Stabilized FEM-BEM Coupling for Maxwell Transmission Problems

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Abstract

We consider the scattering of monochromatic electromagnetic waves at a dielectric object with a non-smooth surface. This paper studies the discretization of this problem by means of coupling finite element methods (FEM) and boundary element methods (BEM). Straightforward symmetric coupling as in [R. Hiptmair, *Coupling of finite elements and boundary elements in electromagnetic scattering*, SIAM J. Num. Anal. 41 (2003), pp. 919-944] suffers from instabilities at wave numbers related to interior Dirichlet eigenvalues, the so-called spurious resonance phenomenon.

A remedy is offered by adopting the idea underlying the widely used combined field integral equations (CFIE). These can be obtained from Robin-type trace operators, which ensure uniqueness of solutions of the associated interior boundary value problem for all frequencies. This implies uniqueness of solutions of the coupled problem. In the spirit of [R. Hiptmair and P. Meury, *Stabilized FEM-BEM Coupling for Helmholtz Transmission Problems*, SIAM J. Numer. Anal. 44 (2006), pp. 2106-2130], in order to get a coercive variational problem, we have to incorporate a regularizing operator into the modified traces.

The discretization of the coupled variational problem is then based on **curl**-conforming finite elements inside the scatterer, div_Γ-conforming boundary elements for the surface currents and curl_Γ-conforming boundary elements for an auxiliary function on the boundary. Adapting a Helmholtz-type splitting to the discrete setting, permits us to show asymptotic optimality of the Galerkin-FEM-BEM solution.

1 Introduction

We consider the electromagnetic scattering of monochromatic incident waves from a penetrable, three-dimensional bounded object $\Omega \subset \mathbb{R}^3$, the scatterer. In applications one usually encounters scatterers with piecewise smooth, Lipschitz continuous boundaries. Thus it is natural to assume the scatterer to be a curvilinear Lipschitz-polyhedron in the parlance of [28, Sect. 1]. For the sake of simplicity, we assume that its surface $\Gamma := \partial\Omega$ is connected. However, with slight changes all theorems can be extended to more general situations. The material parameters ε_r and μ_r may display some spatial variation inside Ω but assume the constant values $\varepsilon_0 > 0$ and $\mu_0 > 0$ in the air region.

Let $\mathbf{E}^s$ denote the complex amplitude of the scattered electric field in the air region $\Omega^+ := \mathbb{R}^3 \backslash \overline{\Omega}$ and $\mathbf{E}$ the total electric field inside the scatterer Ω, which emerge as solutions to the Maxwell

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transmission problem (cf. [49, Sect. 5.6.3])

$$
\begin{aligned}
\mathbf{curl}\, \mu_r(\mathbf{x})^{-1}\, \mathbf{curl}\, \mathbf{E} - \kappa^2 \varepsilon_r(\mathbf{x})\, \mathbf{E} &= \mathbf{F}(\mathbf{x}) \qquad \text{in } \Omega\,, \\
\mathbf{curl}\, \mathbf{curl}\, \mathbf{E}^{\mathrm{s}} - \kappa^2 \mathbf{E}^{\mathrm{s}} &= 0 \qquad \text{in } \Omega^+\,, \\
\boldsymbol{\gamma}_{\mathbf{t}}^+ \mathbf{E}^{\mathrm{s}} - \boldsymbol{\gamma}_{\mathbf{t}}^- \mathbf{E} = \mathbf{g}_D \quad \text{on } \Gamma, \qquad \boldsymbol{\gamma}_N^+ \mathbf{E}^{\mathrm{s}} - \mu_r^{-1} \boldsymbol{\gamma}_N^- \mathbf{E} &= \mathbf{g}_N \quad \text{on } \Gamma\,, \\
\lim_{|\mathbf{x}|\to\infty} \mathbf{curl}\, \mathbf{E} \times \mathbf{x} - i\kappa |\mathbf{x}| \mathbf{E} &= 0\,.
\end{aligned}
\tag{1}
$$

Here, $\kappa := \omega\sqrt{\mu_0 \varepsilon_0} L$ (with $\omega > 0$ the fixed angular frequency of the excitation, L the characteristic length of the scatterer) denotes the normalized wave number and should be considered as a real positive parameter. Furthermore, we write $\boldsymbol{\gamma}_{\mathbf{t}}\mathbf{E}$ for the tangential components of $\mathbf{E}$ on Γ and $\boldsymbol{\gamma}_N \mathbf{E}$ for the "magnetic components" $\mathbf{curl}\, \mathbf{E} \times \mathbf{n}$ on Γ. The exterior unit normal vector field $\mathbf{n}$ on Γ belongs to $\boldsymbol{L}^\infty(\Gamma)$ and is directed from Ω into Ω^+. In the case of excitation by plane electric waves, whose complex amplitude will be denoted by $\mathbf{E}^{\mathrm{inc}}$, the generic jump data $\mathbf{g}_D$ and $\mathbf{g}_N$ evaluate to the following traces

$$
\mathbf{g}_D := -\boldsymbol{\gamma}_{\mathbf{t}} \mathbf{E}^{\mathrm{inc}}\,, \qquad \mathbf{g}_N := -\boldsymbol{\gamma}_N \mathbf{E}^{\mathrm{inc}}\,.
$$

Finally, we designate by $[\gamma \mathbf{U}]_\Gamma := \mathbf{U}_{|\Omega^+} - \mathbf{U}_{|\Omega}$ and $\{\gamma \mathbf{U}\}_\Gamma := \frac{1}{2}\left(\mathbf{U}_{|\Omega^+} + \mathbf{U}_{|\Omega}\right)$ the jump, resp. the average, of some generic trace γ of a function $\mathbf{U}$ across the boundary Γ.

Using Rellich's lemma and unique continuation techniques, the following result can be established (cf. [36, Thm. 3.1]).

Theorem 1.1. *Provided that the relative material parameters μ_r and $\varepsilon_r > 0$ are piecewise smooth and bounded away from zero everywhere in Ω, the problem (1) has a unique solution.*

Boundary element methods (BEM) offer the most flexible way to deal with the homogeneous problem in the unbounded exterior domain Ω^+. They are based on boundary integral operators on the interface Γ. Due to potentially non constant material parameters, the field problem inside Ω may not be amenable to a treatment by means of boundary element methods. Hence, finite element schemes (FEM) have to be used here. Thus the topic of this article comes into focus, namely how to derive and discretize stable coupled variational formulations, and how to analyze the resulting FEM-BEM formulation.

The coupling entails expressing the Dirichlet-to-Neumann (DtN) map of the exterior problem by means of boundary integral operators linking the Cauchy data $\boldsymbol{\gamma}_{\mathbf{t}}\mathbf{E}$ and $\boldsymbol{\gamma}_N \mathbf{E}$ for the electric field. There exists a huge variety of integral formulations for the exterior electromagnetic boundary value problem. A comprehensive survey is given in Nédélec's monograph [49]. In principle all these methods furnish Dirichlet-to-Neumann maps. However, in many cases, in particular with so-called indirect formulations, the resulting operator lacks structural properties of the Dirichlet-to-Neumann map, for instance symmetry. This is obviously the case for second order elliptic problems. If the structure of the DtN map is not preserved, then the linear systems of equations obtained by a Ritz-Galerkin boundary element discretization are adversely affected.

For second order elliptic problems Costabel [27] discovered that the so-called direct boundary integral equation methods provide a remedy. The main idea is to employ the Calderón projector, which acts on the Cauchy data of the problem. For details and theoretical considerations we refer to [21, Sect. 4.5] and [30]. In short, the Calderón projector yields two sets of boundary integral equations. Judiciously combining them yields a version of the Dirichlet-to-Neumann map, which is perfectly suited for a Ritz-Galerkin discretization. Costabel's idea of coupling finite elements with boundary elements is usually referred to as "the symmetric coupling approach". It has been applied to a wide range of strongly elliptic problems; see, among others, [19, 38, 44]. For references to the engineering literature see [53, 55] and the references therein.

Unsurprisingly, the Calderón projector for the Maxwell system has been thoroughly studied, cf. [20, Sect. 1.3.2], [32], [49, Sect. 5.5], and [43, Sect. 3]. The idea of symmetric coupling for the transmission problem was theoretically probed in [4, 1, 2], and in [7] for a related problem involving impedance boundary conditions. All these results employ compactness arguments and the

Fredholm alternative. To this end, most authors have studied the integral operators intrinsically on Γ. They have been successful on smooth interface boundaries, but all efforts to adjust the approach to non-smooth boundaries have been in vain.

The fundamental new insights about the traces of electromagnetic fields, presented in [9, 12, 13, 15], paved the way to further progress. That progress could finally be achieved by remembering a highly effective policy in the modern treatment of boundary integral equations: The guideline is to stay off the boundary as far as possible by studying variational problems instead of the boundary integral operators directly. This policy has demonstrated its efficacy in the work of Costabel [27]. The recent textbook [46] discusses all nuances of this approach for strongly elliptic systems. Moving off the boundary helps steer clear of its awkward geometric features. Thus, the foundation for a theory of electromagnetic boundary integral operators could be laid in [18, 16].

In addition, in order to harness compactness arguments, we have to employ decompositions of the surface vector fields on Γ. The classical composition is the so-called Hodge decomposition [32], which remains a very effective tool on piecewise smooth boundaries, cf. [14], [18], and, in particular [41]. Its counterpart on domains is the Helmholtz decomposition. It is important to realize that there is some leeway in choosing the decomposition, because the exact orthogonality featured by Hodge or Helmholtz decompositions is of minor importance. Instead, we prefer to use related, but simpler, splittings.

Almost all boundary integral equations for the exterior Dirichlet problem in electromagnetic and acoustic scattering are haunted by the presence of “spurious frequencies” [15, 18, 22], for which the equations fail to have unique solutions. Those agree with interior Dirichlet eigenvalues. The symmetrically coupled variational formulation presented in [39] exhibits the same drawback.

In this article, we propose a stabilized method for FEM-BEM coupling based on (mixed) Robin-type boundary conditions to ensure unique solvability of the corresponding interior boundary value problem. The use of complex combinations of boundary integral operators has been an invaluable tool for deriving resonance-free combined field integral equations (CFIE) for electromagnetic scattering from a perfect conductor, cf. [35]. Furthermore, our approach also features regularizing operators, already used to stabilize Maxwell scattering problems in [17], to ensure a Gårding inequality for the sesqui-linear form underlying the variational formulation. In our case, both problems are tackled by introducing modified trace operators.

Based on the generalized traces, stabilized versions of Calderón projectors can be defined for the coupling of domain based variational formulations with boundary integral equations. Thus, we can derive new coupled variational formulations, which feature existence, uniqueness, and stability of solutions for all wave numbers $\kappa > 0$. A similar approach to the one presented here can be found in [54].

To discretize the symmetric and the stabilized coupled variational formulations, we rely on discrete differential forms (edge elements, face elements) on triangulations of both Ω and Γ. The Ritz-Galerkin approach is straightforward, and yet, in the discrete setting another challenge arises. The Helmholtz and Hodge-type decompositions do not directly carry over to the discrete spaces. For pure indirect boundary element formulations (Rumsey's principle) remedies have been explored in [41] and [22]. Direct boundary integral equations were tackled in [18]. All these approaches exploit the fact that appropriate discrete splittings can approximate their continuous counterparts reasonably well. In this paper we adapt the ideas in [18] and [39] to the symmetrically coupled FEM-BEM problem. We will use variants of these results that do not require sophisticated elliptic regularity theory.

The outline of this article is as follows: In the following section we will review the theory of Sobolev spaces and tangential traces. In section 3 we will introduce the potentials, which form the building blocks of the Stratton-Chu representation formula and the boundary integral operators for the electric field equation. In section 4 we construct decompositions of the electric field in Ω. The theoretical results for the symmetrically coupled variational formulation are reviewed in section 5. In section 6 the stabilized coupling strategy is presented. So far, all sections have been merely concerned with the analysis of the continuous variational problems. Then, in section 7, we introduce the finite element and boundary element spaces, which are used for a Ritz-Galerkin discretization of the coupled problem. In section 8 we derive discrete counterparts to the decompositions on the continuous level and establish discrete inf-sup estimates for the underlying sesqui-linear forms. Finally, in section 10 we will establish a priori convergence estimates for the stabilized coupling approach.

2 Traces and Spaces

The main purpose of this section is to define suitable Sobolev spaces, which can be used to derive weak formulations of (1), and review some of their most important properties. The notation and notions we introduce closely follow the ones of [39, Sect. 2].

The natural Hilbert space for an analysis of the Maxwell transmission problem (1) is the space

$$\boldsymbol{H}_{\mathsf{loc}}(\mathbf{curl},\, D) := \left\{ \mathbf{V} \in L^2_{\mathsf{loc}}(D);\, \mathbf{curl}\,\mathbf{V} \in L^2_{\mathsf{loc}}(D) \right\} .$$

Here and below D denotes a generic domain, which can be either Ω or Ω^+. For a thorough examination of these spaces we refer to [33, Chap. 1].

The Sobolev space of scalar functions and their dual spaces, $H^s(\Gamma)$ and $H^{-s}(\Gamma)$, can be defined invariantly for $0 \leq s \leq 1$, see [34, Thm. 1.3.3]. Furthermore, we denote by $\gamma : H^s_{\mathsf{loc}}(D) \mapsto H^{s-1/2}(\Gamma)$, $\frac{1}{2} < s < \frac{3}{2}$, the natural trace operator, cf. [46, Thm. 3.38]. Superscript $+$ and $-$ will be attached to the trace operators, when it is important whether they act from Ω or Ω^+. Furthermore, we denote by $\langle \cdot, \cdot \rangle_\Gamma : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \mapsto \mathbb{C}$ the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, when $L^2(\Gamma)$ is taken as pivot space.

If Γ is a curvilinear Lipschitz polyhedron in the parlance of [28] with smooth components Γ_j, $j = 1, \ldots, N_\Gamma$, we define

$$\begin{aligned} H^s(\Gamma) &:= \left\{ u \in H^1(\Gamma);\, u_{|\Gamma_j} \in H^s(\Gamma_j),\, j = 1, \ldots, N_\Gamma \right\} \quad \text{for } s > 1\,, \\ \boldsymbol{H}^s_{\mathbf{t}}(\Gamma) &:= \left\{ \mathbf{u} \in \boldsymbol{L}^2_{\mathbf{t}}(\Gamma);\, \mathbf{u}_{|\Gamma_j} \in \boldsymbol{H}^s(\Gamma_j),\, j = 1, \ldots, N_\Gamma \right\} \quad \text{for } s \geq 0\,, \end{aligned}$$

where $\boldsymbol{L}^2_{\mathbf{t}}(\Gamma) := \left\{ \mathbf{u} \in \boldsymbol{L}^2(\Gamma);\, \mathbf{u} \cdot \mathbf{n} = 0 \right\}$. We equip all spaces with their natural graph norms.

For any $\mathbf{U} \in \boldsymbol{C}^\infty(\bar{D})$ the tangential components trace $\boldsymbol{\gamma}_{\mathbf{t}}$ and the twisted tangential trace $\boldsymbol{\gamma}_\times$ can be defined a.e. on Γ by

$$\boldsymbol{\gamma}_{\mathbf{t}} \mathbf{U}\,(\mathbf{x}) := \mathbf{n}\,(\mathbf{x}) \times (\mathbf{U}\,(\mathbf{x}) \times \mathbf{n}\,(\mathbf{x}))\,, \qquad \boldsymbol{\gamma}_\times \mathbf{U}\,(\mathbf{x}) := \mathbf{U}\,(\mathbf{x}) \times \mathbf{n}\,(\mathbf{x})\,.$$

For piecewise smooth boundaries their extension onto $\boldsymbol{H}_{\mathsf{loc}}(\mathbf{curl},\, D)$ has been achieved in [9] and [12, Prop. 1.7], and for Lipschitz boundaries in [15, Sect. 2].

Theorem 2.1. *There exist intrinsically defined spaces* $\boldsymbol{H}^{1/2}_{\|}(\Gamma) \subset \boldsymbol{L}^2_{\mathbf{t}}(\Gamma)$ *and* $\boldsymbol{H}^{1/2}_{\perp}(\Gamma) \subset \boldsymbol{L}^2_{\mathbf{t}}(\Gamma)$ *such that the tangential components trace* $\boldsymbol{\gamma}^\pm_{\mathbf{t}} : \boldsymbol{H}^1_{\mathsf{loc}}(D) \mapsto \boldsymbol{H}^{1/2}_{\|}(\Gamma)$ *and the twisted tangential trace* $\boldsymbol{\gamma}^\pm_\times : \boldsymbol{H}^1_{\mathsf{loc}}(D) \mapsto \boldsymbol{H}^{1/2}_{\perp}(\Gamma)$ *are continuous, surjective and possess continuous right inverses.*

Proof. For a proof see [12, Prop, 2.7]. □

Their dual spaces will be denoted by $\boldsymbol{H}^{-1/2}_{\|}(\Gamma)$ and $\boldsymbol{H}^{-1/2}_{\perp}(\Gamma)$ respectively. In what follows

$$\langle \boldsymbol{\lambda}, \boldsymbol{\mu} \rangle_{\mathbf{t}} := \int_\Gamma \boldsymbol{\lambda} \cdot \overline{\boldsymbol{\mu}}\, \mathrm{d}\mathbf{S}\,, \qquad \boldsymbol{\lambda}, \boldsymbol{\mu} \in \boldsymbol{L}^2_{\mathbf{t}}(\Gamma)\,.$$

will stand for the inner product on $\boldsymbol{L}^2_{\mathbf{t}}(\Gamma)$, which can be extended to a sesqui-linear duality pairing

$$\langle \cdot, \cdot \rangle_{\mathbf{t}} : \boldsymbol{H}^{-1/2}_{\|}(\Gamma) \times \boldsymbol{H}^{1/2}_{\|}(\Gamma) \mapsto \mathbb{C}\,, \qquad \langle \cdot, \cdot \rangle_{\mathbf{t}} : \boldsymbol{H}^{-1/2}_{\perp}(\Gamma) \times \boldsymbol{H}^{1/2}_{\perp}(\Gamma) \mapsto \mathbb{C}\,.$$

when $\boldsymbol{L}^2_{\mathbf{t}}(\Gamma)$ is taken as pivot space.

The classical Rellich embedding theorem can be applied to the tangential trace spaces in the following way.

Lemma 2.2. *The embeddings* $\boldsymbol{H}_{\|}^{1/2}(\Gamma) \hookrightarrow \boldsymbol{L}_{\mathbf{t}}^2(\Gamma)$ *and* $\boldsymbol{H}_{\perp}^{1/2}(\Gamma) \hookrightarrow \boldsymbol{L}_{\mathbf{t}}^2(\Gamma)$ *are compact.*

Based on surface differential operators, cf. [12, Sect. 3], we can define

$$\boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma) := \left\{ \mathbf{v} \in \boldsymbol{H}_{\perp}^{-1/2}(\Gamma);\ \mathsf{curl}_\Gamma\, \mathbf{v} \in H^{-1/2}(\Gamma) \right\} ,$$
$$\boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma) := \left\{ \boldsymbol{\zeta} \in \boldsymbol{H}_{\|}^{-1/2}(\Gamma);\ \mathsf{div}_\Gamma\, \boldsymbol{\zeta} \in H^{-1/2}(\Gamma) \right\} .$$

These spaces are endowed with the natural graph norms and are of great importance as suitable trace spaces for vector fields in $\boldsymbol{H}_{\mathsf{loc}}(\mathbf{curl}, D)$ (cf. [12, Thms. 2.7, 2.8], [13, Thm. 4.5] and [9, Sect. 4]):

Theorem 2.3. *The tangential components trace* $\boldsymbol{\gamma}_{\mathbf{t}}^{\pm} : \boldsymbol{H}_{\mathsf{loc}}(\mathbf{curl}, D) \mapsto \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma)$ *and the twisted tangential trace* $\boldsymbol{\gamma}_{\times}^{\pm} : \boldsymbol{H}_{\mathsf{loc}}(\mathbf{curl}, D) \mapsto \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$ *are continuous, surjective with continuous right inverses.*

From this theorem we conclude that $\boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma)$ is exactly the right space for the Dirichlet data $\boldsymbol{\gamma}_{\mathbf{t}}^{-}\mathbf{E}$, $\boldsymbol{\gamma}_{\mathbf{t}}^{+}\mathbf{E}^{\mathrm{s}}$ and $\mathbf{g}_D$ in (1). Thus we adopt the alternative notation $\boldsymbol{\gamma}_D$ for $\boldsymbol{\gamma}_{\mathbf{t}}$ to stress the fact that this is the right "Dirichlet" trace space. As has been demonstrated in [13, Sect. 4], $\boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma)$ and $\boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$ can be put into duality, when $\boldsymbol{L}_{\mathbf{t}}^2(\Gamma)$ is used as pivot space. More precisely, the usual $\boldsymbol{L}_{\mathbf{t}}^2(\Gamma)$-inner product can be extended to a sesqui-linear duality pairing

$$\left\langle \cdot , \cdot \right\rangle_{\mathbf{t}} : \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma) \times \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma) \mapsto \mathbb{C}$$

by means of a Green's formula, $D \in \{\Omega\, , \Omega^+\}$

$$\mp \int_D \mathbf{U} \cdot \mathbf{curl}\, \overline{\mathbf{V}} - \mathbf{curl}\, \mathbf{U} \cdot \overline{\mathbf{V}}\, \mathrm{d}\mathbf{x} = \left\langle \boldsymbol{\gamma}_{\times}^{\pm}\mathbf{U}, \boldsymbol{\gamma}_{\mathbf{t}}^{\pm}\mathbf{V} \right\rangle_{\mathbf{t}} \qquad \forall\, \mathbf{U}, \mathbf{V} \in \boldsymbol{H}_{\mathsf{loc}}(\mathbf{curl}, D)\, ,$$

where an overbar denotes complex conjugation.

