

Bar categories and star operations

E.J. Beggs & S. Majid

Department of Mathematics, University of Wales Swansea
Singleton Parc, Swansea SA2 8PP

&

Department of Mathematics, Queen Mary University of London
327 Mile End Rd, London E1 4NS

&

Isaac Newton Institute, Cambridge CB3 0EH

December 24, 2006

Abstract

We introduce the notion of ‘bar category’ by which we mean a monoidal category equipped with additional structure induced by complex conjugation. Examples of our theory include bimodules over a $*$ -algebra, modules over a conventional Hopf $*$ -algebra and modules over a more general object which call a ‘quasi- $*$ -Hopf algebra’ and for which examples include the standard quantum groups $u_q(g)$ at q a root of unity (these are well-known not to be a usual $*$ -Hopf algebra). We also provide examples of strictly quasiassociative bar categories, including modules over ‘ $*$ -quasiHopf algebras’ and a construction based on finite subgroups $H \subset G$ of a finite group. Inside a bar category one has natural notions of ‘ $*$ -algebra’ and ‘unitary object’ therefore extending these concepts to a variety of new situations. We study braidings and duals in bar categories and \star -braided groups (Hopf algebras) *in* braided-bar categories. Examples include the transmutation $B(H)$ of a quasitriangular Hopf $*$ -algebra and the quantum plane \mathbb{C}_q^2 at certain roots of unity q in the bar category $u_q(\widetilde{su_2})$ -modules. We use our methods to provide a natural quasi-associative C^* -algebra structure on the octonions \mathbb{O} and on a coset example. In the appendix we extend the Tannaka-Krein reconstruction theory to bar categories in relation of Hopf $*$ -algebras.

1 Introduction

One of the key ingredients in an operator algebra approach to noncommutative geometry is that of complex conjugation and its extension to adjoints, $*$ -involutions on algebras and other form of structure. Here commutative unital C^* -algebras correspond by the theorem of Gelfand and Naimark to compact Hausdorff spaces and many geometric ideas extend

from this point of view to noncommutative C^* -algebras. Even if one is not concerned with C^* -completions one may be concerned with the properties of $*$ -algebras at an algebraic level and associated concepts. Thus for example there is an established abstract notion of $*$ -Hopf algebras over \mathbb{C} which in some cases can be completed to a compact quantum group [17]. When one introduces constructions in $*$ -noncommutative geometry one needs $*$ to be respected and this is usually possible in the context of Hilbert spaces and adjoints. However, even in this case the notion of bar category that we introduce is useful as a book keeping device to make explicit the idea of conjugate objects and antilinear maps in various contexts relating to bimodules and conjugate representations.

The framework of bar category moreover allows such concepts to be extended from the familiar case of bimodules over $*$ -algebras or (co)modules over a $*$ -Hopf algebra to other situations where the corresponding notions of conjugation would otherwise not be clear. The most important of these we believe is to standard quantum groups $\mathbb{C}_q[G]$ or $u_q(g)$ at $|q| = 1$. These are not standard $*$ -Hopf algebras but we show that their modules still form a bar category. We also introduce a theory of $*$ -structures for Drinfeld's quasiHopf algebras [10] such that the category of modules is a bar category, and a construction for obtaining these by twisting a $*$ -Hopf algebra. Another application is to quasi-associative algebras [1] where without a categorical framework it would not be clear how to define concepts such as conjugation of bimodules or 'skew-linear inner products' to define Hilbert spaces.

The related idea of star operation \star for objects *in* a bar category is introduced. Quantum planes at $|q| = 1$, the octonions and our coset quasi-associative algebra example are all shown to be star algebras in a very natural way, though the star operations in the first and last case do not square to the identity. The quantum plane example in fact turns out to be a braided \star -Hopf algebra (in a braided bar category). The coset example is closely related to ideas in [6], which is concerned with applications to mathematical physics. Other motivation comes from earlier attempts at such notions in the specific context of braided linear spaces [15, 14].

One of the original motivations for this work was noncommutative Riemannian geometry, where we require $*$ -structures in noncommutative differential forms[8], projective modules etc. and this will appear in a sequel. It is also related to the 'type A' and 'type B' morphisms mentioned in [3].

An outline of the paper is as follows. The basic definition is given in Section 2, with elementary examples such as the bar category **Vect** of vector spaces with conjugation and the bimodule, module and comodule categories. Section 3 contains new examples of bar categories beyond the standard settings. Section 4 covers braiding in bar categories. Section 5 covers \star -algebras and \star -Hopf algebras in (braided) bar categories. Section 6 concludes with the natural notion of dual objects in a bar category. A main result here is that in a bar category a left-dual implies a right-dual.

In the appendix we show that a $*$ -Hopf algebra can be reconstructed from its (bar) category of modules, showing that in some sense, this is the 'correct' notion of bar category [16].

The reader should note that this material is very different, both in definition and application, from the idea of $*$ -autonomous category given in [4]. For one thing, there $*$ is a functor rather than a natural transformation. The $*$ functor in [4] is contravariant, rather than the covariant bar functor described here. The reader should also not confuse ‘quasi- $*$ -Hopf algebras’ and ‘ $*$ -quasiHopf algebras’, both of which we introduce in the paper.

The paper was written during the visit July-December 2006 of the authors to the Isaac Newton Institute. We thank the institute for their support.

2 Bar categories

Here we give the abstract definition that we propose, and the elementary ‘model’ examples that justify the definitions. We recall that a monoidal category $(\mathcal{C}, \otimes, \Phi, 1_{\mathcal{C}}, l, r)$ means a category, with a functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, a natural equivalence $\Phi : ((\otimes) \otimes) \rightarrow (\otimes(\otimes))$ subject to Mac Lane’s pentagon coherence identity and an identity object $1_{\mathcal{C}}$ and associated natural isomorphisms $l : \text{id} \rightarrow \text{id} \otimes 1_{\mathcal{C}}$ and $r : \text{id} \rightarrow 1_{\mathcal{C}} \otimes \text{id}$ compatible with Φ . We refer to [12] for details. We recall that a monoidal functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between monoidal categories means a functor F and a natural equivalence $f : F(\) \otimes F(\) \rightarrow F(\otimes)$ with

$$f_{X,Y \otimes Z} \circ (\text{id} \otimes f_{Y,Z}) \circ \Phi_{F(X),F(Y),F(Z)} = F(\Phi_{X,Y,Z}) \circ c_{X \otimes Y,Z} \circ (f_{X,Y} \otimes \text{id})$$

$$F(1_{\mathcal{C}}) = 1_{\mathcal{D}}, \quad f_{1_{\mathcal{C}},X} \circ l_{F(X)} = F(l_X), \quad f_{X,1_{\mathcal{C}}} \circ r_{F(X)} = F(r_X)$$

for all objects X, Y, Z of \mathcal{C} . Note that in a monoidal category the unit object is unique up to isomorphism and consequently is it conventional without loss of generality to assume strict equality in the last line above. If one wishes to be more precise it would be better to say that there is an isomorphism f^1 with

$$f^1 : 1_{\mathcal{D}} \rightarrow F(1_{\mathcal{C}}), \quad f_{1_{\mathcal{C}},X} \circ (f^1 \otimes \text{id}) \circ l_{F(X)} = F(l_X), \quad f_{X,1_{\mathcal{C}}} \circ (\text{id} \otimes f^1) \circ r_{F(X)} = F(r_X)$$

for all X . Here f^1 if it exists is determined uniquely from (F, f) and we say that F is monoidal with *induced unit isomorphism* f^1 .

