# Explicit Finite Volume Schemes of Arbitrary High Order of Accuracy for Hyperbolic Systems with Stiff Source Terms

Michael Dumbser [a,b] Cedric Enaux [c] Eleuterio F. Toro [b]

[a] *Institut für Aerodynamik und Gasdynamik*
*Pfaffenwaldring 21, D-70550 Stuttgart, Germany*

[b] *Laboratory of Applied Mathematics, University of Trento*
*Via Mesiano 77, F-38050 Trento, Italy*

[c] *CEA, Service NEC, BP 12*
*F-91680 Bruyères-Le-Châtel, France*

**Abstract**

In this article we propose a new class of finite volume schemes of arbitrary accuracy in space and time for systems of hyperbolic balance laws with *stiff* source terms. The new class of schemes is based on a three stage procedure. First, in order to achieve high order accuracy in space, a nonlinear weighted essentially non-oscillatory reconstruction procedure is applied to the cell averages at the current time level. Second, the temporal evolution of the resulting reconstruction polynomials is computed *locally* inside each cell exploiting directly the full system of governing equations. In previous ADER schemes, this was achieved via the Cauchy-Kovalewski procedure, where the governing equation is repeatedly differentiated with respect to space and time to construct a Taylor series expansion of the local solution. As the Cauchy-Kovalewski procedure is based on Taylor series expansions, it is not able to handle systems with stiff source terms since the Taylor series diverges for this case. Therefore, in this article, we present a new strategy that replaces the Cauchy-Kovalewski procedure for high order time interpolation: we present a special local space-time discontinuous Galerkin (DG) finite element scheme that is able to handle arbitrarily stiff source terms in a stable manner. The solution of this space-time DG method can be proven to have several important robustness properties in the presence of stiff source terms. This step is the only part of the entire algorithm which is locally implicit. The third and last step of the proposed ADER finite volume schemes consists of the standard explicit space-time integration over each control volume, using the local space-time DG solutions at the Gaussian integration points for the intercell fluxes and for the space-time integral over the source term. We will show numerical convergence studies for nonlinear systems in one space dimension with both non-stiff and with very stiff source terms up to sixth order of accuracy in space and time. The application of the new method to a large set of different test cases is shown, in particular the stiff scalar model problem of LeVeque and Yee [34], the

relaxation system of Jin and Xin [30] and the full compressible Euler equations with stiff friction source terms.

*Key words:* hyperbolic balance laws, stiff source terms, finite volume schemes, ADER approach, local space-time discontinuous Galerkin method, WENO reconstruction

## 1 Introduction

In this paper, we are concerned with solving numerically one-dimensional hyperbolic systems of *balance* laws (SBL), namely:

$$\frac{\partial}{\partial t}\mathbf{u} + \frac{\partial}{\partial x}\mathbf{f}\left(\mathbf{u}\right) = \mathbf{S}\left(\mathbf{u}, x, t\right), \tag{1}$$

where $\mathbf{u} = \mathbf{u}(x,t)$ is the conservative state, $\mathbf{f}(\mathbf{u})$ is the flux and $\mathbf{S}(\mathbf{u},x,t)$ is the source term. The homogeneous system associated to (1) is the following hyperbolic system of conservation laws (SCL):

$$\frac{\partial}{\partial t}\mathbf{u} + \frac{\partial}{\partial x}\mathbf{f}\left(\mathbf{u}\right) = 0. \tag{2}$$

The definition of hyperbolicity only concerns system (2) above; it means that the Jacobian matrix of $\mathbf{f}(\mathbf{u})$ with respect to $\mathbf{u}$ has real eigenvalues and a set of associated eigenvectors which form a basis of $\mathbb{R}^d$, where $d$ is the dimension of vector $\mathbf{u}$. Divergence-free system associated to (1) is the name we give to the system

$$\frac{\partial}{\partial t}\mathbf{u} = \mathbf{S}\left(\mathbf{u}, x, t\right), \tag{3}$$

which is a system of ordinary differential equations (SODE) since it is assumed that no derivative of $\mathbf{u}$ appears through the function $\mathbf{S}$.

Coming from a wide range of different fields, a large number of physical models can be cast in the SBL form (1). Fluid mechanics is particularly concerned, since compressible fluid dynamics is usually modeled by the Euler system, which is a hyperbolic SCL. In this case, the source term can model the presence of other physical phenomena, such as gravity, geometrical reaction, friction, chemical reaction, etc.

*Email addresses:* michael.dumbser@iag.uni-stuttgart.de (Michael Dumbser), cedric.enaux@centraliens.net (Cedric Enaux), toor@ing.unitn.it (Eleuterio F. Toro).

We now restrict our analysis to source terms of the form $\mathbf{S}(\mathbf{u}, x)$. Compared with SCL, the presence of a source term generally has important consequences on the behaviour of SBL solutions. First, SBL may have non trivial *steady solutions*, namely solutions $\tilde{\mathbf{u}}(x)$ of the following system:

$$\frac{\partial}{\partial x}\mathbf{f}(\tilde{\mathbf{u}}) = \mathbf{S}(\tilde{\mathbf{u}}, x). \tag{4}$$

This situation occurs for instance in the case of an isentropic Euler system with gravity or geometrical reaction. Second, SBL may tend towards reduced systems as we will explain now. At least two process are involved in SBL: a conservative process associated to the homogeneous part (2) with a characteristic speed $\nu_f$, and a dissipative/productive process associated to the divergence-free part (3) with a characteristic speed $\nu_S$. If the time derivative is scaled according to the speed $\nu_f$, the dimensionless form of SBL (1) reads as

$$\frac{\partial}{\partial \bar{t}}\bar{\mathbf{u}} + \frac{\partial}{\partial \bar{x}}\bar{\mathbf{f}}(\bar{\mathbf{u}}) = \frac{1}{\epsilon}\bar{\mathbf{S}}(\bar{\mathbf{u}}, \bar{x}), \tag{5}$$

where bars mean that variables are dimensionless and where $\epsilon \equiv \frac{\nu_f}{\nu_S}$ is the ratio between characteristic speeds. A very small ratio $\epsilon \ll 1$ means that the dissipative/productive process is too fast, compared with the conservative process, to be fully observed. Such a source term is called *stiff* source term. The presence of a stiff source term may make tending the original system towards an asymptotic *reduced system* (see [9]), which can be of different mathematical nature than the original one. This situation occurs for instance in the case of an isentropic Euler system with large friction: the asymptotic limit of the original hyperbolic system is the porous media equation (see [24] and [25]), which is parabolic.

By integrating system (1) over a finite space-time control volume $\mathcal{Q}_i$ one obtains a finite volume formulation for the system of balance laws (1), which usually takes the form

$$\bar{\mathbf{u}}_i^{n+1} = \bar{\mathbf{u}}_i^n - \frac{\Delta t}{\Delta x_i}\left(\mathbf{f}_{i+\frac{1}{2}} - \mathbf{f}_{i-\frac{1}{2}}\right) + \Delta t \bar{\mathbf{S}}_i. \tag{6}$$

This is shown in more detail in the following section. We emphasize that the exact solution of (1) also fulfills (6) exactly if all integrals are computed exactly. The integration of (1) in space and time gives rise to a temporal integral of the flux across the element boundaries $\mathbf{f}_{i+\frac{1}{2}}$ and to a space-time integral $\bar{\mathbf{S}}_i$ of the source term inside $\mathcal{Q}_i$. In practice, to use the finite volume formulation (6) as a numerical tool for real computations, one must replace the integrals of the flux and the source by some suitable numerical approximations, that is to say one must choose a concrete numerical scheme.

For SCL, only a numerical flux must be chosen. In this case, the required prop-

erties are only the classical ones, namely consistency, stability and accuracy. For SBL, a numerical flux and a numerical source must be chosen. Here, the classical properties for each numerical term are also required, but additional properties are needed for the global numerical scheme, that is to say for the pair numerical flux and numerical source:

- the scheme should be *well-balanced*, which means that it is able to preserve steady states numerically.
- the scheme should be robust, even on *coarse grids* if the source term is stiff. A coarse grid is a grid whose size does not take into account the source term. In other words, characteristic space step and characteristic time step are based on the associated homogeneous SCL only.
- the scheme should be *asymptotically-consistent* (or *asymptotic preserving*) if the source term is stiff, which means that it gives the correct asymptotic behaviour even if the source term is underresolved.

In the last three decades, powerful numerical fluxes have been proposed to solve hyperbolic SCL, for example the fluxes of Godunov [18], Osher [16,40] and Roe [45] as well as the various HLL-type fluxes based on the approximate Riemann solver of Harten, Lax and van Leer [23], see e.g. [14,15,53]. A naive approach to solve SBL would consist of using one of these schemes for the flux $\mathbf{f}_{i+\frac{1}{2}}$ and using a space-centered explicit scheme for the source $\bar{\mathbf{S}}_i$. In this case, the source part of method (6) becomes an explicit Euler scheme, which may be unstable, especially in the stiff case. In order to counter that, the numerical source is usually taken implicitly. Hence, the source part of method (6) becomes an implicit Euler scheme, which is unconditionally stable. Anyway, it is well-known that using a classical numerical flux and a space-centered numerical source leads to spurious numerical results; more precisely, the global scheme is neither well-balanced (see [3], [21], [19] for instance), nor asymptotically-consistent (see [41], [7], [5] for instance). Consequently, several interesting approaches have been proposed to overcome that, as we will see in the following paragraph.

To solve SBL, very commonly used approaches are *splitting* schemes. A splitting approach (also called *fractional step method*) consists of solving iteratively the associated SCL with a classical finite volume scheme, and then the associated SODE with a classical numerical tool, like Runge-Kutta or predictor-corrector methods. The accuracy of the global scheme depends on the number and the order of these sub-steps; for example, a three sub-stage Strang splitting (see [47]) is second order accurate in time, but in the non stiff case only, as pointed out in [27]. Although simple and robust, classical first order splittings and Strang splittings lead to global schemes that are neither well-balanced, nor asymptotically-consistent (see [35]) because the coupling between numerical flux and numerical source only occurs through the initial condition of each sub-step. Better splitting schemes for particular SBL have been proposed (see

[1], [7], [38], [8]), and are asymptotically-consistent because at least one of the sub-steps takes into account both flux divergence and source term.

Another simple approach to solve SBL consists of upwinding the source at the interfaces (*USI schemes*). In the original version of USI schemes (see [3], [28]), a classical Riemann solver is first used to evaluate numerically the solution $\mathbf{u}_{i+\frac{1}{2}}$ of the homogeneous Riemann problem (without source term). This solution is then used in a first order finite volume scheme as argument of the flux, but also as argument of the source term in $\bar{\mathbf{S}}_i = \frac{1}{2}(\mathbf{S}(\mathbf{u}_{i-\frac{1}{2}}) + \mathbf{S}(\mathbf{u}_{i+\frac{1}{2}}))$, so that the same numerical information - based on the homogeneous system only - is given to the flux and the source term, which makes the scheme at least approximately well-balanced. More recently, other USI scheme versions have been proposed, which ensure that the scheme is also formally well-balanced (see [43], [4]). The main drawback of these approaches is that the global scheme must be explicit, thus problems may occur in the stiff case. In [5], a new version of USI schemes, designed for stiff relaxation SBL, has been proposed. The resulting scheme is robust, formally asymptotically-consistent and stable under a classical CFL condition, but obtains only first order of accuracy.

In the *well-balanced schemes* (see [21], [33], [19]), the source term is seen as a nonconservative product of a larger system. Due to the presence of this nonconservative product, a particular path has to be chosen instead of the classical Rankine-Hugoniot relations. This path can be chosen in such a way that the well-balanced property is formally imposed to the scheme. Well-balanced schemes are very efficient to maintain steady-states under classical CFL condition $\Delta t \simeq \mathcal{O}(\Delta x)$ (see [19]), but are not designed to capture the good asymptotic behaviour imposed by a stiff source term. More recently, a new version of well-balanced schemes which is also asymptotically-consistent has been proposed for a particular system (see [20]), but the resulting scheme is only stable under a very restrictive parabolic CFL condition of the type $\Delta t \simeq \mathcal{O}(\Delta x^2)$.

To solve SBL numerically, other approaches have been proposed. Sometimes, the Riemann problem considered takes into account the source term presence (*generalized Riemann problem*), but solving such a problem asks for additional hypothesis: see [32] for one possibility and [17] for an other. Both methods give well-balanced schemes, but are not robust enough to deal with stiff source terms. Concerning stiff relaxation systems, another approach consists of solving numerically the asymptotic reduced system instead of solving the original SBL (see [2], [5]). In this case, numerical results can only be obtained in the stiff case, thus some information is lost from the original SBL. In [37] it was pointed out that a semi-discrete discontinuous Galerkin (DG) scheme, i.e. only applying the DG space-discretization and remaining continuous in time, is an asymptotically-consistent scheme for linear systems with stiff relaxation. However, since the main problem of stiff SBL is precisely the *time*

*discretization*, this theoretical result given in [37] is only of very limited use in practice. Another idea proposed in [6] to solve nonlinear SBL is to use a combination of two tools. First, a relaxation scheme (see [31]) is used in order to obtain a linear, but larger, SBL. Second, a well-balanced scheme designed to capture the good asymptotic behaviour is applied on the larger system. The global scheme has many good properties, but is only of first order of accuracy. Finally, some asymptotically-consistent schemes have been derived from the classical upwind flux (see [42], [29], [39]). In these cases, the numerical flux is modified by the presence of the source term.
To our knowledge, a lot of tools to solve numerically SBL have been proposed up to now. The best ones among those are well-balanced and asymptotically-consistent, but none of them allows to reach arbitrary orders of accuracy in space and time while being stable under classical CFL condition.

The aim of this article is now to construct a method for SBL which is at the same time asymptotically consistent and can reach any order of accuracy in space and time under a standard CFL stability condition. The structure of the paper is as follows: First, in section 2 we show the construction of our arbitrary high order finite volume method for systems of balance laws. To assure monotonicity of the numerical solution in the vicinity of discontinuities, we briefly discuss the nonlinear WENO reconstruction operator in section 2.1, which is necessary to obtain a high order polynomial data representation from the cell averages before each time step. Then, the main building block of our finite volume discretization is presented in the subsequent section 2.2, namely a local space-time discontinuous Galerkin scheme used to evolve the reconstruction polynomials in time taking into account simultaneously the flux as well as the stiff source term. Numerical convergence studies are carried out in section 3 for smooth non-stiff and very stiff test problems. Applications of our scheme to various linear and nonlinear systems of balance laws with stiff source terms are shown in section 4 and a summary with conclusions and an outlook regarding future work is given in section 5.