For continuous tangential vector fields $\mathbf{u}$ we define the surface twist operator by

$$\mathsf{R}\big(\mathbf{u}\big)\, (\mathbf{x}) := \mathbf{n}\, (\mathbf{x}) \times \mathbf{u}\, (\mathbf{x})\, ,$$

which gives rise to an isometric mapping $\mathsf{R} : \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma) \mapsto \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$.

We will also need the normal components trace $\boldsymbol{\gamma}_{\mathbf{n}}$ defined by

$$\boldsymbol{\gamma}_{\mathbf{n}}\mathbf{U}\, (\mathbf{x}) := \mathbf{n}\, (\mathbf{x}) \cdot \mathbf{U}\, (\mathbf{x})$$

for almost all $\mathbf{x} \in \Gamma$ and $\mathbf{U} \in \boldsymbol{C}^{\infty}(\bar{\Omega})$. This trace can be extended to a continuous and surjective mapping $\boldsymbol{\gamma}_{\mathbf{n}} : \boldsymbol{H}_{\mathsf{loc}}(\mathsf{div}, \Omega) \mapsto H^{-1/2}(\Gamma)$ (cf. [33, Thm. 2.5]).

Besides the Dirichlet trace $\boldsymbol{\gamma}_D$ the transmission conditions of (1) also feature a second trace, aptly called the Neumann trace $\boldsymbol{\gamma}_N$, which has to be introduced in a weak sense: For

$$\mathbf{U} \in \boldsymbol{H}_{\mathsf{loc}}(\mathbf{curl}^2, D) := \left\{ \mathbf{V} \in \boldsymbol{H}_{\mathsf{loc}}(\mathbf{curl}, D);\ \mathbf{curl}\, \mathbf{curl}\, \mathbf{V} \in \boldsymbol{L}_{\mathsf{loc}}^2(D) \right\} ,$$

we define $\boldsymbol{\gamma}_N^{\pm}\mathbf{U} \in \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$ by

$$\mp \int_D \mathbf{curl}\, \mathbf{U} \cdot \mathbf{curl}\, \overline{\mathbf{V}} - \mathbf{curl}\, \mathbf{curl}\, \mathbf{U} \cdot \overline{\mathbf{V}}\, \mathrm{d}\mathbf{x} = \left\langle \boldsymbol{\gamma}_N^{-}\mathbf{U}, \boldsymbol{\gamma}_D^{-}\mathbf{V} \right\rangle_{\mathbf{t}} \qquad \forall\, \mathbf{V} \in \boldsymbol{H}_{\mathsf{loc}}(\mathbf{curl}, D)\, , \qquad (2)$$

Obviously for smooth vector fields we recover $\boldsymbol{\gamma}_N\mathbf{U} = \boldsymbol{\gamma}_{\times}\, (\mathbf{curl}\, \mathbf{U}) = \mathbf{curl}\, \mathbf{U} \times \mathbf{n}$. The following lemma shows that $\boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$ is exactly the right space for the "Neumann" data in (1).

Lemma 2.4. *The traces* $\boldsymbol{\gamma}_N^{\pm} : \boldsymbol{H}_{\mathsf{loc}}(\mathbf{curl}^2, D) \mapsto \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$ *furnish continuous mappings.*

Proof. For a proof see [38, Lem 3.3]. ∎

There is some analogy between the Helmholtz and the Maxwell case, which is already indicated by the notation we have chosen. In the Helmholtz case fields of total, scattered and incident waves belong to $H^1_{\mathsf{loc}}(\Omega)$, whereas in the Maxwell case they belong to $\boldsymbol{H}_{\mathsf{loc}}(\mathbf{curl}, \Omega)$. For smooth functions U or fields $\mathbf{E}$ and any $\mathbf{x} \in \Gamma$, we can establish the following correspondence between the Dirichlet and Neumann traces:

$$\begin{array}{llcl} \text{Dirichlet trace:} & \boldsymbol{\gamma}_D \mathbf{E}(\mathbf{x}) = \mathbf{n}(\mathbf{x}) \times \mathbf{E}(\mathbf{x}) & \leftrightarrow & \gamma_D U(\mathbf{x}) = U(\mathbf{x}) \,, \\ \text{Neumann trace:} & \boldsymbol{\gamma}_N \mathbf{E}(\mathbf{x}) = \mathbf{curl}\, \mathbf{E}(\mathbf{x}) \times \mathbf{n}(\mathbf{x}) & \leftrightarrow & \gamma_N U(\mathbf{x}) = \mathbf{grad}\, U(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \,, \end{array}$$

Furthermore, in the Helmholtz case, the Dirichlet and Neumann traces are part of the $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$, whereas in the Maxwell case they belong to $\boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma)$ and $\boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$. Nevertheless, the continuity and surjectivity results for both traces are completely analogous.

3 Potentials and Boundary Integral Operators

Any distribution $\mathbf{U} \in \boldsymbol{H}_{\mathsf{loc}}(\mathbf{curl}^2, \Omega^+)$ which satisfies the electric field equation

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - \kappa^2 \mathbf{U} = 0 \quad \text{in } \Omega^+ \,, \tag{3}$$

together with the Silver-Müller radiation condition can be written using the *Stratton-Chu representation formula* (cf. [14, Sect. 3], [20, Chap. 3, Sect. 1.3.2], and [49, Sect. 5.5])

$$\mathbf{U} = \boldsymbol{\Psi}^\kappa_{\mathrm{DL}}\big(\boldsymbol{\gamma}^+_D \mathbf{U}\big) - \boldsymbol{\Psi}^\kappa_{\mathsf{S}}\big(\boldsymbol{\gamma}^+_N \mathbf{U}\big) - \mathbf{grad}\, \Psi^\kappa_{\mathsf{S}}\big(\boldsymbol{\gamma}^+_{\mathbf{n}} \mathbf{U}\big) \quad \text{in } \Omega^+ \,, \tag{4}$$

with the potentials:

$$\begin{array}{rll} \text{scalar single layer potential} & \Psi^\kappa_{\mathsf{S}}\big(\varphi\big)(\mathbf{x}) & := \displaystyle\int_\Gamma G_\kappa\left(|\mathbf{x}-\mathbf{y}|\right) \varphi\left(\mathbf{y}\right) \mathrm{d}\mathbf{S}\left(\mathbf{y}\right) \,, \quad \mathbf{x} \notin \Gamma \,, \\ \text{vectorial single layer potential} & \boldsymbol{\Psi}^\kappa_{\mathsf{S}}\big(\boldsymbol{\mu}\big)(\mathbf{x}) & := \displaystyle\int_\Gamma G_\kappa\left(|\mathbf{x}-\mathbf{y}|\right) \boldsymbol{\mu}\left(\mathbf{y}\right) \mathrm{d}\mathbf{S}\left(\mathbf{y}\right) \,, \quad \mathbf{x} \notin \Gamma \,, \\ \text{Maxwell double layer potential} & \boldsymbol{\Psi}^\kappa_{\mathrm{DL}}\big(\mathbf{u}\big)(\mathbf{x}) & := \left(\mathbf{curl} \circ \boldsymbol{\Psi}^\kappa_{\mathsf{S}} \circ \mathbf{R}\right)\big(\mathbf{u}\big)(\mathbf{x}) \,, \quad \mathbf{x} \notin \Gamma \,, \end{array}$$

based on the Helmholtz kernel

$$G_\kappa\left(\mathbf{z}\right) := \frac{1}{4\pi} \frac{\exp\left(i\kappa|\mathbf{z}|\right)}{|\mathbf{z}|} \,, \qquad \mathbf{z} \neq \mathbf{0} \,.$$

However, a simplification of (4) is possible by observing that [15, Eq. (26)]

$$\mathsf{div}_\Gamma\left(\boldsymbol{\gamma}^+_N \mathbf{U}\right) = \boldsymbol{\gamma}^+_{\mathbf{n}}\left(\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}\right) = \kappa^2 \boldsymbol{\gamma}^+_{\mathbf{n}} \mathbf{U} \quad \text{in } H^{-1/2}(\Gamma)$$

for all $\mathbf{U} \in \boldsymbol{H}_{\mathsf{loc}}(\mathbf{curl}^2, \Omega^+)$ satisfying (3). This makes it possible to get rid of the normal components trace in (4) and we obtain a much simpler version of the representation formula

$$\mathbf{U} = \boldsymbol{\Psi}^\kappa_{\mathrm{DL}}\big(\boldsymbol{\gamma}^+_D \mathbf{U}\big) - \boldsymbol{\Psi}^\kappa_{\mathrm{SL}}\big(\boldsymbol{\gamma}^+_N \mathbf{U}\big) \quad \text{in } \Omega^+ \,, \tag{5}$$

by introducing the Maxwell single layer potential

$$\boldsymbol{\Psi}^\kappa_{\mathrm{SL}}\big(\boldsymbol{\mu}\big)(\mathbf{x}) := \boldsymbol{\Psi}^\kappa_{\mathsf{S}}\big(\boldsymbol{\mu}\big)(\mathbf{x}) + \frac{1}{\kappa^2}\mathbf{grad}\Psi^\kappa_{\mathsf{S}}\big(\mathsf{div}_\Gamma \boldsymbol{\mu}\big)(\mathbf{x}) \,, \quad \mathbf{x} \notin \Gamma \,.$$

Lemma 3.1. *The scalar and vectorial single layer potentials Ψ^κ_{S} and $\boldsymbol{\Psi}^\kappa_{\mathsf{S}}$ give rise to continuous mappings $\Psi^\kappa_{\mathsf{S}} : H^{-1/2}(\Gamma) \mapsto H^1_{\mathsf{loc}}(\mathbb{R}^3)$, $\boldsymbol{\Psi}^\kappa_{\mathsf{S}} : \boldsymbol{H}^{-1/2}_{\parallel}(\Gamma) \mapsto \boldsymbol{H}^1_{\mathsf{loc}}(\mathbb{R}^3)$.*

Proof. For a proof see [27] or [38, Thm. 5.1]. □

Lemma 3.2. *For* $\mathbf{u} \in \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$ *we have* $\mathsf{div}\boldsymbol{\Psi}^\kappa_{\mathsf{S}}(\mathbf{u}) = \Psi^\kappa_{\mathsf{S}}(\mathsf{div}_\Gamma \mathbf{u})$ *in* $L^2_{\mathsf{loc}}(\mathbb{R}^3)$.

Proof. A proof can be found in [45, Lem. 2.3]. □

From this we immediately derive the identities

$$\begin{aligned} \left(\mathbf{curl}\,\mathbf{curl} - \kappa^2 \mathsf{Id}\right) \boldsymbol{\Psi}^\kappa_{\mathsf{S}}(\boldsymbol{\mu}) &= \mathbf{grad}\, \Psi^\kappa_{\mathsf{S}}(\mathsf{div}_\Gamma \boldsymbol{\mu}) && \forall \boldsymbol{\mu} \in \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)\,, \\ \left(\mathbf{curl}\,\mathbf{curl} - \kappa^2 \mathsf{Id}\right) \boldsymbol{\Psi}^\kappa_{\mathrm{DL}}(\mathbf{u}) &= 0 && \forall \mathbf{u} \in \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma)\,, \end{aligned}$$

off the boundary in a point wise sense, and, globally in $\boldsymbol{L}^2_{\mathsf{loc}}(\mathbb{R}^3)$. Thus we conclude that both $\boldsymbol{\Psi}^\kappa_{\mathrm{SL}}$ and $\boldsymbol{\Psi}^\kappa_{\mathrm{DL}}$ are radiating solutions to the electric field equation in $\Omega \cup \Omega^+$.

From these relationships and Lemma 3.1 we immediately derive the following continuity properties.

Lemma 3.3. *The Maxwell single layer potential* $\boldsymbol{\Psi}^\kappa_{\mathrm{SL}} : \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma) \mapsto \boldsymbol{H}_{\mathsf{loc}}(\mathbf{curl}^2, \Omega \cup \Omega^+)$ *and the Maxwell double layer potential* $\boldsymbol{\Psi}^\kappa_{\mathrm{DL}} : \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma) \mapsto \boldsymbol{H}_{\mathsf{loc}}(\mathbf{curl}^2, \Omega \cup \Omega^+)$ *are continuous mappings.*

The potentials also satisfy fundamental jump relations (cf. [25, Thm. 6.11], [49, Thm. 5.5.1] and [38, Sect. 5]).

Lemma 3.4. *The interior and exterior Dirichlet and Neumann traces of the potentials* $\boldsymbol{\Psi}^\kappa_{\mathrm{SL}}$ *and* $\boldsymbol{\Psi}^\kappa_{\mathrm{DL}}$ *are well defined and satisfy*

$$\begin{aligned} \left[\boldsymbol{\gamma}_D \boldsymbol{\Psi}^\kappa_{\mathrm{SL}}(\boldsymbol{\mu})\right]_\Gamma &= 0\,, & \left[\boldsymbol{\gamma}_N \boldsymbol{\Psi}^\kappa_{\mathrm{SL}}(\boldsymbol{\mu})\right]_\Gamma &= -\boldsymbol{\mu}\,, & \forall \boldsymbol{\mu} &\in \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)\,, \\ \left[\boldsymbol{\gamma}_D \boldsymbol{\Psi}^\kappa_{\mathrm{DL}}(\mathbf{u})\right]_\Gamma &= \mathbf{u}\,, & \left[\boldsymbol{\gamma}_N \boldsymbol{\Psi}^\kappa_{\mathrm{DL}}(\mathbf{u})\right]_\Gamma &= 0\,, & \forall \mathbf{u} &\in \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma)\,. \end{aligned}$$

This theorem in conjunction with Lemma 3.2 and $\boldsymbol{\Psi}^\kappa_{\mathsf{S}}(\mathbf{R}(\mathbf{u})) \in \boldsymbol{H}^1_{\mathsf{loc}}(\mathbb{R}^3)$ supplies further jump relations

$$\left[\boldsymbol{\gamma}_{\mathbf{n}} \boldsymbol{\Psi}^\kappa_{\mathrm{DL}}(\mathbf{u})\right]_\Gamma = 0\,, \qquad \left[\gamma\, \mathsf{div}\, \boldsymbol{\Psi}^\kappa_{\mathsf{S}}(\boldsymbol{\mu})\right]_\Gamma = 0\,.$$

By applying averages of Dirichlet and Neumann traces to the potentials of the representation formula we obtain the relevant boundary integral operators for the electric field equation (cf. [39, Lem. 5.1, Thm. 5.2]). Their continuity properties are immediate from Thm. 2.1 and Lemmas 2.4, 3.1.

Lemma 3.5. *The integral operators*

$$\begin{aligned} \mathbf{S}_\kappa &:= \{\boldsymbol{\gamma}_D\}_\Gamma \circ \boldsymbol{\Psi}^\kappa_{\mathsf{S}} && : \boldsymbol{H}^{-1/2}_{\parallel}(\Gamma) \mapsto \boldsymbol{H}^{1/2}_{\parallel}(\Gamma), \\ \mathbf{S}^\times_\kappa &:= \left\{\boldsymbol{\gamma}_\times\right\}_\Gamma \circ \boldsymbol{\Psi}^\kappa_{\mathsf{S}} \circ \mathbf{R} && : \boldsymbol{H}^{-1/2}_{\perp}(\Gamma) \mapsto \boldsymbol{H}^{1/2}_{\perp}(\Gamma), \\ \mathsf{S}_\kappa &:= \{\gamma\}_\Gamma \circ \Psi^\kappa_{\mathsf{S}} && : H^{-1/2}(\Gamma) \mapsto H^{1/2}(\Gamma), \end{aligned}$$

are continuous.

Theorem 3.6. *The following integral operators are continuous:*

$$\begin{aligned} \mathbf{V}_\kappa &:= \{\boldsymbol{\gamma}_D\}_\Gamma \circ \boldsymbol{\Psi}^\kappa_{\mathrm{SL}} && : \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma) \mapsto \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma), \\ \mathbf{K}'_\kappa &:= \{\boldsymbol{\gamma}_N\}_\Gamma \circ \boldsymbol{\Psi}^\kappa_{\mathrm{SL}} && : \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma) \mapsto \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma), \\ \mathbf{K}_\kappa &:= \{\boldsymbol{\gamma}_D\}_\Gamma \circ \boldsymbol{\Psi}^\kappa_{\mathrm{DL}} && : \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma) \mapsto \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma), \\ \mathbf{W}_\kappa &:= \{\boldsymbol{\gamma}_N\}_\Gamma \circ \boldsymbol{\Psi}^\kappa_{\mathrm{DL}} && : \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma) \mapsto \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma). \end{aligned}$$

Beyond continuity, the integral operators possess numerous important properties. In particular, the operators $\mathbf{K}_\kappa$ and $\mathbf{K}'_\kappa$ are closely related as expressed in the following lemma, see [39, Lem. 5.4].

Lemma 3.7. *There exists a compact linear operator* $\mathsf{T}_\kappa : \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma) \mapsto \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$ *such that*

$$\left\langle \mathsf{K}'_\kappa(\boldsymbol{\zeta}), \mathbf{q} \right\rangle_{\mathbf{t}} = \left\langle \boldsymbol{\zeta}, \mathsf{K}_\kappa(\mathbf{q}) \right\rangle_{\mathbf{t}} - \left\langle \mathsf{T}_\kappa(\boldsymbol{\zeta}), \mathbf{q} \right\rangle_{\mathbf{t}} \qquad \forall \boldsymbol{\zeta} \in \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma),\ \mathbf{q} \in \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma).$$

Since we aim to apply the powerful Fredholm alternative argument, compactness properties of the boundary integral operators are of great importance. It will be crucial that we can switch to the "Laplace kernel" G_0 by a compact perturbation only (cf. [18, Thm. 3.12] and [41, Lem. 3.2]).