A natural equivalence means that the natural transformation in question is invertible. We recall that a natural transformation means that the collection of morphisms is functorial with respect to morphisms in the domain of the relevant functors.

2.1 The definition of a bar category

We denote typical objects of a bar category by X, Y, Z etc. We denote by flip the functor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C} \times \mathcal{C}$ sending (X, Y) to (Y, X) .

Definition 2.1.1 *A bar category is a tensor category $(\mathcal{C}, \otimes, 1_{\mathcal{C}}, l, r, \Phi)$ together with the following data:*

- 1) A functor $\text{bar} : \mathcal{C} \rightarrow \mathcal{C}$ (written as $X \mapsto \overline{X}$).
- 2) A natural equivalence bb between the identity and the $\text{bar} \circ \text{bar}$ functors on \mathcal{C} .
- 3) An invertible morphism $\star : 1_{\mathcal{C}} \rightarrow \overline{1_{\mathcal{C}}}$.
- 4) A natural equivalence Υ (capital upsilon) between $\text{bar} \circ \otimes$ and $\otimes \circ (\text{bar} \times \text{bar}) \circ \text{flip}$ from $\mathcal{C} \times \mathcal{C}$ to \mathcal{C} . These are required to obey the rules:
 - a) $\overline{\star\star} = \text{bb}_{1_{\mathcal{C}}} : 1_{\mathcal{C}} \rightarrow \overline{\overline{1_{\mathcal{C}}}}$.
 - b) The following compositions are $r_{\overline{X}}$ and $l_{\overline{X}}$ respectively:

$$\begin{aligned} \overline{X} &\xrightarrow{\overline{l_X}} \overline{X \otimes 1_{\mathcal{C}}} \xrightarrow{\Upsilon_{X,1_{\mathcal{C}}}} \overline{1_{\mathcal{C}} \otimes X} \xrightarrow{\star^{-1} \otimes \text{id}} 1_{\mathcal{C}} \otimes \overline{X} , \\ \overline{X} &\xrightarrow{\overline{r_X}} \overline{1_{\mathcal{C}} \otimes X} \xrightarrow{\Upsilon_{1_{\mathcal{C}},X}} \overline{X \otimes 1_{\mathcal{C}}} \xrightarrow{\text{id} \otimes \star^{-1}} \overline{X} \otimes 1_{\mathcal{C}} . \end{aligned}$$

- c) The following condition is satisfied by Υ and the associator Φ :

$$\overline{\Phi_{X,Y,Z}} \Upsilon_{X \otimes Y, Z}^{-1} (\text{id} \otimes \Upsilon_{X,Y}^{-1}) = \Upsilon_{X,Y \otimes Z}^{-1} (\Upsilon_{Y,Z}^{-1} \otimes \text{id}) \Phi_{\overline{Z}, \overline{Y}, \overline{X}}^{-1} .$$

- d) The following square commutes:

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{\text{bb}_X \otimes \text{bb}_Y} & \overline{\overline{X}} \otimes \overline{\overline{Y}} \\ \text{bb}_{X \otimes Y} \downarrow & & \uparrow \Upsilon_{\overline{Y}, \overline{X}} \\ \overline{\overline{X \otimes Y}} & \xrightarrow{\overline{\Upsilon_{X,Y}}} & \overline{\overline{Y}} \otimes \overline{\overline{X}} \end{array}$$

- e) $\overline{\text{bb}_X} = \text{bb}_{\overline{X}} : \overline{X} \mapsto \overline{\overline{X}}$.

The natural transformation property for Υ is explicitly that, given morphisms $\theta : X \rightarrow B$ and $\phi : Y \rightarrow C$, we have

$$\Upsilon_{B,C}(\overline{\theta \otimes \phi}) = (\overline{\phi} \otimes \overline{\theta}) \Upsilon_{X,Y} : \overline{X \otimes Y} \rightarrow \overline{C} \otimes \overline{B} .$$

The natural transformation property of bb is similarly stated for any morphism θ as

$$\text{bb}_B \circ \theta = \overline{\overline{\theta}} \circ \text{bb}_X .$$

Remark 2.1.2 We can define \mathcal{C}^{op} to be the same monoidal category as \mathcal{C} with opposite tensor product, the same unit object $1_{\mathcal{C}}$ and the induced remaining structures. Then a bar category means:

- 1) a monoidal category \mathcal{C} equipped with a monoidal functor $\text{bar} : \mathcal{C} \rightarrow \mathcal{C}^{\text{op}}$ with associated natural equivalence Υ^{-1} and induced unit isomorphism $\star : 1_{\mathcal{C}} \rightarrow 1_{\mathcal{C}}$
- 2) a natural equivalence $\text{bb} : \text{id} \rightarrow \text{bar} \circ \text{bar}$ such that 4)a),d)e) hold.

2.2 Basic Example: Vector spaces over \mathbb{C}

Given a complex vector space V we define \bar{V} to be the same Abelian group as V but with a conjugate action of \mathbb{C} so

$$\lambda.\bar{v} = \overline{\lambda.v}, \quad v \in V, \lambda \in \mathbb{C}.$$

Here \bar{v} denotes the same element $v \in V$ but viewed in \bar{V} . This defines a functor from the category of complex vector spaces to itself. Objects are sent as above, and a morphism $\phi : V \rightarrow W$ is sent to $\bar{\phi} : \bar{V} \rightarrow \bar{W}$ with $\bar{\phi}(\bar{v}) = \overline{\phi(v)}$ for all $v \in V$.

Vector spaces over any field form a monoidal category with Φ, l, r trivial and unit object the field. In our case the unit object is $1_{\mathcal{C}} = \mathbb{C}$ and \star is given by complex conjugation which we also denote as usual. One may easily verify all the axioms with the trivial choices

$$\text{bb}_V(v) = \bar{v}, \quad \Upsilon_{W,V}(\overline{w \otimes v}) = \bar{w} \otimes \bar{v}. \quad (1)$$

In this way the category **Vect** of complex vector spaces becomes a bar category.

2.3 Basic Example: Bimodules over a star algebra

Let A be an algebra. The category ${}_A\mathcal{M}_A$ of A -bimodules and A -bimodule maps forms a monoidal category in the usual way. The left/right action on a tensor product of bimodules is given by multiplication on the left/right most factor. The associator is trivial and the identity object is A .

Now suppose that A is a \star -algebra over \mathbb{C} (or over some other field with involution) in the sense of an antilinear order reversing involution $a \mapsto a^*$. For an A -bimodule E , define another A -bimodule \bar{E} as follows: As Abelian groups \bar{E} is identified with E (as for the bar-category **Vect**; as there we use an overline to distinguish $\bar{e} \in \bar{E}$ from $e \in E$.)

