathbb{P}^d} \|V_3(\cdot,k) - p\|_{L_2(\Lambda_3)} \le C_n k^{\tau} d^{-n} \ , \qquad \textit{as } d \to \infty$$

with

$$\tau := \max\{0, -(1+3\varepsilon)/2 + (1-3\varepsilon)n/6\ \} \ .$$

Proof. The result is a direct consequence of combining Proposition 6.1 with (6.5). ■

Notice that if $0 < \epsilon < 1/3$ and $n \ge 3(1+3\varepsilon)/(1-3\varepsilon)$, then Theorem 6.3 implies:

$$\inf_{p\in\mathbb{P}^d} \|V_3(\cdot,k) - p\|_{L_2(\Lambda_3)} \le C_n k^{-(1+3\epsilon)/2} \left(\frac{k^{(1-3\varepsilon)/6}}{d}\right)^n \ ,$$

which emphasises the mild dependence of d on k which would be required to preserve the accuracy of the approximation if $k \to \infty$ (and there is no dependence at all if $\epsilon = 1/3$).

Now we turn to the error in approximation by polynomials in the transition zones Λ_1, Λ_2.

Theorem 6.4 *For all $n \ge 2$ and $d \ge n-1$ there exists C_n independent of k such that, for $j = 1,2$,*

$$\inf_{p\in\mathbb{P}^d} \|V_j(\cdot,k) - p\|_{L_2(\Lambda_j)} \ \le\ C_n k^{(\eta-1)/2} \left(\frac{k^{\eta}}{d}\right)^n \ , \quad \textit{as} \quad d \to \infty \ ,$$

where $\eta = \max\{\varepsilon, \delta\}$.

Proof. Without loss of generality we can restrict attention to $\Lambda_1 = [c_{k,\delta}, d_{k,\varepsilon}]$, with

$$c_{k,\delta} = t_1 - c_2 k^{-1/3+\delta}, \qquad d_{k,\varepsilon} = t_1 + c_1 k^{-1/3+\varepsilon}.$$

Applying (6.5) on this domain, and recalling the definition (6.1) of V_1, we have

$$\inf_{p\in\mathbb{P}^d} \|V_1(\cdot,k) - p\|_{L_2(\Lambda_1)} \ \le\ C_n d^{-n} |V(\cdot,k)|_{n,\Lambda_1}. \tag{6.8}$$

By Corollary 5.5, we have

$$|V(\cdot,k)|^2_{n,\Lambda_1} \le C_n k^{2(n-1)/3} \int_{-c_2 k^{-1/3+\delta}}^{c_1 k^{-1/3+\varepsilon}} (c_1 k^{-1/3+\varepsilon} - s)^n (s + c_2 k^{-1/3+\delta})^n \,\mathrm{ds},$$

and therefore

$$|V(\cdot,k)|_{n,\Lambda_1} \le C_n k^{(n-1)/3} k^{(-1/3+\eta)(n+1/2)}. \tag{6.9}$$

Substituting in (6.8), the result follows immediately. ■

The next result shows that v decreases exponentially as $k \to \infty$ in the shadow zone Λ_4, which means that it is safe to approximate it by zero in our numerical scheme.

Theorem 6.5 *There exist positive constants c_0, c_0' such that for all k sufficiently large,*

$$\|v\|_{L_2(\Lambda_4)} \ \le\ c_0' \exp(-c_0 k^{\delta}) \ . \tag{6.10}$$

Remark 6.6 *The stated result can be obtained (formally) by estimating the expansion given in Corollary 5.3 in the shadow region, using the asymptotic result* (5.5) *and ignoring the error term* $R_{L,M}(s,k)$. *The rigorous proof of the exponential decay in the shadow zone is a long-established result in diffraction theory and we refer for example to [21,43,44,33,34,39] for the highly non-trivial proofs. The case of a circle is given in detail in e.g. [41]. We give here a brief account of how Theorem 6.5 follows from the results in the literature.*