Lemma 3.8. *The following integral operators are compact:*

$$\begin{aligned}
\delta \mathsf{S}_\kappa &:= \mathsf{S}_\kappa - \mathsf{S}_0 &&: H^{-1/2}(\Gamma) \mapsto H^{1/2}(\Gamma), \\
\delta \boldsymbol{\mathsf{S}}_\kappa &:= \boldsymbol{\mathsf{S}}_\kappa - \boldsymbol{\mathsf{S}}_0 &&: \boldsymbol{H}_\parallel^{-1/2}(\Gamma) \mapsto \boldsymbol{H}_\parallel^{1/2}(\Gamma), \\
\delta \boldsymbol{\mathsf{S}}_\kappa^\times &:= \boldsymbol{\mathsf{S}}_\kappa^\times - \boldsymbol{\mathsf{S}}_0^\times &&: \boldsymbol{H}_\perp^{-1/2}(\Gamma) \mapsto \boldsymbol{H}_\perp^{1/2}(\Gamma), \\
\delta \boldsymbol{\mathsf{W}}_\kappa &:= \boldsymbol{\mathsf{W}}_\kappa - \boldsymbol{\mathsf{W}}_0 &&: \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma) \mapsto \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma).
\end{aligned}$$

The significance of the case $\kappa = 0$ is highlighted by the following result (cf. [46, Cor. 8.13], [31, Chap. XI, Sect. 2, Thm. 3] and [14, Prop. 4.1])

Lemma 3.9. *The operators* S_0*,* $\boldsymbol{\mathsf{S}}_0$ *and* $\boldsymbol{\mathsf{S}}_0^\times$ *are continuous, self-adjoint and fulfill*

$$\begin{aligned}
\left\langle \mu, \mathsf{S}_0(\mu) \right\rangle_\Gamma &\geq C \left\| \mu \right\|^2_{H^{-1/2}(\Gamma)} && \forall \mu \in H^{-1/2}(\Gamma), \\
\left\langle \boldsymbol{\mu}, \boldsymbol{\mathsf{S}}_0(\boldsymbol{\mu}) \right\rangle_{\mathbf{t}} &\geq C \left\| \boldsymbol{\mu} \right\|^2_{\boldsymbol{H}_\parallel^{-1/2}(\Gamma)} && \forall \boldsymbol{\mu} \in \boldsymbol{H}_\parallel^{-1/2}(\Gamma),\ \mathsf{div}_\Gamma\, \boldsymbol{\mu} = 0\,, \\
\left\langle \mathbf{v}, \boldsymbol{\mathsf{S}}_0^\times(\mathbf{v}) \right\rangle_{\mathbf{t}} &\geq C \left\| \mathbf{v} \right\|^2_{\boldsymbol{H}_\perp^{-1/2}(\Gamma)} && \forall \mathbf{v} \in \boldsymbol{H}_\perp^{-1/2}(\Gamma),\ \mathsf{curl}_\Gamma\, \mathbf{v} = 0\,,
\end{aligned}$$

with constants $C > 0$ *depending only on* Γ*.*

At first glance, the Helmholtz and Maxwell cases seem similar, but there are some apparent differences. The most striking among them is the lack of coercivity of the Maxwell single-layer operator $\boldsymbol{\mathsf{V}}_\kappa$ on $\boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$. In the Helmholtz case, we can combine lemma 3.8 and 3.9 and conclude that the Helmholtz single-layer operator S_κ satisfies a Gårding inequality on $H^{-1/2}(\Gamma)$, i.e. there exists a constant $C > 0$ and a compact operator $\mathsf{T}_\mathsf{S} := \mathsf{S}_0 - \mathsf{S}_\kappa : H^{-1/2}(\Gamma) \mapsto H^{1/2}(\Gamma)$ such that

$$\mathrm{Re}\left\{ \left\langle \vartheta, \mathsf{S}_\kappa(\vartheta) \right\rangle_\Gamma + \left\langle \vartheta, \mathsf{T}_\mathsf{S}(\vartheta) \right\rangle_\Gamma \right\} \geq C \left\| \vartheta \right\|^2_{H^{-1/2}(\Gamma)}, \qquad \forall \vartheta \in H^{-1/2}(\Gamma)\,. \tag{6}$$

However, for the Maxwell single-layer operator $\boldsymbol{\mathsf{V}}_\kappa$ things are different. Since variational formulations are our primary concern, let us inspect the sesqui-linear form associated with $\boldsymbol{\mathsf{V}}_\kappa$, see [16, Sect. 5] for details:

$$\left\langle \boldsymbol{\mu}, \boldsymbol{\mathsf{V}}_\kappa(\boldsymbol{\lambda}) \right\rangle_{\mathbf{t}} = \left\langle \boldsymbol{\mu}, \boldsymbol{\mathsf{S}}_\kappa(\boldsymbol{\lambda}) \right\rangle_{\mathbf{t}} - \kappa^{-2} \left\langle \mathsf{div}_\Gamma\, \boldsymbol{\mu}, \mathsf{S}_\kappa(\mathsf{div}_\Gamma\, \boldsymbol{\lambda}) \right\rangle_\Gamma, \quad \forall \boldsymbol{\lambda}, \boldsymbol{\mu} \in \boldsymbol{H}_\parallel^{-1/2}(\Gamma)\,. \tag{7}$$

Slightly abusing notation, we define $\boldsymbol{\mathsf{V}}_0 := \boldsymbol{\mathsf{S}}_0 + \kappa^{-2}\, \mathbf{grad}_\Gamma \circ \mathsf{S}_0 \circ \mathsf{div}_\Gamma$ and by recalling lemma 3.8 we conclude that $\boldsymbol{\mathsf{V}}_\kappa - \boldsymbol{\mathsf{V}}_0 : \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma) \mapsto \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma)$ is compact. Although, this result is sufficient to establish coercivity in the Helmholtz case (see (6)), it does guarantee a Gårding inequality for the Maxwell single layer operator. According to (7)

$$\left\langle \boldsymbol{\mu}, \boldsymbol{\mathsf{V}}_0(\boldsymbol{\lambda}) \right\rangle_{\mathbf{t}} = \left\langle \boldsymbol{\mu}, \boldsymbol{\mathsf{S}}_0(\boldsymbol{\lambda}) \right\rangle_{\mathbf{t}} - \kappa^{-2} \left\langle \mathsf{div}_\Gamma\, \boldsymbol{\mu}, \mathsf{S}_0(\mathsf{div}_\Gamma\, \boldsymbol{\lambda}) \right\rangle_\Gamma,$$

can be split into a sum of two operators of order minus one and plus one, respectively. In contrast to the Helmholtz case, the operator of order -1 is not elliptic but has an infinite dimensional kernel, which agrees with the kernel of the surface divergence operator div_Γ. This clearly indicates that a Gårding inequality for $\boldsymbol{\mathsf{V}}_\kappa$ on $\boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$ remains elusive.

4 Decompositions

This section provides stable splittings of $\boldsymbol{H}(\mathbf{curl},\,\Omega)$ and $\boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma,\,\Gamma)$, which are needed to establish generalized Gårding inequalities for the sesqui-linear forms underlying the coupled variational formulations. We motivate the splitting idea for two particular cases, namely for an interior source problem and for plane wave scattering from a perfect electric conductor.

Let us first consider the *interior source problem* for the electric wave equation in variational form: For any $\mathbf{F} \in \boldsymbol{L}^2(\Omega)$, find $\mathbf{E} \in \boldsymbol{H}(\mathbf{curl},\,\Omega)$ such that for all $\mathbf{V} \in \boldsymbol{H}(\mathbf{curl},\,\Omega)$ there holds

$$\mathsf{q}_\kappa(\mathbf{E},\mathbf{V}) := \big(\mu_r^{-1}\mathbf{curl}\,\mathbf{E},\,\mathbf{curl}\,\mathbf{V}\big)_0 - \kappa^2\big(\varepsilon_r\mathbf{E},\,\mathbf{V}\big)_0 = \big(\mathbf{F},\,\mathbf{V}\big)_0 .$$

The numerical analysis of the indefinite interior source problem is usually based on the Fredholm alternative, which can only be applied to variational formulations, whose underlying sesqui-linear forms are coercive. Thus establishing a generalized Gårding inequality for $\mathsf{q}_\kappa(\cdot,\cdot)$ on $\boldsymbol{H}(\mathbf{curl},\,\Omega)$ is essential. Unfortunately, due to a lack of compact embedding of $\boldsymbol{H}(\mathbf{curl},\,\Omega)$ into $\boldsymbol{L}^2(\Omega)$, a generalized Gårding inequality remains elusive.

The lack of coercivity can be overcome by a splitting of the fields into two components $\boldsymbol{H}(\mathbf{curl},\,\Omega) = \boldsymbol{X}(\Omega) \oplus \boldsymbol{N}(\Omega)$. In the context of electromagnetic problems this idea has been pioneered by Nédélec and was first applied to integral operators in [32]. Since then, it has emerged as very powerful theoretical tool, see [3, 14, 22] and, in particular, the monograph [49]. The following features of a splitting prove essential:

1. the subspace $\boldsymbol{N}(\Omega)$ in the splitting agrees with the kernel of the **curl**,
2. the compact embedding of the complement subspace $\boldsymbol{X}(\Omega)$ into $\boldsymbol{L}^2(\Omega)$,
3. the splitting is stable
4. extra regularity of vector fields in the complement space $\boldsymbol{X}(\Omega)$.

Thus, any $\mathbf{E} \in \boldsymbol{H}(\mathbf{curl},\,\Omega)$ can be decomposed into two components $\mathbf{E} = \mathbf{E}^0 + \mathbf{E}^\perp$, where $\mathbf{E}^0 \in \boldsymbol{N}(\Omega)$ and $\mathbf{E}^\perp \in \boldsymbol{X}(\Omega)$. This naturally leads us to the definition of a bounded, linear isomorphism $\mathsf{X}_\Omega : \boldsymbol{H}(\mathbf{curl},\,\Omega) \mapsto \boldsymbol{H}(\mathbf{curl},\,\Omega)$, given by $\mathsf{X}_\Omega\big(\mathbf{E}\big) := \mathbf{E}^0 - \mathbf{E}^\perp$, which can be employed to "flip signs" in the decomposition. Due to the compact embedding $\boldsymbol{X}(\Omega) \hookrightarrow \boldsymbol{L}^2(\Omega)$, we conclude that the following terms

$$\big(\varepsilon_r\mathbf{E}^\perp,\,\mathbf{V}^\perp\big)_0\,, \qquad \big(\varepsilon_r\mathbf{E}^\perp,\,\mathbf{V}^0\big)_0\,, \qquad \big(\varepsilon_r\mathbf{E}^0,\,\mathbf{V}^\perp\big)_0\,, \qquad \mathbf{E},\mathbf{V} \in \boldsymbol{H}(\mathbf{curl},\,\Omega)\,,$$

are compact. Thus, there exists a compact sesqui-linear form $\mathsf{c}_\kappa : \boldsymbol{H}(\mathbf{curl},\,\Omega) \times \boldsymbol{H}(\mathbf{curl},\,\Omega) \mapsto \mathbb{C}$, such that the following estimate holds

$$\mathrm{Re}\left\{\mathsf{q}_\kappa(\mathbf{E},\mathsf{X}_\Omega\,(\mathbf{E})) + \mathsf{c}_\kappa\,(\mathbf{E},\mathbf{E})\right\} \geq C\left(\left\|\mathbf{E}^\perp\right\|^2_{\boldsymbol{H}(\mathbf{curl},\,\Omega)} + \left\|\mathbf{E}^0\right\|^2_{\boldsymbol{H}(\mathbf{curl},\,\Omega)}\right)\,,$$

for all $\mathbf{E} \in \boldsymbol{H}(\mathbf{curl},\,\Omega)$. Hence, we immediately derive a generalized Gårding inequality for the sesqui-linear form $\mathsf{q}_\kappa(\cdot,\cdot)$ on the product space $\boldsymbol{X}(\Omega) \times \boldsymbol{N}(\Omega)$.

This motivates the use of the a Helmholtz-type *regular splitting*, whose construction is based on the existence of vector potentials in $\boldsymbol{H}^1(\Omega)$ (cf. [5, Lem. 3.5]):

Lemma 4.1. *There exists a continuous mapping*

$$\mathsf{L} : \boldsymbol{H}(\mathsf{div}\,0,\,\Omega) := \left\{\mathbf{V} \in \boldsymbol{L}^2(\Omega);\ \mathsf{div}\,\mathbf{V} = 0\right\} \mapsto \boldsymbol{H}^1(\Omega)\,,$$

such that $(\mathsf{div} \circ \mathsf{L})\,\big(\mathsf{U}\big) = 0$ *and* $(\mathbf{curl} \circ \mathsf{L})\,\big(\mathsf{U}\big) = \mathsf{U}$ *for all* $\mathsf{U} \in \boldsymbol{H}(\mathsf{div}\,0,\,\mathbb{R}^3)$.

Based on this device we introduce the following operator

$$\mathsf{P} : \boldsymbol{H}(\mathbf{curl},\,\Omega) \mapsto \boldsymbol{H}^1(\Omega)\,, \qquad \mathsf{P}\big(\mathbf{U}\big) := (\mathsf{L} \circ \mathbf{curl})\,\big(\mathbf{U}\big)\,.$$

From the properties of L we immediately derive numerous features of P.

Lemma 4.2. *The operator* P *is a continuous projection that preserves the* **curl** *and satisfies* $\mathrm{Ker}(\mathsf{P}) = \mathrm{Ker}(\mathbf{curl}) \cap \boldsymbol{H}(\mathbf{curl}, \Omega)$.

Since $\mathrm{Ker}(\mathsf{P}) = \mathrm{Ker}(\mathbf{curl}) \cap \boldsymbol{H}(\mathbf{curl}, \Omega)$ it is clear that the following closed subspaces

$$\boldsymbol{X}(\mathbf{curl}, \Omega) := \mathsf{P}\big(\boldsymbol{H}(\mathbf{curl}, \Omega)\big) \quad \text{and} \quad \boldsymbol{N}(\mathbf{curl}, \Omega) := \mathrm{Ker}(\mathbf{curl}) \cap \boldsymbol{H}(\mathbf{curl}, \Omega)$$

provide a stable and direct Helmholtz-type splitting

$$\boldsymbol{H}(\mathbf{curl}, \Omega) = \boldsymbol{X}(\mathbf{curl}, \Omega) \oplus \boldsymbol{N}(\mathbf{curl}, \Omega)\,. \tag{8}$$

For both components we retain the $\boldsymbol{H}(\mathbf{curl}, \Omega)$-norm. The extra regularity of the $\boldsymbol{X}(\mathbf{curl}, \Omega)$-component, which is contained in $\boldsymbol{H}^1(\Omega)$, is essential, since it immediately yields the following compact embedding.

Corollary 4.3. *The embedding* $\boldsymbol{X}(\mathbf{curl}, \Omega) \hookrightarrow \boldsymbol{L}^2(\Omega)$ *is compact.*

Summing up, this provides us with a generalized Gårding inequality for the sesqui-linear form $\mathsf{q}_\kappa(\cdot, \cdot)$ on the product space $\boldsymbol{X}(\mathbf{curl}, \Omega) \times \boldsymbol{N}(\mathbf{curl}, \Omega)$.

It is hardly surprising that the splitting idea has to be adopted for the treatment of boundary integral operators as well. First, we discuss this for a pure scattering problem and related indirect boundary integral equations. Using a single-layer potential ansatz $\mathbf{E} := \boldsymbol{\Psi}^\kappa_{\mathrm{SL}}(\boldsymbol{\mu})$, $\boldsymbol{\mu} \in \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$, as a trial expression for electromagnetic scattering from a perfect electric conductor, we arrive at the following variational problem: For every $\boldsymbol{\gamma}_D \mathbf{E}^{\mathrm{inc}} \in \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma)$, find $\boldsymbol{\mu} \in \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$ such that for all $\boldsymbol{\lambda} \in \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$ there holds

$$\big\langle \boldsymbol{\mu}, \mathsf{V}_\kappa(\boldsymbol{\lambda}) \big\rangle_{\mathbf{t}} = \big\langle \boldsymbol{\mu}, \boldsymbol{\gamma}_D^- \mathbf{E}^{\mathrm{inc}} \big\rangle_{\mathbf{t}}\,.$$

Our considerations at the end of section 3 clearly show, that a Gårding inequality for V_κ on $\boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$ is not available, and, thus an appropriate splitting has to be employed. This time we opt for a Hodge-type splitting of the Neumann trace space into two components: $\boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma) = \boldsymbol{X}(\Gamma) \oplus \boldsymbol{N}(\Gamma)$. In contrast to the splitting employed in [18, Thm. 3.4], we will waive orthogonality for increased regularity in the complement subspace $\boldsymbol{X}(\Gamma)$ and compact embeddings. We require:

1. the subspace $\boldsymbol{N}(\Gamma)$ in the splitting agrees with the kernel of div_Γ,
2. the compact embedding of the complement subspace $\boldsymbol{X}(\Gamma)$ into $\boldsymbol{L}^2_{\mathbf{t}}(\Gamma)$,
3. the splitting is stable
4. extra regularity of vector fields in the complement space $\boldsymbol{X}(\Gamma)$.

Again, any $\boldsymbol{\lambda} \in \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$ can be decomposed into $\boldsymbol{\lambda} = \boldsymbol{\lambda}^\perp + \boldsymbol{\lambda}^0$, where $\boldsymbol{\lambda}^\perp \in \boldsymbol{X}(\Gamma)$ and $\boldsymbol{\lambda}^0 \in \boldsymbol{N}(\Gamma)$. This time, the sign-flip isomorphism $\mathsf{X}_\Gamma : \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma) \mapsto \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$ is defined by $\mathsf{X}_\Gamma(\boldsymbol{\lambda}) := \boldsymbol{\lambda}^\perp - \boldsymbol{\lambda}^0$. Furthermore, the embedding $\boldsymbol{X}(\Gamma) \hookrightarrow \boldsymbol{L}^2_{\mathbf{t}}(\Gamma)$ allows us to identify the following compact sesqui-linear pairings

$$\begin{array}{lll} \big\langle \boldsymbol{\lambda}^\perp, \boldsymbol{\mu}^\perp \big\rangle_{\mathbf{t}}\,, & \big\langle \boldsymbol{\lambda}^\perp, \mathsf{S}_0(\boldsymbol{\mu}^\perp) \big\rangle_{\mathbf{t}}\,, & \\ \big\langle \boldsymbol{\lambda}^\perp, \mathsf{S}_0(\boldsymbol{\mu}^0) \big\rangle_{\mathbf{t}}\,, & \big\langle \boldsymbol{\lambda}^0, \mathsf{S}_0(\boldsymbol{\mu}^\perp) \big\rangle_{\mathbf{t}}\,, & \end{array} \quad \forall \boldsymbol{\lambda}, \boldsymbol{\mu} \in \boldsymbol{H}(\mathbf{curl}, \Omega)\,,$$

which imply existence of a compact sesqui-linear form $\mathsf{c}_\kappa : \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma) \times \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma) \mapsto \mathbb{C}$, such that for all $\boldsymbol{\lambda} \in \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$ the following estimate holds true

$$\mathrm{Re}\left\{ \big\langle \boldsymbol{\lambda}, \mathsf{V}_0\big(\mathsf{X}_\Gamma(\boldsymbol{\lambda})\big) \big\rangle_{\mathbf{t}} + \mathsf{c}_\kappa(\boldsymbol{\lambda}, \boldsymbol{\lambda}) \right\} \geq C \left(\big\| \boldsymbol{\lambda}^\perp \big\|^2_{\boldsymbol{H}_\parallel^{-1/2}(\Gamma)} + \big\| \boldsymbol{\lambda}^0 \big\|^2_{\boldsymbol{H}_\parallel^{-1/2}(\Gamma)} \right)\,,$$

Thus, we immediately derive a generalized Gårding inequality for the sesqui-linear form corresponding to the Maxwell single-layer operator V_κ on the product space $\boldsymbol{X}(\Gamma) \times \boldsymbol{N}(\Gamma)$.

This motivates the construction of the following Hodge-type decomposition of $\boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$: Pick an arbitrary $\boldsymbol{\lambda} \in \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$, set $\omega := \mathsf{div}_\Gamma \boldsymbol{\lambda} \in H^{-1/2}(\Gamma)$ and solve the the Neumann problem

$$\Psi \in \boldsymbol{H}^1(\Omega)/\mathbb{R}: \qquad \Delta\Psi = 0 \ \text{ in } \Omega\,, \qquad \gamma_{\mathbf{n}}^- \mathbf{grad}\,\Psi = \omega \ \text{ on } \Gamma\,.$$

Obviously $\mathbf{W} := \mathbf{grad}\,\Psi \in \boldsymbol{H}(\mathsf{div}\,0, \Omega)$ belongs to the domain of the lifting operator L. Thus we can introduce the operator $\mathsf{J} : H^{-1/2}(\Gamma) \mapsto \boldsymbol{H}^1(\Omega)$ by $\mathsf{J}(\omega) := \mathsf{L}(\mathbf{W})$. Its continuity is straightforward and, thanks to Theorem 2.1, inherited by the mapping

$$\mathsf{P}^\Gamma : \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma) \mapsto \boldsymbol{H}^{1/2}_\perp(\Gamma)\,, \qquad \mathsf{P}^\Gamma := \boldsymbol{\gamma}_\times \circ \mathsf{J} \circ \mathsf{div}_\Gamma\,.$$

Properties of P^Γ matching those of P can be easily established.

Lemma 4.4. *The operator* $\mathsf{P}^\Gamma : \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma) \mapsto \boldsymbol{H}^{1/2}_\perp(\Gamma)$ *is a continuous projection that preserves the* div_Γ *and satisfies* $\mathrm{Ker}(\mathsf{P}^\Gamma) = \mathrm{Ker}(\mathsf{div}_\Gamma) \cap \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$.

By defining the components

$$\boldsymbol{X}(\mathsf{div}_\Gamma, \Gamma) := \mathsf{P}^\Gamma\big(\boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)\big) \quad \text{and} \quad \boldsymbol{N}(\mathsf{div}_\Gamma, \Gamma) := \mathrm{Ker}(\mathsf{div}_\Gamma) \cap \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$$

we arrive at a stable direct decomposition of the space of magnetic traces:

$$\boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma) = \boldsymbol{X}(\mathsf{div}_\Gamma, \Gamma) \oplus \boldsymbol{N}(\mathsf{div}_\Gamma, \Gamma)\,.$$

As before, the extra regularity of $\boldsymbol{X}(\mathsf{div}_\Gamma, \Gamma)$ rewards us with a valuable compact embedding analogous to [18, Thm. 3.4].

Corollary 4.5. *The embedding* $\boldsymbol{X}(\mathsf{div}_\Gamma, \Gamma) \hookrightarrow \boldsymbol{L}^2_{\mathbf{t}}(\Gamma)$ *is compact.*

In short, this provides us with a generalized Gårding inequality for the sesqui-linear form corresponding to the Maxwell single-layer operator V_κ on the product space $\boldsymbol{X}(\mathsf{div}_\Gamma, \Gamma) \times \boldsymbol{N}(\mathsf{div}_\Gamma, \Gamma)$.