## 2 An explicit arbitrary high order accurate finite volume scheme for nonlinear hyperbolic systems with stiff source terms

The aim of this article is to find high order accurate non-oscillatory numerical solutions for hyperbolic systems of balance laws for the vector of conserved quantities $\mathbf{u} = \mathbf{u}(x,t)$ of the form

$$\begin{cases} \text{PDE: } \frac{\partial}{\partial t}\mathbf{u} + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{u}) = \mathbf{S}(\mathbf{u}), \\ \text{IC: } \quad \mathbf{u}(x,0) = \mathbf{u}_0(x), \end{cases} \tag{7}$$

where $\mathbf{f}(\mathbf{u})$ is in general a nonlinear function of the state $\mathbf{u}$ and $\mathbf{S}(\mathbf{u})$ may be a stiff nonlinear source term. To illustrate the general framework of the method in the simplest possible way we restrict ourselves in the whole paper to one space dimension. The extension to multiple space dimensions can be done and will be the topic of future research.

The spatial computational domain $\Omega \subset \mathbb{R}$ is covered completely by pairwise disjoint spatial elements $Q_i = ]x_{i-\frac{1}{2}}; x_{i+\frac{1}{2}}[$, with $\Delta x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ and the cell average of $\mathbf{u}(x,t)$ within $Q_i$ is defined at time $t^n$ as

$$\bar{\mathbf{u}}_i^n = \frac{1}{\Delta x_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{u}(x,t^n) dx. \tag{8}$$

We furthermore define the space-time element spanned by the spatial element $Q_i$ and the time step $\Delta t = t^{n+1} - t^n$ as $\mathcal{Q}_i = Q_i \times ]t^n; t^n + \Delta t[$. The associated relative space and time coordinates $0 \leqslant \xi \leqslant 1$ and $0 \leqslant \tau \leqslant 1$, within one element $\mathcal{Q}_i$ are given by the relations

$$x = x_{i-\frac{1}{2}} + \xi \cdot \Delta x_i, \qquad \text{and} \qquad t = t^n + \tau \cdot \Delta t. \tag{9}$$

In the following, the numerical solution of (7) valid inside each element $\mathcal{Q}_i$ will be denoted with $\mathbf{u}_i(\xi,\tau)$. A standard finite volume discretization of (7) is given after integration of (7) over each space-time element $\mathcal{Q}_i$ as follows:

$$\bar{\mathbf{u}}_i^{n+1} = \bar{\mathbf{u}}_i^n - \frac{\Delta t}{\Delta x_i} \left( \mathbf{f}_{i+\frac{1}{2}} - \mathbf{f}_{i-\frac{1}{2}} \right) + \Delta t \bar{\mathbf{S}}_i, \tag{10}$$

with

$$\mathbf{f}_{i+\frac{1}{2}} = \int_0^1 \mathbf{f}_h(\mathbf{u}_i(1,\tau), \mathbf{u}_{i+1}(0,\tau)) d\tau \quad \text{and } \bar{\mathbf{S}}_i = \int_0^1 \int_0^1 \mathbf{S}(\mathbf{u}_i(\xi,\tau)) d\xi d\tau, \tag{11}$$

where $\mathbf{f}_h(\mathbf{u}_i(1,\tau), \mathbf{u}_{i+1}(0,\tau))$ denotes a numerical flux function (Riemann solver) that depends on the two arguments $\mathbf{u}_i(1,\tau)$ and $\mathbf{u}_{i+1}(0,\tau)$, which are the boundary extrapolated data on the left and on the right side of the element interface $i + \frac{1}{2}$. For an overview of Riemann solvers see [51]. For all computations shown in this paper we use the Rusanov flux, which is also often called the local Lax-Friedrichs flux. The Rusanov flux is a special case of the HLL flux, with a particularly simple wave speed estimate which is taken to be the maximum of the absolute values of the left and right eigenvalues. For an explicit standard first order Godunov-type finite volume scheme, one would now simply have to set $\mathbf{u}_i(\xi,\tau) = \bar{\mathbf{u}}_i^n$ and $\mathbf{u}_{i+1}(\xi,\tau) = \bar{\mathbf{u}}_{i+1}^n$ for the arguments of the numerical flux and inside the source term integral.
We emphasize that formula (10) together with (11) allows the construction of

*arbitrary* high order accurate finite volume schemes, provided the representation of the numerical solution $\mathbf{u}_i(\xi,\tau)$ inside each element and as a consequence the arguments of the numerical flux function and the source term are high order accurate in space and time. Since (10) only computes the time update of the cell averages $\bar{\mathbf{u}}_i^n$ from time $t^n$ to time $t^{n+1}$ we need to reconstruct higher order polynomial data from these cell averages $\bar{\mathbf{u}}^n$ to get better estimates for the arguments of the flux function and the source term in the integrals appearing in (11).
Therefore, as described in detail in the subsequent sections, the necessary steps to construct an arbitrary high order essentially non-oscillatory explicit one-step finite volume scheme are the following: (I) Nonlinear (non-oscillatory) reconstruction of spatial polynomials from the given cell averages at time $t^n$. (II) Local solution of the initial value problem (7) inside each element, where the initial data is given by the spatial reconstruction polynomial at time $t^n$. (III) Numerical integration of the integrals in (11) and update of the cell averages according to (10).

### *2.1 Nonlinear reconstruction technique*

In this section we briefly discuss the proposed nonlinear weighted essentially non-oscillatory (WENO) reconstruction procedure to reconstruct higher order polynomial data within each spatial cell $Q_i$ at time $t^n$ from the given cell averages $\bar{\mathbf{u}}_i^n$. This corresponds to step (I) as outlined at the end of the previous section. We emphasize already at this point that the reconstruction procedure is *nonlinear* and depends strongly on the input data $\bar{\mathbf{u}}_i^n$. Thus, the resulting numerical scheme, even when applied to a completely linear PDE, will be *nonlinear* and thus it will not be possible to give a closed expression of the scheme.

The reconstruction procedure described here for the one-dimensional case follows directly from the guidelines given in [11] for general unstructured two- and three-dimensional meshes. It reconstructs *entire polynomials*, as the original ENO approach proposed by Harten *et al.* in [22]. However, we formally write our method like a WENO scheme [26,36] with a particularly simple choice for the linear weights. The most important difference of our approach compared to classical WENO schemes is that standard WENO methods reconstruct *point values* at the Gaussian integration points instead of an entire polynomial valid inside each element $Q_i$.
Reconstruction is done for each element on a reconstruction stencil $\mathcal{S}_i^s$, which is given by the following union of the element $Q_i$ and its neighbors $Q_j$,

$$\mathcal{S}_i^s = \bigcup_{j=i+s-k}^{i+s+k} Q_j, \tag{12}$$

where $s$ is the stencil shift with respect to the central element $Q_i$ and $k$ is the spatial extension of the stencil to the left and the right. A central reconstruction stencil is given by $s = 0$, an entirely left-sided stencil is given by $s = -k$ and an entirely right-sided stencil is given by $s = k$. In our approach, we always will use the three fixed reconstruction stencils $\mathcal{S}_i^0$, $\mathcal{S}_i^{-k}$ and $\mathcal{S}_i^k$. Given the cell average data $\bar{\mathbf{u}}_i^n$ in all elements $Q_i$ we are looking for a spatial reconstruction polynomial obtained from $\mathcal{S}_i^s$ at time $t^n$ of the form

$$\mathbf{w}_i^s(\xi, t^n) = \sum_{l=0}^{M} \Psi_l(\xi) \hat{\mathbf{w}}_l^{(i,s)}(t^n) := \Psi_l(\xi) \hat{\mathbf{w}}_l^{(i,s)}(t^n), \tag{13}$$

where we use the rescaled Legendre polynomials for the spatial reconstruction basis functions $\Psi_l(\xi)$ such that the $\Psi_l(\xi)$ form an orthogonal basis on the unit interval $I = [0; 1]$. In the following, we will use standard tensor index notation, implying summation over indices appearing twice. The number of polynomial coefficients (degrees of freedom) is $L = M+1$, where $M$ is the degree of the reconstruction polynomial. To compute the reconstruction polynomial $\mathbf{w}_i(\xi, t^n)$ valid for element $Q_i$ we require integral conservation for all elements $Q_j$ inside the stencil $\mathcal{S}_i^s$, i.e.

$$\int_{Q_j} \mathbf{w}_i^s(\xi, t^n) d\xi = \int_{Q_j} \Psi_l(\xi) d\xi \cdot \hat{\mathbf{w}}_l^{(i,s)}(t^n) = \bar{\mathbf{u}}_j^n, \qquad \forall Q_j \in \mathcal{S}_i^s. \tag{14}$$

Equation (14) yields a linear equation system of the form

$$A_{jl} \cdot \hat{\mathbf{w}}_l^{(i,s)}(t^n) = \bar{\mathbf{u}}_j^n \tag{15}$$

for the unknown coefficients $\hat{\mathbf{w}}_l^{(i,s)}(t^n)$ of the reconstruction polynomial on stencil $\mathcal{S}_i^s$. Since we choose $k = M/2$ for even $M$ and $k = (M + 1)/2$ for odd $M$, the number of elements in $\mathcal{S}_i^s$ may become larger than the number of degrees of freedom $L$. In this case, we use a constrained least-squares technique according to [11] to solve (15).

To obtain the final non-oscillatory reconstruction polynomials for each $Q_i$ at time $t^n$, we finally construct a data-dependent nonlinear combination of the polynomials $\mathbf{w}_i^0(\xi, t^n)$, $\mathbf{w}_i^{-k}(\xi, t^n)$ and $\mathbf{w}_i^k(\xi, t^n)$ obtained from the central, left-sided and right-sided stencils as follows:

$$\mathbf{w}_i(\xi, t^n) = \hat{\mathbf{w}}_l^i(t^n) \Psi_l(\xi), \tag{16}$$

with

$$\hat{\mathbf{w}}_l^i(t^n) = \omega_0 \, \hat{\mathbf{w}}_l^{(i,0)}(t^n) + \omega_{-k} \, \hat{\mathbf{w}}_l^{(i,-k)}(t^n) + \omega_k \, \hat{\mathbf{w}}_l^{(i,k)}(t^n). \tag{17}$$

The nonlinear weights $\omega_s$ are given by the relations

$$\omega_s = \frac{\tilde{\omega}_s}{\tilde{\omega}_0 + \tilde{\omega}_{-k} + \tilde{\omega}_k}, \quad \tilde{\omega}_s = \frac{\lambda_s}{(\sigma_s + \epsilon)^r}. \tag{18}$$

In our particular formulation, the oscillation indicators $\sigma_s$ are computed from

$$\sigma_s = \Sigma_{lm}\, \hat{\mathbf{w}}_l^s(t^n)\hat{\mathbf{w}}_m^s(t^n), \qquad \text{with} \qquad \Sigma_{lm} = \sum_{\alpha=1}^{M} \int_0^1 \frac{\partial^\alpha \Psi_l(\xi)}{\partial \xi^\alpha} \cdot \frac{\partial^\alpha \Psi_m(\xi)}{\partial \xi^\alpha} d\xi. \quad (19)$$

Here, $\Sigma_{lm}$ is the universal oscillation indicator matrix for the reference element $Q_E$ that does neither depend on the problem nor on the mesh, see [11]. The parameters $\epsilon$ and $r$ are constants for which we typically choose $\epsilon = 10^{-14}$ and $r = 12$. For the linear weights $\lambda_s$ we choose $\lambda_{-k} = \lambda_k = 1$ and a very large linear weight $\lambda_0$ on the central stencil, typically $\lambda_0 = 10^5$. It has been shown previously [26,36] that the numerical results are quite insensitive to the WENO parameters $\epsilon$ and $r$ and also with respect to the linear weight on the central stencil $\lambda_0$, see [11].
The proposed reconstruction usually uses the accurate and linearly stable central stencil reconstruction in those regions of $\Omega$ where the solution is smooth because of the large linear weight $\lambda_0$. However, due to the strongly nonlinear dependence of the weights $\omega_s$ on the oscillation indicators $\sigma_s$, in the presence of discontinuities the smoother left- or right-sided stencils are preferred, as for standard ENO and WENO methods. For the nonlinear scalar case, the reconstruction operator described above can be directly applied to the cell averages $\bar{u}_i^n$ of the conserved quantity $u$. For nonlinear hyperbolic systems, the reconstruction should be done in characteristic variables [22,12] in order to avoid spurious oscillations that may appear when applying ENO or WENO reconstruction operators component-wise to nonlinear hyperbolic systems.

The result of the reconstruction procedure is a non-oscillatory spatial polynomial $\mathbf{w}_i(\xi, t^n)$ defined at time $t^n$ inside each spatial element $Q_i$. However, we still need to compute the temporal evolution of these polynomials inside each space-time element $\mathcal{Q}_i$ in order to be able to compute the integrals appearing in (11).

### *2.2 The local space-time discontinuous Galerkin scheme*

In previously published ADER finite volume schemes (see e.g. [11,12,50,54]) and also in the original ENO scheme of Harten *et al.* [22] the nonlinear reconstruction step (I) as well as the numerical integration and update step (III) as outlined at the end of section 2 are very similar compared with the new scheme proposed in this article. The only main difference lies in the solution of the local initial value problem (IVP) defined in step (II). In ADER finite volume schemes and also in the original ENO approach the temporal evolution of the reconstruction polynomial is computed using the so-called Cauchy-Kovalewski or Lax-Wendroff procedure. This procedure constructs a local solution of the IVP making the ansatz of a local time Taylor series expanded at time level $t^n$,

where then time derivatives are replaced by spatial derivatives differentiating repeatedly the governing PDE with respect to space and time. The spatial derivatives are obtained from the reconstruction polynomials at time $t^n$.

It is a well known fact that methods based on Taylor series usually do not work in the presence of stiff source terms. Therefore, we propose to replace the Cauchy-Kovalewski procedure by a new local space-time DG scheme in order to solve the local IVP in step (II). In our local space-time DG scheme, the usual integration by parts is done only in time and not in space, which establishes a distinct difference compared to the existing global space-time DG schemes [55]. A comparison of the classical Cauchy-Kovalewski procedure and the new local space-time DG scheme will be shown for a simple case to illustrate the difference in the quality of the solution of the local IVP.

### *2.2.1 Linear scalar model equation*

To illustrate the construction and the theoretical properties of our proposed method, in this section we only consider the simple linear scalar model equation with the linear flux and source functions

$$f(u) = au \qquad \text{and } S(u) = -\nu u, \qquad a > 0, \nu > 0, \tag{20}$$

where the stiffness of the relaxation source term is determined by the parameter $\nu$ and where we suppose periodic boundary conditions for the moment.