Define left and right actions of A on \bar{E} by $a.\bar{e} = \overline{e.a^*}$ and $\bar{e}.a = \overline{a^*.e}$. The functor $\text{bar} : {}_A\mathcal{M}_A \rightarrow {}_A\mathcal{M}_A$ is defined on objects by $E \mapsto \bar{E}$ and on morphisms by $\phi \mapsto \bar{\phi}$, where $\bar{\phi}(\bar{e}) = \overline{\phi(e)}$.

Considering A as an A -bimodule, define the morphism

$$\star : A \rightarrow \bar{A}, \quad a \mapsto \bar{a^*}.$$

Similarly, define $\Upsilon_{F,E} : \overline{F \otimes_A E} \rightarrow \bar{E} \otimes_A \bar{F}$, by $\Upsilon_{F,E}(\overline{f \otimes_A e}) = \bar{e} \otimes_A \bar{f}$. It is easily seen that $\Upsilon_{F,E}$ is an invertible morphism in ${}_A\mathcal{M}_A$:

$$\Upsilon(\overline{(f \otimes_A e).a}) = \Upsilon(\overline{a^*.f \otimes_A e}) = \bar{e} \otimes_A \overline{a^*.f} = \bar{e} \otimes_A \bar{f}.a,$$

and likewise for the left multiplication.

Finally, the isomorphism $\text{bb}_E : E \rightarrow \bar{E}$ is given by $\text{bb}_E(e) = \bar{e}$, and this is a natural transformation between the identity and the bar^2 functor. Hence the monoidal category of bimodules becomes a bar category in a natural way. The \star -structure appears naturally as the \star isomorphism on A viewed as unit object.

2.4 Basic Example: (co)modules over *-Hopf algebras

Let H be a Hopf algebra over a field k in the usual sense, with coproduct $\Delta : H \rightarrow H \otimes H$, counit $\epsilon : H \rightarrow k$, and antipode $S : H \rightarrow H$. We refer to [13] for details and we use the ‘Sweedler notation’ $\Delta h = h_{(1)} \otimes h_{(2)}$ for all $h \in H$. The category \mathcal{M}_H of right H -modules is given the tensor product action of H on tensor products, i.e.

$$(v \otimes w) \triangleleft h = v \triangleleft h_{(1)} \otimes w \triangleleft h_{(2)} .$$

A natural notion over \mathbb{C} (or over any field with involution) is that H is a *-algebra and that Δ is a *-algebra map

$$\Delta \circ * = (* \otimes *) \circ \Delta .$$

From this it follows that S is necessarily invertible and

$$S(h)^* = S^{-1}(h^*) .$$

We say that H in this case is a *Hopf *-algebra*. The notion has been used notably by Woronowicz as an algebraic notion that can in some cases be completed to a compact quantum group[17]. There are other possibilities (see Section 3.2 and 3.3 later).

In this case \mathcal{M}_H can be made into a bar category as follows: As in 2.3 we identify \bar{V} with V as sets, using overline to distinguish $\bar{v} \in \bar{V}$ from $v \in V$. Then the action \triangleleft of H on \bar{V} is given by

$$\bar{v} \triangleleft h = \overline{v \triangleleft S^{-1}(h^*)} .$$

We take the same trivial Υ, bb as in (1) for **Vect** and the same \star on the unit object \mathbb{C} given by complex conjugation. We check that bb is a right module map:

$$\begin{aligned} \text{bb}(v) \triangleleft h &= \overline{\bar{v} \triangleleft h} = \overline{\bar{v} \triangleleft S^{-1}(h^*)} \\ &= \overline{v \triangleleft S^{-1}((S^{-1}(h^*))^*)} \\ &= \overline{v \triangleleft h} = \text{bb}(v \triangleleft h) . \end{aligned}$$

We also check that Υ is a right module map,

$$\begin{aligned} \Upsilon(\overline{(w \otimes v) \triangleleft h}) &= \Upsilon(\overline{(w \otimes v) \triangleleft S^{-1}(h^*)}) \\ &= \Upsilon(\overline{w \triangleleft S^{-1}(h^*)_{(1)} \otimes v \triangleleft S^{-1}(h^*)_{(2)}}) \\ &= \Upsilon(\overline{w \triangleleft S^{-1}(h^*_{(2)}) \otimes v \triangleleft S^{-1}(h^*_{(1)})}) \\ &= \overline{v \triangleleft S^{-1}(h^*_{(1)}) \otimes w \triangleleft S^{-1}(h^*_{(2)})} \\ &= \overline{v \triangleleft h_{(1)} \otimes w \triangleleft h_{(2)}} \\ &= \Upsilon(\overline{w \otimes v} \triangleleft h) . \end{aligned}$$

Analogous results hold for the category ${}_H\mathcal{M}$ of left H -modules. Here tensor product of modules is given now by the left action of Δh . We specify the left action on \bar{V} by

$$h \triangleright \bar{v} = \overline{S^{-1}(h^*) \triangleright v} .$$

and again the trivial forms for $\star, \Upsilon, \text{bb}$.

Since Hopf algebras are in a certain sense ‘self dual’ under arrow reversal there are analogous results for comodule categories. Thus the category \mathcal{M}^H of right H -comodules (say) has objects which are comodules (V, ρ_V) where

$$\rho_V : V \rightarrow V \otimes H, \quad \rho(v) = v_{[0]} \otimes v_{[1]}$$

is a coaction in an explicit notation. Again we identify V and \overline{V} as sets, and define $\star, \text{bb}, \Upsilon$ as in for **Vect**. The coaction on \overline{V} is given by

$$\rho(\overline{v}) = \overline{v_{[0]}} \otimes v_{[1]}^* .$$

We check that bb is a right comodule map:

$$\begin{aligned} \rho \text{bb}(v) &= \rho(\overline{v}) = \overline{v_{[0]}} \otimes (v_{[1]}^*)^* \\ &= \overline{v_{[0]}} \otimes v_{[1]}^{**} = \overline{v_{[0]}} \otimes v_{[1]} \end{aligned}$$

We also check that Υ is a right comodule map,

$$\begin{aligned} (\Upsilon \otimes \text{id})\rho(\overline{w \otimes v}) &= \Upsilon(\overline{(w \otimes v)_{[0]}}) \otimes ((w \otimes v)_{[1]})^* \\ &= \Upsilon(\overline{w_{[0]} \otimes v_{[0]}}) \otimes (w_{[1]} v_{[1]})^* \\ &= \overline{v_{[0]}} \otimes \overline{w_{[0]}} \otimes v_{[1]}^* w_{[1]}^* , \\ \rho \Upsilon(\overline{w \otimes v}) &= \rho(\overline{v \otimes w}) \\ &= \overline{v_{[0]}} \otimes \overline{w_{[0]}} \otimes v_{[1]}^* w_{[1]}^* , \end{aligned}$$

In this way \mathcal{M}^H becomes a bar category when H is a Hopf \ast -algebra. Similarly for the category ${}^H\mathcal{M}$ of left comodules.