5 Symmetric FEM-BEM Coupling

Applying the Green's formula to the electric wave equation in Ω results in the following variational formulation: Find $\mathbf{E} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$ such that

$$\big(\mu_r^{-1}\mathbf{curl}\,\mathbf{E},\, \mathbf{curl}\,\mathbf{V}\big)_0 - \kappa^2\big(\varepsilon_r\mathbf{E},\, \mathbf{V}\big)_0 - \big\langle \mu_r^{-1}\boldsymbol{\gamma}_N^-\mathbf{E}, \boldsymbol{\gamma}_D^-\mathbf{V}\big\rangle_{\mathbf{t}} = \big(\mathbf{F},\, \mathbf{V}\big)_0 \tag{9}$$

for all $\mathbf{V} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$. The coupling to the exterior domain is taken into account by the transmission conditions

$$\mu_r^{-1}\boldsymbol{\gamma}_N^-\mathbf{E} = \boldsymbol{\gamma}_N^+\mathbf{E}^{\mathrm{s}} - \mathbf{g}_N\,, \qquad \boldsymbol{\gamma}_D^+\mathbf{E}^{\mathrm{s}} = \boldsymbol{\gamma}_D^-\mathbf{E} + \mathbf{g}_D\,. \tag{10}$$

In order to incorporate the exterior field on the unbound domain Ω^+ into the variational formulation some realization of the Dirichlet-to-Neumann map has to be provided. It is furnished by the *exterior Calderón projector*, which arises from applying both exterior Dirichlet and Neumann traces to the representation formula (5) (cf. [32, Eq. (29)], [49, Sect. 5.5], [18, Sect. 3.3], and [43, Eq. (24)]). In variational form the resulting identities read

$$\begin{aligned}
\big\langle \boldsymbol{\mu}, \boldsymbol{\gamma}_D^+\mathbf{E}^{\mathrm{s}}\big\rangle_{\mathbf{t}} &= \big\langle \boldsymbol{\mu}, \big(\tfrac{1}{2}\mathsf{Id} + \mathsf{K}_\kappa\big)\,\big(\boldsymbol{\gamma}_D^+\mathbf{E}^{\mathrm{s}}\big)\big\rangle_{\mathbf{t}} - \big\langle \boldsymbol{\mu}, \mathsf{V}_\kappa\big(\boldsymbol{\gamma}_N^+\mathbf{E}^{\mathrm{s}}\big)\big\rangle_{\mathbf{t}}\,, \\
\big\langle \boldsymbol{\gamma}_N^+\mathbf{E}^{\mathrm{s}}, \mathbf{v}\big\rangle_{\mathbf{t}} &= \big\langle \mathsf{W}_\kappa\big(\boldsymbol{\gamma}_D^+\mathbf{E}^{\mathrm{s}}\big), \mathbf{v}\big\rangle_{\mathbf{t}} + \big\langle \big(\tfrac{1}{2}\mathsf{Id} - \mathsf{K}'_\kappa\big)\,\big(\boldsymbol{\gamma}_N^+\mathbf{E}^{\mathrm{s}}\big), \mathbf{v}\big\rangle_{\mathbf{t}}\,,
\end{aligned} \tag{11}$$

for all $\boldsymbol{\mu} \in \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$ and $\mathbf{v} \in \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma)$. Now we can use the transmission conditions (10) and the second equation of the Calderón projector to replace the boundary term in (9). The trick underlying the symmetric coupling approach according to Costabel [26] is to combine the resulting equation together with the first equation of (11) (cf. [32, Sect. 4] for Maxwell equations).

Adopting the abbreviation $\boldsymbol{\lambda} := \gamma_N^+ \mathbf{E}^{\mathrm{s}} \in \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$ we arrive at the following variational formulation: Find $\mathbf{E} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$ and $\boldsymbol{\lambda} \in \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$ such that for all $\mathbf{V} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$ and $\boldsymbol{\mu} \in \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$

$$\begin{array}{rcccl} \mathsf{q}_\kappa(\mathbf{E}, \mathbf{V}) - \left\langle \mathsf{W}_\kappa(\boldsymbol{\gamma}_D^- \mathbf{E}), \boldsymbol{\gamma}_D^- \mathbf{V} \right\rangle_{\mathbf{t}} & + & \left\langle \left(\mathsf{K}'_\kappa - \frac{1}{2}\mathsf{Id}\right)(\boldsymbol{\lambda}), \boldsymbol{\gamma}_D^- \mathbf{V} \right\rangle_{\mathbf{t}} & = & \mathsf{f}_1(\mathbf{V}) \,, \\ \left\langle \boldsymbol{\mu}, \left(\frac{1}{2}\mathsf{Id} - \mathsf{K}_\kappa\right)(\boldsymbol{\gamma}_D^- \mathbf{E}) \right\rangle_{\mathbf{t}} & + & \left\langle \boldsymbol{\mu}, \mathsf{V}_\kappa(\boldsymbol{\lambda}) \right\rangle_{\mathbf{t}} & = & \mathsf{g}_1(\boldsymbol{\mu}) \,, \end{array} \tag{12}$$

with right hand sides

$$\begin{aligned} \mathsf{f}_1(\mathbf{V}) &:= (\mathbf{F}, \mathbf{V})_0 - \left\langle \mathbf{g}_N, \boldsymbol{\gamma}_D^- \mathbf{V} \right\rangle_{\mathbf{t}} + \left\langle \mathsf{W}_\kappa(\mathbf{g}_D), \boldsymbol{\gamma}_D^- \mathbf{V} \right\rangle_{\mathbf{t}} \,, \\ \mathsf{g}_1(\boldsymbol{\mu}) &:= \left\langle \boldsymbol{\mu}, \left(\mathsf{K}_\kappa - \frac{1}{2}\mathsf{Id}\right)(\mathbf{g}_D) \right\rangle_{\mathbf{t}} \,, \end{aligned}$$

and $\mathsf{q}_\kappa(\cdot, \cdot)$ representing the interior sesqui-linear form,

$$\mathsf{q}_\kappa(\mathbf{E}, \mathbf{V}) := \left(\mu_r^{-1} \mathbf{curl}\, \mathbf{E}, \mathbf{curl}\, \mathsf{V}\right)_0 - \kappa^2 \left(\varepsilon_r \mathbf{E}, \mathsf{V}\right)_0 .$$

Lemma 5.1. *Provided that* $\mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - \kappa^2 \mathbf{U} = 0$ *in* Ω *and* $\boldsymbol{\gamma}_D^- \mathbf{U} = 0$ *on* Γ *implies* $\mathbf{U} = 0$*, then any solution of* (12) *provides a solution of* (1) *by retaining* $\mathbf{E} \in \Omega$ *and using the representation formula* (5) *for the Cauchy data* $(\boldsymbol{\gamma}_D^- \mathbf{E} + \mathbf{g}_D, \boldsymbol{\lambda})$ *in* Ω^+.

Proof. For a proof see [39, Lem. 6.1] □

If κ^2 coincides with an interior Dirichlet eigenvalue, then the solution (12) is only unique up to contributions $(0, \boldsymbol{\eta})$, where $\boldsymbol{\eta}$ is contained in the span of Neumann data belonging to interior Dirichlet eigensolutions. In particular, the interior electric field $\mathbf{E}$ and its Dirichlet data $\boldsymbol{\gamma}_D^- \mathbf{E}$ are unique.

Since the sesqui-linear form underlying the variational formulation (12) features the domain-based part $\mathsf{q}_\kappa(\cdot, \cdot)$, as well as the Maxwell single-layer operator V_κ, our considerations from section 4 clearly indicate that coercivity on $\boldsymbol{H}(\mathbf{curl}, \Omega) \times \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$ does not hold. Thus, based on the splittings provided in section 4, we can decompose the trial and test functions in the variational problem (12) according to:

$$\begin{aligned} \mathbf{E} &= \mathbf{E}^\perp + \mathbf{E}^0 \,, \quad \mathbf{E}^\perp \in \boldsymbol{X}(\mathbf{curl}, \Omega),\ \mathbf{E}^0 \in \boldsymbol{N}(\mathbf{curl}, \Omega), \\ \mathbf{V} &= \mathbf{V}^\perp + \mathbf{V}^0 \,, \quad \mathbf{V}^\perp \in \boldsymbol{X}(\mathbf{curl}, \Omega),\ \mathbf{V}^0 \in \boldsymbol{N}(\mathbf{curl}, \Omega), \\ \boldsymbol{\lambda} &= \boldsymbol{\lambda}^\perp + \boldsymbol{\lambda}^0 \,, \quad \boldsymbol{\lambda}^\perp \in \boldsymbol{X}(\mathsf{div}_\Gamma, \Gamma),\ \boldsymbol{\lambda}^0 \in \boldsymbol{N}(\mathsf{div}_\Gamma, \Gamma), \\ \boldsymbol{\mu} &= \boldsymbol{\mu}^\perp + \boldsymbol{\mu}^0 \,, \quad \boldsymbol{\mu}^\perp \in \boldsymbol{X}(\mathsf{div}_\Gamma, \Gamma),\ \boldsymbol{\mu}^0 \in \boldsymbol{N}(\mathsf{div}_\Gamma, \Gamma). \end{aligned}$$

In addition, we sort the unknowns according to their "electric" or "magnetic" nature, grouping them as $(\boldsymbol{\lambda}^\perp, \mathbf{E}^0)$ (electric), $(\boldsymbol{\lambda}^0, \mathbf{E}^\perp)$ (magnetic). Thus we arrive at a variational formulation with a distinct block structure on the Hilbert space

$$\boldsymbol{\mathcal{V}} := \boldsymbol{X}(\mathsf{div}_\Gamma, \Gamma) \times \boldsymbol{N}(\mathbf{curl}, \Omega) \times \boldsymbol{N}(\mathsf{div}_\Gamma, \Gamma) \times \boldsymbol{X}(\mathbf{curl}, \Omega),$$

that is endowed with the natural graph norm: Find $(\boldsymbol{\lambda}^\perp, \mathbf{E}^0, \boldsymbol{\lambda}^0, \mathbf{E}^\perp) \in \boldsymbol{\mathcal{V}}$ such that

$$\widehat{\mathsf{a}}_\kappa^{\mathrm{sym}}\Big((\boldsymbol{\lambda}^\perp, \mathbf{E}^0, \boldsymbol{\lambda}^0, \mathbf{E}^\perp), (\boldsymbol{\mu}^\perp, \mathbf{V}^0, \boldsymbol{\mu}^0, \mathbf{V}^\perp)\Big) = \widehat{\mathsf{f}}_\kappa^{\mathrm{sym}}\Big(\boldsymbol{\mu}^\perp, \mathbf{V}^0, \boldsymbol{\mu}^0, \mathbf{V}^\perp\Big) \tag{13}$$

for all $(\boldsymbol{\mu}^\perp, \mathbf{V}^0, \boldsymbol{\mu}^0, \mathbf{V}^\perp) \in \boldsymbol{\mathcal{V}}$. The sesqui-linear form $\widehat{\mathsf{a}}_\kappa^{\mathrm{sym}} : \boldsymbol{\mathcal{V}} \times \boldsymbol{\mathcal{V}} \mapsto \mathbb{C}$ and the linear form $\widehat{\mathsf{f}}_\kappa^{\mathrm{sym}} : \boldsymbol{\mathcal{V}} \mapsto \mathbb{C}$ are defined by

$$\begin{aligned} &\widehat{\mathsf{a}}_\kappa^{\mathrm{sym}}\Big((\boldsymbol{\lambda}^\perp, \mathbf{E}^0, \boldsymbol{\lambda}^0, \mathbf{E}^\perp), (\boldsymbol{\mu}^\perp, \mathbf{V}^0, \boldsymbol{\mu}^0, \mathbf{V}^\perp)\Big) := \\ &\qquad\qquad \mathsf{a}_\kappa^{\mathrm{sym}}\Big((\mathbf{E}^\perp + \mathbf{E}^0, \boldsymbol{\lambda}^\perp + \boldsymbol{\lambda}^0), (\mathbf{V}^\perp - \mathbf{V}^0, -\boldsymbol{\mu}^\perp + \boldsymbol{\mu}^0)\Big) \,, \\ &\widehat{\mathsf{f}}_\kappa^{\mathrm{sym}}\Big(\boldsymbol{\mu}^\perp, \mathbf{V}^0, \boldsymbol{\mu}^0, \mathbf{V}^\perp\Big) := \mathsf{f}_\kappa^{\mathrm{sym}}\Big(\mathbf{V}^\perp - \mathbf{V}^0, -\boldsymbol{\mu}^\perp + \boldsymbol{\mu}^0\Big) \,, \end{aligned}$$

where $\mathsf{a}_\kappa^{\mathrm{sym}}$ and $\mathsf{f}_\kappa^{\mathrm{sym}}$ are the sesqui-linear and the linear form underlying the variational equation (12) (see [39, Eq. 8.1] for a detailed structure of $\mathsf{a}_\kappa^{\mathrm{sym}}$). Evidently, the split variational formulation (13) produces exactly the same solutions as (12) for $\boldsymbol{\lambda} = \boldsymbol{\lambda}^\perp + \boldsymbol{\lambda}^0$ and $\mathbf{E} = \mathbf{E}^\perp + \mathbf{E}^0$, provided that unique solutions exist. The decomposition of test and trial spaces provides us with the following powerful theorem.

Theorem 5.2. *The sesqui-linear form $\widehat{\mathsf{a}}_\kappa^{\mathrm{sym}} : \boldsymbol{\mathcal{V}} \times \boldsymbol{\mathcal{V}} \mapsto \mathbb{C}$ satisfies a generalized Gårding inequality; that is, it can be written as a sum $\widehat{\mathsf{a}}_\kappa^{\mathrm{sym}} = \mathsf{a}_{\mathrm{E}} + \mathsf{a}_{\mathrm{C}}$ of a $\boldsymbol{\mathcal{V}}$-elliptic sesqui-linear form $\mathsf{a}_{\mathrm{E}} : \boldsymbol{\mathcal{V}} \times \boldsymbol{\mathcal{V}} \mapsto \mathbb{C}$ and a compact sesqui-linear form $\mathsf{a}_{\mathrm{C}} : \boldsymbol{\mathcal{V}} \times \boldsymbol{\mathcal{V}} \mapsto \mathbb{C}$.*

Proof. For a proof see [39, Thm. 8.3]. □

6 Stable FEM-BEM Coupling

As pointed out in Lemma 5.1, the existence of spurious modes is directly linked to the fact that for certain κ there exist non-trivial interior Maxwell solutions $\mathbf{E}$ satisfying $\boldsymbol{\gamma}_D^- \mathbf{E} = 0$. On the other hand, there are complex Robin-type boundary conditions which ensure unique solvability of the corresponding boundary value problem, namely,

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - \kappa^2 \mathbf{E} = 0 \quad \text{in } \Omega\,, \qquad \boldsymbol{\gamma}_D^- \mathbf{E} + i\eta\, \boldsymbol{\gamma}_N^- \mathbf{E} = 0 \quad \text{on } \Gamma\,, \tag{14}$$

for some $\eta \in \mathbb{R} \setminus \{0\}$. Testing equation (2) with $\mathbf{V} := \mathbf{E}$ yields the identity

$$i\eta \left\| \boldsymbol{\gamma}_N^- \mathbf{E} \right\|^2_{\boldsymbol{L}^2_{\mathbf{t}}(\Gamma)} = \left\langle \boldsymbol{\gamma}_N^- \mathbf{E}, \boldsymbol{\gamma}_D^- \mathbf{E} \right\rangle_{\mathbf{t}} = \int_\Omega |\mathbf{curl}\,\mathbf{E}|^2 - \kappa^2 |\mathbf{E}|^2 \,\mathrm{d}\mathbf{x} \in \mathbb{R}\,.$$

Considering the imaginary part only, we arrive at $\boldsymbol{\gamma}_N^- \mathbf{E} = 0$ and $\boldsymbol{\gamma}_D^- \mathbf{E} = 0$ immediately follows from the boundary condition. Since both traces are equal to zero we conclude that $\mathbf{E}$ must vanish on Ω, which establishes uniqueness of solutions to the boundary value problem (14). Note that we can rely on a Robin-type boundary operator to state the transmission conditions of (1), as long as we are able to recover the conventional traces.

In order to obtain a stable coupled variational formulation for the Maxwell transmission problem (1), we will make use of the idea of complex linear combinations of traces underlying the boundary value problem (14). Recalling the Calderón projector in its operator form, we arrive at the following two equations

$$\boldsymbol{\gamma}_D^+ \mathbf{E}^{\mathrm{s}} = \left(\tfrac{1}{2}\mathsf{Id} + \mathsf{K}_\kappa\right)\left(\boldsymbol{\gamma}_D^+ \mathbf{E}^{\mathrm{s}}\right) - \mathsf{V}_\kappa\left(\boldsymbol{\gamma}_N^+ \mathbf{E}^{\mathrm{s}}\right) \qquad \text{in } \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma)\,, \tag{15}$$

$$\boldsymbol{\gamma}_N^+ \mathbf{E}^{\mathrm{s}} = \mathsf{W}_\kappa\left(\boldsymbol{\gamma}_D^+ \mathbf{E}^{\mathrm{s}}\right) + \left(\tfrac{1}{2}\mathsf{Id} - \mathsf{K}'_\kappa\right)\left(\boldsymbol{\gamma}_N^+ \mathbf{E}^{\mathrm{s}}\right) \qquad \text{in } \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)\,. \tag{16}$$

Unfortunately, the trace spaces $\boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma)$ and $\boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$ do not match, which means that complex linear combinations of Dirichlet and Neumann traces are not well defined. Thus, we cannot simply work with the natural trace spaces of problem (1), but have to do a lifting of both traces onto $\boldsymbol{L}^2_{\mathbf{t}}(\Gamma)$. Now for $\eta > 0$, the complex linear combination (15) + $i\eta$ (16) yields the identity

$$0 = \left(\mathsf{K}_\kappa - \tfrac{1}{2}\mathsf{Id} + i\eta\, \mathsf{W}_\kappa\right)\left(\boldsymbol{\gamma}_D^+ \mathbf{E}^{\mathrm{s}}\right) - \left(\mathsf{V}_\kappa + i\eta\, \left(\tfrac{1}{2}\mathsf{Id} + \mathsf{K}'_\kappa\right)\right)\left(\boldsymbol{\gamma}_N^+ \mathbf{E}^{\mathrm{s}}\right) \in \boldsymbol{L}^2_{\mathbf{t}}(\Gamma)\,, \tag{17}$$

which can be used to replace either (15) or (16) in a coupled variational formulation. Moreover, in order to get meaningful tangential trace operators, we have to replace $\boldsymbol{H}(\mathbf{curl}, \Omega)$ by the Hilbert space

$$\boldsymbol{X} := \left\{ \mathbf{U} \in \boldsymbol{H}(\mathbf{curl}, \Omega);\ \boldsymbol{\gamma}_D^- \mathbf{U} \in \boldsymbol{L}^2_{\mathbf{t}}(\Gamma) \right\}\,,$$

which is Hilbert space with respect to the graph norm on $\boldsymbol{H}(\mathbf{curl}, \Omega)$, cf. [47, Chapt. 4].

