

# THE PERIODIC MAGNETIC SCHRÖDINGER OPERATORS: SPECTRAL GAPS AND TUNNELING EFFECT

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ABSTRACT. A periodic Schrödinger operator on a noncompact Riemannian manifold  $M$  such that  $H^1(M, \mathbb{R}) = 0$  endowed with a properly discontinuous cocompact isometric action of a discrete group are considered. Under some additional conditions on the magnetic field existence of an arbitrary large number of gaps in the spectrum of such an operator in the semiclassical limit is established. The proofs are based on the study of the tunneling effect in the corresponding quantum system.

## INTRODUCTION

Let  $M$  be a noncompact oriented manifold of dimension  $n \geq 2$  equipped with a properly discontinuous action of a finitely generated, discrete group  $\Gamma$  such that  $M/\Gamma$  is compact. Suppose that  $H^1(M, \mathbb{R}) = 0$ , i.e. any closed 1-form on  $M$  is exact. Let  $g$  be a  $\Gamma$ -invariant Riemannian metric and  $\mathbf{B}$  a real-valued  $\Gamma$ -invariant closed 2-form on  $M$ . Assume that  $\mathbf{B}$  is exact and choose a real-valued 1-form  $\mathbf{A}$  on  $M$  such that  $d\mathbf{A} = \mathbf{B}$ .

Thus, one has a natural mapping

$$u \mapsto ih du + \mathbf{A}u$$

from  $C_c^\infty(M)$  to the space  $\Omega_c^1(M)$  of smooth, compactly supported one-forms on  $M$ . The Riemannian metric allows to define scalar products in these spaces and consider the adjoint operator

$$(ih d + \mathbf{A})^* : \Omega_0^1(M) \rightarrow C_0^\infty(M).$$

A Schrödinger operator with magnetic potential  $\mathbf{A}$  is defined by the formula

$$H^h = (ih d + \mathbf{A})^*(ih d + \mathbf{A}).$$

Here  $h > 0$  is a semiclassical parameter, which is assumed to be small.

Choose local coordinates  $X = (X_1, \dots, X_n)$  on  $M$ . Write the 1-form  $\mathbf{A}$  in the local coordinates as

$$\mathbf{A} = \sum_{j=1}^n A_j(X) dX_j,$$

the matrix of the Riemannian metric  $g$  as

$$g(X) = (g_{jl}(X))_{1 \leq j, l \leq n}$$

and its inverse as

$$g(X)^{-1} = (g^{jl}(X))_{1 \leq j, l \leq n}.$$

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Denote  $|g(X)| = \det(g(X))$ . Then the magnetic field  $\mathbf{B}$  is given by the following formula

$$\mathbf{B} = \sum_{j < k} B_{jk} dX_j \wedge dX_k, \quad B_{jk} = \frac{\partial A_k}{\partial X_j} - \frac{\partial A_j}{\partial X_k}.$$

Moreover, the operator  $H^h$  has the form

$$H^h = \frac{1}{\sqrt{|g(X)|}} \sum_{1 \leq j, l \leq n} \left( ih \frac{\partial}{\partial X_j} + A_j(X) \right) \left[ \sqrt{|g(X)|} g^{jl}(X) \left( ih \frac{\partial}{\partial X_l} + A_l(X) \right) \right].$$

For any  $x \in M$ , denote by  $B(x)$  the anti-symmetric linear operator on the tangent space  $T_x M$  associated with the 2-form  $\mathbf{B}$ :

$$g_x(B(x)u, v) = \mathbf{B}_x(u, v), \quad u, v \in T_x M.$$

Recall that the intensity of the magnetic field is defined as

$$\mathrm{Tr}^+(B(x)) = \sum_{\substack{\lambda_j(x) > 0 \\ i\lambda_j(x) \in \sigma(B(x))}} \lambda_j(x) = \frac{1}{2} \mathrm{Tr}([B^*(x) \cdot B(x)]^{1/2}).$$

It turns out that in many problems the function  $x \mapsto h \cdot \mathrm{Tr}^+(B(x))$  can be considered as a magnetic potential, that is, as a magnetic analogue of the electric potential  $V$  in a Schrödinger operator  $-h^2 \Delta + V$ .

In this paper we will always assume that the magnetic field has a periodic set of compact potential wells. More precisely, put

$$b_0 = \min\{\mathrm{Tr}^+(B(x)) : x \in M\}$$

and assume that there exist a (connected) fundamental domain  $\mathcal{F}$  and a constant  $\epsilon_0 > 0$  such that

$$(1) \quad \mathrm{Tr}^+(B(x)) \geq b_0 + \epsilon_0, \quad x \in \partial \mathcal{F}.$$

For any  $\epsilon_1 \leq \epsilon_0$ , put

$$U_{\epsilon_1} = \{x \in \mathcal{F} : \mathrm{Tr}^+(B(x)) < b_0 + \epsilon_1\}.$$

Thus  $U_{\epsilon_1}$  is an open subset of  $\mathcal{F}$  such that  $U_{\epsilon_1} \cap \partial \mathcal{F} = \emptyset$  and, for  $\epsilon_1 < \epsilon_0$ ,  $\overline{U_{\epsilon_1}}$  is compact and included in the interior of  $\mathcal{F}$ . Any connected component of  $U_{\epsilon_1}$  with  $\epsilon_1 < \epsilon_0$  and also any its translation under the action of an element of  $\Gamma$  can be understood as a magnetic well. These magnetic wells are separated by potential barriers, which are getting higher and higher when  $h \rightarrow 0$  (in the semiclassical limit).