2.5 Functors between bar categories

Definition 2.5.1 *Given two bar categories \mathcal{C} and \mathcal{D} , a functor from \mathcal{C} to \mathcal{D} is a monoidal functor $F : \mathcal{C} \rightarrow \mathcal{D}$ with the additional properties:*

(1) *There is a natural equivalence fb between the functors $\text{bar } F$ and $F \text{ bar}$, i.e. $\text{fb}_Y : \overline{F(Y)} \rightarrow F(\overline{Y})$.*

(2) *For the unit, $F(\overline{1}) = \overline{F(1)}$, and for the $\star : 1 \rightarrow \overline{1}$ morphism, $\star = F(\star)$.*

(3) *$F(\text{bb}_Y) = \text{fb}_{\overline{Y}} \circ \overline{\text{fb}_Y} \circ \text{bb}_{F(Y)}$.*

(4) *The following diagram commutes, where we recall that the definition of a monoidal functor involves a natural equivalence $f_{X,Y} : F(X) \otimes F(Y) \rightarrow F(X \otimes Y)$ which respects the associator:*

$$\begin{array}{ccccc} \overline{F(X \otimes Y)} & \xrightarrow{\overline{f_{X,Y}^{-1}}} & \overline{F(X) \otimes F(Y)} & \xrightarrow{\Upsilon} & \overline{F(Y) \otimes F(X)} \\ \text{fb}_{X \otimes Y} \downarrow & & & & \downarrow \text{fb}_Y \otimes \text{fb}_X \\ \overline{F(\overline{X \otimes Y})} & \xrightarrow{F(\Upsilon)} & \overline{F(\overline{Y} \otimes \overline{X})} & \xrightarrow{\overline{f_{\overline{Y}, \overline{X}}^{-1}}} & \overline{F(\overline{Y}) \otimes F(\overline{X})} \end{array}$$

Proposition 2.5.2 *Given a monoidal bar functor (F, f) between bar categories \mathcal{C} and \mathcal{D} as in 2.5.1, the following diagram commutes:*

$$\begin{array}{ccc} \overline{F(X)} & \xrightarrow{\text{fb}} & F(\overline{X}) \\ \overline{F(\text{bb})} \downarrow & & \downarrow \text{bb} \\ \overline{F(\overline{X})} & \xrightarrow{\overline{\text{fb}^{-1}}} & \overline{\overline{F(\overline{X})}} \end{array}$$

Proof: By using (3) from 2.5.1 and (e) from 2.1.1, the diagram can be rewritten as:

$$\begin{array}{ccc} \overline{F(X)} & \xrightarrow{\text{fb}} & F(\overline{X}) \\ \text{bb} \downarrow & & \downarrow \text{bb} \\ \overline{\overline{F(X)}} & \xrightarrow{\overline{\text{fb}}} & \overline{\overline{F(\overline{X})}} \end{array}$$

This commutes as bb is a natural transformation between the identity and bar^2 . \square

Clearly, an example of a bar functor is the forgetful functor to **Vect** from any of the above four bar categories associated to a Hopf \ast -algebra H . This forgets the (co)action of H . In the appendix we prove the converse that if a bar functor to **Vect** when restricted to objects with duals is representable, it factors through such a construction for some Hopf \ast -algebra which may be reconstructed.

Proposition 2.5.3 *If the monoidal functor (F, f) from \mathcal{C} to \mathcal{D} is a 1-1 correspondence of both objects and morphisms, and \mathcal{C} is a bar category, then we can make \mathcal{D} into a bar category so that (F, f) is a functor of bar categories as follows: Define*

- 1) $\overline{F(X)} = F(\overline{X})$ for all objects X in \mathcal{C} .
- 2) fb is a natural transformation from $F \circ \text{bar}$ to itself.
- 3) $\overline{F(\phi)} = \text{fb}_Y^{-1} \circ F(\overline{\phi}) \circ \text{fb}_X$ for all morphisms $\phi : X \rightarrow Y$ in \mathcal{C} .
- 4) $\star = F(\star)$ on the unit object.
- 5) $\text{bb}_{F(X)} = \overline{\text{fb}_X^{-1}} \text{fb}_{\overline{X}}^{-1} F(\text{bb}_X)$ for all objects X in \mathcal{C} .
- 6) $\Upsilon_{F(X), F(Y)} = (\text{fb}_Y^{-1} \otimes \text{fb}_X^{-1}) f_{\overline{Y}, \overline{X}}^{-1} F(\Upsilon_{X, Y}) \text{fb}_{X \otimes Y} \overline{f_{X, Y}}$ for all objects X, Y in \mathcal{C} .

Proof:

2.6 \star -Objects in a bar-category

In any bar category it makes sense to ask if an object X is isomorphic to \overline{X} .

Definition 2.6.1 *An object X in a bar category is called a star object if it is equipped with a morphism $\star_X : X \rightarrow \overline{X}$ with the following properties:*

- a) $\overline{\star_X} \circ \star_X = \text{bb}_X$.
b) $(\star_X \otimes \text{id})r_X = r_{\overline{X}} \star_X$ and $(\text{id} \otimes \star_X)l_X = l_{\overline{X}} \star_X$.

Clearly $1_{\mathcal{C}}$ is always an example.

Example 2.6.2 In the bar category \mathbf{Vect} a \star object structure on a vector space V is given by an antilinear map $J : V \rightarrow V$ such that $J^2 = \text{id}$. Here $\star(v) = \overline{J(v)}$ is J viewed as a linear map $V \rightarrow \overline{V}$. We will usually write $J(v) = v^*$.

Example 2.6.3 In ${}_H\mathcal{M}$ (see 2.4) a \star structure is given in the same way as in \mathbf{Vect} but with J covariant in the sense $J(h \triangleright v) = (Sh)^* \triangleright J(v)$. This is the notion of a ‘ \star -action’ or unitary action in [13] (where J is denoted by $*$).

Example 2.6.4 In ${}_A\mathcal{M}_A$ (see 2.3) we again use the identification of \overline{X} as a bimodule with the Abelian group X but reverse bimodule structures. Then $\star_X : X \rightarrow \overline{X}$ takes the form $e \mapsto \overline{J(e)}$ where $J^2 = \text{id}$ and now $J : X \rightarrow X$ is a skew bimodule map in the sense $J(a.e) = J(e).a^*$ and $J(e.a) = a^*.J(e)$ for $a \in A$ and $e \in X$.

We can also write some other related definitions in any bar category.

Definition 2.6.5 An inner product on an object X a bar category means a morphism $\langle , \rangle : \overline{X} \otimes X \rightarrow 1_{\mathcal{C}}$ such that

$$\star \circ \langle , \rangle = \overline{\langle , \rangle} \circ \Upsilon_{\overline{X}, X}^{-1} \circ (\text{id} \otimes \text{bb}_X)$$

(This is just the usual conjugate symmetry for many examples.) It will also be convenient to have the opposite definition with the bar on the second copy of X .

Example 2.6.6 In \mathbf{Vect} this becomes a usual sesquilinear inner product \langle , \rangle on a vector space V that is antilinear in its first input and which is symmetric in the skew sense

$$\overline{\langle v, w \rangle} = \langle w, v \rangle, \quad v, w \in V$$

where the bar on the left denotes usual complex conjugation (it is \star in \mathbf{Vect}). One should add a positivity requirement to have a (pre)-Hilbert space.