The space of basis and test functions $V_h$ of the local space-time DG scheme is defined to be the space spanned by piecewise polynomials given by the space-time tensor products of the scaled Legendre polynomials $\Psi_i(\xi)$ and $\Psi_j(\tau)$ of degree $0 \leqslant i, j \leqslant M$, i.e.

$$\Phi_k = \Phi_k(\xi, \tau) = \Psi_i(\xi) \cdot \Psi_j(\tau). \tag{21}$$

In eqn. (21) the index $k$ with $1 \leqslant k = k(i, j) \leqslant N_d$ is a mono-index ranging from 1 to the number of degrees of freedom $N_d = (M + 1)^2$, computed from the index pair $(i, j)$. As already defined above, $0 \leqslant \xi \leqslant 1$ and $0 \leqslant \tau \leqslant 1$ are the spatial and the temporal coordinates in the space-time reference element $\mathcal{Q}_E = [0; 1] \times [0; 1] \in \mathbb{R}^2$. In the following, we will use the following two scalar products of two functions $f(\xi, \tau)$ and $g(\xi, \tau)$,

$$\langle f, g \rangle = \int_0^1 \int_0^1 f(\xi, \tau) \cdot g(\xi, \tau) d\xi d\tau, \quad [f(\xi, \tau), g(\xi, \tau)] = \int_0^1 f(\xi, \tau) \cdot g(\xi, \tau) d\xi, \tag{22}$$

where the first one denotes the space-time scalar product over the space-time reference element $\mathcal{Q}_E$ and the second one is the purely spatial scalar product over the spatial reference element $Q_E = [0; 1]$. The local numerical solution $u_i$

of (7) inside each space-time control volume $\mathcal{Q}_i$ is approximated within the reference element $\mathcal{Q}_E$ using the basis functions $\Phi_k$ as follows:

$$u_i = u_i(\xi,\tau) = \sum_{l=1}^{N^d} \Phi_l\left(\xi,\tau\right) \cdot \hat{u}_l^i := \Phi_l\left(\xi,\tau\right) \hat{u}_l^i, \tag{23}$$

where we again use the classical Einstein summation convention for tensor calculus, which implies summation over all indices appearing twice.

The proposed local space-time discontinuous Galerkin finite element method is now obtained by first rewriting the governing PDE (7) together with the assumption (20) in terms of the variables in the reference element, i.e.

$$\frac{\partial}{\partial\tau} u + a^* \frac{\partial}{\partial\xi} u = -\nu^* u, \tag{24}$$

with $a^* = \Delta t \xi_x \cdot a$ and $\nu^* = \Delta t \cdot \nu$.

Multiplication of the modified governing PDE with test functions $\Phi_k \in V_h$ and integration over the reference element $\mathcal{Q}_E$ yields

$$\left\langle \Phi_k, \frac{\partial}{\partial\tau} u_i \right\rangle + a^* \left\langle \Phi_k, \frac{\partial}{\partial\xi} u_i \right\rangle = -\nu^* \left\langle \Phi_k, u_i \right\rangle. \tag{25}$$

For the space-time discontinuous Galerkin scheme presented in [55] one would now have to integrate both terms on the left hand side by parts in space and time in order to introduce the information from the neighbor elements and to shift the spatial derivative operator onto the test function. For our purposes, however, the integration by parts in space is *not* required since we want to keep a *local* formulation that does not need any information from the neighbor elements but for which it is sufficient to provide an initial condition. Therefore, we only use the integration by parts in time for the first term that contains the time derivative, and obtain

$$\begin{aligned}\left[\Phi_k(\xi,1), u_i(\xi,1)\right] - \left[\Phi_k(\xi,0), w_i(\xi,t^n)\right] - \left\langle \frac{\partial}{\partial\tau}\Phi_k, u_i \right\rangle + \\ a^* \left\langle \Phi_k,\ \frac{\partial}{\partial\xi} u_i \right\rangle = -\nu^* \left\langle \Phi_k, u_i \right\rangle.\end{aligned} \tag{26}$$

The spatial scalar products appearing in (26) correspond to the fluxes in time direction. Due to the causality principle the future has no influence on the past, i.e. we can take the numerical solution *inside* the element itself for the flux at relative time $\tau = 1$, whereas the flux at relative time $\tau = 0$ will be completely defined by the initial condition $w_i(\xi,t^n) = \Psi_m(\xi)\hat{w}_m^i(t^n)$. The initial condition is hence given by the reconstruction polynomials obtained from the reconstruction operator applied to the cell averages at the current time $t^n$.

We recall that the $\Psi_m(\xi)$ are the reconstruction basis functions introduced in section 2.1. Please note that due to the use of a *discontinuous* Galerkin approximation, in general $w_i(\xi,t^n) \neq u_i(\xi,0^+)$, i.e. the reconstruction polynomials at $t=t^n$ do not necessarily agree with the boundary extrapolated polynomial $u_i(\xi,0^+)$ of the space-time DG solution inside element $\mathcal{Q}_i$ at $\tau=0^+$. Inserting the time fluxes and the ansatz for the numerical solution (23) into (26) yields

$$[\Phi_k(\xi,1),\Phi_l(\xi,1)]\,\hat{u}_l^i - [\Phi_k(\xi,0),\Psi_m(\xi)]\,\hat{w}_m^i(t^n) - \left\langle \frac{\partial}{\partial\tau}\Phi_k,\Phi_l \right\rangle \hat{u}_l^i + a^* \left\langle \Phi_k,\, \frac{\partial}{\partial\xi}\Phi_l \right\rangle \hat{u}_l^i = -\nu^* \left\langle \Phi_k,\Phi_l \right\rangle \hat{u}_l^i. \tag{27}$$

Introducing the element mass matrix $M_{kl} = \langle \Phi_k,\Phi_l \rangle$, the stiffness matrices with respect to time $K_{kl}^{\tau} = \left\langle \frac{\partial}{\partial\tau}\Phi_k,\Phi_l \right\rangle$ and space $K_{kl}^{\xi} = \left\langle \Phi_k,\frac{\partial}{\partial\xi}\Phi_l \right\rangle$ as well as the flux matrices $F_{km}^0 = [\Phi_k(\xi,0),\Psi(\xi)]$ and $F_{kl}^1 = [\Phi_k(\xi,1),\Phi_l(\xi,1)]$ for relative time $\tau=0$ and $\tau=1$, respectively, we obtain the following equation system for the unknowns $\hat{u}_l^i$:

$$Y_{kl}\,\hat{u}_l^i = F_{km}^0 \hat{w}_m^i(t^n). \tag{28}$$

The system has a unique solution if the system matrix

$$Y_{kl} = F_{kl}^1 - K_{kl}^{\tau} + a^* K_{kl}^{\xi} + \nu^* M_{kl} \tag{29}$$

is invertible. It will be shown in the following that this is always the case for any value of $\nu$. Therefore, the solution of (28) can be written as

$$\hat{u}_l^i = Y_{kl}^{-1} \cdot F_{km}^0 \hat{w}_m^i(t^n). \tag{30}$$

We note that the local space-time DG scheme (27) requires the solution of the linear equation system (28) and thus is *locally implicit*. Due to the local character of the method, the computation of the space-time degrees of freedom $\hat{u}_l^i$ can be done independently for each cell $\mathcal{Q}_i$, without considering neighbor elements. However, we emphasize that this is the only locally implicit part of the entire algorithm. The resulting finite volume scheme (10) is completely explicit.

### *2.2.2 Properties of the local space-time discontinuous Galerkin scheme*

**Well-posedness** The mass matrix $M_{kl}$ is diagonal since the functions $\Phi_k(\xi,\tau)$ form an orthogonal basis on the space-time reference element $\mathcal{Q}_E$ with respect to the scalar product $<\cdot,\cdot>$. Therefore, the system matrix $Y_{kl}$ is diagonally dominant in the case $\nu \to \infty$ and thus the system (28) will be always well posed, especially in the stiff limit.

**Boundedness of the solution** In the following we want to show via asymptotic analysis that the solution $u_i$ of (27) remains bounded in the limit $\nu \to \infty$. Even more, we will show that the discrete solution of (27) tends to zero as $\nu \to \infty$. We therefore write (27) with $\Phi_k \in V_h$ and $\nu^* = 1/\epsilon$, where $\epsilon > 0$ is a small parameter and $\nu \to \infty$ for $\epsilon \to 0$, as follows:

$$[\Phi_k(\xi,1), u_i(\xi,1)] - [\Phi_k(\xi,0), w_i(\xi,t^n)] - \left\langle \frac{\partial}{\partial\tau}\Phi_k, u_i \right\rangle + a^* \left\langle \Phi_k, \frac{\partial}{\partial\xi} u_i \right\rangle = -\frac{1}{\epsilon} \langle \Phi_k, u_i \rangle . \tag{31}$$

Furthermore, we write a series expansion for $u^i = u^i(\xi,\tau)$ in terms of the small parameter $\epsilon$ with $u_i^{(0)} \in V_h$, $u_i^{(1)} \in V_h$, and $u_i^{(2)} \in V_h$ as

$$u_i = u_i^{(0)} + \epsilon u_i^{(1)} + \epsilon^2 u_i^{(2)} + \mathcal{O}(\epsilon^3). \tag{32}$$

After inserting (32) into (31) we obtain the following equation system in terms of powers of $\epsilon$:

$$\begin{aligned}
&\epsilon^{-1} \left\langle \Phi_k, u_i^{(0)} \right\rangle + \\
&\epsilon^0 \left( \left[ \Phi_k(\xi,1), u_i^{(0)}(\xi,1) \right] - \left\langle \frac{\partial}{\partial\tau}\Phi_k, u_i^{(0)} \right\rangle + a^* \left\langle \Phi_k, \frac{\partial}{\partial\xi} u_i^{(0)} \right\rangle \right) + \\
&\epsilon^0 \left( \left\langle \Phi_k, u_i^{(1)} \right\rangle - [\Phi_k(\xi,0), w_i(\xi,t^n)] \right) + \\
&\epsilon^1 \left( \left[ \Phi_k(\xi,1), u_i^{(0)}(\xi,1) \right] \right) + \\
&\epsilon^1 \left( \left\langle \Phi_k, u_i^{(2)} \right\rangle - \left\langle \frac{\partial}{\partial\tau}\Phi_k, u_i^{(1)} \right\rangle + a^* \left\langle \Phi_k, \frac{\partial}{\partial\xi} u_i^{(1)} \right\rangle \right) + \mathcal{O}(\epsilon^2) = 0. \\
&\qquad\qquad \forall \Phi_k \in V_h
\end{aligned} \tag{33}$$

Since eqn. (33) must be fulfilled for *any* value of $\epsilon > 0$, all coefficients after the terms in $\epsilon$ must vanish. From the leading term $\epsilon^{-1}$ we therefore obtain

$$\left\langle \Phi_k, u_i^{(0)} \right\rangle = 0 \qquad \forall \Phi_k \in V_h \qquad \Rightarrow \quad u_i^{(0)} = 0. \tag{34}$$

Inserting (34) in (33) and considering the coefficients of the terms $\epsilon^0$ we obtain

$$\left\langle \Phi_k, u_i^{(1)} \right\rangle - [\Phi_k(\xi,0), w_i(\xi,t^n)] = 0, \tag{35}$$

and from the term $\epsilon^1$ we get

$$\left\langle \Phi_k, u_i^{(2)} \right\rangle - \left\langle \frac{\partial}{\partial\tau}\Phi_k, u_i^{(1)} \right\rangle + a^* \left\langle \Phi_k, \frac{\partial}{\partial\xi} u_i^{(1)} \right\rangle = 0. \tag{36}$$

Equations (35) and (36) connect $u_i^{(1)}$ and $u_i^{(2)}$ with the initial condition $w_i(\xi,t^n)$. Since $w_i(\xi,t^n)$ does not depend explicitly on $\epsilon$, from eqns. (34) - (36) and the

ansatz (32) follows

$$\lim_{\epsilon \to 0} u_i(\xi,\tau) = \lim_{\epsilon \to 0} \left( \epsilon u_i^{(1)} + \epsilon^2 u_i^{(2)} + \mathcal{O}(\epsilon^3) \right) = 0. \tag{37}$$

From (37) follows the boundedness of $u_i(\xi,\tau)$ in $\mathcal{Q}_i$ in the stiff limit $\epsilon \to 0$. This property is necessary to eliminate the stiffness from the numerical flux in (11) in the limit $\epsilon \to 0$. For our model problem (20) the numerical flux $f_h(u_i(1,\tau), u_{i+1}(0,\tau))$ in (11) is a *linear* function of its two arguments $u_i(1,\tau)$ and $u_{i+1}(0,\tau)$. Since (37) is valid independently for all elements $\mathcal{Q}_i$, we get $f_{i+\frac{1}{2}} \to 0$ and $f_{i-\frac{1}{2}} \to 0$ for $\epsilon \to 0$.

**Boundedness of the source term integral** Even more important for the robustness of the finite volume scheme (10) is the boundedness of the source space-time integral $\bar{S}_i$ defined in (11). For the linear scalar model equation (20) the source space-time integral reads with $1/\epsilon = \nu^* = \Delta t\, \nu$ together with the asymptotic ansatz (32) and eqn. (34) as

$$\bar{S}_i = -\nu \left\langle 1, u_i(\xi,\tau) \right\rangle = -\Delta t^{-1} \frac{1}{\epsilon} \left\langle 1, u_i(\xi,\tau) \right\rangle = -\frac{1}{\Delta t} \left\langle 1, u_i^{(1)} + \epsilon^1 u_i^{(2)} + \mathcal{O}(\epsilon^2) \right\rangle. \tag{38}$$

Using (35) and the conservation property (14) of the reconstruction operator, i.e. $[1, w_i(\xi,t^n)] = \bar{u}_i^n$, we finally obtain

$$\bar{S}_i = -\frac{1}{\Delta t} \left( [1, w_i(\xi,t^n)] + \mathcal{O}(\epsilon) \right) = -\frac{1}{\Delta t} \left( \bar{u}_i^n + \mathcal{O}(\epsilon) \right), \tag{39}$$

and in the stiff limit $\epsilon \to 0$ we have

$$\lim_{\epsilon \to 0} \bar{S}_i = -\frac{\bar{u}_i^n}{\Delta t}. \tag{40}$$

Inserting (37) and (40) into (10) and (11), we obtain the following finite volume scheme in the stiff limit:

$$\lim_{\epsilon \to 0} \bar{u}_i^{n+1} = \bar{u}_i^n + \frac{\Delta t}{\Delta x} \lim_{\epsilon \to 0} \left( f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}} \right) + \Delta t \lim_{\epsilon \to 0} \bar{S}_i = 0. \tag{41}$$

This means that for *any* bounded initial condition $\bar{u}_i^n$ and for any bounded $\Delta t$ (e.g. bounded by the standard CFL condition) our finite volume scheme of any order of accuracy captures the stiff limit of (7) with (20) *exactly*.