For any linear operator  $T$  in a Hilbert space, we will denote by  $\sigma(T)$  its spectrum. By a gap in the spectrum of a self-adjoint operator  $T$  we will mean any connected component of the complement of  $\sigma(T)$  in  $\mathbb{R}$ , that is, any maximal interval  $(a, b)$  such that

$$(a, b) \cap \sigma(T) = \emptyset.$$

We will consider the magnetic Schrödinger operator  $H^h$  as an unbounded self-adjoint operator in the Hilbert space  $L^2(M)$ . The main object of our investigation is the gaps in the spectrum of this operator.

The problem of existence of gaps in the spectra of second order periodic differential operators has been extensively studied recently. Some related results on spectral gaps for periodic magnetic Schrödinger operators can be found for example in [2, 7, 15, 16, 17, 18, 19, 20, 21, 22, 23, 25] (see also the references therein).

In this paper, we will be interested in spectral gaps located below the top of potential barriers, that is, on the interval  $[0, h(b_0 + \epsilon_0)]$ . For such energy levels of a quantum particle, the important role is played by the tunneling effect, that is, by the possibility for a particle with such an energy to pass through a potential barrier. We will demonstrate how the investigation of the tunneling effect allows to obtain some information on gaps in the spectrum of the Schrödinger operator in the given energy interval.

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### 1. TUNNELING EFFECT AND LOCALIZATION OF THE SPECTRUM

In this Section, we briefly recall the key result on localization of the spectrum of the magnetic Schrödinger operator  $H^h$  obtained in [7], which follows from the semiclassical analysis of the tunneling effect for the corresponding quantum system.

For any domain  $W$  in  $M$ , denote by  $H_W^h$  the unbounded self-adjoint operator in the Hilbert space  $L^2(W)$  defined by the operator  $H^h$  in  $\bar{W}$  with the Dirichlet boundary conditions. The operator  $H_W^h$  is generated by the quadratic form

$$u \mapsto q_W^h[u] := \int_W |(ihd + \mathbf{A})u|^2 dx$$

with the domain

$$\text{Dom}(q_W^h) = \{u \in L^2(W) : (ihd + \mathbf{A})u \in L^2\Omega^1(W), u|_{\partial W} = 0\},$$

where  $L^2\Omega^1(W)$  denotes the Hilbert space of  $L^2$  differential 1-forms on  $W$ ,  $dx$  is the Riemannian volume form on  $M$ .

Assume now that the operator  $H^h$  satisfies the condition (1). Fix  $\epsilon_1 > 0$  and  $\epsilon_2 > 0$  such that  $\epsilon_1 < \epsilon_2 < \epsilon_0$ , and consider the operator  $H_D^h$  associated with the domain  $D = \bar{U}_{\epsilon_2}$ . The operator  $H_D^h$  has discrete spectrum.

The investigation of the tunneling effect allows to prove that the spectrum of  $H^h$  on the interval  $[0, h(b_0 + \epsilon_1)]$  is localized in an exponentially small neighborhood of the spectrum of  $H_D^h$ .

**Theorem 1.1** ([7]). *Under the assumption (1), for any  $\epsilon_1 < \epsilon_2 < \epsilon_0$ , there exist  $C, c, h_0 > 0$  such that, for any  $h \in (0, h_0]$*

$$\sigma(H^h) \cap [0, h(b_0 + \epsilon_1)] \subset \{\lambda \in [0, h(b_0 + \epsilon_2)] : \text{dist}(\lambda, \sigma(H_D^h)) < Ce^{-c/\sqrt{h}}\},$$

$$\sigma(H_D^h) \cap [0, h(b_0 + \epsilon_1)] \subset \{\lambda \in [0, h(b_0 + \epsilon_2)] : \text{dist}(\lambda, \sigma(H^h)) < Ce^{-c/\sqrt{h}}\}.$$

A slightly weaker version of this theorem (which uses the largest absolute value of the eigenvalues of  $B(x)$  instead of  $\text{Tr}^+(B(x))$ ) was proved by Nakamura in [25].

The proof of Theorem 1.1 uses the approach to the study of the tunneling effect in multi-well problems developed by Helffer and Sjöstrand for Schrödinger operators with electric potentials (see for instance [12, 13]) and extended to magnetic Schrödinger operators in [14, 9]. Since the operator  $H^h$  is not with compact resolvent, we do work not with individual eigenvalues, but with resolvents and use the strategy developed in [13, 15, 4] for the case of electric potential and in [5] for the case of magnetic field.