Thus, introducing the new variable $\boldsymbol{\lambda} := \boldsymbol{\gamma}_N^+ \mathbf{E}^{\mathrm{s}} \in \boldsymbol{L}^2_{\mathbf{t}}(\Gamma)$ together with the equations (9), (10), (16), and (17), we arrive at the following coupled variational formulation: Find $\mathbf{E} \in \boldsymbol{X}$ and $\boldsymbol{\lambda} \in \boldsymbol{L}^2_{\mathbf{t}}(\Gamma)$, such that for all $\mathbf{V} \in \boldsymbol{X}$ and $\boldsymbol{\mu} \in \boldsymbol{L}^2_{\mathbf{t}}(\Gamma)$ there holds

$$\begin{aligned} \mathsf{q}_\kappa(\mathbf{E}, \mathbf{V}) - \left\langle \mathsf{W}_\kappa\left(\boldsymbol{\gamma}_D^- \mathbf{E}\right), \boldsymbol{\gamma}_D^- \mathbf{V} \right\rangle_{\mathbf{t}} + \left\langle \left(\mathsf{K}'_\kappa - \tfrac{1}{2}\mathsf{Id}\right)(\boldsymbol{\lambda}), \boldsymbol{\gamma}_D^- \mathbf{V} \right\rangle_{\mathbf{t}} &= \mathsf{f}_2(\mathbf{V})\,, \\ \left\langle \boldsymbol{\mu}, \left(\mathsf{V}_\kappa + i\eta\, \left(\tfrac{1}{2}\mathsf{Id} + \mathsf{K}_\kappa\right)\right)\left(\boldsymbol{\gamma}_D^- \mathbf{E}\right) \right\rangle_{\mathbf{t}} + \left\langle \boldsymbol{\mu}, \left(\tfrac{1}{2}\mathsf{Id} - \mathsf{K}_\kappa - i\eta\, \mathsf{W}_\kappa\right)(\boldsymbol{\lambda}) \right\rangle_{\mathbf{t}} &= \mathsf{g}_2(\boldsymbol{\mu})\,, \end{aligned} \tag{18}$$

where the right hand sides are given by

$$\begin{aligned} \mathsf{f}_2(\mathbf{V}) &= (\mathbf{F},\mathbf{V})_0 - \langle \mathbf{g}_N, \boldsymbol{\gamma}_D^- \mathbf{V}\rangle_{\mathbf{t}} + \langle \mathsf{W}_\kappa(\mathbf{g}_D), \boldsymbol{\gamma}_D^- \mathbf{V}\rangle_{\mathbf{t}}\,, \\ \mathsf{g}_2(\boldsymbol{\mu}) &= \langle \boldsymbol{\mu}, \left(\mathsf{K}_\kappa - \tfrac{1}{2}\mathsf{Id} + i\eta\,\mathsf{W}_\kappa\right)(\mathbf{g}_D)\rangle_{\mathbf{t}}\,. \end{aligned}$$

At first sight, this variational formulation looks promising, since it shares a similar internal structure as the symmetric formulation and in addition there is hope that the additional terms might suppress internal resonances. However, due to a lack of compactness of the boundary integral operators

$$\tfrac{1}{2}\mathsf{Id} + \mathsf{K}_\kappa : \boldsymbol{L}^2_{\mathbf{t}}(\Gamma) \mapsto \boldsymbol{L}^2_{\mathbf{t}}(\Gamma)\,, \qquad \mathsf{W}_\kappa : \boldsymbol{L}^2_{\mathbf{t}}(\Gamma) \mapsto \boldsymbol{L}^2_{\mathbf{t}}(\Gamma)\,,$$

on non-smooth domains, the sesqui-linear form underlying the variational formulation (18) is in general not coercive. Unfortunately, switching to smooth domains (18) does not provide us with a stable variational formulation either, since W_κ is not even compact on smooth boundaries. This bars us from applying the Fredholm alternative on either smooth or non-smooth domains and prevents us from establishing existence and uniqueness of solutions and asymptotic quasi-optimality error estimates. Hence, simple complex combination of Dirichlet and Neumann traces is not enough to stabilize coupled variational formulations.

The problem concerning the non-matching Dirichlet and Neumann trace spaces and the lack of coercivity can be overcome by introducing a special trace transformation operator

$$\boldsymbol{\mathcal{T}} : \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma,\Gamma) \times \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma,\Gamma) \mapsto \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma,\Gamma) \times \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma,\Gamma)$$

defined by

$$\boldsymbol{\mathcal{T}}\begin{bmatrix}\mathbf{u}\\ \boldsymbol{\mu}\end{bmatrix} := \begin{bmatrix}\mathbf{u} + i\eta\,\mathsf{M}(\boldsymbol{\mu})\\ \boldsymbol{\mu}\end{bmatrix}\,, \qquad \eta > 0\,, \tag{19}$$

for all $\mathbf{u} \in \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma,\Gamma)$ and $\boldsymbol{\mu} \in \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma,\Gamma)$.

The main ingredient here is a *regularising operator*

$$\mathsf{M} : \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma,\Gamma) \mapsto \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma,\Gamma)\,,$$

which satisfies the following assumption.

Assumption 6.1. *We suppose that*

1. $\mathsf{M} : \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma,\Gamma) \mapsto \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma,\Gamma)$ *is compact, and*
2. $\mathrm{Re}\left\{\langle \boldsymbol{\mu}, \mathsf{M}(\boldsymbol{\mu})\rangle_{\mathbf{t}}\right\} > 0$ *for all* $\boldsymbol{\mu} \in \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma,\Gamma) \setminus \{0\}$.

After a straightforward application of the trace transformation operator (19) to the exterior Calderón projector (11) the first equation changes into

$$\begin{aligned} \langle \boldsymbol{\mu}, \gamma_D^+\mathbf{E}^{\mathrm{s}} + i\eta\,\mathsf{M}(\gamma_N^+\mathbf{E}^{\mathrm{s}})\rangle_{\mathbf{t}} &= \langle \boldsymbol{\mu}, \left(\tfrac{1}{2}\mathsf{Id} + \mathsf{K}_\kappa + i\eta\,\mathsf{M}\circ\mathsf{W}_\kappa\right)(\gamma_D^+\mathbf{E}^{\mathrm{s}})\rangle_{\mathbf{t}} \\ &\quad + \langle \boldsymbol{\mu}, \left(i\eta\,\mathsf{M}\circ\left(\tfrac{1}{2}\mathsf{Id} - \mathsf{K}'_\kappa\right) - \mathsf{V}_\kappa\right)(\gamma_N^+\mathbf{E}^{\mathrm{s}})\rangle_{\mathbf{t}}\,. \end{aligned}$$

A simple algebraic transformation yields the following variational identities for the Dirichlet and Neumann traces $\boldsymbol{\gamma}_D^+\mathbf{E}^{\mathrm{s}}$ and $\boldsymbol{\gamma}_N^+\mathbf{E}^{\mathrm{s}}$ (cf. [40, Sect. 6] for the Helmholtz case)

$$\begin{aligned} \langle \boldsymbol{\mu}, \boldsymbol{\gamma}_D^+\mathbf{E}^{\mathrm{s}}\rangle_{\mathbf{t}} &= \langle \boldsymbol{\mu}, \left(\tfrac{1}{2}\mathsf{Id} + \mathsf{K}_\kappa + i\eta\,\mathsf{M}\circ\mathsf{W}_\kappa\right)(\gamma_D^+\mathbf{E}^{\mathrm{s}})\rangle_{\mathbf{t}} \\ &\quad - \langle \boldsymbol{\mu}, \left(\mathsf{V}_\kappa + i\eta\,\mathsf{M}\circ\left(\tfrac{1}{2}\mathsf{Id} + \mathsf{K}'_\kappa\right)\right)(\gamma_N^+\mathbf{E}^{\mathrm{s}})\rangle_{\mathbf{t}}\,, \\ \langle \gamma_N^+\mathbf{E}^{\mathrm{s}}, \mathbf{v}\rangle_{\mathbf{t}} &= \langle \mathsf{W}_\kappa(\gamma_D^+\mathbf{E}^{\mathrm{s}}), \mathbf{v}\rangle_{\mathbf{t}} + \langle \left(\tfrac{1}{2}\mathsf{Id} - \mathsf{K}'_\kappa\right)(\gamma_N^+\mathbf{E}^{\mathrm{s}}), \mathbf{v}\rangle_{\mathbf{t}}\,, \end{aligned} \tag{20}$$

for all $\boldsymbol{\mu} \in \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma,\Gamma)$ and $\mathbf{v} \in \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma,\Gamma)$. The identity (20) provides an alternative realization of the Dirichlet-to-Neumann map.

For the construction of regularizing operators we strongly rely on the techniques already established for the regularisation of Maxwell scattering problems (cf. [17, Sect. 4]). A crucial tool in the construction of a suitable regularizing operator will be the following trace space

$$\boldsymbol{H}(\mathsf{curl}_\Gamma,\Gamma) := \left\{\boldsymbol{\mu} \in \boldsymbol{L}^2_{\mathbf{t}}(\Gamma);\ \mathsf{curl}_\Gamma\,\boldsymbol{\mu} \in L^2(\Gamma)\right\} \subset \boldsymbol{L}^2_{\mathbf{t}}(\Gamma)\,.$$

Lemma 6.2. *The space* $\boldsymbol{H}(\mathsf{curl}_\Gamma, \Gamma) \subset \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma)$ *is a dense subspace.*

Proof. We start from the dense inclusions $\boldsymbol{C}^\infty(\Omega) \subset \boldsymbol{H}^1(\Omega)$ and $\boldsymbol{H}^1(\Omega) \subset \boldsymbol{H}(\mathbf{curl}, \Omega)$. By definition and due to theorem 2.3 we conclude that $\mathbf{V}_\gamma := \boldsymbol{\gamma}_D\big(\boldsymbol{C}^\infty(\Omega)\big) \subset \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma)$ is dense. Since the following inclusions hold $\mathbf{V}_\gamma \subset \boldsymbol{H}(\mathsf{curl}_\Gamma, \Gamma) \subset \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma)$, the statement is proved. □

Lemma 6.3. *The embedding* $\boldsymbol{H}(\mathsf{curl}_\Gamma, \Gamma) \hookrightarrow \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma)$ *is compact.*

Proof. Our is similar to the one given in [17, Lem. 2.5]. Let $\{\mathbf{u}_n\}_{n\in\mathbb{N}} \subset \boldsymbol{H}(\mathsf{curl}_\Gamma, \Gamma)$ such that $\|\mathbf{u}_n\|_{\boldsymbol{H}(\mathsf{curl}_\Gamma, \Gamma)} \leq 1$, for all $n \in \mathbb{N}$. The compact embedding $\boldsymbol{L}^2_{\mathbf{t}}(\Gamma) \hookrightarrow \boldsymbol{H}_\perp^{-1/2}(\Gamma)$ directly implies, that there exists $\mathbf{u} \in \boldsymbol{H}_\perp^{-1/2}(\Gamma)$ and a subsequence $\mathbf{u}_{n_k}$ of $\mathbf{u}_n$ such that $\mathbf{u}_{n_k} \to \mathbf{u}$ strongly in $\boldsymbol{H}_\perp^{-1/2}(\Gamma)$.

Due to the continuity of the operator $\mathsf{curl}_\Gamma : \boldsymbol{H}_\perp^{-1/2}(\Gamma) \mapsto H^{-3/2}(\Gamma)$ we get $\mathsf{curl}_\Gamma \mathbf{u}_{n_k} \to \mathsf{curl}_\Gamma \mathbf{u}$ strongly in $H^{-3/2}(\Gamma)$ (see [12] for the proof and a definition of $H^{-3/2}(\Gamma)$).

On the other hand we know that $\|\mathsf{curl}_\Gamma \mathbf{u}_{n_k}\|_{L^2(\Gamma)} \leq 1$, which implies up to extraction of a subsequence $\mathsf{curl}_\Gamma \mathbf{u}_{n_k}$, is strongly converging to an element in $\boldsymbol{H}_\perp^{-1/2}(\Gamma)$. By uniqueness of the limit we conclude that $\mathsf{curl}_\Gamma \mathbf{u} \in \boldsymbol{H}_\perp^{-1/2}(\Gamma)$, and, up to selecting a proper subsequence $\mathbf{u}_{n_k} \to \mathbf{u} \in \boldsymbol{H}_\perp^{-1/2}(\Gamma)$, strongly. □

A simple eligible operator M can be introduced through a variational definition: For any $\boldsymbol{\zeta} \in \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$ find $\mathsf{M}(\boldsymbol{\zeta}) \in \boldsymbol{H}(\mathsf{curl}_\Gamma, \Gamma)$ such that

$$\big\langle \mathsf{M}(\boldsymbol{\zeta}), \mathbf{q} \big\rangle_{\mathbf{t}} + \big\langle \mathsf{curl}_\Gamma \mathsf{M}(\boldsymbol{\zeta}), \mathsf{curl}_\Gamma \mathbf{q} \big\rangle_{\mathbf{t}} = \big\langle \boldsymbol{\zeta}, \mathbf{q} \big\rangle_{\mathbf{t}}, \qquad \forall \mathbf{q} \in \boldsymbol{H}(\mathsf{curl}_\Gamma, \Gamma). \tag{21}$$

In order to simplify notations we introduce the associated sesqui-linear form

$$\mathsf{b}(\mathbf{p}, \mathbf{q}) := \big\langle \mathbf{p}, \mathbf{q} \big\rangle_{\mathbf{t}} + \big\langle \mathsf{curl}_\Gamma \mathbf{p}, \mathsf{curl}_\Gamma \mathbf{q} \big\rangle_{\mathbf{t}}. \tag{22}$$

Compactness of $\mathsf{M} : \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma) \mapsto \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma)$ immediately follows from lemma 6.3.

Lemma 6.4. *The regularisation operator* $\mathsf{M} : \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma) \mapsto \boldsymbol{H}(\mathsf{curl}_\Gamma, \Gamma)$ *defined by* (21) *is injective and thus item* ***2.*** *of assumption 6.1 holds true for all* $\boldsymbol{\mu} \in \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$.

Proof. The proof closely follows the one given in [17, Sect. 4]. Assume that $\mathsf{M}(\boldsymbol{\zeta}) = 0$ from which we conclude that $\big\langle \boldsymbol{\zeta}, \mathbf{q} \big\rangle_{\mathbf{t}} = 0$ for all $\mathbf{q} \in \boldsymbol{H}(\mathsf{curl}_\Gamma, \Gamma)$. Now choose $\boldsymbol{\eta} \in \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma)$ such that

$$\big\langle \boldsymbol{\zeta}, \boldsymbol{\eta} \big\rangle_{\mathbf{t}} = \|\boldsymbol{\zeta}\|^2_{\boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)}$$

and since $\boldsymbol{H}(\mathsf{curl}_\Gamma, \Gamma) \subset \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma)$ is dense, there exists a sequence $\{\boldsymbol{\eta}_k\}_{k\in\mathbb{N}} \subset \boldsymbol{H}(\mathsf{curl}_\Gamma, \Gamma)$ such that $\boldsymbol{\eta}_k \to \boldsymbol{\eta}$ strongly in $\boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma, \Gamma)$. From the definition of the regularisation operator we infer $0 = \big\langle \boldsymbol{\zeta}, \boldsymbol{\eta}_k \big\rangle_{\mathbf{t}}$, for all $k \in \mathbb{N}$. Thus taking the limit yields $\boldsymbol{\zeta} = 0$, which finishes the proof. □

Thus we conclude that both items of assumption 6.1 are satisfied and hence M given by the implicit definition (21) qualifies as a regularising operator.

Using the abbreviation $\boldsymbol{\lambda} := \boldsymbol{\gamma}_N^+ \mathbf{E}^{\mathrm{s}} \in \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$ and the same trick as in section 5, to couple the boundary integral equations (20) together with the variational problem on Ω, we finally end up with the following formulation: Find $\mathbf{E} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$ and $\boldsymbol{\vartheta} \in \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$ such that for all $\mathbf{V} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$ and $\boldsymbol{\mu} \in \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$

$$\begin{aligned} \mathsf{q}_\kappa(\mathbf{E}, \mathbf{V}) - \big\langle \mathsf{W}_\kappa\big(\boldsymbol{\gamma}_D^- \mathbf{E}\big), \boldsymbol{\gamma}_D^- \mathbf{V} \big\rangle_{\mathbf{t}} + \big\langle \big(\mathsf{K}'_\kappa - \tfrac{1}{2}\mathsf{Id}\big)\,(\boldsymbol{\lambda}), \boldsymbol{\gamma}_D^- \mathbf{V} \big\rangle_{\mathbf{t}} &= \mathsf{f}_3(\mathbf{V}), \\ \big\langle \boldsymbol{\mu}, \big(\tfrac{1}{2}\mathsf{Id} - \mathsf{K}_\kappa - i\eta\,\mathsf{M} \circ \mathsf{W}_\kappa\big)\,\big(\boldsymbol{\gamma}_D^- \mathbf{E}\big) \big\rangle_{\mathbf{t}} \qquad & \\ + \big\langle \boldsymbol{\mu}, \big(\mathsf{V}_\kappa + i\eta\,\mathsf{M} \circ \big(\tfrac{1}{2}\mathsf{Id} + \mathsf{K}'_\kappa\big)\big)\,(\boldsymbol{\lambda}) \big\rangle_{\mathbf{t}} &= \mathsf{g}_3(\boldsymbol{\mu}), \end{aligned} \tag{23}$$

with the right hand sides given by

$$\begin{aligned}
\mathsf{f}_3(\mathbf{V}) &:= (\mathbf{F},\,\mathbf{V})_0 - \langle \mathbf{g}_N, \boldsymbol{\gamma}_D^- \mathbf{V}\rangle_{\mathbf{t}} + \langle \mathsf{W}_\kappa(\mathbf{g}_D), \boldsymbol{\gamma}_D^- \mathbf{V}\rangle_{\mathbf{t}}\,,\\
\mathsf{g}_3(\boldsymbol{\mu}) &:= \left\langle \boldsymbol{\mu}, \left(\mathsf{K}_\kappa - \tfrac{1}{2}\mathsf{Id} + i\eta\,\mathsf{M}\circ\mathsf{W}_\kappa\right)(\mathbf{g}_D)\right\rangle .
\end{aligned}$$

Summing up, due to the compactness of the regularisation operator M we conclude that all additional off-diagonal terms in the sesqui-linear form underlying the regularised variational formulation, compared to the symmetric formulation, are compact. In combination with theorem 5.2 this results again in a Gårding inequality on $\boldsymbol{\mathcal{V}}$. It remains to establish uniqueness of solutions, which amounts to confirming that (23) is really immune to spurious resonances.

Lemma 6.5. *Any solution of* (23) *provides a solution of* (1) *by retaining* $\mathbf{E}$ *in* Ω *and using the representation formula* (5) *for the Cauchy data* $(\boldsymbol{\gamma}_D^-\mathbf{E} + \mathbf{g}_D,\, \boldsymbol{\lambda})$ *in* Ω^+.

Proof. Our approach is based on [52, Sect. 4.3] and [18, Sect. 5]. Testing with $\mathbf{V}$ that is compactly supported in Ω confirms that $\mathbf{E}$ satisfies (1) in Ω. We conclude (9) for any admissible $\mathbf{V}$. This renders (23) equivalent to

$$\begin{aligned}
\langle \boldsymbol{\xi}, \boldsymbol{\gamma}_D^-\mathbf{V}\rangle_{\mathbf{t}} - \langle \mathsf{W}_\kappa(\mathbf{u}), \boldsymbol{\gamma}_D^-\mathbf{V}\rangle_{\mathbf{t}} + \left\langle \left(\mathsf{K}'_\kappa - \tfrac{1}{2}\mathsf{Id}\right)(\boldsymbol{\lambda}), \boldsymbol{\gamma}_D^-\mathbf{V}\right\rangle_{\mathbf{t}} &= 0\,,\\
\left\langle \boldsymbol{\mu}, \left(\tfrac{1}{2}\mathsf{Id} - \mathsf{K}_\kappa - i\eta\,\mathsf{M}\circ\mathsf{W}_\kappa\right)(\mathbf{u})\right\rangle_{\mathbf{t}} - \left\langle \boldsymbol{\mu}, \left(\mathsf{V}_\kappa + i\eta\,\mathsf{M}\circ\left(\tfrac{1}{2}\mathsf{Id} + \mathsf{K}'_\kappa\right)\right)(\boldsymbol{\lambda})\right\rangle_{\mathbf{t}} &= 0\,,
\end{aligned}$$

with $\boldsymbol{\xi} := \mu_r^{-1}\boldsymbol{\gamma}_N^-\mathbf{E} - \mathbf{g}_N$ and $\mathbf{u} := \boldsymbol{\gamma}_D^-\mathbf{E} - \mathbf{g}_D$. Translated into operator notation this yields

$$\left(\boldsymbol{\mathcal{T}} \circ \begin{bmatrix} \tfrac{1}{2}\mathsf{Id} - \mathsf{K}_\kappa & \mathsf{V}_\kappa \\ -\mathsf{W}_\kappa & \tfrac{1}{2}\mathsf{Id} + \mathsf{K}'_\kappa \end{bmatrix}\right) \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{\lambda} - \boldsymbol{\xi} \end{bmatrix}, \tag{24}$$

where the second operator in the product we recognize as an *interior Calderón projector* [18, Sect. 3.3]. By applying the trace transformation operator (19) to the Dirichlet and Neumann traces of the following function

$$\mathbf{U}(\mathbf{x}) := \boldsymbol{\Psi}_{\mathrm{SL}}^\kappa(\boldsymbol{\lambda})(\mathbf{x}) - \boldsymbol{\Psi}_{\mathrm{DL}}^\kappa(\mathbf{u})(\mathbf{x})\,, \qquad \mathbf{x} \in \Omega\,,$$

we obtain the traces

$$\boldsymbol{\gamma}_D^-\mathbf{U} + i\eta\,\mathsf{M}(\boldsymbol{\gamma}_N^-\mathbf{U}) = 0\,, \qquad \boldsymbol{\gamma}_N^-\mathbf{U} = \boldsymbol{\lambda} - \boldsymbol{\xi}\,.$$

Furthermore, since $\mathbf{U}$ is a solution to the boundary value problem

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - \kappa^2\mathbf{U} = 0 \text{ in } \Omega\,, \qquad \boldsymbol{\gamma}_D^-\mathbf{U} + i\eta\,\mathsf{M}(\boldsymbol{\gamma}_N^-\mathbf{U}) = 0 \text{ on } \Gamma\,,$$

integration by parts together with the "test function" $\mathbf{V} := \overline{\mathbf{U}}$ yields

$$i\eta\,\langle \boldsymbol{\gamma}_N^-\mathbf{U}, \mathsf{M}(\boldsymbol{\gamma}_N^-\mathbf{U})\rangle_{\mathbf{t}} = \langle \boldsymbol{\gamma}_N^-\mathbf{U}, \boldsymbol{\gamma}_D^-\mathbf{U}\rangle_{\mathbf{t}} = \int_\Omega |\mathbf{curl}\,\mathbf{U}|^2 - \kappa^2\,|\mathbf{U}|^2\,\mathrm{d}\mathbf{x} \in \mathbb{R}\,.$$

Considering the imaginary part of the previous equation we finally arrive at

$$0 = \eta\,\mathrm{Re}\left\{\langle \boldsymbol{\gamma}_N^-\mathbf{U}, \mathsf{M}(\boldsymbol{\gamma}_N^-\mathbf{U})\rangle_{\mathbf{t}}\right\}$$

and item **2.** of assumption 6.1 immediately implies $\boldsymbol{\lambda} = \boldsymbol{\xi}$. From (24) we conclude that $(\mathbf{u},\,\boldsymbol{\lambda})$ belong to the kernel of the interior Calderón projector, which implies that they represent Cauchy data of an exterior Maxwell solution. Hence, due to the following definition

$$\mathbf{W}(\mathbf{x}) := \boldsymbol{\Psi}_{\mathrm{DL}}^\kappa(\mathbf{u})(\mathbf{x}) - \boldsymbol{\Psi}_{\mathrm{SL}}^\kappa(\boldsymbol{\lambda})(\mathbf{x})\,, \qquad \mathbf{x} \in \Omega^+\,.$$

we obtain a pair of solutions $(\mathbf{E},\,\mathbf{W})$ to the Maxwell transmission problem (1). Finally, uniqueness of solutions to the transmission problem carries over to the variational formulation (23). □

Eventually, existence of solutions to the variational problem (23) follows from their uniqueness and a Fredholm argument, see [46, Thm. 2.33].