To establish (6.10) *it is sufficient to prove that in the shadow* ($s \in \Lambda_4$)

$$|v(s,k)| \le C_1(1+k)^{2/3} \exp\left(-C_2 k^{1/3}\rho(s)\right), \tag{6.11}$$

where $\rho(s) := \min\{t_1 - s, s - t_2 + 2\pi\}$ *and* C_1, C_2 *are independent of* k *and* s. *In turn,* (6.11) *can be derived for example from [44, Thm 2]. We remark that [44, Thm 2] states the result in a form in effect implying* (6.11) *for a point source at an arbitrary distance from the scatterer rather than for a plane wave incidence. However, the error term obtained in [44, (1.13),(1.11)] is "uniform" with respect to the location of the source (see, [21, 43,44]) and this allows the extension to the case of a plane wave incidence using the procedure outlined in the proof of Corollary 5.3 above.*

We note in passing that the results on the exponential decay (in two-dimensional problems) in [43,44] do not require the contour to be analytic but only sufficiently smooth. There are also extensions to arbitrary dimension, but these require analytic scattering surfaces and (as stated) are only valid in the "deep shadow" (i.e. a bounded distance away from the shadow boundary) – see, e.g. [39, Thm 3] which uses the ideas of [33].

We can now prove our final result which provides an estimate for the error in the Galerkin method.

Theorem 6.7 *Let* $\tilde{v}$ *be the Galerkin solution as described in* §3, *with* Λ_j *as defined in* (6.2) – (6.4) *and with oscillatory functions* e_j *as given in* (6.1). *Choose the parameters* $\varepsilon = 1/9$ *and* $0 < \delta \le \varepsilon$ *and suppose that polynomials of degree* d_I *are used in the illuminated zone and* d_T *in the transition zones. Then for all* $n \ge 6$ *with* $n \le d_I + 1$ *and* $n \le d_T + 1$, *there exists a constant* C_n *so that*

$$\|v - \tilde{v}\| \le C_n \left(\frac{B_k}{\alpha_k}\right) k \left\{ k^{-2/3} \left(\frac{k^{1/9}}{d_I}\right)^n + k^{-4/9}\left(\frac{k^{1/9}}{d_T}\right)^n + \exp(-c_0 k^{\delta}) \right\} .$$

Proof. The proof is a consequence of Corollary 3.2, combined with Theorems 6.3, 6.4 and 6.5. ∎

From this we have the following simple consequence, which shows that at worst the number of degrees of freedom should increase with $k^{1/9}$ in order to maintain the accuracy of the method as $k \to \infty$.

Corollary 6.8 *Under the conditions of Theorem 6.7, suppose that* $d_I = d_T = d$. *Suppose also that (as has been proven for circular domains in* §4) $B_k/\alpha_k = \mathcal{O}(k^{1/3})$ *as* $k \to \infty$. *Then, for* $n \ge 6$, *with* $n \le d+1$,

$$\|v - \tilde{v}\| \le C_n k \left\{ k^{-1/9}\left(\frac{k^{1/9}}{d}\right)^n + k^{1/3}\exp(-c_0 k^{\delta}) \right\} .$$

7 Numerical Experiments

In this section we present some numerical experiments which illustrate the theoretical results given above. With the notation of §3, let t_1, t_2 denote the tangency points, and introduce the transition zones:

$$\Lambda_1 := [t_1 - \beta_1 k^{-1/3+\delta}, t_1 + \alpha_1 k^{-1/3+\varepsilon}], \qquad \Lambda_2 := [t_2 - \alpha_2 k^{-1/3+\varepsilon}, t_2 + \beta_2 k^{-1/3+\delta}],$$

and the illuminated and shadow parts as follows

$$\Lambda_3 := [t_1 + \alpha_3 k^{-1/3+\varepsilon}, t_2 - \alpha_4 k^{-1/3+\varepsilon}] , \qquad \Lambda_4 := [t_2 + \beta_2 k^{-1/3+\delta}, 2\pi] \cup [0, t_1 - \beta_1 k^{-1/3+\delta}]$$

where the parameters ε, $\delta, \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \alpha_4$ have to be specified. For all of our experiments, we choose $\epsilon = 1/9$ and $\delta = 0$ (even though the proof of Theorem 6.5 is for positive δ). Below we shall in fact choose $\alpha_1 > \alpha_3$ and $\alpha_2 > \alpha_4$, which means that the transition zones overlap with the illuminated zones. For the non-standard Galerkin method, we employ a smooth partition of unity (cf. (3.1)), constructed in terms of translations, reflections and dilations of the basic function

$$\chi(x) := \frac{1}{2}\Big[\varphi(x) + 1 - \varphi(1-x)\Big]$$

where

$$\varphi(x) = \begin{cases} 1 & x \le 0, \\ \exp\left(\frac{2\exp(-1/x)}{x-1}\right) & 0 < x < 1, \\ 0, & x \ge 1. \end{cases}$$

which gives a smooth transition between the constant functions 1 and 0 (see [11]).