Example 2.6.7 In bimodules ${}_A\mathcal{M}_A$ over a \star -algebra A it means an A -valued inner product \langle , \rangle on a bimodule E obeying

$$\langle e, f \rangle^* = \langle f, e \rangle, \quad \langle e, f \rangle a = \langle e, f.a \rangle, \quad \langle a^*.e, f \rangle = \langle e, a.f \rangle$$

The first two conditions are those of a right (pre) Hilbert module if one adds a positivity requirement. Given the first, the second condition can also be written as $a\langle e, f \rangle = \langle e.a^*, f \rangle$. The last condition applies in our bimodule setting and says that additional left action of A is adjointed by \langle , \rangle in a way that agrees with $*$ (a ‘unitary’ representation in some sense). From our point of view it is the condition that the map $E \times E \rightarrow A$ descends to $E \otimes_A E \rightarrow A$.

Example 2.6.8 In the case ${}_H\mathcal{M}$ for a $*$ -Hopf algebra it means a sesquilinear inner product as for **Vect** which is covariant under the action of H on $\overline{V} \otimes V$. A short computation shows that this last is equivalent to

$$\langle h^* \triangleright v, w \rangle = \langle w, h \triangleright v \rangle, \quad h \in H, \quad v, w \in V$$

i.e. that the action of H is adjointed by $\langle \cdot, \cdot \rangle$ in a way that agrees with $*$ (i.e. ‘unitary’ in some sense).

Definition 2.6.9 B is a star algebra in the bar category \mathcal{C} if there is a product $\mu : B \otimes B \rightarrow B$ making B into an algebra in \mathcal{C} , and if B is also a star object so that

$$\overline{\mu} \Upsilon_{B,B}^{-1} (\star_B \otimes \star_B) = \star_B \mu : B \otimes B \rightarrow \overline{B}.$$

If $\eta : 1_{\mathcal{C}} \rightarrow B$ is a unit for the product, we require $\overline{\eta} \circ \star = \star_B \circ \eta$.

It is intended here that the product is associative with respect to a possibly non-trivial associator Φ in the category.

Example 2.6.10 In the case of **Vect** this means that B is an ordinary $*$ -algebra with $\star(b) = \overline{b^*}$, i.e. the antilinear map J above is antimultiplicative with respect to the product (and in this context denoted $*$). An example is of course bounded operators $B(\mathcal{H})$ on a Hilbert space \mathcal{H} , with $*$ given by the adjoint operation.

Example 2.6.11 An algebra B in ${}_A\mathcal{M}_A$ means a bimodule B and a product map $\mu : B \otimes_A B \rightarrow B$ which need be associative only when one takes the tensor product over A . A unit means a bimodule map $A \rightarrow B$, determined by the image of the identity, and forming a unit for μ . We provide the notion of a \star -involution on such an algebraic structure.

Example 2.6.12 In the case ${}_H\mathcal{M}$, an algebra B means an H -module algebra in the sense $h \triangleright (bc) = (h_{(1)} \triangleright b)(h_{(2)} \triangleright c)$ and $h \triangleright 1 = \epsilon(h)1$. We require further that B is a $*$ -algebra in the usual sense as for **Vect** and that $(h \triangleright b)^* = (Sh)^* \triangleright b^*$ as above. This is the usual notion for a $*$ -Hopf algebra acting on a $*$ -algebra [13]. Here we again have $\star(b) = \overline{b^*}$.

There is obviously a dual notion of a coalgebra C in a bar category \mathcal{C} as a coalgebra with coproduct $\Delta : C \rightarrow C \otimes C$ and counit $\epsilon : C \rightarrow 1_{\mathcal{C}}$ which are coassociative and counital in the monoidal category, now with C a \star -object such that

$$\Upsilon_{C,C} \circ \overline{\Delta} \circ \star_C = (\star_C \otimes \star_C) \circ \Delta, \quad \star \circ \epsilon = \overline{\epsilon} \circ \star_C.$$

This is not the only possible equation, we could add a braiding or similar map.

Example 2.6.13 A coalgebra in ${}_A\mathcal{M}_A$ means a ‘coring’[7]. Hence the above is a natural notion of \star -structure on a coring.

This completes the basic definitions and ‘elementary examples’ in the sense that one does not need the notion of a bar category to formulate these examples. Rather it is a useful idea to unify various notions and to keep track of the book-keeping.

3 New examples of bar categories

Until now we have tested our ideas out on ‘standard’ settings where one knows in any case how conjugation etc. should proceed. In that case our bar notion serves only to unify some slightly different contexts. In this section we give some genuinely new examples where the definitions would not be so clear without a categorical context. The first one is only slightly new (it is known in some form) but included for completeness.

3.1 Example: Crossed or Drinfeld-Yetter modules over $*$ -Hopf algebras

For any Hopf algebra the category of right ‘crossed’ or Drinfeld-Yetter modules $\overset{x}{\mathcal{M}}_H^H$ for H consists of right H -modules and right H -comodules satisfying the consistency condition

$$\rho(v \triangleleft h) = v_{[0]} \triangleleft h_{(2)} \otimes S(h_{(1)}) v_{[1]} h_{(3)} \quad \forall v \in V, \forall a \in H. \quad (2)$$

In the case of H finite-dimensional a crossed module is just the same as a right module for the Drinfeld quantum double Hopf algebra $D(H) = H \bowtie H^{*\text{op}}$ and its properties tend to be in line with those known for this. In the finite-dimensional case it was shown [15] that $D(H)$ for H a $*$ -Hopf algebra inherits a natural Hopf $*$ -algebra structure. In the general case:

Proposition 3.1.1 *Let H be a $*$ -Hopf algebra. The category of right H -crossed module or ‘Drinfeld-Yetter’ modules is a bar category with \bar{V} defined as for \mathcal{M}_H and \mathcal{M}^H separately, i.e. $\bar{V} = V$ as an Abelian group and with action and coaction on \bar{V} given by*

$$\rho(\bar{v}) = \bar{v}_{[0]} \otimes v_{[1]}^*, \quad \bar{v} \triangleleft h = \overline{v \triangleleft S^{-1}(h^*)}.$$

Proof: We only have to check that \bar{V} is again a crossed or Drinfeld-Yetter module, i.e.

$$\begin{aligned} \rho(\bar{v} \triangleleft h) &= \overline{\rho(v \triangleleft S^{-1}(h^*))} \\ &= \overline{(v \triangleleft S^{-1}(h^*))_{[0]} \otimes ((v \triangleleft S^{-1}(h^*))_{[1]})^*} \\ &= \overline{v_{[0]} \triangleleft S^{-1}(h^*)_{(2)} \otimes (S(S^{-1}(h^*)_{(1)}) v_{[1]} S^{-1}(h^*)_{(3)})^*} \\ &= \overline{v_{[0]} \triangleleft S^{-1}(h^*)_{(2)} \otimes (h^*_{(3)} v_{[1]} S^{-1}(h^*_{(1)}))^*} \\ &= \overline{v_{[0]} \triangleleft h_{(2)} \otimes S(h_{(1)}) v_{[1]}^* h_{(3)}}. \end{aligned}$$

The rest of the axioms hold as inherited from **Vect**.