### *2.2.3 Comparison of the Cauchy-Kovalewski procedure with the local space-time DG scheme for a linear scalar ODE*

As mentioned already before, in the original ENO approach of Harten *et al.* [22] and also for ADER finite volume schemes [11,12,50,52], the time-accurate

temporal evolution of the reconstruction polynomials $w_i(\xi, t^n)$ is predicted inside each element within one time step using the Cauchy-Kovalewski procedure, where the local solution is computed inside each element via a temporal Taylor series in which the time derivatives are replaced by space derivatives using repeated differentiation of the governing partial differential equation (7). As an initial condition for this procedure, the reconstruction polynomials $w_i(\xi, t^n)$ at time $t = t^n$ are taken. In other words, we are looking for a *local* solution $u_i(\xi, \tau)$ of the initial value problem for (7) *inside* each space-time element $\mathcal{Q}_i$, where the initial condition is given by the reconstruction polynomials, i.e. $u(x(\xi), 0) = w_i(\xi, t^n)$.
Neglecting convection for the moment, i.e. setting $a = 0$, the PDE (7) with (20) reduces to the simple linear ordinary differential equation

$$\frac{\partial}{\partial t} u = -\nu u, \qquad t \in \mathbb{R}_0^+, \tag{42}$$

whose solution is given by

$$u(t, \nu) = u(0) e^{-\nu t}. \tag{43}$$

For very large values of $\nu$, the solution (43) tends to the discontinuous limit solution

$$\lim_{\nu \to \infty} u(t, \nu) = \begin{cases} u(0) & \text{if } t = 0, \\ 0 & \text{if } t > 0. \end{cases} \tag{44}$$

It is obvious that a Taylor series expanded at time $t = 0$ is not able to approximate such a discontinuous solution as given by (44). For this reason, the Cauchy-Kovalewski method can not be applied in this case to construct a local solution to (7) since it is based essentially on the applicability of the Taylor series expansion in time. In order to construct a polynomial approximation to (43) that is at the same time high order accurate *and* is able to capture the stiff limit (44), we have propose the new local space-time discontinuous Galerkin scheme (28) applied to (7) locally inside each element. The solution $u_i(\xi, \tau)$ of this local space-time DG scheme applied to all elements $\mathcal{Q}_i$ is used in the finite volume scheme (10) to compute the numerical fluxes at the element interfaces and to compute the space-time integral of the source term in (11).
scheme is able to capture the limit (44) correctly. The correct behaviour of the local space-time DG scheme in the stiff limit $\nu \to \infty$ compared to the divergence of the classical Cauchy-Kovalewski procedure, which is normally used in ADER finite volume and ADER discontinuous Galerkin schemes [10,13,49], is illustrated in Fig. 1. We show the numerical solutions obtained with sixth order schemes (basis polynomials $\Phi_k$ of maximal degree five) applied to (42) for increasing values of $\nu$ with the initial condition $u(0) = 1$. This figure shows very clearly that the Cauchy-Kovalewski procedure already fails for the small value of $\nu = 3$. For very large values of $\nu$, the solution of the Cauchy-

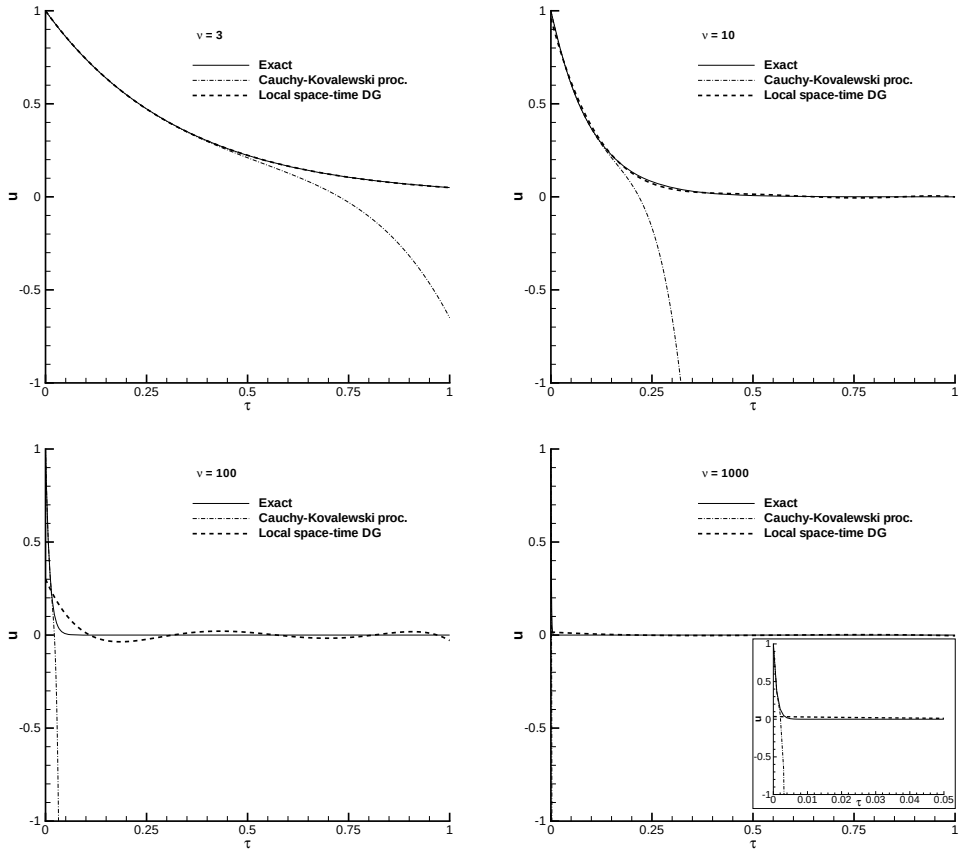


Fig. 1. Exact solution and sixth order numerical solutions of (42) using the Cauchy-Kovalewski procedure based on a temporal Taylor series expansion and the new local space-time DG scheme for $\nu = 3$ (top left), $\nu = 10$, (top right), $\nu = 100$ (bottom left) and $\nu = 1000$ (bottom right).

Kovalewski procedure is only correct in a very small interval $[0; \delta]$ $(0 < \delta \ll 1)$ and diverges quickly in the remaining interval $]\delta; 1]$, whereas our new local space-time DG scheme apparently converges to the correct solution (44) of (42) for $\nu \to \infty$ in the interval $]\delta; 1]$. We note furthermore that the local space-time DG scheme produces an almost continuous solution at $\tau = 0$ for small values of $\nu$ and exhibits an increasing jump at $\tau = 0$ for increasing $\nu$ in order to capture correctly the discontinuous behaviour of (43). Note that in all cases shown $u(0) = 1$.

### *2.2.4 General linear hyperbolic systems with stiff source terms*

The extension of the proposed local space-time discontinuous Galerkin scheme to general linear hyperbolic systems is straightforward. We consider linear systems of the form

$$\frac{\partial}{\partial t} u_p + A_{pq} \frac{\partial}{\partial x} u_q = -E_{pq} u_q, \tag{45}$$

where $u_p$ is the vector of state of $n$ unknowns, $A_{pq}$ is a $n \times n$ matrix with real eigenvalues and with a complete set of eigenvectors. The $n \times n$ matrix $E_{pq}$ must be positive definite. We then rewrite the system (45) in reference coordinates $\xi$ and $\tau$ and obtain

$$\frac{\partial}{\partial t} u_q + A^*_{pq} \frac{\partial}{\partial x} u_q = -E^*_{pq} u_q, \tag{46}$$

with $A^*_{pq} = \Delta t \xi_x \cdot A_{pq}$ and $E^*_{pq} = \Delta t \cdot E_{pq}$. The same steps as described in Section 2.2.1 can be applied and we finally obtain the equation system

$$\left( \delta_{pq} (F^1_{kl} - K^\tau_{kl}) + A^*_{pq} K^\xi_{kl} + E^*_{pq} M_{kl} \right) \hat{u}^i_{ql} = \delta_{pr} \, F^0_{km} \hat{w}^i_{rm}(t^n). \tag{47}$$

Here, $\delta_{pq}$ is the classical Kronecker symbol. The combination of the indices $k$,$l$,$m$ for the degrees of freedom with the indices $p$ and $q$ for the variables can be interpreted as sub-array syntax. Using the sub-array syntax we denote the system matrix

$$Y_{pqkl} = \delta_{pq} (F^1_{kl} - K^\tau_{kl}) + A^*_{pq} K^\xi_{kl} + E^*_{pq} M_{kl}, \tag{48}$$

and formally write the solution of (47) as

$$\hat{u}^i_{ql} = Y^{-1}_{pqkl} \cdot \delta_{pr} \, F^0_{km} \hat{w}^i_{rm}(t^n). \tag{49}$$

Eqn. (47) is a local linear equation system that can be solved for each element $\mathcal{Q}_i$ independently and thus also leads to a locally implicit scheme for $u_i(\xi, \tau)$ and does not need any information from the neighboring elements.

### *2.2.5 General nonlinear hyperbolic systems with stiff source terms*

For the construction of the space-time DG scheme for general nonlinear hyperbolic systems of conservation laws with source terms of the form (7), we also first re-write the system (7) in coordinates of the reference element as follows:

$$\frac{\partial}{\partial \tau} \mathbf{u} + \frac{\partial}{\partial \xi} \mathbf{f}^*(\mathbf{u}) = \mathbf{S}^*(\mathbf{u}), \tag{50}$$

with the modified flux and source function $\mathbf{f}^* = \mathbf{f}^*(\mathbf{u}) = \Delta t \xi_x \mathbf{f}(\mathbf{u})$ and $\mathbf{S}^* = \mathbf{S}^*(\mathbf{u}) = \Delta t \, \mathbf{S}(\mathbf{u})$. We then multiply with the test functions $\Phi_k(\xi, \tau)$, integrate over $\mathcal{Q}_E$ and subsequently integrate the first term containing the time derivative by parts, as in the linear case, in order to obtain

$$[\Phi_k(\xi, 1), \mathbf{u}(\xi, 1)] - [\Phi_k(\xi, 0), \mathbf{u}(\xi, 0)] - \left\langle \frac{\partial}{\partial \tau} \Phi_k, \mathbf{u} \right\rangle + \left\langle \Phi_k, \, \frac{\partial}{\partial \xi} \mathbf{f}^* \right\rangle = \langle \Phi_k, \mathbf{S}^* \rangle \, . \tag{51}$$

Inserting the numerical fluxes in $\tau$ direction as well as the ansatz for the numerical solution (23) into (51) yields the following nonlinear system of equations for the unknowns $\hat{\mathbf{u}}^i_l$

$$[\Phi_k(\xi,1),\Phi_l(\xi,1)]\,\hat{\mathbf{u}}_l^i - [\Phi_k(\xi,0),\Psi_m(\xi)]\,\hat{\mathbf{w}}_m^i(t^n) -$$
$$\left\langle \frac{\partial}{\partial\tau}\Phi_k,\Phi_l \right\rangle \hat{\mathbf{u}}_l^i + \left\langle \Phi_k,\, \frac{\partial}{\partial\xi}\mathbf{f}^*(\Phi_l\,\hat{\mathbf{u}}_l^i) \right\rangle - \left\langle \Phi_k, \mathbf{S}^*(\Phi_l\,\hat{\mathbf{u}}_l^i) \right\rangle = 0. \tag{52}$$

The necessity to solve the local nonlinear system of equations (52) adds further complications to our algorithm compared to the case of linear systems. Due to the locally implicit character of the local space-time DG scheme, the use of a Newton algorithm or other strategies for finding roots of nonlinear equation systems becomes necessary. In this paper, we apply the following strategy: To compute the solution of (52) we first linearize the nonlinear system (50) with respect to the initial condition given by $\mathbf{w}^i(\xi,t^n)$, then we solve the resulting linear equation system (47) exactly using Gauss-Jordan elimination and obtain as result a first guess $\hat{\mathbf{u}}_l^{(i,1)}$ of the solution $\hat{\mathbf{u}}_l^i$ of (52). Linearizing about $\hat{\mathbf{u}}_l^{(i,1)}\Phi_l(\xi,1)$ and solving the resulting linear system again exactly yields the second guess values $\hat{\mathbf{u}}_l^{(i,2)}$. This procedure is usually repeated for a total number of three times. The third guess values $\hat{\mathbf{u}}_l^{(i,3)}$ are then the starting point of a standard multivariate Newton method for nonlinear systems of equations as described e.g. in chapter 9.7 of [44]. We remark that for the solution of (52) the initial guess used as input for the Newton method remains very crucial even for so-called globally convergence Newton methods as described in [44]. We note that for nonlinear systems with stiff source terms, most of the computational time of our algorithm is spent in the solution of (52). Since the authors are not experts in the field of efficiently solving nonlinear systems of equations, there may be other, much more efficient techniques to solve (52). However, the main scope of this article is not to solve (52) efficiently but to validate the general approach.

We finally would like to point out that for the special case $\mathbf{f}(\mathbf{u}) = 0$ the scheme (52) automatically reduces to a standard discontinuous Galerkin method for the nonlinear system of first order ordinary differential equations (ODE)

$$\frac{\partial}{\partial t}\mathbf{u} = \mathbf{S}(\mathbf{u}). \tag{53}$$

At this point, we would like to summarize again the necessary steps for our proposed explicit arbitrary high order accurate finite volume schemes for hyperbolic systems with stiff source terms:

(I) Compute the degrees of freedom $\hat{\mathbf{w}}_l^i(t^n)$ of the weighted essentially non-oscillatory (WENO) reconstruction polynomials $\mathbf{w}_i(\xi,t^n)$ at time $t^n$ from the given cell averages $\bar{\mathbf{u}}_i^n$ of the finite volume scheme using (15)-(19).
(II) Compute the solution $\hat{\mathbf{u}}_l^i$ of the local space-time discontinuous Galerkin

method (52), where the initial condition is given by the reconstructed degrees of freedom $\mathbf{w}_l^i(\xi, t^n)$ at time $t^n$.

(III) Use the solution $\mathbf{u}_i(\xi,\tau) = \Phi_l(\xi,\tau)\hat{\mathbf{u}}_l^i$ to compute the arguments for the source term and the numerical flux in (11) that are needed for the explicit finite volume scheme (10). The integrals appearing in eqn. (11) are computed using classical Gaussian quadrature formulae, see e.g. [48] for details. Update the cell averages according to (10) to the new time $t^{n+1}$ and restart with step (I).

At the end of this section we would like to add a very important remark concerning systems of balance laws: In order to obtain a correct coupling of the flux and the source term inside the local space-time DG scheme (52) together with the finite volume discretization (10), numerical schemes of order of accuracy of at least *two* must be used, which means that the polynomial degree of the basis and test functions $\Phi_k$ must be at least *one*. This is due to the fact that the first order version of the local space-time DG scheme, i.e. the one with polynomial degree zero for the basis and test functions $\Phi_k$, does *not couple* source and flux in the local solution $\mathbf{u}_i(\xi,\tau)$ because the term $<\Phi_k, \frac{\partial}{\partial\xi}\mathbf{f}^*(\Phi_l\hat{\mathbf{u}}_l^i)>$ in (52) vanishes in this case.