The method is then specified by the triple $\mathbf{d} = (d_1, d_2, d_3)$, i.e. the degrees of the polynomials utilized in, respectively, the transition zones Λ_1 and Λ_2 and in the illuminated region Λ_3.

The accuracy of the method relies heavily on the choice of the parameters α_i, β_i, and especially on the length of the shadow zone, where the solution is approximated by zero. We have sought a good choice of these constants experimentally by working with a relatively low-frequency (mainly to measure the size of shadow zone). That is, we have solved numerically our problem for k not very large in order to measure the size of the shadow zone. We have then chosen constants β_1 and β_2 in such a way that the shadow zone Λ_4 was large enough to consider the solution negligible in this zone, and used such values in the experiments for higher values of k.

With regard to the assembly of the Galerkin matrix, here we use only standard quadrature methods which are not in themselves $k-$robust. However the application and analysis of oscillatory integration procedures is the subject of ongoing work. By analysing here the Galerkin method itself we answer the fundamental questions regarding $k-$ robust stability and consistency which are prerequisites for the formulation and analysis of fully practical algorithms.

We have first tested our method for the circle, where the exact solution can be computed (by Fourier analysis), with $k = 50,\ 100,\ 200,\ 400$ and 800. In this case we have taken

$$\Lambda_1 = [\tfrac{\pi}{2} - \tfrac{7\pi}{5}k^{-1/3}, \tfrac{\pi}{2} + \tfrac{7\pi}{15}k^{-1/3+\varepsilon}], \quad \Lambda_2 = [\tfrac{3\pi}{2} - \tfrac{7\pi}{15}k^{-1/3+\varepsilon}, \tfrac{3\pi}{2} + \tfrac{7\pi}{5}k^{-1/3}]$$

and

$$\Lambda_3 = [\tfrac{\pi}{2} + \tfrac{\pi}{5}k^{-1/3+\varepsilon}, \tfrac{3\pi}{2} - \tfrac{\pi}{5}k^{-1/3+\varepsilon}].$$

That is, in the notation given in the beginning of this section,

$$\alpha_1 = \alpha_2 = 7\pi/15, \quad \beta_1 = \beta_2 = 7\pi/5, \quad \alpha_3 = \alpha_4 = \pi/5.$$

Notice that Λ_3 overlaps Λ_1 in a subinterval of width $4\pi/15k^{-1/3+\varepsilon}$ and in this region both cut-off functions χ_3 and χ_1 are non-zero. Similarly Λ_3 overlaps Λ_2.

To test the correctness of Corollary 6.8 we computed the norm of the error $\|v - \tilde{v}\|$ by using Simpson rule with a sufficiently high number of nodes. Since the exact solution grows as k when $k \to \infty$, we give in Table 1 values of the quantity $k^{-1}\|v - \tilde{v}\|$ for different choices of the degree of the polynomials and the wave number k. We observe that for fixed k the error decreases quickly until it stagnates, when the (exponentially small with respect to k) error in the shadow zone (where the zero approximation has been used) is not converging to zero. We see also that the convergence not only depends on k as k grows but in fact the error often in fact decreases with k.