3.2 Example: Bimodules over flip- $*$ -Hopf algebras

Sometimes one has Hopf algebras which are simply not $*$ -Hopf algebras but for which $*$ still behaves well with respect to the coproduct. The most common ‘nonstandard’ variant is:

Definition 3.2.1 A flip- $*$ -Hopf algebra is a Hopf algebra which is a $*$ -algebra and for which the coproduct and counit obey

$$\Delta \circ * = (* \otimes *) \tau \circ \Delta, \quad \overline{\epsilon(h)} = \epsilon(h^*)$$

In this case it follows that $S(h)^* = S(h^*)$.

A simple commutative example is the polynomial algebra $\mathbb{C}[u^i_j]$ ($1 \leq i, j \leq n$) with the relation $\det(u) = 1$, coproduct, counit and $*$

$$\Delta u^i_j = \sum_k u^i_k \otimes u^k_j, \quad \epsilon u^i_j = \delta^i_j, \quad (u^i_j)^* = u^j_i.$$

This is a non-standard real form of $\mathbb{C}[SL_n]$ which does not correspond to any group but to set of hermitian matrices of determinant 1. For $n = 2$ it is the $*$ -algebra of regular functions on the mass hyperboloid in Minkowski space $\mathbb{R}^{1,3}$ (and it is the $q = 1$ limit of the hyperboloid in an approach to q -deformed Minkowski space based on braided groups[13].)

Proposition 3.2.2 For any Hopf algebra one has a category ${}_H\mathcal{M}_H$ of H -bimodules with tensor product defined by Δ . If H is a flip- $*$ -Hopf algebra this can be made into a bar category by $\overline{V} = V$ as Abelian groups, right and left actions on \overline{V} given by

$$h \triangleright \overline{v} = \overline{v \triangleleft h^*}, \quad \overline{v} \triangleleft h = \overline{h^* \triangleright v}.$$

and bb and Υ as in (1).

Proof: We need to check that bb and Υ give morphisms, leaving some of this calculation to the reader.

$$\begin{aligned} h \triangleright \text{bb}(v) &= h \triangleright \overline{\overline{v}} \\ &= \overline{\overline{v \triangleleft h^*}} = \overline{h^{**} \triangleright v} \end{aligned}$$

$$\begin{aligned} h \triangleright \Upsilon(\overline{v \otimes w}) &= h \triangleright (\overline{w \otimes v}) \\ &= \overline{h_{(1)} \triangleright \overline{w} \otimes h_{(2)} \triangleright \overline{v}} \\ &= \overline{w \triangleleft h_{(1)}^* \otimes v \triangleleft h_{(2)}^*} \\ &= \overline{w \triangleleft h_{(2)}^* \otimes v \triangleleft h_{(1)}^*}, \\ \Upsilon(h \triangleright \overline{v \otimes w}) &= \Upsilon(\overline{(v \otimes w) \triangleleft h^*}) \\ &= \overline{\Upsilon(v \triangleleft h_{(1)}^* \otimes w \triangleleft h_{(2)}^*)} \\ &= \overline{w \triangleleft h_{(2)}^* \otimes v \triangleleft h_{(1)}^*}. \end{aligned}$$

3.3 Example: Modules over quasi-* Hopf algebras

We recall [13] that a 2-cocycle on a Hopf algebra H is an invertible element $\mathcal{F} \in H \otimes H$ such that

$$\mathcal{F}_{12}(\Delta \otimes \text{id})(\mathcal{F}) = \mathcal{F}_{23}(\text{id} \otimes \Delta)(\mathcal{F}), \quad (\epsilon \otimes \text{id})\mathcal{F} = 1$$

Given such an element, one may conjugate the coproduct of H to a new Hopf algebra $H^{\mathcal{F}}$ which an equivalent category of modules. Here we give a different application of such a cocycle. It is not directly related to but to but in the same spirit as the notion of quasi-* Hopf algebra introduced in [15] to describe q -deformed inhomogeneous quantum groups for real q . Our version will be more applicable to $|q| = 1$.

Definition 3.3.1 *A quasi-* Hopf algebra is a quadruple $(H, *, \mathcal{F}, \gamma)$ where H is a Hopf algebra and *-algebra, $\mathcal{F} \in H \otimes H$ is a 2-cocycle and $\gamma \in H$ is an invertible element such that*

$$\Delta(h^*) = \mathcal{F}^{-1}(\Delta h)^* \otimes^* \mathcal{F}, \quad \epsilon(h)^* = \epsilon(h^*), \quad S^{-1}(h^*) = \gamma^{-1}(Sh)^* \gamma, \quad \gamma \otimes \gamma = \tilde{\mathcal{F}}_{21} \mathcal{F} \Delta \gamma$$

for all $h \in H$, where $\tilde{\mathcal{F}} = (* \otimes *) (S \otimes S)(\mathcal{F})$.

Remark 3.3.2 *If $\tilde{\mathcal{F}} = \mathcal{F}_{21}^{-1}$ we say that the quasi-* Hopf algebra is of real type. In this case γ is grouplike, i.e. $\Delta \gamma = \gamma \otimes \gamma$.*

Theorem 3.3.3 *The category ${}_H \mathcal{M}$ of left modules of a quasi-* Hopf algebra $(H, *, \mathcal{F}, \gamma)$ can be made into a bar category as follows: Again, for V an object in ${}_H \mathcal{M}$, we define \overline{V} to be V as a set, with*

$$h \triangleright \overline{v} = \overline{(Sh)^* \triangleright v}.$$

We define $\Upsilon_{V,W} : \overline{V \otimes W} \rightarrow \overline{V} \otimes \overline{W}$ and $\text{bb}_V : V \rightarrow \overline{\overline{V}}$ by

$$\begin{aligned} \Upsilon_{V,W}(\overline{v \otimes w}) &= \overline{\mathcal{F}^{(2)} \triangleright w \otimes \mathcal{F}^{(1)} \triangleright v}, \\ \text{bb}_V(v) &= \overline{\overline{\gamma \triangleright v}}. \end{aligned} \tag{3}$$

Proof: First show that $\text{bb}_V : V \rightarrow \overline{\overline{V}}$ defined in (3) is a morphism in ${}_H \mathcal{M}$.

$$\begin{aligned} g \triangleright \text{bb}_V(v) &= \overline{g \triangleright \overline{\gamma \triangleright v}} \\ &= \overline{S(g)^* \triangleright \overline{\gamma \triangleright v}} \\ &= \overline{S(S(g)^*)^* \gamma \triangleright v}, \end{aligned}$$

and by definition of γ we have $\gamma g = S(S(g)^*)^* \gamma$. Next show that $\Upsilon_{V,W} : \overline{V \otimes W} \rightarrow \overline{V} \otimes \overline{W}$ defined in (3) is a morphism in ${}_H \mathcal{M}$.

$$g \triangleright (\overline{v \otimes w}) = \overline{S(g)^* \triangleright (v \otimes w)}$$

$$= \overline{S(g)^*_{(1)\triangleright v} \otimes S(g)^*_{(2)\triangleright w}} .$$