## 3 Numerical convergence studies

To assess the convergence behaviour of our method numerically, we solve the following nonlinear hyperbolic system with source terms, for which a non-trivial exact reference solution is known by construction:

$$\begin{aligned}
&\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = -\nu\left(u - u_e\right) + \frac{\partial}{\partial t}u_e + \frac{\partial}{\partial x}\left(\frac{1}{2}v_e^2\right), \\
&\frac{\partial}{\partial t}v + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = -\nu\left(v - v_e\right) + \frac{\partial}{\partial t}v_e + \frac{\partial}{\partial x}\left(\frac{1}{2}u_e^2\right).
\end{aligned} \tag{54}$$

It is easy to see that any differentiable function pair $u_e(x,t), v_e(x,t)$ satisfies eqn. (54). For our convergence studies, we choose the following smooth reference solution:

$$u_e(x,t) = U_0 + A_u \sin\left(kx - \omega t\right), \qquad v_e(x,t) = V_0 + A_v \cos\left(kx - \omega t\right). \tag{55}$$

In particular, we choose the following parameters for the reference solution used in the numerical convergence studies: $U_0 = 4$, $V_0 = 6$, $A_u = 0.1$, $A_v = 0.3$, $k = \omega = 2\pi$. Equation (54) is solved on the computational domain $\Omega = [0;1]$ with periodic boundary conditions. The Courant number is set in the following test cases to CFL $= 0.5$.

First, we assess the capability of our scheme to maintain the balance between

the nonlinear advection operator on the left hand side and the source terms on the right hand side in the non-stiff case for $\nu = 10$. The initial conditions for $u$ and $v$ are in this case $u(x,0) = u_e(x,0)$ and $v(x,0) = v_e(x,0)$. We compute the problem for half a period, i.e. up to the final output time $t = 0.5$. The numerical convergence results obtained for the variable $v$ with the proposed ADER finite volume schemes from second to sixth order of accuracy in space and time are shown in Table 1, where $N_G$ denotes the number of grid cells used to discretize the domain $\Omega$. The errors and the associated convergence rates between two successive grid refinements are shown in $L^1$, $L^2$ and $L^\infty$ norm. The error norms are computed numerically according to

$$\left\| w - v_e \right\|_p = \left( \int_0^1 \left[ w(x,t) - v_e(x,t) \right]^p dx \right)^{\frac{1}{p}}, \tag{56}$$

using Gaussian quadrature rules of appropriate order. We emphasize that the norms are computed comparing the second component of the reconstructed solution against the exact reference solution $v_e$. As an approximation for the infinity norm, we take the maximum of the error obtained in any of the Gaussian integration points.
The results presented in Table 1 show clearly that the method converges with the designed order of accuracy. They furthermore indicate the capability of the method to maintain a good balance between the source terms on the right hand side of the governing equation and the nonlinear convection on the left hand side, respectively.

Second, we assess the accuracy and the robustness of the proposed schemes in the presence of a very stiff source term. Therefore, we choose in this second test case $\nu = 10^8$. Since we know that the stiff relaxation source term in the governing equation (54) will cause the solution to relax to the equilibrium given by the reference solution $u_e, v_e$ from any initial condition, we choose a constant initial condition $u(x,0) = 10$, $v(x,0) = 2$, which is far from the equilibrium $u_e, v_e$ and as a consequence in the first time steps the source is very stiff. Once the equilibrium $u = u_e, v = v_e$ has been reached, the scheme must be able to maintain it. The numerical convergence rates obtained for variable $v$ with ADER finite volume schemes from second to sixth order for this very stiff case ($\nu = 10^8$) are shown in Table 2. From the results we can clearly conclude that the method is at the same time able to treat stiff source terms robustly and maintains an excellent balance between flux divergence and source term at the designed order of accuracy in space and time. To our knowledge, this is the first finite volume scheme ever presented in the research literature on stiff source terms that achieves arbitrary high order of accuracy in space and time. This is, of course, only valid for sufficiently smooth solutions.

Table 1
Numerical convergence rates for the non-stiff case ($\nu = 10$) obtained with ADER finite volume schemes from second to sixth order of accuracy in space and time.

| $N_G$ | $L^1$ | $L^2$ | $L^\infty$ | $\mathcal{O}_{L^1}$ | $\mathcal{O}_{L^2}$ | $\mathcal{O}_{L^\infty}$ |
|---|---|---|---|---|---|---|
| ADER-FV $\mathcal{O}2$, $(M = 1)$. $\nu = 10$ | | | | | | |
| 8 | 3.1079E-02 | 3.3731E-02 | 5.3694E-02 | | | |
| 16 | 6.4558E-03 | 7.8656E-03 | 1.5286E-02 | 2.3 | 2.1 | 1.8 |
| 32 | 1.1027E-03 | 1.5591E-03 | 4.6096E-03 | 2.5 | 2.3 | 1.7 |
| 64 | 1.9859E-04 | 3.2959E-04 | 1.1626E-03 | 2.5 | 2.2 | 2.0 |
| 128 | 2.8261E-05 | 5.5964E-05 | 2.8027E-04 | 2.8 | 2.6 | 2.1 |
| ADER-FV $\mathcal{O}3$, $(M = 2)$. $\nu = 10$ | | | | | | |
| 8 | 4.0967E-03 | 5.3548E-03 | 1.0574E-02 | | | |
| 16 | 5.4254E-04 | 7.0971E-04 | 1.4511E-03 | 2.9 | 2.9 | 2.9 |
| 32 | 6.9171E-05 | 8.9516E-05 | 1.8292E-04 | 3.0 | 3.0 | 3.0 |
| 64 | 8.6332E-06 | 1.1171E-05 | 2.2846E-05 | 3.0 | 3.0 | 3.0 |
| 128 | 1.0816E-06 | 1.3965E-06 | 2.8546E-06 | 3.0 | 3.0 | 3.0 |
| ADER-FV $\mathcal{O}4$, $(M = 3)$. $\nu = 10$ | | | | | | |
| 4 | 1.5831E-02 | 2.0495E-02 | 4.1219E-02 | | | |
| 8 | 1.1568E-03 | 1.3030E-03 | 2.2840E-03 | 3.8 | 4.0 | 4.2 |
| 16 | 6.8436E-05 | 7.6848E-05 | 1.3577E-04 | 4.1 | 4.1 | 4.1 |
| 32 | 4.1739E-06 | 4.6990E-06 | 8.8561E-06 | 4.0 | 4.0 | 3.9 |
| 64 | 2.5792E-07 | 2.9389E-07 | 5.4790E-07 | 4.0 | 4.0 | 4.0 |
| ADER-FV $\mathcal{O}5$, $(M = 4)$. $\nu = 10$ | | | | | | |
| 4 | 1.3054E-02 | 1.5158E-02 | 2.4062E-02 | | | |
| 8 | 4.9450E-04 | 6.3210E-04 | 1.2255E-03 | 4.7 | 4.6 | 4.3 |
| 16 | 1.6178E-05 | 2.1234E-05 | 4.3206E-05 | 4.9 | 4.9 | 4.8 |
| 20 | 5.3607E-06 | 7.0209E-06 | 1.4525E-05 | 4.9 | 5.0 | 4.9 |
| 32 | 5.3921E-07 | 6.8677E-07 | 1.4518E-06 | 4.9 | 4.9 | 4.9 |
| ADER-FV $\mathcal{O}6$, $(M = 5)$. $\nu = 10$ | | | | | | |
| 4 | 8.3790E-03 | 9.9571E-03 | 2.2749E-02 | | | |
| 8 | 1.6979E-04 | 2.0617E-04 | 5.0498E-04 | 5.6 | 5.6 | 5.5 |
| 12 | 1.5335E-05 | 1.8985E-05 | 4.7928E-05 | 5.9 | 5.9 | 5.8 |
| 16 | 2.7810E-06 | 3.4639E-06 | 9.0072E-06 | 5.9 | 5.9 | 5.8 |
| 20 | 7.5279E-07 | 9.5828E-07 | 2.5537E-06 | 5.9 | 5.8 | 5.6 |

Table 2
Numerical convergence rates for the very stiff case ($\nu = 10^8$) obtained with ADER finite volume schemes from second to sixth order of accuracy in space and time.

| $N_G$ | $L^1$ | $L^2$ | $L^\infty$ | $\mathcal{O}_{L^1}$ | $\mathcal{O}_{L^2}$ | $\mathcal{O}_{L^\infty}$ |
|---|---|---|---|---|---|---|
| ADER-FV $\mathcal{O}2$, $(M = 1)$. $\nu = 10^8$ | | | | | | |
| 8 | 2.9784E-02 | 3.0049E-02 | 3.4246E-02 | | | |
| 16 | 6.3522E-03 | 7.2830E-03 | 1.1337E-02 | 2.2 | 2.0 | 1.6 |
| 32 | 5.2567E-04 | 8.5936E-04 | 1.7792E-03 | 3.6 | 3.1 | 2.7 |
| 64 | 1.2096E-04 | 2.1170E-04 | 4.3802E-04 | 2.1 | 2.0 | 2.0 |
| 128 | 1.5717E-05 | 3.8232E-05 | 1.0892E-04 | 2.9 | 2.5 | 2.0 |
| ADER-FV $\mathcal{O}3$, $(M = 2)$. $\nu = 10^8$ | | | | | | |
| 8 | 3.5814E-03 | 5.0870E-03 | 9.2163E-03 | | | |
| 16 | 4.5652E-04 | 6.7004E-04 | 1.2552E-03 | 3.0 | 2.9 | 2.9 |
| 32 | 5.7309E-05 | 8.4607E-05 | 1.6027E-04 | 3.0 | 3.0 | 3.0 |
| 64 | 7.1382E-06 | 1.0613E-05 | 2.0140E-05 | 3.0 | 3.0 | 3.0 |
| 128 | 8.9658E-07 | 1.3275E-06 | 2.5379E-06 | 3.0 | 3.0 | 3.0 |
| ADER-FV $\mathcal{O}4$, $(M = 3)$. $\nu = 10^8$ | | | | | | |
| 4 | 1.4142E-02 | 1.9636E-02 | 3.8569E-02 | | | |
| 8 | 1.0485E-03 | 1.2385E-03 | 2.3951E-03 | 3.8 | 4.0 | 4.0 |
| 16 | 6.4253E-05 | 7.5030E-05 | 1.4553E-04 | 4.0 | 4.0 | 4.0 |
| 32 | 3.9752E-06 | 4.6373E-06 | 9.0331E-06 | 4.0 | 4.0 | 4.0 |
| 64 | 2.4920E-07 | 2.8917E-07 | 5.5709E-07 | 4.0 | 4.0 | 4.0 |
| ADER-FV $\mathcal{O}5$, $(M = 4)$. $\nu = 10^8$ | | | | | | |
| 4 | 1.3054E-02 | 1.5158E-02 | 2.4062E-02 | | | |
| 8 | 4.9450E-04 | 6.3210E-04 | 1.2255E-03 | 4.7 | 4.6 | 4.3 |
| 16 | 1.6179E-05 | 2.1235E-05 | 4.3216E-05 | 4.9 | 4.9 | 4.8 |
| 32 | 5.3935E-07 | 6.8713E-07 | 1.4690E-06 | 4.9 | 4.9 | 4.9 |
| 64 | 2.0147E-08 | 2.5747E-08 | 6.4216E-08 | 4.7 | 4.7 | 4.5 |
| ADER-FV $\mathcal{O}6$, $(M = 5)$. $\nu = 10^8$ | | | | | | |
| 4 | 8.3790E-03 | 9.9571E-03 | 2.2749E-02 | | | |
| 8 | 1.6980E-04 | 2.0617E-04 | 5.0498E-04 | 5.6 | 5.6 | 5.5 |
| 12 | 1.5336E-05 | 1.8986E-05 | 4.7918E-05 | 5.9 | 5.9 | 5.8 |
| 16 | 2.7812E-06 | 3.4641E-06 | 8.9977E-06 | 5.9 | 5.9 | 5.8 |
| 20 | 7.5301E-07 | 9.5840E-07 | 2.5566E-06 | 5.9 | 5.8 | 5.6 |

## 4 Applications

### *4.1 Model system with linear flux and nonlinear source term*

We consider the following class of linear advection systems with non linear relaxation:

$$\begin{cases} \dfrac{\partial}{\partial t}u + \dfrac{\partial}{\partial x}v = 0, \\ \dfrac{\partial}{\partial t}v + \dfrac{\partial}{\partial x}u = -\dfrac{1}{\epsilon}\dfrac{v}{\alpha(u)}, \end{cases} \tag{57}$$

where $\epsilon$ is a positive parameter and $\alpha(u)$ is any given function which satisfies $\alpha(u) > 0 \quad \forall u$ in the domain of interest and does not depend on $\epsilon$. System (57) can be seen as a dimensionless system where $\epsilon$ is the ratio between the characteristic time of the relaxation process over the characteristic time of the pure advection process. If the relaxation process is much faster than the advection process, namely if $\epsilon \ll 1$ which corresponds to the stiff case, it is possible to obtain the asymptotic limit of the system. An asymptotic expansion of the relaxating variable formally reads as

$$v = v_0 + \epsilon v_1 + \mathcal{O}\left(\epsilon^2\right), \tag{58}$$

where $v_0$ and $v_1$ are unknown functions of $x$ and $t$. Here we recall that expansion (58) should not be interpreted as a convergent mathematical series, but as a truncated formal expansion. Injecting (58) into system (57), we find iteratively that $v_0 = 0$ and $v_1 = -\alpha(u)\dfrac{\partial}{\partial x}u$ for an arbitrary small value of $\epsilon$. Thus, as $\epsilon \to 0$, variable $v$ is given by the following equation:

$$v = -\epsilon\alpha(u)\frac{\partial}{\partial x}u + \mathcal{O}\left(\epsilon^2\right), \tag{59}$$

and the asymptotic limit of system (57) reads as

$$\frac{\partial}{\partial t}u = \epsilon\frac{\partial}{\partial x}\left[\alpha(u)\frac{\partial}{\partial x}u\right] + \mathcal{O}\left(\epsilon^2\right), \tag{60}$$

which is a non linear diffusion equation.

**Example 1: $\alpha(u) = 1$.** We consider first the linear case $\alpha(u) = 1$. According to (60), as $\epsilon \to 0$, system (57) reduces to the well-known heat equation:

$$\frac{\partial}{\partial t}u = \epsilon\frac{\partial^2}{\partial x^2}u, \tag{61}$$

where $\epsilon$ plays the role of a (small) diffusion coefficient. Suppose that the domain of interest is $x \in \mathbb{R}$ and that an initial condition $u(x, 0) = u_0(x)$ is given.