	$k = 50$	$k = 100$	$k = 200$	$k = 400$	$k = 800$
$\mathbf{d} = (4,4,4), \#\mathbf{d} = 15$	1.65E − 02	1.48E − 02	1.42E − 02	1.43E − 02	1.49E − 02
$\mathbf{d} = (8,8,8), \#\mathbf{d} = 27$	2.82E − 03	2.15E − 03	1.65E − 03	1.28E − 03	9.79E − 04
$\mathbf{d} = (12,12,12), \#\mathbf{d} = 39$	6.83E − 04	4.33E − 04	3.52E − 04	2.83E − 04	2.29E − 04
$\mathbf{d} = (16,16,16), \#\mathbf{d} = 51$	6.49E − 04	1.14E − 04	7.28E − 05	7.03E − 05	4.97E − 05
$\mathbf{d} = (20,20,20), \#\mathbf{d} = 63$	7.30E − 04	1.27E − 04	8.76E − 05	4.82E − 05	3.30E − 05
$\mathbf{d} = (24,24,24), \#\mathbf{d} = 75$	1.04E − 03	1.50E − 04	1.13E − 04	4.89E − 05	3.60E − 05
$\mathbf{d} = (28,28,28), \#\mathbf{d} = 87$	1.58E − 03	2.32E − 04	1.36E − 04	5.03E − 05	5.12E − 05
$\mathbf{d} = (32,32,32), \#\mathbf{d} = 99$	1.48E − 03	2.93E − 04	1.59E − 04	5.48E − 05	6.34E − 05

Table 1 $k^{-1}\|v - \tilde{v}\|$ for unit circle. $\#\mathbf{d}$ denotes the dimension of the finite dimensional approximating space.

In Figure 3 we plot the scaled error $k^{-1}\|v - \tilde{v}\|$ versus the degrees of freedom for this series of experiments. Since the vertical axis is logartihmic, this clearly shows exponential convergence with respect to the degrees of freedom, up to the point where the small error in the shadow region dominates. There is then some small but apparently bounded oscillation as the number of degrees of freedom increases. (Note that the small error in the shadow region could be reduced by introducing a sufficiently accurate approximation to the field there, rather than approximating by zero as we have done here.)

The numerical solution for this experiment is depicted in Figure 4. The top panel in this figure illustrates the computed approximation to the function v, while the bottom panel illustrates the computed "slowly varying" function $V(\cdot, k)$, in this case for $k = 400$. We observe that the error stabilises as $k \to \infty$, which is actually better than the theory predicts.

Finally, to illustrate that our method is applicable to general convex objects, we have tested it when Γ is the ellipse $\{(2\cos t, \sin t) \mid t \in [0, 2\pi]\}$ and with the incident wave is travelling in the direction $\hat{\mathbf{a}} = (3, 1)$. In this case we have taken

$$\alpha_1 = \alpha_2 = 13L/20, \quad \beta_1 = \beta_2 = L/2, \quad \alpha_3 = \alpha_4 = 7L/20,$$

L being the length of the curve, and t_1, t_2 the tangency points.

Although we do not have the exact solution to this problem in analytic form, we can compare our solution with results obtained by applying the Nyström method of Kress (see [19, §3.4]), which is exponentially convergent for fixed k, but not robust as k increases. For each k an "exact solution" $\tilde{v}$ is computed by Kress's method with 6000 degrees of freedom and the relative error $\|v - \widetilde{v}\|/\|\widetilde{v}\|$ is computed. The results are presented in Table 2 below.

Note that, in contrast to the circle case, there is always a small loss of accuracy here for very large k. It is not clear yet if this is an indication of the actual sharpness of our convergence theory or simply an indication that the "exact" solution $\widetilde{v}$ computed by the Nyström method is inaccurate for these values of k.

We also illustrate the results for the ellipse in Figure 5 simply by showing the numerical approximation of v (top panel) and the computed V (bottom panel). Note the symmetry has been lost here.

	$k = 50$	$k = 100$	$k = 200$	$k = 400$
$\mathbf{d} = (4, 4, 4), \#\mathbf{d} = 15$	3.52E − 03	3.37E − 03	3.47E − 03	3.62E − 03
$\mathbf{d} = (8, 8, 8), \#\mathbf{d} = 27$	1.65E − 03	9.29E − 04	8.22E − 04	1.02E − 03
$\mathbf{d} = (12, 12, 12), \#\mathbf{d} = 39$	1.60E − 03	7.22E − 04	2.88E − 04	4.44E − 04
$\mathbf{d} = (16, 16, 16), \#\mathbf{d} = 51$	1.61E − 03	7.03E − 04	9.35E − 05	2.96E − 04
$\mathbf{d} = (20, 20, 20), \#\mathbf{d} = 63$	1.76E − 03	7.22E − 04	4.04E − 05	2.85E − 04

Table 2 Approximate relative errors in v for the ellipse. $\#\mathbf{d}$ denotes the dimension of the finite dimensional approximating space.