The idea of the proof is to construct an approximate resolvent  $R^h(z)$  of the operator  $H^h$  for any  $z$ , which is not exponentially close to the spectrum of  $H_D^h$ ,

starting from the resolvent of  $H_D^h$  and the resolvent of the Dirichlet realization of  $H^h$  in the complement to the wells. The proof of the fact that the error of the approximation is exponentially small is based on Agmon-type weighted estimates (cf. [1] and their semi-classical versions in [12] for the case of Schrödinger operators and [9] for the case of magnetic Schrödinger operators).

## 2. QUASIMODES AND SPECTRAL GAPS

Theorem 1.1 reduces the investigation of gaps in the spectrum of the operator  $H^h$  to the study of the eigenvalue distribution for the operator  $H_D^h$ . Actually, it turns out that, in order to show the existence of arbitrary large number of gaps in the spectrum of  $H^h$  on some interval, it suffices to construct arbitrarily long sequences of approximate eigenvalues of  $H_D^h$  on this interval located far enough from each other. This observation is formulated more precisely in the following theorem.

**Theorem 2.1.** *Let  $N \geq 1$ . Suppose that there is a subset  $\mu_0^h < \mu_1^h < \dots < \mu_N^h$  of an interval  $I(h) \subset [0, h(b_0 + \epsilon_1))$  such that*

(1) *There exist constants  $c > 0$  and  $M \geq 1$  such that*

$$(2) \quad \begin{aligned} \mu_j^h - \mu_{j-1}^h &> ch^M, \quad j = 1, \dots, N, \\ \text{dist}(\mu_0^h, \partial I(h)) &> ch^M, \quad \text{dist}(\mu_N^h, \partial I(h)) > ch^M, \end{aligned}$$

*for any  $h > 0$  small enough;*

(2) *Each  $\mu_j^h, j = 0, 1, \dots, N$ , is an approximate eigenvalue of the operator  $H_D^h$ : for some  $v_j^h \in C_c^\infty(D)$  we have*

$$(3) \quad \|H_D^h v_j^h - \mu_j^h v_j^h\| = \alpha_j(h) \|v_j^h\|,$$

*where  $\alpha_j(h) = o(h^M)$  as  $h \rightarrow 0$ .*

*Then the spectrum of  $H^h$  on the interval  $I(h)$  has at least  $N$  gaps for any sufficiently small  $h > 0$ .*

*Proof.* Recall the following well-known estimate, which holds for any self-adjoint operator  $A$  in a Hilbert space:

$$\|(A - \lambda I)^{-1}\| = 1/d(\lambda, \sigma(A)), \quad \lambda \notin \sigma(A).$$

By this fact and (3), it follows that, for any  $j = 0, 1, \dots, N$ , there exists  $\lambda_j^h \in \sigma(H_D^h) \cap I(h)$  such that

$$(4) \quad \lambda_j^h - \mu_j^h = o(h^M), \quad h \rightarrow 0.$$

By (2) and (4), we have

$$\lambda_j^h - \lambda_{j-1}^h > ch^M, \quad j = 1, \dots, N,$$

for any  $h > 0$  small enough.

Recall also a rough estimate for the number  $N_h(\alpha, \beta)$  of eigenvalues of  $H_D^h$  on an arbitrary interval  $(h\alpha, h\beta)$  (see, for instance, [9, Lemma 4.2]): for some  $C$  and  $h_0$

$$(5) \quad N_h(\alpha, \beta) \leq Ch^{-n}, \quad \forall h \in (0, h_0].$$

**Lemma 2.2.** *Let  $M > 0$  and  $c > 0$ . There exist  $C > 0$  and  $h_1 > 0$  such that, if  $\alpha^h$  and  $\beta^h$  are two points in the spectrum of  $H^h$  on the interval  $I(h)$  with  $\beta^h - \alpha^h > ch^M$ , then, for any  $h \in (0, h_1]$ , the spectrum of  $H^h$  has at least one gap in the interval  $(\alpha^h, \beta^h)$  of length larger than  $Ch^{M+n}$ .*

*Proof.* Divide the interval  $(\alpha^h, \beta^h)$  in  $[Dh^{-n}]$  equal subintervals with some constant  $D > 0$  (here  $[a]$  denotes the smallest integer larger than  $a$ ). By (5), it follows that if  $D$  is large enough, there exists a constant  $h_2 > 0$  such that, for any  $h \in (0, h_2]$ , at least one of these intervals does not meet the spectrum of  $H_D^h$ . Consider this interval and divide it in three equal parts. By Theorem 1.1, there exists a constant  $h_1 > 0$  such that, for any  $h \in (0, h_1]$ , the central subinterval of this partition does not meet the spectrum of  $H^h$ . Since  $\alpha^h$  and  $\beta^h$  belong to the spectrum of  $H^h$ , it is clear that there exists a gap in  $(\alpha^h, \beta^h)$ , containing this subinterval. This proves the lemma.  $\square$