Then an analytical solution of equation (61) is known, namely the following Green function:

$$u(x,t) = \frac{1}{\sqrt{4\pi\epsilon t}} \int_{-\infty}^{+\infty} u_0(\xi) e^{-\frac{(x-\xi)^2}{4\epsilon t}} d\xi. \tag{62}$$

We now solve the system (57) up to the final output time $t = 50$ in the computational domain $\Omega = [-\frac{1}{2}; \frac{1}{2}]$ with $\alpha(u) = 1$ using ADER-FV schemes from second to fifth order of accuracy on 100 cells. The local space-time DG scheme for linear systems can be directly applied according to (47). We take a stiffness parameter of $\epsilon = 10^{-4}$ and as initial condition we choose

$$u_0(x) = \begin{cases} 1000 & \text{if } x \leqslant 0, \\ 1 & \text{if } x > 0. \end{cases} \tag{63}$$

The boundary conditions are chosen to be transmissive. The analytical solution of the heat equation (61) with the initial condition (63) is given in terms of the error function $\mathrm{erf}(x)$ at $t = 50$ with $\epsilon = 10^{-4}$ as

$$u(x,50) = \frac{1001}{2} - \frac{999}{2} \cdot \mathrm{erf}(5\sqrt{2}\, x), \tag{64}$$

against which the numerical solutions will be compared. The Courant number is set in all computations to $\mathrm{CFL} = \Delta t/\Delta x = 0.9$. The final output time $t = 50$ is quite large and is reached with the chosen combination of mesh and Courant number after 5556 iterations. Hence, we expect that the low order schemes will add more spurious numerical diffusion in this test case compared to the high order schemes since we compute a large number of time steps. This conjecture is indeed confirmed by our numerical results that are depicted in Fig. 2. We can see that generally all the schemes capture the exact reference solution (64) quite well. However, the higher order schemes produce better results than the lower order methods. This means that even in the diffusion limit of the stiff system (57) higher order schemes may produce better results than lower order methods. We finally would like to emphasize that our numerical results have been obtained using a standard *explicit* one-step high order finite volume scheme, where we only use a particular procedure in order to predict the local time-evolution of the reconstructed polynomials. This is achieved via our new local space-time discontinuous Galerkin scheme proposed in this article. The rest of the finite volume scheme is standard.

**Example 2: $\alpha(u) = u(1-u)$.** We now consider a special non linear case, namely $\alpha(u) = u(1-u)$. Hypothesis $\alpha(u) > 0$ is valid if and only if $u \in ]0;1[$. In this region, according to (60), as $\epsilon \to 0$, system (57) reduces to the following

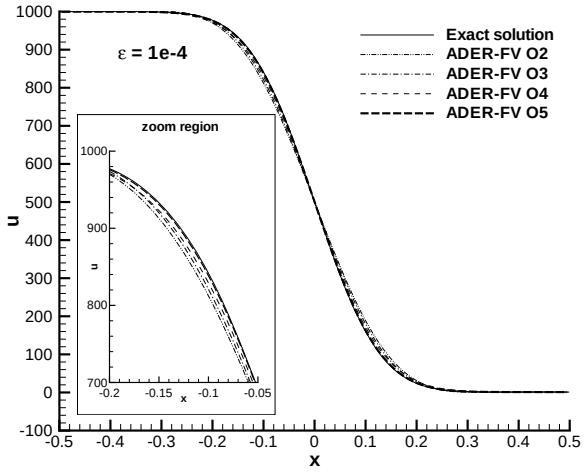


Fig. 2. Exact and numerical solutions obtained on 100 cells using ADER-FV schemes of second to fifth order of accuracy in space and time for the model system with linear flux and stiff linear source ($\epsilon = 10^{-4}$) at time $t = 50$.

non linear diffusion equation:

$$\frac{\partial}{\partial t}u = \epsilon \frac{\partial}{\partial x}\left[u(1-u)\frac{\partial}{\partial x}u\right]. \tag{65}$$

Suppose that the domain of interest is $\Omega = [-0.5; 0.5]$ for a given $L > 0$, and that Neumann boundary conditions $\left(\frac{\partial}{\partial x}u\right)(-0.5, t) = \left(\frac{\partial}{\partial x}u\right)(0.5, t) = 0$ are imposed $\forall t > 0$. Then, for the initial condition we use

$$u(x,0) = \begin{cases} 1^- \text{ if } x \leqslant 0, & \text{where } 1^- = \lim_{\delta \to 0} 1 - \delta, \\ \\ 0^+ \text{ if } x > 0, & \text{where } 0^+ = \lim_{\delta \to 0} \delta, \end{cases} \tag{66}$$

where for small $\delta > 0$, an analytical solution of equation (65) is known, namely:

$$u(x,t) = \min(1, \max(0, \frac{1}{2}\left(1 - \frac{x}{\sqrt{\epsilon t}}\right))). \tag{67}$$

We solve this test problem for $\delta = 10^{-6}$ with ADER-FV schemes of second, third and fifth order of accuracy using 100 cells in the computational domain $\Omega$ up to time $t = 10$ with $\epsilon = 10^{-3}$ and a Courant number of CFL = 0.25. The numerical results and the exact reference solution for the stiff limit are depicted in Fig. 3. We note that all methods agree very well with the reference solution and that even in this case, with discontinuities in the first derivative of the solution, we can clearly see an improvement with increasing order of accuracy.

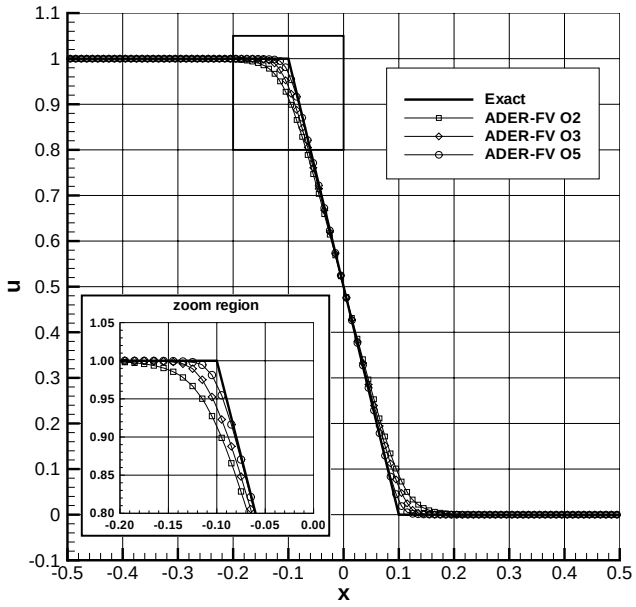


Fig. 3. Exact and numerical solutions obtained on 100 cells using ADER-FV schemes of second, third and fifth order of accuracy in space and time for the model system with linear flux and stiff nonlinear source ($\epsilon = 10^{-3}$) at time $t = 10$.

### *4.2 Model system with nonlinear flux and linear source term*

We now consider the following class of non linear advection systems with linear relaxation:

$$\begin{cases} \dfrac{\partial}{\partial t}u + \dfrac{\partial}{\partial x}v = 0, \\ \\ \dfrac{\partial}{\partial t}v + \dfrac{\partial}{\partial x}f(u) = -\dfrac{1}{\epsilon}v, \end{cases} \tag{68}$$

where $\epsilon$ is a positive parameter and $f(u)$ is any given function which satisfies $f'(u) \geq 0 \quad \forall u$ in the domain of interest and does not depend on $\epsilon$. Injecting asymptotic expansion (58) into system (68), we find iteratively that $v_0 = 0$ and $v_1 = -f'(u)\dfrac{\partial}{\partial x}u$ for an arbitrary small value of $\epsilon$. Thus, as $\epsilon \to 0$, variable $v$ is given by the following equation:

$$v = -\epsilon f'(u)\frac{\partial}{\partial x}u + \mathcal{O}\left(\epsilon^2\right), \tag{69}$$

and the asymptotic limit of system (68) reads as

$$\frac{\partial}{\partial t}u = \epsilon\frac{\partial}{\partial x}\left[f'(u)\frac{\partial}{\partial x}u\right] + \mathcal{O}\left(\epsilon^2\right), \tag{70}$$

which is a non linear diffusion equation. Note that in the previous section, we had $\alpha(u) > 0$, while here we have $f'(u) \geq 0$.

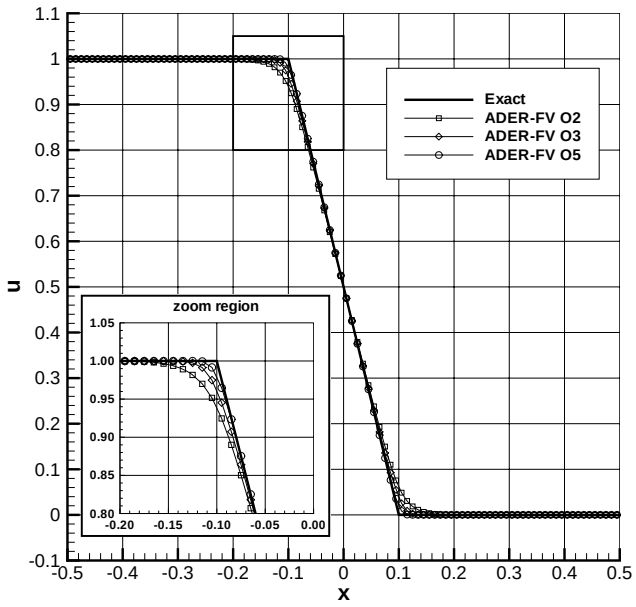


Fig. 4. Exact and numerical solutions obtained on 100 cells using ADER-FV schemes of second, third and fifth order of accuracy in space and time for the model system with nonlinear flux and stiff linear source ($\epsilon = 10^{-3}$) at time $t = 10$.

**Example 3: $f'(u) = u(1-u)$.** We consider the special non linear case $f'(u) = u(1-u)$, which leads to $f(u) = \frac{1}{2}u^2 - \frac{1}{3}u^3$. The hypothesis $f'(u) > 0$ is verified if and only if $u \in [0;1]$. In this region, according to (70), as $\epsilon \to 0$, system (68) reduces to the non linear diffusion equation (65). Suppose that the domain of interest is $x \in [-0.5; 0.5]$ for a given $L > 0$, and that Neumann boundary conditions $\left(\frac{\partial}{\partial x}u\right)(-0.5,t) = \left(\frac{\partial}{\partial x}u\right)(0.5,t) = 0$ are imposed $\forall t > 0$. Then, an analytical solution of equation (65) is (67), which is compatible with the initial condition:

$$u(x,0) = \begin{cases} 1^- \text{ if } x \leqslant 0, & \text{where } 1^- = \lim_{\delta \to 0} 1 - \delta, \\ \\ 0^+ \text{ if } x > 0, & \text{where } 0^+ = \lim_{\delta \to 0} \delta, \end{cases} \tag{71}$$

We solve this test problem for $\delta = 10^{-4}$ with ADER-FV schemes of second, third and fifth order of accuracy using 100 cells in the computational domain $\Omega$ up to time $t = 10$ with $\epsilon = 10^{-3}$ and a Courant number of CFL = 0.25. The numerical results and the exact reference solution for the stiff limit are depicted in Fig. 4. As in the previous test problem, all methods agree again very well with the reference solution and an improvement with increasing order of accuracy is also visible.

In this section we apply our method to the Euler equations of compressible gas dynamics with stiff friction. The full Euler system with friction reads as (7) with the vector of conservative variables $\mathbf{u}$, the flux $\mathbf{f}(\mathbf{u})$ and the (stiff) source term $\mathbf{S}(\mathbf{u})$ as

$$\mathbf{u} = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix}, \qquad \mathbf{f}(\mathbf{u}) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ u(\rho E + p) \end{pmatrix}, \qquad \mathbf{S}(\mathbf{u}) = -\nu \begin{pmatrix} 0 \\ \rho u \\ \rho u^2 \end{pmatrix}. \tag{72}$$

The system still needs to be closed by an equation of state (EOS) of the form $p = p(\mathbf{u})$. For the following numerical calculations we consider a computational domain $\Omega = [0; 1]$ with the Dirichlet boundary conditions $\mathbf{u}(0, t) = \mathbf{u}(0, 0)$ and $\mathbf{u}(1, t) = \mathbf{u}(1, 0)$ and the initial condition

$$\mathbf{u}(x, 0) = \begin{cases} (1.65,\ 0,\ 5.039849068) \text{ if } x \leqslant 0.25, \\ (0.01,\ 0,\ 0.003962233) \text{ if } x > 0.25. \end{cases} \tag{73}$$

For the stiffness parameter $\nu$ we take

$$\nu(x, t) = \begin{cases} 0 & \text{if } x \leqslant 0.25, \\ 1500 & \text{if } x > 0.25, \end{cases} \tag{74}$$

which means that we solve the Euler equations *without* any friction in the left quarter of $\Omega$ ($x \in [0; 0.25]$) and with *stiff* friction in the region $x \in [0.25; 1]$. This setup corresponds to an interface of an inviscid compressible gas with a porous medium into which the gas may penetrate. For all the following computations we set the Courant number to CFL = 0.9.

**Isentropic Euler system with stiff friction** Under the assumption that the flow is completely isentropic, we can write the equation of state as

$$p(\mathbf{u}) = k\rho^{\gamma}. \tag{75}$$

In this case, the pressure does not depend on the total energy $\rho E$ and thus the energy equation in (7) and (72) can be omitted. We set $k = 1$ and $\gamma = 1.4$. We solve (7) with (72) and the EOS (75) up to $t = 2.0$ using 100 cells and ADER-FV schemes from second to fourth order of accuracy. The reference solution is computed with a second order ADER-FV scheme on 10000 cells. The results are depicted in Fig. 5a. We clearly see that on a fixed grid the diffusion limit of the Euler equations is captured better by the higher order schemes. Although the results of the second order method are still of acceptable accuracy, we nevertheless observe that the second order method adds too much numerical diffusion. A very similar testcase has previously been proposed by

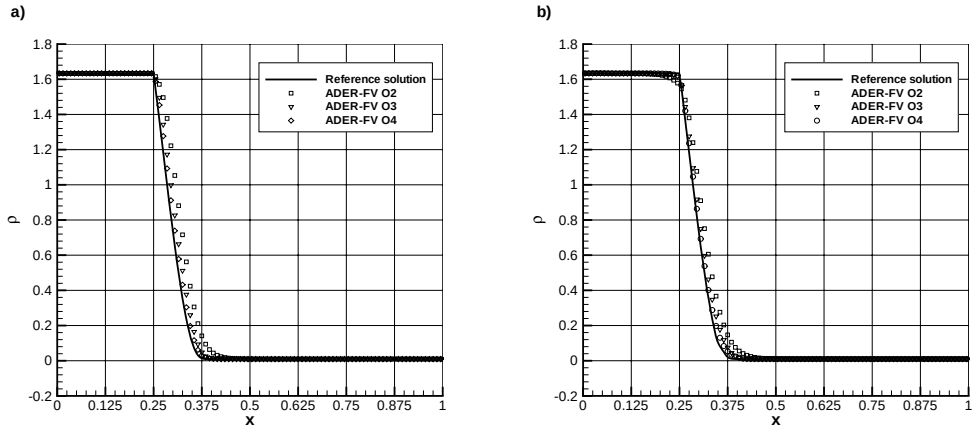


Fig. 5. Reference solution and numerical solutions at time $t = 2$ obtained on 100 cells using ADER-FV schemes of second, third and fourth order of accuracy in space and time for the Euler system with stiff friction. a) Isentropic Euler system (left) and b) Full Euler system with ideal gas EOS (right).