References

1. Abboud, T., Nédélec, J.-C., Zhou, B.: Méthode des équations intégrales pour les hautes fréquencies. C.R. Acad. Sci. Paris, **318** Série I, 165–170 (1994)
2. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions. Dover, New York (1972)
3. Amini, S.: On the choice of the coupling parameter in boundary integral formulations of the exterior acoustics problem. Appl. Anal. **35**, 75–92 (1990)
4. Arden, S., Chandler-Wilde, S. N., Langdon, S.: A collocation method for high frequency scattering by convex polygons. Reading University Numerical Analysis Report 7/05, Reading, United Kingdom, to appear in J. Comp. Appl. Math.
5. Babich, V.M., Buldyrev, V.S.: Short-wavelength Diffraction Theory. Springer-Verlag, Berlin (1991)
6. Babich, V.M., Dement'ev, D.B., Samokish, B.A., Smyshlyaev, V.P.: On evaluation of the diffraction coefficients for arbitrary "nonsingular" directions of a smooth convex cone, SIAM J. Appl. Math. **60**, 536-573, (2000)
7. Babich, V.M., Kirpichnikova, N.Y.: The Boundary-Layer Method in Diffraction Problems. Springer-Verlag, Berlin (1979)
8. Bonner, B.D., Graham, I.G., Smyshlyaev, V.P.: The computation of conical diffraction coefficients in high-frequency acoustic wave scattering. SIAM J. Numer. Anal. **43**, 1202–1230 (2005)
9. Bonner, B.D.: Calculating Conical Diffraction Coefficients. PhD Thesis, University of Bath, United Kingdom (2003)
10. Brakhage, H., Werner, P.: Über das Dirichletsche Aussenraumproblem für die Helmholtzsche Schwingungsgleichung. Arch. der Math., **16**, 325–329 (1965)
11. Bruno, O.P., Geuzaine, C. A., Monro, J.A., Reitich, F.: Prescribed error tolerances within fixed computational times for scattering problems of arbitrarily high frequency: the convex case. Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **362**, 629–645 (2004).
12. Bruno, O.P., Geuzaine, C.A., Reitich, F.: On the $\mathcal{O}(1)$ solution of multiple-scattering problems. IEEE Trans. Magn. **41**, 1488-1491 (2005).
13. Buffa, A., Sauter, S.A.: Stabilisation of the acoustic single layer potential on non-smooth domains. SIAM J. Sci Comput., to appear.
14. Burton, A.J., Miller, G.F.: The application of integral methods for the numerical solution of boundary value problems. Proc. R. Soc. London, Ser. A **232**, 201–210 (1971)
15. Buslaev, V. S.: Short-wave asymptotic behaviour in the problem of diffraction by smooth convex contours (in Russian). Trudy Mat. Inst. Steklov. **73**, 14–117 (1964). Abbreviated English translation: On the shortwave asymptotic limit in the problem of diffraction by convex bodies. Soviet Physics Doklady **7**, 685–687 (1963)