Although (23) provides us with a stable variational formulation, it cannot be discretized by means of a straightforward Galerkin scheme, due to the various operator products. This suggests to introduce the auxiliary variable

$$\mathbf{p} := \mathsf{M}\big(\left(\tfrac{1}{2}\mathsf{Id} + \mathsf{K}'_\kappa\right) (\boldsymbol{\lambda}) - \mathsf{W}_\kappa\big(\gamma_D^- \mathbf{E} + \mathbf{g}_D\big)\big) \in \boldsymbol{H}(\mathsf{curl}_\Gamma, \Gamma)\,, \tag{25}$$

which converts (23) into the following mixed variational formulation: Find $\mathbf{E} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$, $\boldsymbol{\lambda} \in \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$, and $\mathbf{p} \in \boldsymbol{H}(\mathsf{curl}_\Gamma, \Gamma)$ such that for all $\mathbf{V} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$, $\boldsymbol{\mu} \in \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$, and $\mathbf{q} \in \boldsymbol{H}(\mathsf{curl}_\Gamma, \Gamma)$

$$\begin{aligned}
\mathsf{q}_\kappa(\mathbf{E}, \mathbf{V}) - \big\langle \mathsf{W}_\kappa\big(\gamma_D^- \mathbf{E}\big), \gamma_D^- \mathbf{V}\big\rangle_{\mathbf{t}} + \big\langle \big(\mathsf{K}'_\kappa - \tfrac{1}{2}\mathsf{Id}\big)\,(\boldsymbol{\lambda}), \gamma_D^- \mathbf{V}\big\rangle_{\mathbf{t}} &= \mathsf{f}_4(\mathbf{V})\,, \\
\big\langle \boldsymbol{\mu}, \big(\tfrac{1}{2}\mathsf{Id} - \mathsf{K}_\kappa\big)\,\big(\gamma_D^- \mathbf{E}\big)\big\rangle_{\mathbf{t}} + \big\langle \boldsymbol{\mu}, \mathsf{V}_\kappa(\boldsymbol{\lambda})\big\rangle_{\mathbf{t}} - i\eta \langle \boldsymbol{\mu}, \mathbf{p}\rangle_{\mathbf{t}} &= \mathsf{g}_4(\boldsymbol{\mu})\,, \\
\big\langle \mathsf{W}_\kappa\big(\gamma_D^- \mathbf{E}\big), \mathbf{q}\big\rangle_{\mathbf{t}} - \big\langle \big(\tfrac{1}{2}\mathsf{Id} + \mathsf{K}'_\kappa\big)\,(\boldsymbol{\lambda}), \mathbf{q}\big\rangle_{\mathbf{t}} + \mathsf{b}(\mathbf{p}, \mathbf{q}) &= \mathsf{h}_4(\mathbf{q})\,,
\end{aligned} \tag{26}$$

with right hand sides given by

$$\begin{aligned}
\mathsf{f}_4(\mathbf{V}) &:= (\mathbf{F}, \mathbf{V})_0 - \langle \mathbf{g}_N, \gamma_D^- \mathbf{V}\rangle_{\mathbf{t}} + \big\langle \mathsf{W}_\kappa(\mathbf{g}_D), \gamma_D^- \mathbf{V}\big\rangle_{\mathbf{t}}\,, \\
\mathsf{g}_4(\boldsymbol{\mu}) &:= \big\langle \boldsymbol{\mu}, \big(\mathsf{K}_\kappa - \tfrac{1}{2}\mathsf{Id}\big)\,(\mathbf{g}_D)\big\rangle_{\mathbf{t}}\,, \\
\mathsf{h}_4(\mathbf{q}) &:= -\big\langle \mathsf{W}_\kappa(\mathbf{g}_D), \mathbf{q}\big\rangle_{\mathbf{t}}\,.
\end{aligned}$$

An essential feature of the stabilized variational formulation is that the auxiliary unknown $\mathbf{p}$ obtained from the solutions $(\mathbf{E}, \boldsymbol{\lambda}, \mathbf{p})$ to the mixed variational formulation (26) can be recast into the following expression

$$\mathbf{p} = \mathsf{M}\big(\left(\tfrac{1}{2}\mathsf{Id} + \mathsf{K}'_\kappa\right) (\boldsymbol{\lambda}) - \mathsf{W}_\kappa\big(\gamma_D^- \mathbf{E} + \mathbf{g}_D\big)\big)\,.$$

At second glance, we realize that $\mathbf{p} = \mathbf{0}$, if $(\mathbf{E}, \boldsymbol{\vartheta})$ solves (26). This follows directly from lemma 6.5 and (11). Summing up, $\mathbf{p}$ is a "dummy variable".

Again using the splittings from section 4 and grouping the components into electric $(\boldsymbol{\lambda}^\perp, \mathbf{E}^0)$, magnetic $(\boldsymbol{\lambda}^0, \mathbf{E}^\perp)$, and auxiliary ones $\mathbf{p}$ we arrive at a variational formulation on the Hilbert space

$$\boldsymbol{\mathcal{W}} := \boldsymbol{\mathcal{V}} \times \boldsymbol{H}(\mathsf{curl}_\Gamma, \Gamma)\,,$$

that is endowed with the natural graph norm: Find $(\boldsymbol{\lambda}^\perp, \mathbf{E}^0, \boldsymbol{\lambda}^0, \mathbf{E}^\perp, \mathbf{p}) \in \boldsymbol{\mathcal{W}}$ such that

$$\widehat{\mathsf{a}}_\kappa^{\mathrm{reg}}\Big((\boldsymbol{\lambda}^\perp, \mathbf{E}^0, \boldsymbol{\lambda}^0, \mathbf{E}^\perp, \mathbf{p}), (\boldsymbol{\mu}^\perp, \mathbf{V}^0, \boldsymbol{\mu}^0, \mathbf{V}^\perp, \mathbf{q})\Big) = \widehat{\mathsf{f}}_\kappa^{\mathrm{reg}}\Big(\boldsymbol{\mu}^\perp, \mathbf{V}^0, \boldsymbol{\mu}^0, \mathbf{V}^\perp, \mathbf{q}\Big)\,, \tag{27}$$

for all $(\boldsymbol{\mu}^\perp, \mathbf{V}^0, \boldsymbol{\mu}^0, \mathbf{V}^\perp, \mathbf{q}) \in \boldsymbol{\mathcal{W}}$. Again, the sesqui-linear form and the linear form of the split variational equation are related to those underlying (26), namely $\mathsf{a}_\kappa^{\mathrm{reg}}$ and $\mathsf{f}_\kappa^{\mathrm{reg}}$, through the following equations

$$\begin{aligned}
&\widehat{\mathsf{a}}_\kappa^{\mathrm{reg}}\Big((\boldsymbol{\lambda}^\perp, \mathbf{E}^0, \boldsymbol{\lambda}^0, \mathbf{E}^\perp, \mathbf{p}), (\boldsymbol{\mu}^\perp, \mathbf{V}^0, \boldsymbol{\mu}^0, \mathbf{V}^\perp, \mathbf{q})\Big) := \\
&\qquad\qquad \mathsf{a}_\kappa^{\mathrm{reg}}\Big((\mathbf{E}^\perp + \mathbf{E}^0, \boldsymbol{\lambda}^\perp + \boldsymbol{\lambda}^0, \mathbf{p}), (\mathbf{V}^\perp - \mathbf{V}^0, -\boldsymbol{\mu}^\perp + \boldsymbol{\mu}^0, \mathbf{q})\Big)\,, \\
&\widehat{\mathsf{f}}_\kappa^{\mathrm{reg}}\Big(\boldsymbol{\mu}^\perp, \mathbf{V}^0, \boldsymbol{\mu}^0, \mathbf{V}^\perp, \mathbf{q}\Big) := \mathsf{f}_\kappa^{\mathrm{reg}}\Big(\mathbf{V}^\perp - \mathbf{V}^0, -\boldsymbol{\mu}^\perp + \boldsymbol{\mu}^0, \mathbf{q}\Big)\,.
\end{aligned}$$

In order to settle the issue of existence and uniqueness of solutions of (26) we first observe that by the very definition of M in (21) and (25) the first two components of $(\mathbf{U}, \boldsymbol{\lambda}, \mathbf{p})$ of (26) will also solve (23) and thus lemma 6.5 ensures uniqueness.

The next lemma tells us that we do not need to worry about the new terms introduced into the variational equations.

Lemma 6.6. *The following sesqui-linear forms are compact*

$$\begin{aligned}
\langle\cdot,\cdot\rangle_{\mathbf{t}} &: \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma,\Gamma)\times\boldsymbol{H}(\mathsf{curl}_\Gamma,\Gamma) \mapsto \mathbb{C}\,,\\
\langle\mathsf{W}_\kappa(\,\cdot\,),\cdot\rangle_{\mathbf{t}} &: \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma,\Gamma)\times\boldsymbol{H}(\mathsf{curl}_\Gamma,\Gamma) \mapsto \mathbb{C}\,,\\
\left\langle\left(\tfrac{1}{2}\mathsf{Id}+\mathsf{K}'_\kappa\right)(\,\cdot\,),\cdot\right\rangle_{\mathbf{t}} &: \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma,\Gamma)\times\boldsymbol{H}(\mathsf{curl}_\Gamma,\Gamma) \mapsto \mathbb{C}\,.
\end{aligned}$$

Proof. It is sufficient to note that the sesqui-linear forms

$$\langle\cdot,\cdot\rangle_{\mathbf{t}},\ \left\langle\left(\tfrac{1}{2}\mathsf{Id}+\mathsf{K}'_\kappa\right)(\,\cdot\,),\cdot\right\rangle_{\mathbf{t}} : \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma,\Gamma)\times\boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma,\Gamma)\mapsto\mathbb{C}$$

and

$$\langle\mathsf{W}_\kappa(\,\cdot\,),\cdot\rangle_{\mathbf{t}} : \boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma,\Gamma)\times\boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma,\Gamma)\mapsto\mathbb{C}$$

are continuous and that the injection $\boldsymbol{H}(\mathsf{curl}_\Gamma,\Gamma)\hookrightarrow\boldsymbol{H}^{-1/2}(\mathsf{curl}_\Gamma,\Gamma)$ is compact due to lemma 6.3. □

As an immediate consequence of this result we note that all additional off-diagonal terms of (26) are compact. Furthermore, the sesqui-linear form

$$\mathsf{b}(\cdot,\cdot):\boldsymbol{H}(\mathsf{curl}_\Gamma,\Gamma)\times\boldsymbol{H}(\mathsf{curl}_\Gamma,\Gamma)\mapsto\mathbb{C}$$

is clearly elliptic, since it gives rise to an inner product on $\boldsymbol{H}(\mathsf{curl}_\Gamma,\Gamma)$. In combination with theorem 5.2 we conclude that the sesqui-linear form $\widehat{\mathsf{a}}_\kappa^{\mathrm{reg}}:\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{W}}\mapsto\mathbb{C}$ from (27) satisfies a Gårding inequality on the Hilbert space $\boldsymbol{\mathcal{W}}$.

Again, a Fredholm argument ensures the existence of solutions from the uniqueness result. Thus we have obtained a well-posed variational formulation which yields weak solutions to the transmission problem and which is amenable to standard Galerkin discretizations.

Summing up, the sesqui-linear form $\mathsf{a}_\kappa^{\mathrm{reg}}$ underlying the stabilized variational formulation (26) on the non-split Sobolev spaces can be decomposed into the following block structure

$$\begin{array}{ccccc}
\boxed{\mathsf{q}_\kappa(\mathbf{E},\mathbf{V})-\langle\mathsf{W}_\kappa(\gamma_D^-\mathbf{E}),\gamma_D^-\mathbf{V}\rangle_{\mathbf{t}}} & + & \left\langle\left(\mathsf{K}'_\kappa-\tfrac{1}{2}\mathsf{Id}\right)(\boldsymbol{\lambda}),\gamma_D^-\mathbf{V}\right\rangle_{\mathbf{t}} & & \\
\left\langle\boldsymbol{\mu},\left(\tfrac{1}{2}\mathsf{Id}-\mathsf{K}_\kappa\right)(\gamma_D^-\mathbf{E})\right\rangle_{\mathbf{t}} & + & \boxed{\langle\boldsymbol{\mu},\mathsf{V}_\kappa(\boldsymbol{\lambda})\rangle_{\mathbf{t}}} & - & i\eta\langle\boldsymbol{\mu},\mathbf{p}\rangle_{\mathbf{t}}\\
\langle\mathsf{W}_\kappa(\gamma_D^-\mathbf{E}),\mathbf{q}\rangle_{\mathbf{t}} & - & \left\langle\left(\tfrac{1}{2}\mathsf{Id}+\mathsf{K}'_\kappa\right)(\boldsymbol{\lambda}),\mathbf{q}\right\rangle_{\mathbf{t}} & + & \boxed{\mathsf{b}(\mathbf{p},\mathbf{q})}
\end{array}$$

From lemma 6.6 we obtain that the dotted frames mark compact sesqui-linear forms. Furthermore, lemma 3.7 implies that the operator $\mathsf{K}_\kappa^*-\mathsf{K}'_\kappa:\boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma,\Gamma)\mapsto\boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma,\Gamma)$ is compact and thus up to some compact perturbation, the corresponding sesqui-linear forms cancel out. Finally, we conclude that all sesqui-linear forms enclosed by solid boxes are skew-symmetric. Thus, we conclude that up to compact perturbations the sesqui-linear form $\mathsf{a}_\kappa^{\mathrm{reg}}$ has a block skew-symmetric structure.

7 Galerkin Discretization

We equip (the curvilinear polyhedron) Ω with a family of tetrahedral, shape regular triangulations $\{\Omega_h\}_h$. The parameter h designates the mesh width, that is the length of the longest edge. Let $\mathbb{H}$ stand for the collection of mesh widths occurring in $\{\Omega_h\}_h$ and, moreover, assume that $\mathbb{H}\subset\mathbb{R}_+$ forms a decreasing series tending to zero. The set $\mathcal{T}_h$ will include all tetrahedra of Ω_h. Restricting Ω_h, $h\in\mathbb{H}$, to Γ gives a sequence $\{\Gamma_h\}_h$ of surface meshes. They inherit shape regularity from $\{\Omega_h\}_h$. We suppose that all Γ_h are aligned with edges of Γ.

Discrete electric fields should be modeled by discrete 1-forms (edge elements). They can be represented by piecewise polynomial vector fields: For a *fixed* degree ν, $\nu\in\mathbb{N}_0$, and any tetrahedron $T\in\mathcal{T}_h$ the local spaces are given by (cf. [48])

$$\boldsymbol{\mathcal{E}}^1_{\nu+1}(T):=\left\{\mathbf{V}\in(\mathcal{P}_{\nu+1}(T))^3\,;\ \mathbf{V}(\mathbf{x})\cdot\mathbf{x}=0\ \forall\mathbf{x}\in T\right\},$$

where $\mathcal{P}_{\nu+1}(T)$ is the space of multivariate polynomials of total degree ν on T. This gives rise to the global finite element space

$$\boldsymbol{\mathcal{E}}^1_{\nu+1}(\Omega_h) := \left\{ \mathbf{U} \in \boldsymbol{H}(\mathbf{curl}, \Omega);\ \mathbf{U}_{|T} \in \boldsymbol{\mathcal{E}}^1_{\nu+1}(T)\ \forall T \in \Omega_h \right\}.$$

This renders degrees of freedom based on moments (of tangential components) on edges, faces, and the elements themselves well defined. See [37] and [24] for details and proof of unisolvence. The discrete 1-forms on $\{\Omega_h\}_h$ form an affine family of finite elements in the sense of [23] with respect to the pullback of 1-forms. Based on the degrees of freedom, we can introduce nodal interpolation operators $\boldsymbol{\Pi}^1_{\nu+1}$ onto $\boldsymbol{\mathcal{E}}^1_{\nu+1}(\Omega_h)$. To begin with they are defined only for continuous vector fields but can be generalized to less regular settings

Lemma 7.1. *If $s > \frac{1}{2}$, then for all $\mathbf{U} \in \boldsymbol{H}^s(\Omega)$ such that $\mathbf{curl}\,\mathbf{U} \in \boldsymbol{H}^s(\Omega)$*

$$\begin{aligned} \left\|\mathbf{U} - \boldsymbol{\Pi}^1_{\nu+1}(\mathbf{U})\right\|_{\boldsymbol{L}^2(\Omega)} &\le Ch^{\min\{\nu+1,s\}} \left(\left|\mathbf{U}\right|_{\boldsymbol{H}^s(\Omega)} + \left|\mathbf{curl}\,\mathbf{U}\right|_{\boldsymbol{H}^s(\Omega)} \right), \\ \left\|\mathbf{curl}\left(\mathbf{U} - \boldsymbol{\Pi}^1_{\nu+1}(\mathbf{U})\right)\right\|_{\boldsymbol{L}^2(\Omega)} &\le Ch^{\min\{\nu+1,s\}} \left|\mathbf{curl}\,\mathbf{U}\right|_{\boldsymbol{H}^s(\Omega)}, \end{aligned}$$

with constants $C > 0$ depending only on Ω, ν, s and the shape-regularity of the meshes.

Proof. For a proof see [24, Lem. 3.2, Lem. 3.3]. □

The reason why we want to use the nodal interpolation operator $\boldsymbol{\Pi}^1_{\nu+1}$, although it fails to be defined on the entire space $\boldsymbol{H}(\mathbf{curl}, \Omega)$, is its exceptional algebraic properties. In order to explain them, we need to introduce the $\boldsymbol{H}(\mathsf{div}, \Omega)$-conforming spaces $\boldsymbol{\mathcal{F}}_\nu(\Omega_h)$ of discrete 2-forms, also known as face elements, cf. [8, Chap. 3] and [48]. Suitable degrees of freedom for this space are provided by moments of face fluxes and weighted integrals over elements. They introduce the nodal interpolation operators $\boldsymbol{\Pi}^2_\nu$ onto $\boldsymbol{\mathcal{F}}_\nu(\Omega_h)$. A straightforward application of the Stokes theorem, cf. [37], confirms the following *commuting diagram property*

$$\mathbf{curl} \circ \boldsymbol{\Pi}^1_{\nu+1} = \boldsymbol{\Pi}^2_\nu \circ \mathbf{curl}, \tag{28}$$

which is valid for all vector fields contained in the domain $\mathrm{Dom}\left(\boldsymbol{\Pi}^1_{\nu+1}\right)$ of $\boldsymbol{\Pi}^1_{\nu+1}$. From relation (28) we conclude that $\boldsymbol{\Pi}^1_{\nu+1}$ leaves the kernel of the $\mathbf{curl}$ invariant.