By Lemma 2.2, each interval  $(\lambda_j^h, \lambda_{j+1}^h)$ ,  $j = 0, 1, \dots, N-1$  contains at least one gap in the spectrum of  $H^h$  of length  $\geq Ch^{M+n}$ , and the spectrum of  $H^h$  on the interval  $I(h)$  has at least  $N$  gaps of length  $\geq Ch^{M+n}$  for any  $h$  small enough.  $\square$

### 3. A GENERAL CASE

As a first application of Theorem 2.1, we show that the spectrum of the Schrödinger operator  $H^h$  with the periodic magnetic field, having magnetic wells, has gaps (and, moreover, an arbitrarily large number of gaps) on the interval  $[0, h(b_0 + \epsilon_0)]$  in the semiclassical limit  $h \rightarrow 0$ . Under some additional generic assumption, this result was obtained in [7].

**Theorem 3.1.** *Under the assumption (1), there exists, for any natural  $N$ ,  $h_0 > 0$  such that, for any  $h \in (0, h_0]$ , the spectrum of  $H^h$  in the interval  $[0, h(b_0 + \epsilon_0)]$  has at least  $N$  gaps.*

*Proof.* Keep notation of Section 1. Fix some natural  $N$ . Choose some

$$b_0 < \mu_0 < \mu_1 < \dots < \mu_N < b_0 + \epsilon_1.$$

For any  $j = 0, 1, \dots, N$ , take any  $x_j \in D$  such that

$$\text{Tr}^+(B(x_j)) = \mu_j.$$

Choose a local chart  $f_j : U_j \rightarrow \mathbb{R}^n$  defined in a neighborhood  $U_j$  of  $x_j$  with local coordinates  $X = (X_1, X_2, \dots, X_n) \in \mathbb{R}^n$ . Suppose that  $f_j(U_j)$  is a ball  $B = B(0, r)$  in  $\mathbb{R}^n$ ,  $f_j(x_j) = 0$ , the Riemannian metric at  $x_j$  becomes the standard Euclidean metric on  $\mathbb{R}^n$  and

$$\mathbf{B}(x_j) = \sum_{k=1}^{d_j} \mu_{jk} dX_{2k-1} \wedge dX_{2k}.$$

Let  $\varphi_j$  be a smooth function on  $B$  such that

$$|\mathbf{A}(X) - d\varphi_j(X) - A_j^q(X)| \leq C|X|^2,$$

where

$$A_j^q(X) = \frac{1}{2} \sum_{k=1}^{d_j} \mu_{jk} (X_{2k-1} dX_{2k} - X_{2k} dX_{2k-1}).$$

Write  $X'' = (X_{2d_j+1}, \dots, X_n)$ . Let  $\chi_j$  be a smooth function on  $D$  with support in a neighborhood of  $x_j$  and satisfying near  $x_j$ ,  $\chi_j(x) \equiv 1$ . Let  $v_j^h \in C_c^\infty(D)$  be defined as

$$v_j^h(x) = \chi_j(x) \exp\left(-i\frac{\varphi_j(x)}{h}\right) \exp\left(-\frac{1}{4h} \sum_{k=1}^{d_j} \mu_{jk}(X_{2k-1}^2 + X_{2k}^2)\right) \exp\left(-\frac{|X''|^2}{h^{2/3}}\right).$$

It is shown in the proof of Theorem 2.2 from [9] that

$$\|(H_D^h - h\mu_j)v_j^h\| \leq Ch^{4/3}\|v_j^h\|.$$

So the result follows from Theorem 2.1 with  $\mu_j^h = h\mu_j$  and  $M = 1$ .  $\square$

#### 4. POTENTIAL WELLS WITH THE REGULAR POINT BOTTOM

One can get a more precise information on location and asymptotic behavior of gaps in the spectrum of a magnetic Schrödinger operator with magnetic wells, imposing additional conditions on the bottom of the magnetic well. In this Section, we consider a case when the bottom of the magnetic well contains zero-dimensional components, that is, isolated points, and, moreover, the magnetic field behaves regularly near these points. More precisely, we will assume that, for some integer  $k > 0$ , if  $B(x_0) = 0$ , then there exists a positive constant  $C$  such that for all  $x$  in some neighborhood of  $x_0$  the estimate holds:

$$(6) \quad C^{-1}d(x, x_0)^k \leq \text{Tr}^+(B(x)) \leq Cd(x, x_0)^k$$

(here  $d(x, y)$  denotes the geodesic distance between  $x$  and  $y$ ).

In this case, the important role is played by a differential operator  $K_{\bar{x}_0}^h$  in  $\mathbb{R}^n$ , which is in some sense an approximation to the operator  $H^h$  near  $x_0$ . Recall its definition [9].