16. Buslaev, V. S. Formulas for the short-wave asymptotic behavior in the diffraction problem by convex bodies. (in Russian) Vestnik Leningrad. Univ. **17** (13), 5–21 (1962)
17. Buslaev, V.S.: The asymptotic behavior of the spectral characteristics of exterior problems for the Schrödinger operator (in Russian). Izv. Akad. Nauk SSSR Ser. Mat. **39**, 149–235 (1975); English translation: Math. USSR–Izv. **9**, 139-223 (1975)
18. Chazarain, J.: Construction de la paramétrix du problème mixte hyperbolique pour l'equation des ondes. C. R. Acad. Sc. Paris **276**, 1213–1215 (1973).
19. Colton, D., Kress, R.: Inverse Acoustic and Electromagnetic Scattering. Springer-Verlag, New York (1998)
20. Ecevit, F.: Integral equation formulations of electromagnetic and acoustic scattering problems: convergence of multiple scattering iterations and high-frequency asymptotic expansions. PhD Thesis, University of Minnesota, (2005)
21. Filippov, V.B.: Rigorous justification of the shortwave asymptotic theory of diffraction in the shadow zone. J. Sov. Math. **6**, 577-626 (1976)
22. Fock, V.A.: Electromagnetic Diffraction and Propagation Problems. Pergamon Press, New York (1965)
23. Ganesh, M., Graham, I.G.: A high-order algorithm for obstacle scattering in three dimensions. J. Comp. Phys., **198**, 211-242 (2004)
24. Ganesh, M., Langdon, S., Sloan, I.H.: Efficient evaluation of highly oscillatory acoustic scattering surface integrals. Reading University Numerical Analysis Report 6/05, Reading, United Kingdom (2005)
25. Giebermann, K.: Schnelle Summationsverfahren zur numerischen Lösung von Integralgleichungen für Streuprobleme im $\mathbb{R}^3$. PhD Thesis, University of Karlsruhe (1997)
26. Giladi, E., Keller, J.B.: An asymptotically derived boundary element method for the Helmholtz equation. In 20th Annual Review of Progress in Applied Computational Electromagnetics, Syracuse, New York (2004)
27. Hörmander, L.: The Analysis of Linear Differential Operators. I, Distribution Theory and Fourier Analysis. Springer, Berlin, 1983.
28. Kress, R.: Minimizing the condition number of boundary integral operators in acoustic and electromagnetic scattering. Q. Jl. Mech. appl. Math. **38**, 323-341 (1985)
29. Kress, R., Spassov, W.T.: On the condition number of boundary integral operators for the exterior Dirichlet problem for the Helmholtz equation. Numer. Math. **42**, 77-85 (1983)
30. Landau, L.J.: Bessel functions: monotonicity and bounds. J. London Math. Soc. **61**, 197–215 (2000)
31. Langdon, S., Chandler-Wilde, S.N.: A wavenumber independent boundary element method for an acoustic scattering problem. Isaac Newton Institute for Mathematical Sciences Preprint NI03049-CPD, 2003, to appear in SIAM J. Numer. Anal.
32. Langdon, S., Chandler-Wilde, S. N.: Implementation of a boundary element method for high frequency scattering by convex polygons. In Chen, K. (ed.) Proc. 5th U.K. Conf. on Boundary Integral Methods, Liverpool, 2005, 2–11. University of Liverpool (2005)
33. Lebeau, G.: Régularité Gevrey 3 pour la diffraction. Comm. in Partial Differential Equations **9**, 1437–1494 (1984)
34. Hargé, T., Lebeau, G.: Diffraction par un convexe. Invent. Math. **118**, 161-196 (1994)
35. Ludwig, D.: Uniform asymptotic expansion of the field scattered by a convex onject at high frequencies. Comm. Pure Appl. Math. **20**, 103-138 (1967)
36. Morawetz, C.S., Ludwig, D.: An inequality for the reduced wave equation and the justification of geometrical optics. Comm. Pure Appl. Math. **21**, 187-203 (1968)
37. Melrose, R.B., Taylor, M.E.: Near peak scattering and the corrected Kirchhoff approximation for a convex obstacle. Adv. in Math. **55**, 242-315 (1985)
38. Nédélec, J.-C.: Acoustic and Electromagnetic Equations. Springer-Verlag, New York (2001)
39. Popov, G.: Some estimates of Green's functions in the shadow. Osaka J. Math. **24**, 1-12 (1987)
40. Schwab, C.: $p-$ and $hp-$ finite element methods. Theory and applications in solid and fluid mechanics. Oxford University Press, Oxford (1998)
41. Ursell, F.: Creeping modes in a shadow. Proc. Camb. Phil. Soc. **68**, 171–191 (1968)
42. Watson, G.N.: A treatise on the theory of Bessel functions. Cambridge University Press, Cambridge (1995)
43. Zayaev, A.B., Filippov, V.P.: Rigorous justification of the asymptotic solutions of "sliding-wave" type. J. Sov. Math. **30**, 2395–2406 (1985)
44. Zayaev, A.B., Filippov, V.P.: Rigorous justification of the Friedlander-Keller formulas. J. Sov. Math. **32**, 134–143 (1986)
45. Zworski, M.: High frequency scatering by a convex obstacle. Duke Math. J. **61**, 545–634 (1990)