To pick a suitable discrete trial space for $\boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$ we also adopt the perspective of differential forms. Be aware that $\boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$ is the trace space for magnetic fields, and keep in mind that those can also be described by 1-forms. This suggests that $\boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$ should be approximated by traces of discrete 1-forms on the surface. In other words, as $\boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma, \Gamma)$-conforming boundary element space we chose $\boldsymbol{\gamma}_\times \boldsymbol{\mathcal{E}}^1_{\nu+1}(\Omega_h)$. Elementary computations reveal that this generates exactly the two-dimensional face elements $\boldsymbol{\mathcal{F}}_\nu(\Gamma_h)$ on the surface mesh, see [50]. The degrees of freedom are also inherited from $\boldsymbol{\mathcal{E}}^1_{\nu+1}(\Omega_h)$. By construction, the induced nodal interpolation operator $\boldsymbol{\Phi}^2_\nu$ satisfies

$$\boldsymbol{\Phi}^2_\nu \circ \boldsymbol{\gamma}_\times = \boldsymbol{\gamma}_\times \circ \boldsymbol{\Pi}^1_{\nu+1},$$

which, due to (28), implies another commuting diagram property,

$$\mathsf{div}_\Gamma \circ \boldsymbol{\Phi}^2_\nu = \mathsf{Q}^\Gamma_\nu \circ \mathsf{div}_\Gamma,$$

for sufficiently smooth tangential surface vector fields. Here, Q^Γ_ν is the plain $L^2(\Gamma)$-orthogonal projection onto the space $\mathcal{Q}_\nu(\Gamma_h)$ of discontinuous, piecewise polynomials of degree ν on Γ_h. Invariance of $\mathrm{Ker}\left(\mathsf{div}_\Gamma\right) \cap \mathrm{Dom}\left(\boldsymbol{\Phi}^2_\nu\right)$ under $\boldsymbol{\Phi}^2_\nu$ is immediate.

From the results of [50] and [41, Sect. 5] we obtain the following interpolation error estimates.

Lemma 7.2. *If $\boldsymbol{\mu} \in \boldsymbol{H}^s_{\mathbf{t}}(\Gamma)$, $\mathsf{div}_\Gamma\,\boldsymbol{\mu} \in H^s(\Gamma)$ for some $s > 0$, then*

$$\begin{aligned} \left\|\boldsymbol{\mu} - \boldsymbol{\Phi}^2_\nu(\boldsymbol{\mu})\right\|_{\boldsymbol{L}^2(\Gamma)} &\le Ch^{\min\{s,\nu+1\}} \left(\left|\boldsymbol{\mu}\right|_{\boldsymbol{H}^s_{\mathbf{t}}(\Gamma)} + \left|\mathsf{div}_\Gamma\,\boldsymbol{\mu}\right|_{H^s(\Gamma)} \right), \\ \left\|\mathsf{div}_\Gamma\left(\boldsymbol{\mu} - \boldsymbol{\Phi}^2_\nu(\boldsymbol{\mu})\right)\right\|_{L^2(\Gamma)} &\le Ch^{\min\{s,\nu+1\}} \left|\mathsf{div}_\Gamma\,\boldsymbol{\mu}\right|_{H^s(\Gamma)}. \end{aligned}$$

Finally, it remains to choose a suitable conforming discrete trial space for $\boldsymbol{H}(\mathsf{curl}_\Gamma, \Gamma)$. Picking an arbitrary function $\mathbf{u}_h$ from a such trial space, it must feature $\mathbf{q}_h \in \boldsymbol{L}^2_{\mathbf{t}}(\Gamma)$, as well as $\mathsf{curl}_\Gamma \mathbf{q}_h \in L^2(\Gamma)$. Thus a suitable choice would be to take $\boldsymbol{\gamma}_{\mathbf{t}} \boldsymbol{\mathcal{E}}^1_1(\Omega_h)$, which creates exactly the space of $\boldsymbol{H}(\mathsf{curl}_\Gamma, \Gamma)$-conforming surface edge elements $\boldsymbol{\mathcal{E}}^1_1(\Gamma_h)$. The degrees of freedom are again inherited from $\boldsymbol{\mathcal{E}}^1_1(\Omega_h)$. For sufficiently smooth vector fields, the induced nodal interpolation operator $\boldsymbol{\Phi}^1_1$ satisfies the following commuting diagram property

$$\mathsf{curl}_\Gamma \circ \boldsymbol{\Phi}^1_1 = \mathsf{Q}^\Gamma_0 \circ \mathsf{curl}_\Gamma \, .$$

Again, we conclude invariance of $\mathrm{Ker}\left(\mathsf{curl}_\Gamma\right) \cap \mathrm{Dom}\left(\boldsymbol{\Phi}^1_1\right)$ under $\boldsymbol{\Phi}^1_1$.

Lemma 7.3. *If* $\mathbf{q} \in \boldsymbol{H}^s_{\mathbf{t}}(\Gamma)$, $\mathsf{curl}_\Gamma \mathbf{q} \in H^s(\Gamma)$ *for some* $0 < s \leq 1$, *then*

$$\begin{aligned} \left\| \mathbf{q} - \boldsymbol{\Phi}^1_1(\mathbf{q}) \right\|_{\boldsymbol{L}^2(\Gamma)} &\leq Ch^s \left(\|\mathbf{q}\|_{\boldsymbol{H}^s_{\mathbf{t}}(\Gamma)} + \left|\mathsf{curl}_\Gamma \mathbf{q}\right|_{H^s(\Gamma)} \right) , \\ \left\| \mathsf{curl}_\Gamma \left(\mathbf{q} - \boldsymbol{\Phi}^1_1(\mathbf{q})\right) \right\|_{L^2(\Gamma)} &\leq Ch^s \left|\mathsf{curl}_\Gamma \mathbf{q}\right|_{H^s(\Gamma)} . \end{aligned}$$

Based on the conforming finite element spaces, the Galerkin discretization of the variational problems (26) and (27) is straightforward: Find $\mathbf{E}_h \in \boldsymbol{\mathcal{E}}^1_{\nu+1}(\Omega_h)$, $\boldsymbol{\vartheta}_h \in \boldsymbol{\mathcal{F}}_\nu(\Gamma_h)$, and $\mathbf{p}_h \in \boldsymbol{\mathcal{E}}^1_1(\Gamma_h)$ such that for all $\mathbf{V}_h \in \boldsymbol{\mathcal{E}}^1_{\nu+1}(\Omega_h)$, $\boldsymbol{\mu}_h \in \boldsymbol{\mathcal{F}}_\nu(\Gamma_h)$, and $\mathbf{q}_h \in \boldsymbol{\mathcal{E}}^1_1(\Gamma_h)$

$$\begin{aligned} \mathsf{q}_\kappa(\mathbf{E}_h, \mathbf{V}_h) - \left\langle \mathbf{W}_\kappa(\gamma^-_D \mathbf{E}_h), \gamma^-_D \mathbf{V}_h \right\rangle_{\mathbf{t}} + \left\langle \left(\mathbf{K}'_\kappa - \tfrac{1}{2}\mathsf{Id}\right)(\boldsymbol{\lambda}_h), \gamma^-_D \mathbf{V}_h \right\rangle_{\mathbf{t}} &= \mathsf{f}_4(\mathbf{V}_h) , \\ \left\langle \boldsymbol{\mu}_h, \left(\tfrac{1}{2}\mathsf{Id} - \mathbf{K}_\kappa\right)(\gamma^-_D \mathbf{E}_h) \right\rangle_{\mathbf{t}} + \left\langle \boldsymbol{\mu}_h, \mathbf{V}_\kappa(\boldsymbol{\lambda}_h) \right\rangle_{\mathbf{t}} + i\eta \left\langle \boldsymbol{\mu}_h, \mathbf{p}_h \right\rangle_{\mathbf{t}} &= \mathsf{g}_4(\boldsymbol{\mu}_h) , \\ \left\langle \mathbf{W}_\kappa(\gamma^-_D \mathbf{E}_h), \mathbf{q}_h \right\rangle_{\mathbf{t}} - \left\langle \left(\tfrac{1}{2}\mathsf{Id} + \mathbf{K}'_\kappa\right)(\boldsymbol{\lambda}_h), \mathbf{q}_h \right\rangle_{\mathbf{t}} + \mathsf{b}(\mathbf{p}, \mathbf{q}_h) &= \mathsf{h}_4(\mathbf{q}_h) . \end{aligned} \tag{29}$$

Remark 7.4. Why do we have to worry about approximating the auxiliary variable $\mathbf{p}$ at all, though it vanishes and apparently the choice of boundary elements does not affect the convergence of Galerkin solutions? The reason is that the convergence of discrete solutions hinges on sufficiently good approximation properties of the underlying finite element and boundary element spaces. Hence the use of lowest order boundary elements is sufficient to ensure optimal convergence rates for the discretization error.

8 Discrete Decompositions

The splitting idea, which was used to prove coercivity on the continuous level, has to be adopted for an analysis on the discrete level as well. We follow a simple guideline, which boils down to applying nodal interpolation operators to the Helmholtz-type splittings in section 4.

First we construct a discrete counterpart of $\boldsymbol{X}(\mathbf{curl}, \Omega)$. We strongly rely on the projector already introduced in section 4. According to the recipe outlined above, it is formally defined as $\mathsf{P}_h := \boldsymbol{\Pi}^1_{\nu+1} \circ \mathsf{P}$. However, even on $\mathsf{P}(\boldsymbol{H}(\mathbf{curl}, \Omega))$ the nodal interpolation operator $\boldsymbol{\Pi}^1_{\nu+1}$ fails to be bounded, because the smoothness of the **curl** is not controlled. Nonetheless, we aim to apply P_h to finite element functions only, the following lemma saves the idea.

Lemma 8.1. *If* $\mathbf{U} \in H^s(\Omega)$ *and* $\mathbf{curl}\,\mathbf{U} \in \boldsymbol{\mathcal{F}}_\nu(\Omega_h)$, *for some* $s \geq 1$, *then* $\mathbf{U} \in \mathrm{Dom}\left(\boldsymbol{\Pi}^1_{\nu+1}\right)$ *and*

$$\left\| \mathbf{U} - \boldsymbol{\Pi}^1_{\nu+1}(\mathbf{U}) \right\|_{\boldsymbol{L}^2(\Omega)} \leq Ch^{\min\{\nu+1, s\}} \left| \mathbf{U} \right|_{H^s(\Omega)} ,$$

with $C > 0$ *depending only on* Ω, ν, *and the shape regularity of* Ω_h.

Proof. For a proof see [42, Lem. 2.1]. □

Due to the commuting diagram property (28) and lemma 4.2, we conclude that $\mathbf{curl}\,\mathsf{P}(\mathbf{U}_h) \in \boldsymbol{\mathcal{F}}_\nu(\Omega_h)$ for $\mathbf{U}_h \in \boldsymbol{\mathcal{E}}^1_{\nu+1}(\Omega_h)$,

Lemma 8.2. *The operator* $\mathsf{P}_h : \boldsymbol{\mathcal{E}}^1_{\nu+1}(\Omega_h) \mapsto \boldsymbol{\mathcal{E}}^1_{\nu+1}(\Omega_h)$ *is a h-uniformly continuous projection and preserves the* **curl**, *and* $\mathrm{Ker}(\mathsf{P}_h) = \mathrm{Ker}(\mathbf{curl}) \cap \boldsymbol{\mathcal{E}}^1_{\nu+1}(\Omega_h)$.

Thus by defining

$$\boldsymbol{X}_h(\mathbf{curl},\,\Omega_h) := \mathsf{P}_h\big(\boldsymbol{\mathcal{E}}^1_{\nu+1}(\Omega_h)\big)\,, \qquad \boldsymbol{N}_h(\mathbf{curl},\,\Omega_h) := \mathrm{Ker}(\mathbf{curl}) \cap \boldsymbol{\mathcal{E}}^1_{\nu+1}(\Omega_h)\,,$$

we instantly get a h-uniformly $\boldsymbol{H}(\mathbf{curl},\,\Omega)$-stable direct splitting

$$\boldsymbol{\mathcal{E}}^1_{\nu+1}(\Omega_h) = \boldsymbol{X}_h(\mathbf{curl},\,\Omega_h) \oplus \boldsymbol{N}_h(\mathbf{curl},\,\Omega_h)\,.$$

The following result makes it possible to pursue the same strategy in the case of the face elements space $\boldsymbol{\mathcal{F}}_\nu(\Gamma_h)$, cf. [41, Lem. 6.2].

Lemma 8.3. *If* $\boldsymbol{\mu} \in \boldsymbol{H}^s_{\mathbf{t}}(\Gamma)$ *and* $\mathsf{div}_\Gamma\,\boldsymbol{\mu} \in \mathcal{Q}_\nu(\Gamma_h)$*, for some* $s \geq \frac{1}{2}$*, then* $\boldsymbol{\mu} \in \mathrm{Dom}\,\big(\boldsymbol{\Phi}^2_\nu\big)$ *and*

$$\big\|\boldsymbol{\mu} - \boldsymbol{\Phi}^2_\nu(\boldsymbol{\mu})\big\|_{\boldsymbol{L}^2(\Gamma)} \leq C h^{\min\{\nu+1,s\}} \big\|\boldsymbol{\mu}\big\|_{\boldsymbol{H}^s_{\mathbf{t}}(\Gamma)}\,,$$

with $C > 0$ *only depending on* Γ*,* ν*,* s*, and the shape-regularity of the meshes.*

Thus, we can define the operator

$$\mathsf{P}^\Gamma_h : \boldsymbol{\mathcal{F}}_\nu(\Gamma_h) \mapsto \boldsymbol{\mathcal{F}}_\nu(\Gamma_h)\,, \qquad \mathsf{P}^\Gamma_h := \boldsymbol{\Phi}^2_\nu \circ \mathsf{P}^\Gamma\,,$$

and obtain properties similar to those of $\mathsf{P}_h : \boldsymbol{\mathcal{E}}^1_{\nu+1}(\Omega_h) \mapsto \boldsymbol{\mathcal{E}}^1_{\nu+1}(\Omega_h)$.

Lemma 8.4. *The mapping* $\mathsf{P}^\Gamma_h : \boldsymbol{\mathcal{F}}_\nu(\Gamma_h) \mapsto \boldsymbol{\mathcal{F}}_\nu(\Gamma_h)$ *is a* h*-uniformly continuous projector, which preserves* div_Γ *and fulfills* $\mathrm{Ker}(\mathsf{P}^\Gamma_h) = \mathrm{Ker}(\mathsf{div}_\Gamma) \cap \boldsymbol{\mathcal{F}}_\nu(\Gamma_h)$*.*

The projector P^Γ_h furnishes the desired h-uniformly $\boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma,\,\Gamma)$-stable splitting of the discrete Neumann trace space

$$\boldsymbol{\mathcal{F}}_\nu(\Gamma_h) = \boldsymbol{X}_h(\mathsf{div}_\Gamma,\,\Gamma_h) \oplus \boldsymbol{N}_h(\mathsf{div}_\Gamma,\,\Gamma_h)\,,$$

with

$$\boldsymbol{X}_h(\mathsf{div}_\Gamma,\,\Gamma_h) := \mathsf{P}^\Gamma_h\big(\boldsymbol{\mathcal{F}}_\nu(\Gamma_h)\big)\,, \qquad \boldsymbol{N}_h(\mathsf{div}_\Gamma,\,\Gamma_h) := \mathrm{Ker}(\mathsf{div}_\Gamma) \cap \boldsymbol{\mathcal{F}}_\nu(\Gamma_h)\,.$$

Notice that vector fields in $\boldsymbol{\mathcal{E}}^1_{\nu+1}(\Omega_h)$ are by no means continuous across inter element faces. On the other hand, any vector field in $\boldsymbol{H}^1(\Omega)$ must possess continuous components. Furthermore, there are elements in $\boldsymbol{X}_h(\mathsf{div}_\Gamma,\,\Gamma_h)$ that are not twisted tangential traces of continuous vector fields. In short,

$$\begin{aligned}\boldsymbol{X}_h(\mathbf{curl},\,\Omega_h) &\not\subset \boldsymbol{X}(\mathbf{curl},\,\Omega)\,, \quad \boldsymbol{X}_h(\mathsf{div}_\Gamma,\,\Gamma_h) \not\subset \boldsymbol{X}(\mathsf{div}_\Gamma,\,\Gamma)\,,\\ \boldsymbol{N}_h(\mathbf{curl},\,\Omega_h) &\subset \boldsymbol{N}(\mathbf{curl},\,\Omega)\,, \quad \boldsymbol{N}_h(\mathsf{div}_\Gamma,\,\Gamma_h) \subset \boldsymbol{N}(\mathsf{div}_\Gamma,\,\Gamma)\,.\end{aligned}$$

Thus by choosing

$$\boldsymbol{\mathcal{W}}_h := \boldsymbol{X}_h(\mathsf{div}_\Gamma,\,\Gamma_h) \times \boldsymbol{N}_h(\mathbf{curl},\,\Omega_h) \times \boldsymbol{N}_h(\mathsf{div}_\Gamma,\,\Gamma_h) \times \boldsymbol{X}_h(\mathbf{curl},\,\Omega_h) \times \boldsymbol{\mathcal{E}}^1_1(\Gamma_h)\,,$$

as a discrete approximation space for $\boldsymbol{\mathcal{W}}$, we have made a *nonconforming* choice, since $\boldsymbol{\mathcal{W}}_h \not\subset \boldsymbol{\mathcal{W}}$. Note that this is a special type of non-conformity, since it does not arise from the choice of discrete spaces, but from the way they are split. However, the Gårding inequality for $\widehat{\mathsf{a}}$ was only established with respect to the split space $\boldsymbol{\mathcal{W}}$. This prevents us from applying the well-known results about convergence of conforming Galerkin discretizations of coercive variational problems [51].

9 Discrete Inf-Sup Estimates

We start by recalling the main results of the abstract convergence theory from [18, Sec. 4.1] (see also [10, 11, 22]).

Let W be a Hilbert space with an W-stable decomposition $W = X \oplus N$, such that for any $w \in W$ we have uniquely determined $u \in X$, $v \in N$ with $w = u + v$ and

$$C^{-1}\big\|w\big\|_W \leq \big\|u\big\|_W + \big\|v\big\|_W \leq C\,\big\|w\big\|_W\,.$$

Based on the splitting we can define the isomorphism an $\mathsf{X} : W \mapsto W$ by $\mathsf{X}\big(w\big) := \overline{u - v}$.

Assumption 9.1. *Consider a sequence of closed subspaces $W_h \subset W$ with decompositions $W_h = X_h \oplus N_h$, satisfying the following assumptions:*

1. *The family W_h is approximating in W, i.e.*

$$\lim_{h\to 0} \inf_{w_h \in W_h} \|w - w_h\|_W = 0\,.$$

2. *W_h satisfies a gap property, i.e. there exist two subsets X_h, N_h of W_h such that*

$$\delta_h := \max\{\delta(X, X_h), \delta(N, N_h)\} \to 0\,, \quad \text{as } h \to 0,$$

where

$$\delta(X, X_h) := \sup_{u_h \in X_h} \inf_{u \in X} \frac{\|u_h - u\|_W}{\|u_h\|_W}\,, \quad \text{and} \quad \delta(N, N_h) := \sup_{v_h \in N_h} \inf_{v \in N} \frac{\|v_h - v\|_W}{\|v_h\|_W}\,.$$

Remark 9.2. In the particular case in which $N_h \subset N$, we have of course $\delta(N_h, N) = 0$. This means that the condition $\delta(W, W_h) \to 0$ for $h \to 0$ implies that N_h is approximating in N, i.e.

$$\lim_{h\to 0} \inf_{v_h \in N_h} \|v - v_h\|_W = 0\,.$$

The following theorem provides us with a discrete inf-sup estimate on the conforming, discrete trial space, cf. [18, Thm. 4.1] and [10, Th,. 3.7].

Theorem 9.3. *Assume that $\mathsf{A} : W \mapsto W^*$ is continuous and that there exists a compact operator $\mathsf{T} : W \mapsto W^*$ and a constants $\alpha > 0$ such that for all $w \in W$*

$$\mathrm{Re}\left\{\langle (\mathsf{A} + \mathsf{T})(w), \mathsf{X}(w)\rangle\right\} \geq \alpha \|w\|_W^2\,,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between W^* and W. Assume further that A is one-to-one and let $\{W_h\}_h$ denote a sequence of subspaces of W satisfying assumption 9.1.

Then there exists $h_0 > 0$ such that for all $h < h_0$ the following inf-sup estimate holds

$$\sup_{v_h \in W_h} \frac{|\langle A(w_h), v_h\rangle|}{\|v_h\|} \geq \frac{\alpha}{2}\|w_h\|_X \qquad \text{for all } w_h \in W_h\,. \tag{30}$$

It is well known that the discrete in-sup condition implies that the discrete Galerkin equation

$$\langle A(w_h), v_h\rangle = \langle f, v_h\rangle \qquad \text{for all } v_h \in W_h\,,$$

has unique solutions for all right hand sides $f \in W^*$ and that the discretization error is quasi-optimal, i.e. there exists a constant $C > 0$ such that

$$\|w - w_h\|_W \leq C \inf_{v_h \in W_h} \|w - v_h\|_W\,,$$

where $w \in W$ satisfies $A(w) = f$.