Let  $\bar{x}_0$  be a zero of  $B$ . Choose local coordinates  $f : U(\bar{x}_0) \rightarrow \mathbb{R}^n$  on  $M$ , defined in a sufficiently small neighborhood  $U(\bar{x}_0)$  of  $\bar{x}_0$ . Suppose that  $f(\bar{x}_0) = 0$ , and the image  $f(U(\bar{x}_0))$  is a ball  $B(0, r)$  in  $\mathbb{R}^n$  centered at the origin.

Write the 2-form  $\mathbf{B}$  in the local coordinates as

$$\mathbf{B}(X) = \sum_{1 \leq l < m \leq n} b_{lm}(X) dX_l \wedge dX_m, \quad X = (X_1, \dots, X_n) \in B(0, r).$$

Let  $\mathbf{B}^0$  be the closed 2-form in  $\mathbb{R}^n$  with polynomial components defined by the formula

$$\mathbf{B}^0(X) = \sum_{1 \leq l < m \leq n} \sum_{|\alpha|=k} \frac{X^\alpha}{\alpha!} \frac{\partial^\alpha b_{lm}}{\partial X^\alpha}(0) dX_l \wedge dX_m, \quad X \in \mathbb{R}^n.$$

One can find a 1-form  $\mathbf{A}^0$  on  $\mathbb{R}^n$  with polynomial components such that

$$d\mathbf{A}^0(X) = \mathbf{B}^0(X), \quad X \in \mathbb{R}^n.$$

Let  $K_{\bar{x}_0}^h$  be a self-adjoint differential operator in  $L^2(\mathbb{R}^n)$  with polynomial coefficients given by the formula

$$K_{\bar{x}_0}^h = (ihd + \mathbf{A}^0)^*(ihd + \mathbf{A}^0),$$

where the adjoints are taken with respect to the Hilbert structure in  $L^2(\mathbb{R}^n)$  given by the flat Riemannian metric  $(g_{lm}(0))$  in  $\mathbb{R}^n$ . If  $\mathbf{A}^0$  is written as

$$\mathbf{A}^0 = A_1^0 dX_1 + \dots + A_n^0 dX_n,$$

then  $K_{\bar{x}_0}^h$  is given by the formula

$$K_{\bar{x}_0}^h = \sum_{1 \leq l, m \leq n} g^{lm}(0) \left( ih \frac{\partial}{\partial X_l} + A_l^0(X) \right) \left( ih \frac{\partial}{\partial X_m} + A_m^0(X) \right).$$

The operators  $K_{\bar{x}_0}^h$  have discrete spectrum (cf, for instance, [11, 8]). Using the simple dilation  $X \mapsto h^{\frac{1}{k+2}} X$ , one can show that the operator  $K_{\bar{x}_0}^h$  is unitarily equivalent to  $h^{\frac{2k+2}{k+2}} K_{\bar{x}_0}^1$ . Thus,  $h^{-\frac{2k+2}{k+2}} K_{\bar{x}_0}^h$  has discrete spectrum, independent of  $h$ .

**Theorem 4.1.** *Suppose that the operator  $H^h$  satisfies the condition (1) with some  $\epsilon_0 > 0$  and there exists a zero  $\bar{x}_0$  of  $B$ ,  $B(\bar{x}_0) = 0$ , satisfying the assumption (6) for some integer  $k > 0$ . Then, for any natural  $N$ , there exist constants  $C > 0$  and  $h_0 > 0$  such that the part of the spectrum of  $H^h$ , contained in the interval  $[0, Ch^{\frac{2k+2}{k+2}}]$ , has at least  $N$  gaps for any  $h \in (0, h_0)$ .*

*Proof.* Fix  $\epsilon_1$  and  $\epsilon_2$  such that  $0 < \epsilon_1 < \epsilon_2 < \epsilon_0$  and consider the operator  $H_D^h$  associated with the domain  $D = \overline{U_{\epsilon_2}}$ . Denote by  $\lambda_1 < \lambda_2 < \lambda_3 < \dots$  the eigenvalues of the operator  $K_{\bar{x}_0}^1$  (not taking into account multiplicities).

For any  $j \in \mathbb{N}$ , let  $w_j^h \in L^2(\mathbb{R}^n)$  be any eigenfunction of  $K_{\bar{x}_0}^h$  corresponding to the eigenvalue  $h^{\frac{2k+2}{k+2}} \lambda_j$ . Let  $\chi$  be a compactly supported cut-off function in the neighborhood  $U(\bar{x}_0)$  of  $\bar{x}_0$  as above equal to 1 in a neighborhood of  $\bar{x}_0$ . Define

$$v_j^h(x) = \chi(x) w_j^h(x).$$

As shown in the proof of Theorem 2.5 in [9], we have

$$\| (H_D^h - h^{\frac{2k+2}{k+2}} \lambda_j) v_j^h \| \leq C_j h^{\frac{2k+3}{k+2}} \|v_j^h\|.$$