Bouchut *et al.* [5]. Unfortunately they did not specify all parameters of their test problem. Especially, the initial and boundary conditions were not given, so it was not possible to compute exactly the same test case. However, the results are qualitatively similar compared to ours.

**Full Euler system with stiff friction** Using the ideal gas law, the equation of state reads as

$$p(\mathbf{u}) = (\gamma - 1)(\rho E - \frac{1}{2}\rho u^2), \tag{76}$$

where $\gamma = 1.4$. In this case, we must consider the full Euler system including the energy equation. In the initial condition (73) the total energy $\rho E$ is chosen such that the pressure according to (76) is equal to the pressure obtained in the isentropic case from eqn. (75). We therefore expect the results to be very similar to the previous ones. Hence, we solve (7) with (72) and the EOS (76) up to $t = 2.0$ using the initial condition (73). The computational domain is discretized with 100 cells using ADER-FV schemes from second to fourth order of accuracy. The reference solution is computed again with a second order ADER-FV scheme on 10000 cells. The results are depicted in Fig. 5b. Similar to the isentropic case, we observe that also the diffusion limit of the full Euler equations is captured better by the higher order schemes.

We finally would like to remark that even some first order numerical methods designed for stiff systems of balance laws may encounter problems with this test case for the full Euler equations since they may produce negative values for the total energy $\rho E$.

The relaxation system of Jin and Xin [30] reads as follows:

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{u} + \dfrac{\partial}{\partial x}\mathbf{v} = 0, \\ \dfrac{\partial}{\partial t}\mathbf{v} + A\dfrac{\partial}{\partial x}\mathbf{u} = -\dfrac{1}{\epsilon}\left(\mathbf{v} - \mathbf{f}(\mathbf{u})\right), \end{cases} \tag{77}$$

where $\mathbf{f}(\mathbf{u})$ is a given function, $\epsilon$ is a positive arbitrary small parameter and $A$ is a constant matrix. The advection part of this system is linear, while the relaxation part is not linear in general, due to the presence of $\mathbf{f}(\mathbf{u})$. It is easy to check that the asymptotic limit of system (77) is:

$$\frac{\partial}{\partial t}\mathbf{u} + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{u}) = \epsilon\frac{\partial}{\partial x}\left[\Big(A - J^2(\mathbf{u})\Big)\frac{\partial}{\partial x}\mathbf{u}\right], \tag{78}$$

where $J = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}$. System (78) is a hyperbolic system of conservation laws with non linear diffusion if and only if:

$$A \geq J^2(\mathbf{u}) \quad \forall \mathbf{u}. \tag{79}$$

We now apply our proposed ADER finite volume schemes for hyperbolic systems with stiff source terms to the relaxation system of Jin and Xin with the following definitions of the vector $\mathbf{u}$ and the flux function $\mathbf{f}(\mathbf{u})$ appearing in the source term:

$$\mathbf{u} = (\rho, \rho u, \rho E), \qquad \mathbf{f}(\mathbf{u}) = \Big(\rho u, \rho u^2 + p, u(\rho E + p)\Big), \tag{80}$$

with the equation of state that closes the system,

$$p = (\gamma - 1)\left(\rho E - \frac{1}{2}\rho u^2\right). \tag{81}$$

With this choice eqn. (77) converges to the compressible Euler equations in the stiff limit. The stiffness parameter $\nu = 1/\epsilon$ is set in all the following test cases to $\nu = 10^{12}$, which leads to a very stiff source term. The matrix $A$ is chosen to be the simple diagonal matrix $A = \mathrm{diag}(a_m, a_m, a_m)$, which is kept constant in space and time.

**Shock tube problems** We consider initial value problems for (77) where the initial condition for $\mathbf{u}$ has the form

$$\mathbf{u}(x,0) = \begin{cases} (\rho_L, (\rho u)_L, (\rho E)_L) & \text{if } x \leqslant x_c, \\ (\rho_R, (\rho u)_R, (\rho E)_R) & \text{if } x > x_c. \end{cases} \tag{82}$$

Table 3
Initial states left and right, simulation end times and initial position $x_c$ of the discontinuity for the 1D shock tube problems computed with the relaxation system of Jin and Xin.

| Case | $\rho_L$ | $u_L$ | $p_L$ | $\rho_R$ | $u_R$ | $p_R$ | $t_{\text{end}}$ | $x_c$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1.0 | 0.75 | 1.0 | 0.125 | 0.0 | 0.1 | 0.20 | 0.5 |
| 2 | 1.0 | -2.0 | 0.4 | 1.0 | 2.0 | 0.4 | 0.15 | 0.5 |
| 3 | 0.445 | 0.698 | 3.528 | 0.5 | 0.0 | 0.571 | 0.14 | 0.5 |
| 4 | 5.99924 | 19.5975 | 460.895 | 5.99242 | -6.19633 | 46.0950 | 0.035 | 0.4 |
| 5 | 1.0 | 0.0 | 1000. | 1.0 | 0.0 | 0.01 | 0.012 | 0.5 |
| 6 | 1.0 | -19.59745 | 1000. | 1.0 | -19.59745 | 0.01 | 0.012 | 0.8 |

The initial condition for $\mathbf{v}$ is simply $\mathbf{v}(x,0) = 0$. For the Euler equations of compressible gas dynamics the exact solution of those Riemann problems can be computed analytically and will serve in the following for validation of our numerical method when applied to the relaxation system of Jin and Xin. We compute the solution of the initial value problem (77) and (82) using second to fourth order ADER-FV schemes for six different cases of shock tube problems. All initial conditions as well as the final output times $t_{\text{end}}$ and the initial position of the discontinuity $x_c$ are listed in Table 3. The values $a_m$ defining the matrix $A$ are given for each test case in Table 4. For all computations we use a constant Courant number of CFL = 0.75. The exact solution and the numerical solutions obtained by our proposed method are shown for all six shock tube problems in Figures 6-8, where also the number of mesh cells is indicated. For most of the test cases we note an excellent agreement with the exact solution and most of the numerical solutions are monotone, thanks to the nonlinear WENO reconstruction procedure. We note that the reconstruction is done in the characteristic variables of the compressible Euler equations and *not* in the characteristic variables of the advection operator of the relaxation system of Jin and Xin. This is necessary to suppress unphysical oscillations. Using this particular characteristic reconstruction, small spurious oscillations are only visible for shock tube problems number two and four. We note that these test cases can be even difficult to compute with standard high order finite volume schemes for the compressible Euler equations. Much more numerical difficulties arise in the relaxation system of Jin and Xin due to the very stiff source term. However, the numerical results confirm that our method produces essentially non-oscillatory results, maintains high accuracy even for hyperbolic systems with stiff source terms and has the correct behaviour in the stiff limit.

Table 4
Entries $a_m$ of the diagonal matrix $A$ for the 1D shock tube problems.

| Case | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $a_m$ | 7 | 12 | 12 | 900 | 1500 | 9000 |

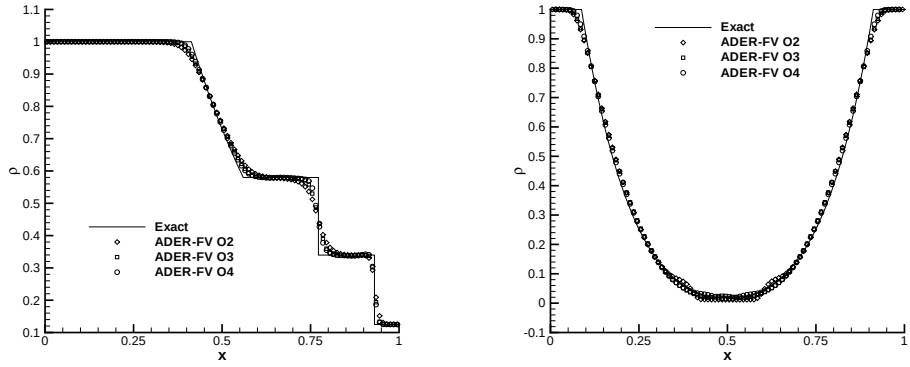


Fig. 6. Exact and numerical solutions for shock tube problems number one (left) and two (right), obtained with the relaxation system of Jin and Xin on 100 cells.

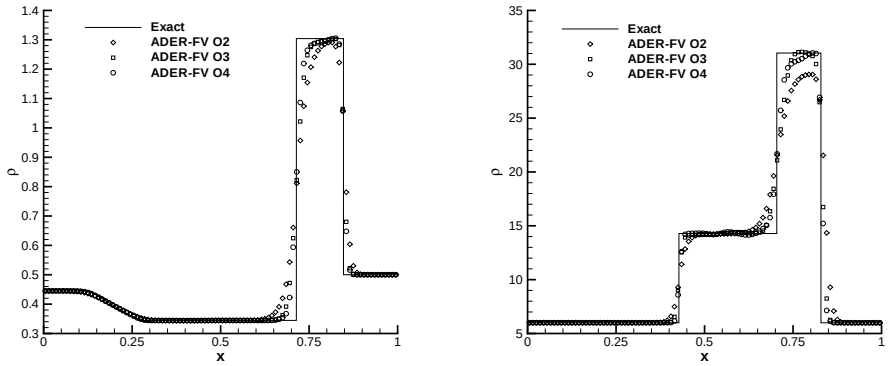


Fig. 7. Exact and numerical solutions for shock tube problems number three (left) and four (right), obtained with the relaxation system of Jin and Xin on 100 cells.

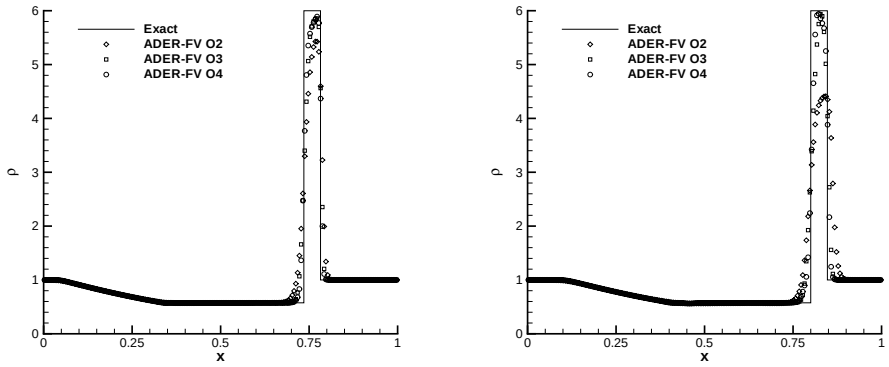


Fig. 8. Exact and numerical solutions for shock tube problems number five (left) and six (right), obtained with the relaxation system of Jin and Xin on 200 cells.

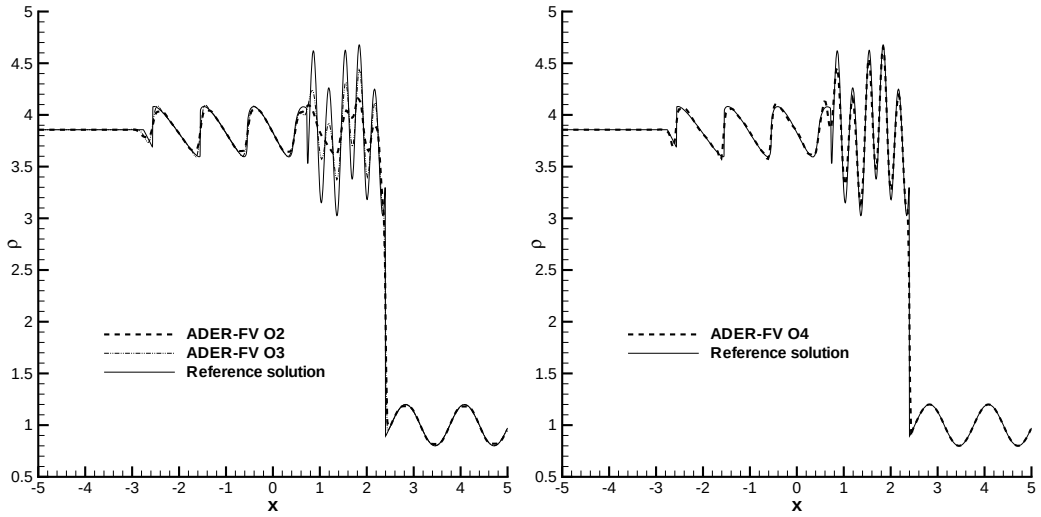


Fig. 9. Reference solution and numerical solutions at $t = 1.8$ for the shock-density interaction test case obtained with the relaxation system of Jin and Xin on 400 cells using ADER-FV schemes of second, third and fourth order of accuracy.

**Shock-density interaction** We now consider a test problem proposed originally by Shu and Osher [46] for the compressible Euler equations in the more general framework of the stiff relaxation system of Jin and Xin (77) in order to emphasize the advantages of high order methods. The computational domain is $\Omega = [-5; 5]$ and the initial condition for $\mathbf{u}$ is given by

$$(\rho, u, p)\,(x, 0) = \begin{cases} (3.8571, 2.6294, 10.333) & \text{if } x < -4, \\ (1 + 0.2\sin(5x),\ \ 0,\ \ 1) & \text{if } x \geq -4, \end{cases} \tag{83}$$

Furthermore we set $\mathbf{v}(x, 0) = 0$. This leads to a shock wave with Mach number $M = 3$ running into the sinusoidal density fluctuation. The interaction of the shock with the density fluctuation generates sound waves and high-frequency entropy fluctuations that are very difficult to capture with low order schemes on coarse meshes. In Fig. 9 we show the numerical results obtained with ADER-FV schemes from second to fourth order of accuracy on 400 cells at the final output time $t = 1.8$. We choose $a_m = 25$ and a Courant number of CFL $= 0.75$.

One can clearly see that the second order method is not at all able to resolve the high frequency entropy waves. The third order scheme already resolves the whole frequency content but is still too dissipative since the amplitudes of the entropy waves are not yet captured, see Fig. 9 on the left. Only the fourth order scheme is able to resolve the whole solution quite well on this relatively coarse mesh, see Fig. 9 on the right. The reference solution was computed using a second order TVD finite volume scheme for the compressible Euler equations on 10000 cells.