From 3.3.1, this gives, where \mathcal{F}^- is an independent copy of \mathcal{F}^{-1} ,

$$\begin{aligned} g \triangleright (\overline{v \otimes w}) &= \overline{\mathcal{F}^{-(1)} S(g_{(2)})^* \mathcal{F}^{(1)\triangleright v} \otimes \mathcal{F}^{-(2)} S(g_{(1)})^* \mathcal{F}^{(2)\triangleright w}} , \\ \Upsilon(g \triangleright (\overline{v \otimes w})) &= \overline{S(g_{(1)})^* \mathcal{F}^{(2)\triangleright w} \otimes S(g_{(2)})^* \mathcal{F}^{(1)\triangleright v}} \\ &= g \triangleright (\overline{\mathcal{F}^{(2)\triangleright w} \otimes \mathcal{F}^{(1)\triangleright v}}) \\ &= g \triangleright \Upsilon(\overline{v \otimes w}) . \end{aligned}$$

Next show that the compatibility condition between Υ and the associator is satisfied.

$$\Upsilon^{-1}(\overline{w \otimes \overline{v}}) = \overline{\mathcal{F}^{-(1)\triangleright v} \otimes \mathcal{F}^{-(2)\triangleright w}}$$

we calculate

$$\begin{aligned} \Upsilon_{V \otimes U, W}^{-1}(\text{id} \otimes \Upsilon_{V, U}^{-1})(\overline{w \otimes (\overline{u \otimes \overline{v}})}) &= \overline{\Upsilon_{V \otimes U, W}^{-1}(\overline{w \otimes \mathcal{F}^{-(1)\triangleright v} \otimes \mathcal{F}^{-(2)\triangleright u}})} \\ &= \overline{\tilde{\mathcal{F}}^{-(1)\triangleright (\mathcal{F}^{-(1)\triangleright v} \otimes \mathcal{F}^{-(2)\triangleright u})} \otimes \tilde{\mathcal{F}}^{-(2)\triangleright w}} \\ &= \overline{(\tilde{\mathcal{F}}^{-(1)}_{(1)} \mathcal{F}^{-(1)\triangleright v} \otimes \tilde{\mathcal{F}}^{-(1)}_{(2)} \mathcal{F}^{-(2)\triangleright u}) \otimes \tilde{\mathcal{F}}^{-(2)\triangleright w}} , \\ \Upsilon_{V, U \otimes W}^{-1}(\Upsilon_{U, W}^{-1} \otimes \text{id})(\overline{(\overline{w \otimes \overline{u}}) \otimes \overline{v}}) &= \overline{\Upsilon_{V, U \otimes W}^{-1}(\overline{\mathcal{F}^{-(1)\triangleright u} \otimes \mathcal{F}^{-(2)\triangleright w} \otimes \overline{v}})} \\ &= \overline{\tilde{\mathcal{F}}^{-(1)\triangleright v} \otimes \tilde{\mathcal{F}}^{-(2)\triangleright (\mathcal{F}^{-(1)\triangleright u} \otimes \mathcal{F}^{-(2)\triangleright w})}} \\ &= \overline{\tilde{\mathcal{F}}^{-(1)\triangleright v} \otimes (\tilde{\mathcal{F}}^{-(2)}_{(1)} \mathcal{F}^{-(1)\triangleright u} \otimes \tilde{\mathcal{F}}^{-(2)}_{(2)} \mathcal{F}^{-(2)\triangleright w})} . \end{aligned}$$

As the original category has trivial associator, this is implied by

$$\tilde{\mathcal{F}}^{-(1)}_{(1)} \mathcal{F}^{-(1)} \otimes \tilde{\mathcal{F}}^{-(1)}_{(2)} \mathcal{F}^{-(2)} \otimes \tilde{\mathcal{F}}^{-(2)} = \tilde{\mathcal{F}}^{-(1)} \otimes \tilde{\mathcal{F}}^{-(2)}_{(1)} \mathcal{F}^{-(1)} \otimes \tilde{\mathcal{F}}^{-(2)}_{(2)} \mathcal{F}^{-(2)} ,$$

which is the cocycle condition for $\mathcal{F} \in H \otimes H$. Finally, show that Υ satisfies the compatibility condition with bb. Begin with

$$\begin{aligned} \Upsilon \circ \overline{\Upsilon}(\overline{\overline{v \otimes w}}) &= \overline{\Upsilon(\overline{\overline{\mathcal{F}^{(2)\triangleright w} \otimes \mathcal{F}^{(1)\triangleright v}}})} \\ &= \overline{\tilde{\mathcal{F}}^{(2)\triangleright (\overline{\mathcal{F}^{(1)\triangleright v}}} \otimes \overline{\mathcal{F}^{(2)\triangleright w}})} \otimes \tilde{\mathcal{F}}^{(1)\triangleright (\overline{\mathcal{F}^{(2)\triangleright w}})}} \\ &= \overline{S(\tilde{\mathcal{F}}^{(2)})^* \mathcal{F}^{(1)\triangleright v} \otimes S(\tilde{\mathcal{F}}^{(1)})^* \mathcal{F}^{(2)\triangleright w}} \end{aligned}$$

Next we have

$$\begin{aligned} \text{bb}(v \otimes w) &= \overline{\overline{y \triangleright (v \otimes w)}} \\ &= \overline{\gamma_{(1)\triangleright v} \otimes \gamma_{(2)\triangleright w}} , \end{aligned}$$

so the condition becomes

$$\overline{\overline{\gamma \triangleright \overline{v}} \otimes \overline{\overline{\gamma \triangleright \overline{w}}}} = \overline{\overline{S(\tilde{\mathcal{F}}^{(2)})^* \mathcal{F}^{(1)} \gamma_{(1)\triangleright v} \otimes S(\tilde{\mathcal{F}}^{(1)})^* \mathcal{F}^{(2)} \gamma_{(2)\triangleright w}}} ,$$

which is satisfied by 3.3.1. \square

To give a simple class of examples, we recall [9] that a Hopf algebra H is quasitriangular if there is an invertible element $\mathcal{R} \in H \otimes H$ such that

$$(\Delta \otimes \text{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (\text{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}, \quad \tau \circ \Delta = \mathcal{R}(\Delta)\mathcal{R}^{-1}$$

(which then obeys the YBE or braid relations in $H^{\otimes 3}$). In this case the elements

$$u = (S\mathcal{R}^{(2)})\mathcal{R}^{(1)}, \quad v = \mathcal{R}^{(1)}(S\mathcal{R}^{(2)})$$

implement S^2 and S^{-2} respectively by conjugation. A ribbon Hopf algebra is a quasitriangular one for which the central element vu has a square root $\nu \in H$ (the ribbon element) such that

$$\Delta\nu = \mathcal{R}^{-1}\mathcal{R}_{21}^{-1}(\nu \otimes \nu), \quad \epsilon\nu = 1, \quad S\nu = \nu$$

An exposition with required identities can be found in [13].