For a given natural  $N$ , choose any constant  $C > \lambda_{N+1}$ . Then the result follows from Theorem 2.1 with  $\mu_j^h = h^{\frac{2k+2}{k+2}} \lambda_j$ ,  $j = 1, \dots, N+1$ .  $\square$

## 5. POTENTIAL WELLS WITH THE ONE-DIMENSIONAL BOTTOM

In this section we consider the case when the manifold  $M$  is an oriented two-dimensional Riemannian manifold, and the zero set of the periodic magnetic field  $\mathbf{B}$  contains a one-dimensional non-degenerate compact component. More precisely, suppose that:

- $b_0 = 0$ ;
- the zero set of the magnetic field  $\mathbf{B}$  has a connected component  $\gamma$ , which is a bounded smooth curve;
- there are constants  $k \in \mathbb{N}$  and  $C > 0$  such that for all  $x$  in some neighborhood of  $\gamma$  the estimates hold:

$$(7) \quad C^{-1} d(x, \gamma)^k \leq |B(x)| \leq C d(x, \gamma)^k.$$

In particular, for  $k = 1$  the last condition means that  $\nabla B$  does not vanish on  $\gamma$ .

On compact manifolds, this model was introduced for the first time by Montgomery [24] and was further studied in [9, 26, 6]. In this paper we consider the problem of existence of gaps for this model in a simple particular case. Namely, we assume that the leading term of the Taylor expansion of the magnetic field  $B$  at  $\gamma$  is constant along  $\gamma$ . More precisely, write the volume 2-form  $\mathbf{B}$  as

$$\mathbf{B} = B(x)\omega, \quad x \in M,$$

where  $\omega$  is the Riemannian volume form on  $M$ . Denote by  $N$  the external unit normal vector to  $\gamma$ . Let  $\tilde{N}$  denote an arbitrary extension of  $N$  to a smooth vector field on  $M$ . Consider the function  $\beta_1$  on  $M$  given by the formula

$$\beta_1(x) = \tilde{N}^k B(x), \quad x \in M.$$

By (7), it is easy to see that

$$\beta_1(x) \neq 0, \quad x \in \gamma.$$

Our assumption is that the restriction of  $\beta_1$  to  $\gamma$  (which is independent of the choice of smooth extension  $\tilde{N}$ ) is constant along  $\gamma$ :

$$(8) \quad \beta_1(x) \equiv \beta_1 = \text{const}, \quad x \in \gamma.$$

For  $k = 1$ , this condition means that the length of the gradient  $|\nabla B|$  is constant along  $\gamma$ .

**Theorem 5.1.** *Under the given assumptions, for any natural  $N$  there exist constants  $C > 0$  and  $h_0 > 0$  such that the part of the spectrum of  $H^h$  contained in the interval  $[0, Ch^{\frac{2k+2}{k+2}}]$  has at least  $N$  gaps for any  $h \in (0, h_0)$ .*

*Proof.* Choose a normal coordinate system  $(x, y)$  in a tubular neighborhood  $U$  of  $\gamma$ . Thus,  $x \in [0, L] \cong S_L^1 = \mathbb{R}/L\mathbb{Z}$  is the natural parameter along  $\gamma$  ( $L$  is the length of  $\gamma$ ),  $\gamma$  is given by the equation  $y = 0$ , and  $y \in (-\varepsilon_0, \varepsilon_0)$  is the natural parameter along the geodesic, passing through the point on  $\gamma$  with the coordinates  $(x, 0)$  orthogonal to  $\gamma$ . It is well known that in such coordinates the metric  $g$  has the form

$$g = a(x, y)^2 dx^2 + dy^2,$$

where

$$a(x, 0) = 1, \quad \frac{\partial a}{\partial y}(x, 0) = 0.$$

Write

$$\mathbf{A} = A_1 dx + A_2 dy$$

and

$$\mathbf{B} = b(x, y) dx \wedge dy, \quad b = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}.$$

The external unit normal vector to  $\gamma$  has the form

$$N = \frac{\partial}{\partial y},$$

and one can take a smooth extension  $\tilde{N}$  as

$$\tilde{N} = \frac{\partial}{\partial y}.$$

Thus, we have

$$\frac{\partial^j b}{\partial y^j}(x, 0) = 0, \quad j = 0, 1, \dots, k-1.$$

and

$$\beta_1 = \frac{\partial^k}{\partial y^k} (a^{-1}b) \Big|_{y=0} = \frac{\partial^k b}{\partial y^k}(x, 0) \neq 0.$$

As above, denote by  $H_D^h$  the unbounded self-adjoint operator in  $L^2(D)$  given by the operator  $H^h$  in the domain  $D = \bar{U}$  with the Dirichlet boundary conditions and by  $\lambda_1^h < \lambda_2^h < \dots < \lambda_N^h < \dots$  the eigenvalues of  $H_D^h$ .