### 4.5 The scalar model problem of LeVeque and Yee

The model problem proposed by LeVeque and Yee [34] is a scalar linear advection problem with a non linear reaction source term, which can be stiff. The governing PDE reads as

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}u = -\nu u\,(u-1)\left(u-\frac{1}{2}\right), \tag{84}$$

where $\nu$ is a given positive coefficient. The computational domain is $\Omega = [0;1]$ with transmissive boundary conditions. The following initial condition is considered:

$$u(x,0) = \begin{cases} 1 & \text{if } x \leqslant 0.3, \\ 0 & \text{if } x > 0.3. \end{cases} \tag{85}$$

With this particular initial condition, the source term is zero. Thus, an analytical solution of problem (84)-(85) is known, namely: $u(x,t) = u(x-t,0)$. which means that the initial profile of $u$ is advected with constant speed 1. LeVeque and Yee have pointed out that neither a Mac Cormack predictor-corrector method, nor a Strang splitting method give the physically correct advection speed in the stiff case. We further note that also a standard ADER finite volume scheme [54,50] using the usual Cauchy-Kovalewski procedure instead of our local space-time discontinuous Galerkin scheme to compute $u_i(\xi,\tau)$ from the reconstructed polynomials $w_i(\xi,t^n)$ will produce the wrong advection speed. We now solve the above mentioned test problem up to $t = 0.3$ using 100 cells and a Courant number of CFL $= 0.75$. Following LeVeque and Yee we take the following values for the stiffness parameter: $\nu = 1$, $\nu = 10$, $\nu = 100$, $\nu = 1000$. The numerical results obtained with our new ADER-FV schemes from second to sixth order of accuracy are depicted in Fig. 10 for the non-stiff as well as for the stiff case. We note an excellent agreement with the exact solution in all cases. In particular, the advection speed of $a = 1$ is captured correctly. Furthermore, we can clearly see that our numerical solution is essentially non-oscillatory and that the resolution of the discontinuity is improved in the non-stiff case when using higher order schemes. For the stiff case, we observe less numerical diffusion than in the non-stiff case. This is due to the reaction source term, which has two stable equilibrium solutions at $u = 0$ and at $u = 1$. Any numerical dissipation generated by the numerical scheme will lead to a smearing of the discontinuity and will subsequently lead to intermediate values of $u$ that do not correspond to either of the stable equilibria. The stiff reaction source term will immediately try to push the solution back towards the closest equilibrium, which in the end leads to a generally sharper profile for the stiff case compared to the non-stiff case.

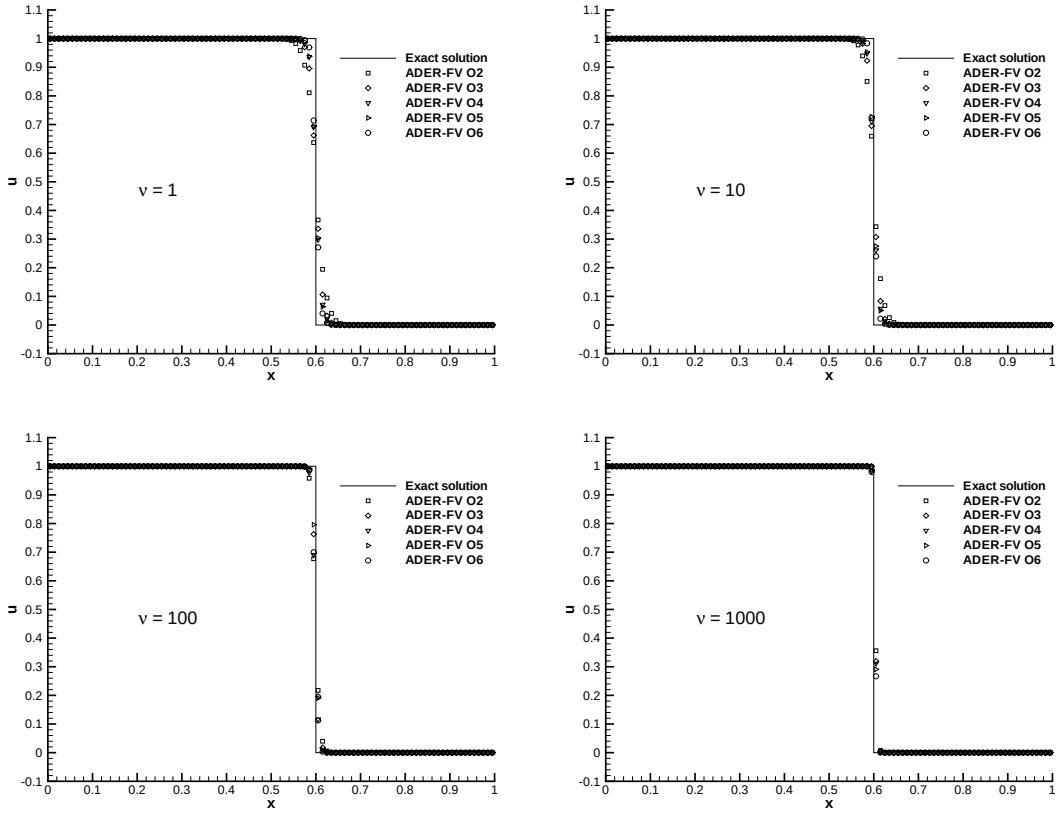


Fig. 10. Exact solution and numerical solutions for the model problem of LeVeque and Yee at $t = 0.3$ using 100 cells for the non-stiff case (top row) and the stiff case (bottom row). $\nu = 1$ (top left), $\nu = 10$, (top right), $\nu = 100$ (bottom left), $\nu = 1000$, (bottom right).

## 5 Summary and conclusions

In this article we have developed a new unsplit explicit essentially non-oscillatory one-step finite volume scheme of arbitrary high order of accuracy in space and time for nonlinear hyperbolic systems with stiff source terms. In continuity with previous work of the authors on schemes of arbitrary high order of accuracy in space and time, we call our new method also ADER (**a**rbitrary high order **der**ivatives) finite volume scheme. The essentially non-oscillatory character of the method is obtained via a special nonlinear WENO reconstruction procedure that produces entire reconstruction polynomials in terms of orthogonal basis functions, instead of point values that are generated usually by standard WENO schemes. The spatial reconstruction polynomials obtained from the particular WENO reconstruction operator are used as initial conditions for a local space-time discontinuous Galerkin scheme which solves an initial value problem for the governing PDE *locally* inside each element without considering the neighboring elements. The local space-time DG scheme leads in general to a *local* system of nonlinear equations that has to be solved

individually for each element. In this article, we use a standard globally convergent Newton algorithm [44] to solve the problem in the nonlinear case. As initial guess we solve an associated linear problem after linearization of the governing PDE. In the case of a purely linear governing PDE, the linear equation system resulting from the local space-time DG scheme can be solved exactly. We emphasize that the local space-time DG scheme is the only locally implicit part appearing in the proposed ADER-FV schemes. The Newton algorithm only has to iterate over the space-time degrees of freedom inside *one element* and does not have to consider the degrees of freedom in the neighboring elements. The resulting ADER finite volume scheme, built upon the solution of the local space-time DG scheme, is then completely explicit. Compared to previous ADER-FV schemes for hyperbolic systems with non-stiff source terms, the new local space-time DG scheme has replaced the usual Cauchy-Kovalewski procedure, which is not able to treat stiff problems. The rest of the scheme remains the same.

We have shown via asymptotic analysis that the new local space-time DG scheme is able to capture correctly the stiff limit in the case of a linear scalar PDE with stiff source term. Subsequently, the boundedness of the space-time integral of the source term $\bar{S}_i$ has been demonstrated in the stiff limit. From this result, it has been deduced that the resulting ADER finite volume scheme is also consistent with the stiff limit for any bounded time step $\Delta t$.
Numerical convergence studies for our proposed method have been carried out up to sixth order of accuracy in space and time for a nonlinear hyperbolic system with source terms. A non-stiff case and also a very stiff case have been considered. The ADER-FV schemes then have been applied to several stiff linear and nonlinear model systems for which an analytical reference solution is available. In all the cases our schemes were able to capture very well the stiff limit of these equations and to maintain also the monotonicity of the solution, even using schemes of very high order of accuracy. Usually, the results obtained with the higher order schemes were better than the results obtained with the lower order methods, even for test problems with discontinuities in the state or in the derivatives. We emphasize that for our explicit one-step finite volume method even for very stiff problems the time step is only restricted by the standard CFL stability condition and *not* by the stiffness of the source term.
We then have shown an application of ADER-FV schemes of up to fourth order of accuracy to the isentropic and the full Euler equations with stiff friction, where a reference solution has been computed on a very fine mesh. As in the previous cases of the model systems, the ADER-FV schemes also captured the stiff diffusion limit of the Euler equations correctly. Again, the higher order schemes produced better results with less numerical diffusion than the low order schemes.
Subsequently we have applied our new algorithm to a very interesting stiff system of hyperbolic balance laws, namely the stiff relaxation system of Jin

and Xin [30], which rewrites a nonlinear hyperbolic system as an enlarged linear hyperbolic system with stiff source terms. For the application of our high order algorithm we have chosen the relaxation system corresponding to the compressible one-dimensional Euler equations. We have computed a large number of one-dimensional shock tube problems, for which exact reference solutions are known. For the relaxation system of Jin and Xin, a particularly large value has been chosen for the stiffness parameter in order to minimize the dissipation induced by the right hand side of the limit equation for $\mathbf{u}$. We were able to show the benefit of very high order numerical methods even for the stiff relaxation system of Jin and Xin on the standard shock-density interaction test problem proposed originally by Shu and Osher for the compressible Euler equations [46]. To our knowledge, up to now no numerical method of accuracy greater than two in space and time has been applied yet to the stiff relaxation system of Jin and Xin.
The most difficult test case treated in this article is the stiff nonlinear scalar test problem propose by LeVeque and Yee [34]. It was shown in [34] that robust standard schemes such as the Mac Cormack scheme or the Strang splitting produce oscillatory results for small values of the stiffness parameter and that these methods do not obtain the correct speed of the discontinuity in the very stiff case. However, in this article, for all values of the stiffness parameter studied by LeVeque and Yee, our new ADER-FV schemes were able to produce monotone results with the correct speed of the discontinuity, even in the very stiff case.

At this point, we would like to summarize the advantages and the disadvantages of our proposed ADER-FV scheme for hyperbolic systems with stiff source terms. Among the clear advantages of our scheme is the fact that it can reach any desired order of accuracy greater or equal two in space and time simultaneously. To our knowledge, this has not yet been achieved by any other numerical method for hyperbolic systems with stiff source terms. Furthermore, the unsplit finite volume discretization (10) mimics the underlying physics of the governing equation since it is based directly on an integral formulation of the governing PDE (7). The same is true for the (also unsplit) local space-time DG scheme, which directly solves a weak formulation of the local initial value problem (7) in space-time. The underlying $L^2$-projection of the local space-time DG scheme that leads to the Galerkin orthogonality property guarantees the optimality of the numerical solution $\mathbf{u}_i(\xi,\tau)$ in $L^2$-norm. Since the local space-time DG scheme can be solved individually for each element $\mathcal{Q}_i$ we suppose that it is easier to solve than a globally implicit scheme that must take into account all $Q_i \in \Omega$. Numerical evidence has shown that our scheme seems also to be consistent with the stiff limit of the governing PDE. A final advantage is the explicit one-step character of our finite volume discretization that is based on the solution $\mathbf{u}_i(\xi,\tau)$ of the local space-time DG method. However, a first disadvantage of our proposed method is the high computational effort associated with the Newton algorithm that has to be used in the case of non-

linear systems. Compared to the rest of the algorithm, the Newton solver is by far the most expensive part of the scheme. Future work has to be done to make the Newton algorithm for the nonlinear case more efficient and robust. Other, more advanced, techniques can be tried in the future. Seen from a theoretical point of view, another disadvantage of our scheme may be the fact that the first order version of the method does not work since it does not provide the correct coupling of source terms and fluxes which only comes in via a higher order discretization in space and time.

Further extensions and applications of the proposed ADER-FV schemes for stiff problems will concern the extension to multiple space dimensions and the application to two-fluid flow. Of particular interest will be the application to colliding plasma flows with stiff friction and very stiff temperature relaxation. Further applications may also involve chemically reacting flows and kinetic models for compressible gas dynamics. Last but not least, we would like to mention that for the special case $\mathbf{f}(\mathbf{u}) = 0$ our local space-time DG scheme reduces to a solver of stiff nonlinear systems of ordinary differential equations (ODE) allowing the development of irregular solutions in time.

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## A Reconstruction basis functions

The rescaled Legendre polynomials, which constitute an orthogonal basis on the unit interval $I = [0; 1]$, are given up to polynomial degree $M = 5$ by:

$$\begin{aligned}
\Psi_0(\xi) &= 1 \\
\Psi_1(\xi) &= 2\,\xi - 1 \\
\Psi_2(\xi) &= 6\,\xi^2 - 6\,\xi + 1 \\
\Psi_3(\xi) &= 20\,\xi^3 - 30\,\xi^2 + 12\,\xi - 1 \\
\Psi_4(\xi) &= 70\,\xi^4 - 140\,\xi^3 + 90\,\xi^2 - 20\,\xi + 1 \\
\Psi_5(\xi) &= 252\,\xi^5 - 630\,\xi^4 + 560\,\xi^3 - 210\,\xi^2 + 30\,\xi - 1
\end{aligned} \tag{A.1}$$

## B Mass-, flux- and stiffness matrices for the local space-time DG method for the linear case

For polynomial degree $M = 1$, the mass matrix of the local space-time Discontinuous Galerkin scheme is

$$M_{kl} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{9} \end{pmatrix}, \tag{B.1}$$

the flux matrices for $\tau = 0$ and $\tau = 1$ are

$$F^0_{kl} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \\ -1 & 0 \\ 0 & -\frac{1}{3} \end{pmatrix}, \qquad F^1_{kl} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 1 & 0 & 1 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}, \tag{B.2}$$

and the temporal and spatial stiffness matrices are given by

$$K^\tau_{kl} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 \end{pmatrix}, \qquad K^\xi_{kl} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{B.3}$$

For polynomial degree $M = 2$, the mass matrix of the local space-time Discontinuous Galerkin scheme is

$$M_{kl} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{9} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{15} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{15} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{25} \end{pmatrix}, \tag{B.4}$$

the flux matrices for $\tau = 0$ and $\tau = 1$ are

$$F^0_{kl} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \\ -1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{5} \\ 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}, \qquad F^1_{kl} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} & 0 & 0 & \frac{1}{5} & 0 & 0 & \frac{1}{5} \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} & 0 & 0 & \frac{1}{5} & 0 & 0 & \frac{1}{5} \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} & 0 & 0 & \frac{1}{5} & 0 & 0 & \frac{1}{5} \end{pmatrix}, \tag{B.5}$$

and the temporal and spatial stiffness matrices are given by

$$K^\tau_{kl} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{5} & 0 & 0 & 0 \end{pmatrix}, \qquad K^\xi_{kl} = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{B.6}$$