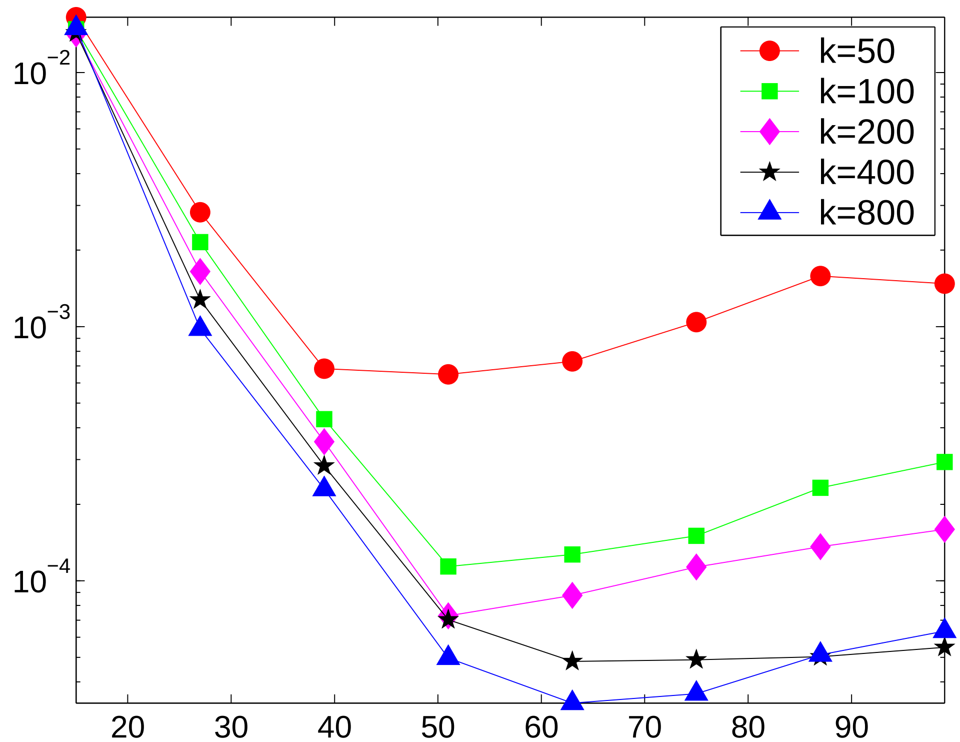

Fig. 3 Scaled error $k^{-1}\|v - \tilde{v}\|$ versus degrees of freedom for different values of k