Adding an exact one form  $d\phi(x)$  to  $\mathbf{A}$ , without loss of generality, we can assume that

$$A_1(x, 0) = \alpha_1 \equiv \text{const.}$$

Consider the self-adjoint operator  $H^{h,0}$  in  $L^2(S_L^1 \times \mathbb{R})$  defined by the formula

$$H^{h,0} = -h^2 \frac{\partial^2}{\partial y^2} + \left( ih \frac{\partial}{\partial x} + \alpha_1 + \frac{1}{(k+1)!} \beta_1 y^{k+1} \right)^2, \quad x \in S_L^1, \quad y \in \mathbb{R}.$$

By [9, Theorem 2.7], the operator  $H^{h,0}$  in  $L^2(S_L^1 \times \mathbb{R})$  has discrete spectrum. We construct some eigenfunctions of  $H^{h,0}$ , using separation of variables. Consider a function  $u \in L^2(S_L^1 \times \mathbb{R})$  of the form

$$u(x, y) = e^{2\pi i \frac{p(h)}{L} x} v(y), \quad x \in S_L^1, \quad y \in \mathbb{R},$$

with some  $v \in L^2(\mathbb{R}, dy) \cap C^\infty(\mathbb{R})$  and  $p(h) \in \mathbb{Z}$ . Then

$$H^{h,0} u(x, y) = e^{2\pi i \frac{p(h)}{L} x} H(h, \beta(h)) v(y),$$

where

$$\beta(h) = \frac{2\pi h p(h)}{L} - \alpha_1$$

and

$$H(h, \beta) = -h^2 \frac{\partial^2}{\partial y^2} + \left( \beta - \frac{1}{(k+1)!} y^{k+1} \right)^2.$$

For any  $\alpha > 0$  the dilation operator

$$(T(\alpha)f)(y) = \sqrt{\alpha} f(\alpha y), \quad f \in L^2(\mathbb{R}, dy),$$

is a unitary operator in  $L^2(\mathbb{R}, dy)$ , satisfying the conditions

$$\frac{\partial}{\partial y} T(\alpha) = \alpha T(\alpha) \frac{\partial}{\partial y}, \quad y^s T(\alpha) = \alpha^{-s} T(\alpha) y^s.$$

Using these relations, it is easy to check that the identity

$$H(h, \beta) T(\alpha) = \alpha^{-(2k+2)} T(\alpha) H(\alpha^{k+2} h, \alpha^{k+1} \beta)$$

holds for any  $h > 0, \alpha > 0, \beta > 0$ , and, in particular,

$$H(h, \beta) T(h^{-\frac{1}{k+2}}) = h^{\frac{2k+2}{k+2}} T(h^{-\frac{1}{k+2}}) H(1, h^{-\frac{k+1}{k+2}} \beta).$$

For any fixed  $b \in \mathbb{R}$ , the operator  $H(1, b)$  has simple discrete spectrum

$$\mu_1(b) < \mu_2(b) < \dots, \quad \mu_j(b) \rightarrow +\infty,$$

where  $\mu_j(b)$  are continuous functions. Fix a natural  $N$ . Then there exist an interval  $(b_1, b_2)$  and a system of disjoint intervals  $(c_j, C_j), j = 1, 2, \dots, N, c_1 < C_1 < c_2 < \dots < C_{N-1} < c_N < C_N$  such that, for any  $b \in (b_1, b_2)$ , the inclusions  $\mu_j(b) \in (c_j, C_j), j = 1, 2, \dots, N$  hold. Choose  $p(h) \in \mathbb{Z}$  so that

$$b_1 < h^{-\frac{k+1}{k+2}} \beta(h) < b_2,$$

or, equivalently,

$$\frac{L}{2\pi} (\alpha_1 h^{-1} + b_1 h^{-\frac{1}{k+2}}) < p(h) < \frac{L}{2\pi} (\alpha_1 h^{-1} + b_2 h^{-\frac{1}{k+2}}).$$

Such a  $p(h)$ , clearly, always exists.

Let  $v_j \in L^2(\mathbb{R}, dy) \cap C^\infty(\mathbb{R})$  be the eigenfunction of  $H(1, h^{-1/(k+2)}\beta(h))$ , corresponding to the eigenvalue  $\mu_j(h^{-1/(k+2)}\beta(h))$ :

$$H\left(1, h^{-1/(k+2)}\beta(h)\right)v_j = \mu_j\left(h^{-1/(k+2)}\beta(h)\right)v_j.$$