Remark 9.4. The approximation and the gap property of assumption 9.1 of the family of subspaces $W_h \subset W$ are equivalent to the existence of two bounded, linear operators, namely an *interpolation operator* $\Pi_h : W \mapsto W_h$, and a *bridge mapping* $\mathsf{B}_h : W_h \mapsto W$, which satisfy

$$\forall w \in W: \; \|w - \Pi_h(w)\|_W \to 0\,, \qquad \|\mathsf{Id} - \mathsf{B}_h\| \to 0\,, \tag{31}$$

as $h \to 0$, see [18, Sect. 4.1] and [39, Sect. 11]. Provided we have two such operators on hand, then the following two estimates are straightforward

$$\begin{aligned} \inf_{w_h \in W_h} \|w - w_h\|_W &\leq C\|w - \Pi_h(w)\|_W \to 0 && \text{for } h \to 0\,, \\ \delta(W, W_h) := \sup_{w_h \in W_h} \inf_{w \in W} \frac{\|w_h - w\|_W}{\|w_h\|_W} &\leq C \sup_{w_h \in W_h} \frac{\|w_h - \mathsf{B}_h(w_h)\|_W}{\|w_h\|_W} \to 0 && \text{for } h \to 0\,. \end{aligned}$$

Thus, item 1 and 2 of assumption 9.1 hold. In finite element/boundary element framework the existence of an interpolation estimate is straightforward, since interpolation error estimates are well established.

We return to particular setting of the coupled variational formulation and start by splitting the continuous trial space $\boldsymbol{\mathcal{W}} = \boldsymbol{\mathcal{X}} \oplus \boldsymbol{\mathcal{N}}$ into the following components

$$\begin{aligned} \boldsymbol{\mathcal{X}} &:= \boldsymbol{X}(\mathsf{div}_\Gamma, \Gamma) \times \{\mathbf{0}\} \times \{\mathbf{0}\} \times \boldsymbol{X}(\mathbf{curl}, \Omega) \times \boldsymbol{H}(\mathsf{curl}_\Gamma, \Gamma)\,, \\ \boldsymbol{\mathcal{N}} &:= \{\mathbf{0}\} \times \boldsymbol{N}(\mathbf{curl}, \Omega) \times \boldsymbol{N}(\mathsf{div}_\Gamma, \Gamma) \times \{\mathbf{0}\} \times \{\mathbf{0}\}\ , \end{aligned}$$

and note that the discrete trial $\boldsymbol{\mathcal{W}}_h$ space is approximating in $\boldsymbol{\mathcal{W}}$ due to lemma 7.1, lemma 7.2 and lemma 7.3. Thus part 1 of assumption 9.1 holds true.

Furthermore, we split the discrete trial space $\boldsymbol{\mathcal{W}}_h = \boldsymbol{\mathcal{X}}_h \oplus \boldsymbol{\mathcal{N}}_h$ according to

$$\begin{aligned} \boldsymbol{\mathcal{X}}_h &:= \boldsymbol{X}_h(\mathsf{div}_\Gamma, \Gamma_h) \times \{\mathbf{0}\} \times \{\mathbf{0}\} \times \boldsymbol{X}_h(\mathbf{curl}, \Omega_h) \times \boldsymbol{\mathcal{E}}_1^1(\Gamma_h)\,, \\ \boldsymbol{\mathcal{N}}_h &:= \{\mathbf{0}\} \times \boldsymbol{N}_h(\mathbf{curl}, \Omega_h) \times \boldsymbol{N}_h(\mathsf{div}_\Gamma, \Gamma_h) \times \{\mathbf{0}\} \times \{\mathbf{0}\}\ , \end{aligned}$$

into a non-conforming component $\boldsymbol{\mathcal{X}}_h \not\subset \boldsymbol{\mathcal{X}}$ and a conforming component $\boldsymbol{\mathcal{N}}_h \subset \boldsymbol{\mathcal{N}}$, for which $\delta(\boldsymbol{\mathcal{N}}, \boldsymbol{\mathcal{N}}_h) = 0$ holds. According to remark 9.4, we can rely on a suitable bridge mapping to establish the gap property 2 of assumption 9.1. A suitable operator can be constructed in a componentwise fashion on the discrete trial spaces $\boldsymbol{X}_h(\mathbf{curl}, \Omega_h)$ and $\boldsymbol{X}_h(\mathsf{div}_\Gamma, \Gamma_h)$, see [39, Sect. 11].

First, we define the bridge mapping $\mathsf{B}_\Omega : \boldsymbol{X}_h(\mathbf{curl}, \Omega_h) \mapsto \boldsymbol{X}(\mathbf{curl}, \Omega)$ by $\mathsf{B}_\Omega(\mathbf{U}_h) := \mathsf{P}(\mathbf{U}_h)$, $\mathbf{U}_h \in \boldsymbol{X}_h(\mathbf{curl}, \Omega_h)$. The projection properties from lemma 8.2 and lemma 4.2 yields

$$\left(\boldsymbol{\Pi}^1_{\nu+1} \circ \mathsf{B}_\Omega\right)(\mathbf{U}_h) = \mathbf{U}_h\,, \qquad \mathbf{curl}\,\mathsf{B}_\Omega(\mathbf{U}_h) = \mathbf{curl}\,\mathbf{U}_h \in \boldsymbol{\mathcal{F}}_\nu(\Omega_h)\,,$$

for all $\mathbf{U}_h \in \boldsymbol{X}_h(\mathbf{curl}, \Omega_h)$. Thus, lemma 8.1 permits us to estimate

$$\begin{aligned} \left\|\mathbf{U}_h - \mathsf{B}_\Omega \mathbf{U}_h\right\|_{\boldsymbol{L}^2(\Omega)} = \left\|\left(\boldsymbol{\Pi}^1_{\nu+1} - \mathsf{Id}\right)\mathsf{B}_\Omega(\mathbf{U}_h)\right\|_{\boldsymbol{L}^2(\Omega)} &\leq Ch\left\|\mathsf{B}_\Omega \mathbf{U}_h\right\|_{\boldsymbol{H}^1(\Omega)} \\ &\leq Ch\left\|\mathbf{curl}\,\mathbf{U}_h\right\|_{\boldsymbol{L}^2(\Omega)}\,, \end{aligned} \tag{32}$$

where the constant $C > 0$ only depends on Ω, ν, and the shape regularity of Ω_h.

The same construction can also be used for $\boldsymbol{X}_h(\mathsf{div}_\Gamma, \Gamma_h)$. We introduce the bridge mapping $\mathsf{B}_\Gamma : \boldsymbol{X}_h(\mathsf{div}_\Gamma, \Gamma_h) \mapsto \boldsymbol{X}(\mathsf{div}_\Gamma, \Gamma)$ by $\mathsf{B}_\Gamma(\boldsymbol{\mu}_h) := \mathsf{P}^\Gamma_h(\boldsymbol{\mu}_h)$, $\boldsymbol{\mu}_h \in \boldsymbol{X}_h(\mathsf{div}_\Gamma, \Gamma_h)$. As above, using the lemma 8.4 and lemma 4.4, we obtain

$$\left(\boldsymbol{\Phi}^2_\nu \circ \mathsf{B}_\Gamma\right)(\boldsymbol{\mu}_h) = \boldsymbol{\mu}_h\,, \qquad \mathsf{div}_\Gamma\,\mathsf{B}_\Gamma(\boldsymbol{\mu}_h) = \mathsf{div}_\Gamma\,\boldsymbol{\mu}_h \in \mathcal{Q}_\nu(\Gamma_h)\,,$$

for all $\boldsymbol{\mu}_h \in \boldsymbol{X}_h(\mathsf{div}_\Gamma, \Gamma_h)$. A straightforward application of lemma 8.3 yields the following estimate

$$\begin{aligned} \left\|\boldsymbol{\mu}_h - \mathsf{B}_\Gamma(\boldsymbol{\mu}_h)\right\|_{\boldsymbol{L}^2(\Gamma)} = \left\|\left(\boldsymbol{\Phi}^2_\nu - \mathsf{Id}\right)(\boldsymbol{\mu}_h)\right\|_{\boldsymbol{L}^2(\Gamma)} &\leq Ch^{1/2}\left\|\mathsf{B}_\Gamma(\boldsymbol{\mu}_h)\right\|_{\boldsymbol{H}^{1/2}_\perp(\Gamma)} \\ &\leq Ch^{1/2}\left\|\mathsf{div}_\Gamma\,\boldsymbol{\mu}_h\right\|_{H^{-1/2}(\Gamma)}\,, \end{aligned} \tag{33}$$

with $C > 0$ depending only on Γ, ν and the shape regularity of the meshes. Thus we define the bridge mapping

$$\mathsf{B} : \boldsymbol{\mathcal{W}}_h \mapsto \boldsymbol{\mathcal{W}}\,, \qquad \mathsf{B}\left(\boldsymbol{\mu}_h^\perp, \mathbf{U}_h^0, \boldsymbol{\mu}_h^0, \mathbf{U}_h^\perp, \mathbf{p}_h\right) := \left(\mathsf{B}_\Gamma(\boldsymbol{\mu}_h^\perp), \mathbf{U}_h^0, \boldsymbol{\mu}_h^0, \mathsf{B}_\Omega(\mathbf{U}_h^\perp), \mathbf{p}_h\right)\ ,$$

for $\boldsymbol{\mu}_h^\perp \in \boldsymbol{X}_h(\mathsf{div}_\Gamma, \Gamma_h)$, $\boldsymbol{\mu}_h^0 \in \boldsymbol{N}_h(\mathsf{div}_\Gamma, \Gamma_h)$, $\mathbf{U}_h^\perp \in \boldsymbol{X}_h(\mathbf{curl}, \Omega_h)$, $\mathbf{U}_h^0 \in \boldsymbol{N}_h(\mathbf{curl}, \Omega_h)$, and $\mathbf{p} \in \boldsymbol{\mathcal{E}}_1^1(\Gamma_h)$. Combining the estimates (32), (33) we end up with

$$\delta(\boldsymbol{\mathcal{W}}, \boldsymbol{\mathcal{W}}_h) \leq \sup_{\mathbf{w}_h \in \boldsymbol{\mathcal{W}}_h} \frac{\left\|\mathbf{w}_h - \mathsf{B}(\mathbf{w}_h)\right\|_{\boldsymbol{\mathcal{W}}}}{\left\|\mathbf{w}_h\right\|_{\boldsymbol{\mathcal{W}}}} \leq Ch^{1/2} \to 0\,,\ \text{as } h \to 0\,,$$

thus we have established the desired gap property for the space $\boldsymbol{\mathcal{W}}_h$.

10 Convergence

The discrete inf-sup estimate established in section 9 paves the way for a quasi optimal asymptotic estimate of the discretization error.

Theorem 10.1. *There exists a mesh width $h_0 \in \mathbb{H}$, depending only on Ω, κ, ν and the shape regularity of the meshes Ω_h, such that for every $h < h_0$ the discrete problem* (29) *has a unique solution $(\mathbf{E}_h, \boldsymbol{\vartheta}_h, \mathbf{p}_h) \in \boldsymbol{\mathcal{E}}^1_{\nu+1}(\Omega_h) \times \boldsymbol{\mathcal{F}}_\nu(\Gamma_h) \times \boldsymbol{\mathcal{E}}^1_1(\Gamma_h)$, which is quasi optimal in the following sense*

$$\begin{aligned}\|\mathbf{E}-\mathbf{E}_h\|_{\boldsymbol{H}(\mathbf{curl},\,\Omega)} &+ \|\boldsymbol{\lambda}-\boldsymbol{\lambda}_h\|_{\boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma,\,\Gamma)} + \|\mathbf{p}-\mathbf{p}_h\|_{\boldsymbol{H}(\mathsf{curl}_\Gamma,\,\Gamma)} \\ &\le C \inf \left\{ \begin{array}{l} \|\mathbf{E}-\mathbf{V}_h\|_{\boldsymbol{H}(\mathbf{curl},\,\Omega)} + \|\boldsymbol{\lambda}-\boldsymbol{\mu}_h\|_{\boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma,\,\Gamma)}; \\ \mathbf{V}_h \in \boldsymbol{\mathcal{E}}^1_{\nu+1}(\Omega_h),\ \boldsymbol{\mu}_h \in \boldsymbol{\mathcal{F}}_\nu(\Gamma_h) \end{array} \right\},\end{aligned}$$

with a constant $C > 0$ independent of $(\mathbf{E}, \boldsymbol{\vartheta}, \mathbf{p})$ and $h \in \mathbb{H}$.

Proof. In order to prove the quasi optimality estimate we have to verify all assumptions from theorem 9.3. The roles of W, W_h are now played by $\boldsymbol{\mathcal{W}}, \boldsymbol{\mathcal{W}}_h$ and the spaces X, X_h and N, N_h have to be replaced by $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{X}}_h$ and $\boldsymbol{\mathcal{N}}, \boldsymbol{\mathcal{N}}_h$. Continuity, compactness and injectivity can be obtained directly from section 6. The approximation and the gap property for $\boldsymbol{\mathcal{W}}_h$ have already been established in section 9. Eventually, we obtain h-uniform stability according to (30) for $\widehat{\mathsf{a}}$ on the family $\boldsymbol{\mathcal{W}}_h$, $h \in \mathbb{H}$, provided that h is sufficiently small.

Furthermore, from lemma 8.4 and 8.2 we obtain the following h-uniform equivalence of norms

$$\|(\boldsymbol{\mu}_h^\perp, \mathbf{V}_h^0, \boldsymbol{\mu}_h^0, \mathbf{V}_h^\perp, \mathbf{q}_h)\|_{\boldsymbol{\mathcal{W}}} \asymp \|(\mathbf{V}_h^\perp + \mathbf{V}_h^0, \boldsymbol{\mu}_h^\perp + \boldsymbol{\mu}_h^0, \mathbf{q}_h)\|_{\boldsymbol{\mathcal{Y}}} \qquad \forall\, (\boldsymbol{\mu}_h^\perp, \mathbf{V}_h^0, \boldsymbol{\mu}_h^0, \mathbf{V}_h^\perp, \mathbf{q}_h) \in \boldsymbol{\mathcal{W}}_h\,,$$

where the space $\boldsymbol{\mathcal{Y}}$ is defined by $\boldsymbol{\mathcal{Y}} := \boldsymbol{H}(\mathbf{curl},\,\Omega) \times \boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma,\,\Gamma) \times \boldsymbol{H}(\mathsf{curl}_\Gamma,\,\Gamma)$ and endowed with its natural graph norm.

Based on the discrete inf-sup estimate on the space $\boldsymbol{\mathcal{W}}_h$, we get for $h_0 < h$

$$\begin{aligned} &\sup_{(\mathbf{V}_h,\boldsymbol{\mu}_h,\mathbf{q}_h)\in\boldsymbol{\mathcal{E}}^1_{\nu+1}(\Omega_h)\times\boldsymbol{\mathcal{F}}_\nu(\Gamma_h)\times\boldsymbol{\mathcal{E}}^1_1(\Gamma_h)} \frac{\left|\mathsf{a}_\kappa^{\mathrm{reg}}\Big((\mathbf{E}_h, \boldsymbol{\lambda}_h, \mathbf{p}_h), (\mathbf{V}_h, \boldsymbol{\mu}_h, \mathbf{q}_h)\Big)\right|}{\|(\mathbf{V}_h, \boldsymbol{\mu}_h, \mathbf{q}_h)\|_{\boldsymbol{\mathcal{Y}}}} \\ &\ge C \sup_{(\boldsymbol{\mu}_h^\perp, \mathbf{V}_h^0, \boldsymbol{\mu}_h^0, \mathbf{V}_h^\perp, \mathbf{q}_h)\in\boldsymbol{\mathcal{W}}_h} \frac{\left|\widehat{\mathsf{a}}_\kappa^{\mathrm{reg}}\Big((\boldsymbol{\lambda}_h^\perp, \mathbf{E}_h^0, \boldsymbol{\lambda}_h^0, \mathbf{E}_h^\perp, \mathbf{p}_h), (\boldsymbol{\mu}_h^\perp, \mathbf{V}_h^0, \boldsymbol{\mu}_h^0, \mathbf{V}_h^\perp, \mathbf{q}_h)\Big)\right|}{\|(\boldsymbol{\mu}_h^\perp, \mathbf{V}_h^0, \boldsymbol{\mu}_h^0, \mathbf{V}_h^\perp, \mathbf{q}_h)\|_{\boldsymbol{\mathcal{W}}}} \\ &\ge C\Big\|\Big(\boldsymbol{\lambda}_h^\perp, \mathbf{E}_h^0, \boldsymbol{\lambda}_h^0, \mathbf{E}_h^\perp, \mathbf{p}_h\Big)\Big\|_{\boldsymbol{\mathcal{W}}} \ge C\|(\mathbf{E}_h, \boldsymbol{\lambda}_h, \mathbf{p}_h)\|_{\boldsymbol{\mathcal{Y}}}\,, \end{aligned}$$

with constants independent of the functions and $h \in \mathbb{H}$. From Babuška's theory [6], we conclude the error estimate from the theorem. □

Note that the auxiliary variable $\mathbf{p}$ does not show up on the right hand side of the quasi optimality estimate. This is guaranteed by $\mathbf{p} = \mathbf{0} \in \boldsymbol{\mathcal{E}}^1_1(\Gamma_h)$. However, we can not simply drop the auxiliary variable from (26), since the quasi-optimality estimate in theorem 10.1 is an asymptotic estimate, which only holds under sufficient approximation properties of the space $\boldsymbol{\mathcal{W}}_h$.

The main prerequisite for establishing orders of convergence of best approximations in finite element spaces are assumptions on the regularity of solutions of the continuous problem (26). We will assume that both the electric and the magnetic fields $\mathbf{E}$, $\mathbf{H} := (i\omega\mu_r)^{-1}\mathbf{curl}\,\mathbf{E}$ belong to $H^s(\Omega)$ for some $s > 0$. We point out that the regularity of solutions of Maxwell's equations depends on the discontinuities of the material parameters ε_r and μ_r (cf. [29]).

It is reasonable to demand that the discontinuities of ε_r and μ_r be resolved by the meshes Ω_h. That is, if Ω_i, $i = 0, \ldots, M$, $M \in \mathbb{M}$, are subdomains of Ω on which both material parameters are smooth, then $\Omega_{h|\Omega_i}$ must supply a valid triangulation of Ω_i. Then we can exploit $\mathbf{curl}\,\mathbf{E} = i\kappa\mu_r\mathbf{H}$ to see that $\mathbf{curl}\,\mathbf{E}$ is locally in $H^s(\Omega_i)$, $i = 1, \ldots, M$. Globally $\mathbf{curl}\,\mathbf{E}$ is at least contained in $H^{\min\{s,1/4\}}(\Omega)$.

Lemma 10.2. *If* $\mathbf{E}, \mathbf{H} \in H^s(\Omega)$ *for some* $s > 0$*, and if the jumps of* ε_r *and* μ_r *are resolved by all triangulations, we find a constant* $C > 0$ *depending on only on* ε_r*,* μ_r*,* Ω*,* ν *and the shape regularity of the meshes* Ω_h *such that*

$$\left\| \mathbf{E} - \mathbf{\Pi}^1_{\nu+1}(\mathbf{E}) \right\|_{\boldsymbol{H}(\mathbf{curl},\,\Omega)} \leq C h^{\min\{\nu+1,s\}} \left(\left\| \mathbf{E} \right\|_{H^s(\Omega)} + \sum_{i=1}^{M} \left\| \mathbf{H} \right\|_{H^s(\Omega_i)} \right) .$$

Proof. For a proof see [39, Lem. 12.2]. □

Lemma 10.3. *Assume that the meshes* Ω_h *resolve the discontinuities of both* ε_r *and* μ_r*, that* $\mathbf{H}, \mathbf{E} \in H^s(\Omega)$ *and that* $\mathbf{E}^{inc}$ *is smooth. Then*

$$\left\| \boldsymbol{\vartheta} - \mathbf{\Phi}^2_{\nu}(\boldsymbol{\vartheta}) \right\|_{\boldsymbol{H}^{-1/2}(\mathsf{div}_\Gamma,\,\Gamma)} \leq C h^{\min\{\nu+1,s\}} \left(\left\| \mathbf{E} \right\|_{H^s(\Omega)} + \sum_{i=1}^{M} \left\| \mathbf{H} \right\|_{H^s(\Omega_i)} + \left\| \mathbf{E}^{inc} \right\|_{H^s(\Omega)} \right) ,$$

where $C > 0$ *depends neither on* $\mathbf{H}$*,* $\mathbf{E}$*,* $\mathbf{E}^{inc}$ *nor on* $h \in \mathbb{H}$

Proof. For a proof see [39, Lem. 12.3]. □

The two previous lemma along with the quasi optimality estimate from theorem 10.1 imply the convergence of the Galerkin solutions in $\boldsymbol{\mathcal{E}}^1_\nu(\Omega_h) \times \boldsymbol{\mathcal{F}}_\nu(\Gamma_h) \times \boldsymbol{\mathcal{E}}^1_1(\Gamma_h)$ of the order $O\left(h^{\min\{\nu+1,s\}}\right)$ in the natural energy norms as $h \to 0$.

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