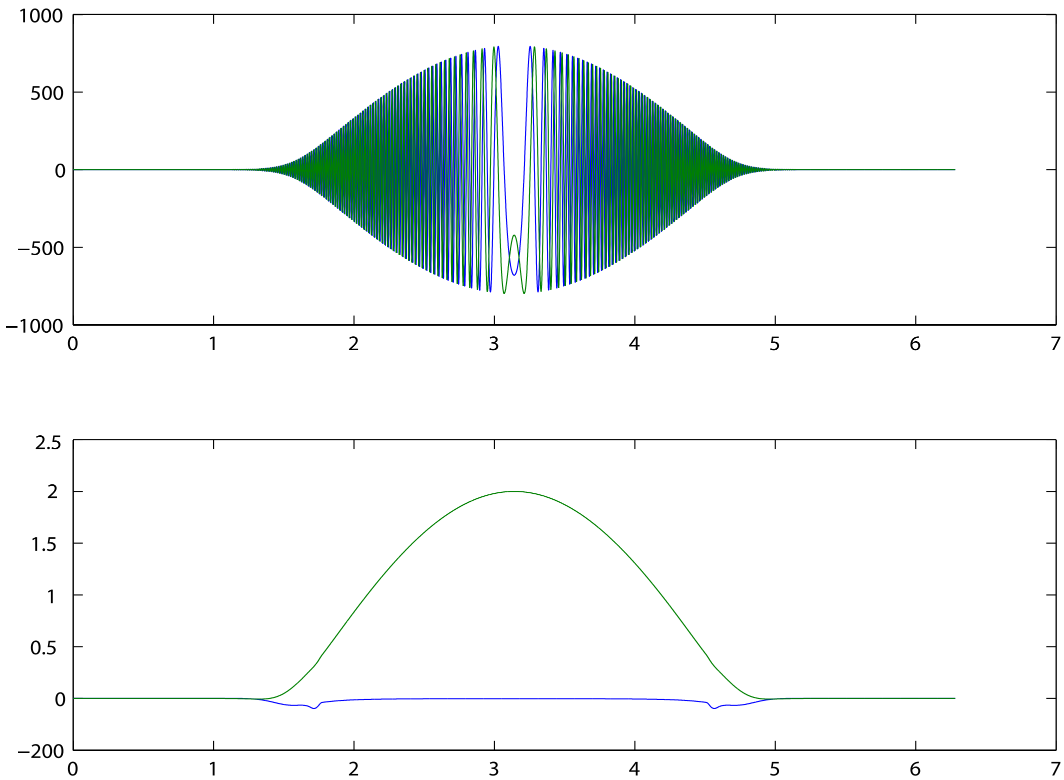

Fig. 4 Computed approximations, real and imaginary part, of v and $V(s, 400)$ for the unit circle with $k = 400$.

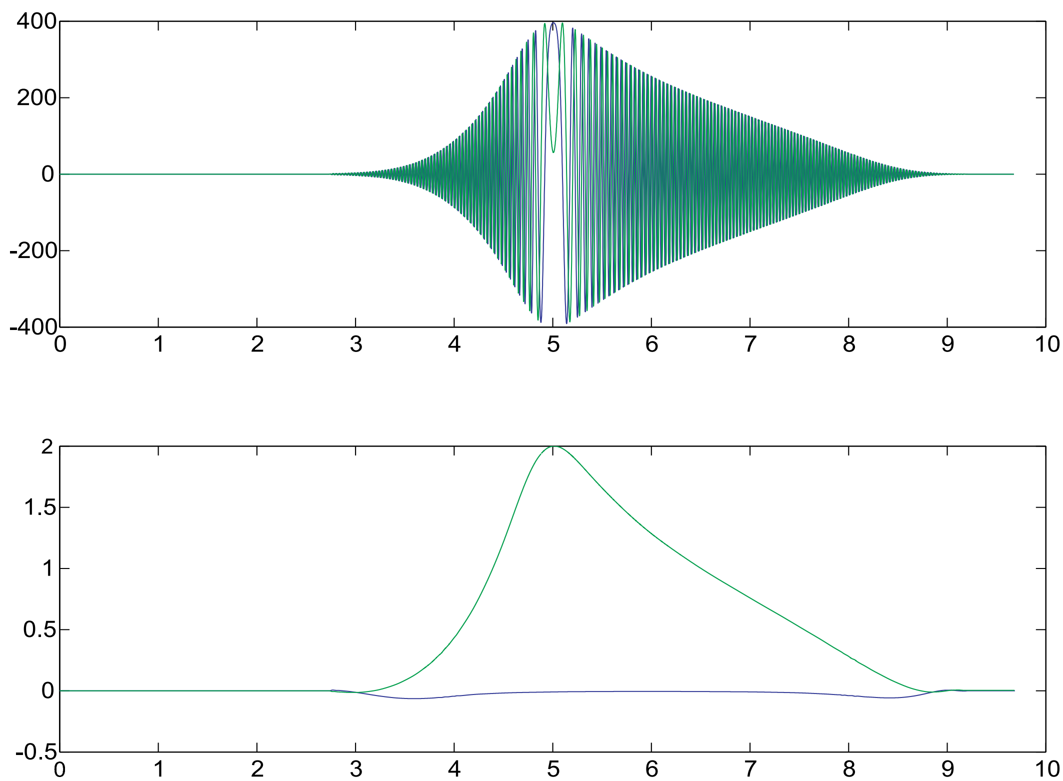

Fig. 5 Computed approximations of v and $V(s, 200)$ for the ellipse with $k = 200$.