Then  $v_j^h = T(h^{-1/(k+2)})v_j$  is an eigenfunction of  $H(h, \beta(h))$  with the corresponding eigenvalue

$$\mu_j^h = h^{\frac{2k+2}{k+2}}\mu_j(\beta(h)),$$

and, therefore, the function

$$u_j^h(x, y) = e^{2\pi i \frac{p(h)}{L}x} v_j^h(y)$$

is an eigenfunction of  $H^{h,0}$  with the same eigenvalue:

$$H^{h,0}u_j^h = \mu_j^h u_j^h.$$

Thus, we have shown that, for any natural  $N$ , there exists  $N$  eigenvalues  $\mu_1^h < \mu_2^h < \dots < \mu_N^h$  of  $H^{h,0}$  (generally speaking, not consecutive) such that

$$\mu_{j+1}^h - \mu_j^h > Ch^{\frac{2k+2}{k+2}}, \quad j = 1, 2, \dots, N-1.$$

The normal coordinate system  $(x, y)$  defined above gives a diffeomorphism

$$\Theta : S_L^1 \times (-\varepsilon_0, \varepsilon_0) \rightarrow \Omega,$$

onto a tubular neighborhood  $\Omega$  of  $\gamma$ . Let  $\chi_0$  be a cut off function supported in  $S_L^1 \times (-\varepsilon_0, \varepsilon_0)$  and equal to 1 in a neighborhood of  $S_L^1 \times \{0\}$ . Then, by [9, Theorem 2.7], we have

$$\|(H_D^h - \mu_j^h)(\chi_0 u_j^h) \circ \Theta^{-1}\| \leq Ch^{\frac{2k+2}{k+2}} \|\chi_0 u_j^h\|, \quad j = 1, 2, \dots, N.$$

Theorem 2.1 completes the proof.  $\square$

## 6. CONCLUDING REMARKS

In [21], one considered the case when all the zeroes of  $B$  are isolated points, satisfying the assumption (6) for some integer  $k > 0$ . In this case one can get a more precise information about location of spectral gaps.

**Theorem 6.1** ([21]). *Suppose that there exists at least one zero of  $B$ , and all the zeroes of  $B$  are isolated points, satisfying the assumption (6) for some integer  $k > 0$ . Then there exists an increasing sequence  $\{\lambda_m, m \in \mathbb{N}\}$ , satisfying the condition:  $\lambda_m \rightarrow \infty$  as  $m \rightarrow \infty$ , such that, for any  $a$  and  $b$  such that  $\lambda_m < a < b < \lambda_{m+1}$  for some  $m$ ,*

$$[ah^{\frac{2k+2}{k+2}}, bh^{\frac{2k+2}{k+2}}] \cap \sigma(H^h) = \emptyset,$$

for any  $h > 0$  small enough.

As a direct consequence of Theorem 6.1, we get another proof of Theorem 4.1 in this particular case.

The numbers  $\{\lambda_m, m \in \mathbb{N}\}$  in Theorem 6.1 are the eigenvalues of a so called model operator  $K^h$ , which is defined as follows. Choose a fundamental domain  $\mathcal{F} \subset M$  so that  $B$  does not vanishes on the boundary of  $\mathcal{F}$ . Let  $\{\bar{x}_j | j = 1, \dots, K\}$  denote all the zeroes of  $B$  in  $\mathcal{F}$ ;  $\bar{x}_i \neq \bar{x}_j$ , if  $i \neq j$ . The operator  $K^h$  is a self-adjoint operator in  $L^2(\mathbb{R}^n)^K$  given by

$$K^h = \bigoplus_{1 \leq j \leq K} K_{\bar{x}_j}^h.$$

The proof of Theorem 6.1 given in [21] makes use of abstract functional-analytic methods developed in [20] (see also a survey paper [22]). These methods were developed for the study of similar questions for a periodic magnetic Schrödinger operator

$$H_\mu = (i d + \mathbf{A})^*(i d + \mathbf{A}) + \mu^{-2}V(x),$$

with a  $\Gamma$ -invariant Morse potential  $V \geq 0$  on the universal covering  $M$  of a compact manifold in the strong electric field limit ( $\mu \rightarrow 0$ ), where, as above, we assume that the 2-form  $\mathbf{B} = d\mathbf{A}$  is  $\Gamma$ -invariant (here  $\Gamma = \pi_1(M)$ ). Indeed, they allow to obtain stronger results than the existence of gaps in the spectrum, namely, to prove Murrey-von Neumann equivalence of the corresponding spectral projections of  $H_\mu$  and the associated model operator (see [20, 22] for details).

Observe, however, that, using Theorem 1.1 and making full use of Theorem 2.5 in [9] (that provides a more complete information on the whole spectrum of  $H_D^h$ , not only the construction of some approximate eigenvalues as it was needed for the proof of Theorem 4.1), one can easily give another proof of Theorem 6.1.

On the other side, it is much easier to construct approximate eigenvalues, showing the existence of some eigenvalues for  $H_D^h$ , than to localize completely its spectrum. Therefore, the scheme of a proof of existence of spectral gaps suggested in this paper could be very efficient for treating cases where we know how to construct approximate eigenvalues, but the treatment of the whole spectrum could be more difficult. One could mention two cases where this idea should definitely work:

- (1) The two-dimensional magnetic Dirichlet problem studied in [10] (see, in particular, [10, Theorem 7.3]).
- (2) The more generic Montgomery model considered in [6].

These cases will be discussed elsewhere.

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