Proposition 3.3.4 *A ribbon Hopf algebra (H, \mathcal{R}, ν) which is a flip- $*$ -Hopf algebra such that $\mathcal{R}^{*\otimes*} = \mathcal{R}_{21}^{-1}$ can be viewed as a quasi- $*$ -Hopf algebra with $\mathcal{F} = \mathcal{R}$ and $\gamma = v^{-1}\nu$.*

Proof: Since we have assumed both parts of

$$(\Delta g)^{*\otimes*} = \tau\Delta(g^*), \quad \tau\Delta g = \mathcal{R}(\Delta g)\mathcal{R}^{-1},$$

where τ is transposition, it is clear that $\mathcal{F} = \mathcal{R}$ plays the role required. It remains only to find γ . We start by noting that for a flip- $*$ -Hopf algebra, $S(h^*) = (Sh)^*$ while $(S \otimes S)(\mathcal{R}) = \mathcal{R}$ for any quasitriangular Hopf algebra[13]. Hence under the reality condition on \mathcal{R} the defining properties for $\gamma \in H$ becomes, for all $g \in H$

$$\gamma^{-*}g\gamma^* = S^2(g), \quad \gamma \otimes \gamma = \Delta\gamma \tag{4}$$

From general analysis on quasi-triangular Hopf algebras [13], the first equation in (4) is satisfied by $\gamma = u^{-*}$ or $\gamma = v^*$. However these choices may not satisfy the second equation. Since $\mathcal{R}^{*\otimes*} = \mathcal{R}_{21}^{-1}$, we have $u^* = u^{-1}$ and $v^* = v^{-1}$ hence if we have a ribbon element, $\gamma = v^{-1}\nu$ is grouplike and still implements S^2 as required.

Example 3.3.5 *The anyon Hopf algebra [13] for q a primitive l -th root of unity ($l = 2k + 1 > 1$ an odd integer) has a single invertible generator K for which*

$$K^l = 1, \quad \Delta K = K \otimes K, \quad \epsilon(K) = 1 \quad S(K) = K^{-1}.$$

This Hopf algebra is ribbon, with quasitriangular and associated elements

$$\mathcal{R}_K = \frac{1}{l} \sum_{a,b=0}^{l-1} q^{-2ab} K^a \otimes K^b, \quad \nu = u = v = \frac{1}{l} \sum_{a,r=0}^{l-1} q^{-2a(a-r)} K^r \tag{5}$$

We take the star structure to be $K^* = K^{-1}$, and then $\mathcal{R}^{*\otimes*} = \mathcal{R}^{-1} = \mathcal{R}_{21}^{-1}$ and we have a quasi- $*$ Hopf algebra as an example of 3.3.4 with $\gamma = 1$.

The bar category of modules of this quasi- $*$ Hopf algebra is that of anyonic or \mathbb{Z}_l -graded complex vector spaces (here $K \triangleright v = q^{|v|}v$ acting in an element of degree $|v|$). Morphisms are degree-preserving linear maps and

$$\text{bb}_V(v) = \bar{v}, \quad \Upsilon_{V,W}(\overline{v \otimes w}) = q^{2|v||w|} \bar{w} \otimes \bar{v}$$

on homogeneous elements.

Example 3.3.6 As another example of the structure given in 3.3.4, take the example of $u_q(SU(2))$, where q is a primitive l th root of unity (for odd integer $l > 1$). The algebra is defined as generated by $1, K^{\pm 1}, E$ and F , subject to relations

$$K E = q^2 E K, \quad K F = q^{-2} F K, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}, \quad E^l = 0 = F^l, \quad K^l = 1.$$

The coproduct on the generators is given by

$$\Delta E = E \otimes K + 1 \otimes E, \quad \Delta F = F \otimes 1 + K^{-1} \otimes F, \quad \Delta K = K \otimes K.$$

The counit is defined by

$$\epsilon(1) = \epsilon(K^{\pm 1}) = 1, \quad \epsilon(E) = \epsilon(F) = 0,$$

and the antipode by

$$S(K^{\pm 1}) = K^{\mp 1}, \quad S(E) = -E K^{-1}, \quad S(F) = -K F.$$

This Hopf algebra is known to be ribbon with quasitriangular and related structures

$$\begin{aligned} \mathcal{R} &= \mathcal{R}_K \sum_{n=0}^{l-1} \frac{(q - q^{-1})^n}{[n; q^{-2}]!} E^n \otimes F^n, \\ \nu &= \frac{1}{l} \left(\sum_{m=0}^{l-1} q^{2m^2} \right) \sum_{n,a=0}^{l-1} \frac{(q - q^{-1})^n}{[n; q^{-2}]!} q^{-(l+1)(n-a-1)^2/2} E^n K^a F^n, \end{aligned}$$

where \mathcal{R}_K is defined in (5). It is also known that this is made into a flip- $*$ -Hopf algebra by

$$(q^{\pm H})^* = q^{\mp H}, \quad E^* = -F, \quad F^* = -E.$$

Applying the star to \mathcal{R} gives $\mathcal{R}^{*\otimes*} = (\mathcal{R}^{-1})_{21}$ as required to apply the above results. Hence we have a quasi- $*$ Hopf algebra with $\gamma = v^{-1}\nu$.

Example 3.3.7 *As yet another example of the structure given in 3.3.4, take the example of $u_q(\widetilde{SU(2)})$, where q is a primitive l th root of unity (for odd integer $l > 1$). This is a central extension of $u_q(SU(2))$ by an invertible generator \tilde{K} satisfying the anyonic relations. Then we have a new quasitriangular structure:*

$$\tilde{\mathcal{R}} = \mathcal{R}_{\tilde{K}} \mathcal{R}_K \sum_{n=0}^{l-1} \frac{(q - q^{-1})^n}{[n; q^{-2}]!} E^n \otimes F^n .$$

and the same element γ as in 3.3.6.

3.4 Example: Modules over $*$ -quasi Hopf algebras and twisting

We first recall the Drinfeld twist theory. Take ${}_H\mathcal{M}$ to be the category of left modules of a Hopf algebra H . For a cochain $\mathcal{F} \in H \otimes H$ we can twist the coproduct by $\Delta_{\mathcal{F}} = \mathcal{F} \Delta \mathcal{F}^{-1}$ to give a quasiHopf algebra [10] which we denote $H^{\mathcal{F}}$. For simplicity we assume that $(\text{id} \otimes \epsilon)\mathcal{F} = (\epsilon \otimes \text{id})\mathcal{F} = 1$. This construction can be thought of as twisting the tensor product in ${}_H\mathcal{M}$ to a new category $\mathcal{C} = {}_H\mathcal{F}\mathcal{M}$ connected by a monoidal ‘twisting functor’ $F : {}_H\mathcal{M} \rightarrow \mathcal{C}$ [16, 13]. Here F is the identity on objects and morphisms, \mathcal{C} has a tensor product, and the tensor products on the two categories are related by

$$f_{X,Y} : F(X) \otimes F(Y) \rightarrow F(X \otimes Y) , \quad F(x) \otimes F(y) \mapsto F(\mathcal{F}^{-(1)} \triangleright x \otimes \mathcal{F}^{-(2)} \triangleright y) .$$

Here since F is the identity on objects we extend its notation to the identity operation on elements, which is a useful convention to allow us to give concise formulae. Thus $F(x)$ denotes $x \in X$ viewed in $F(X)$. In terms of this notation, $h \triangleright F(x) = F(h \triangleright x)$. A brief calculation shows that, even though ${}_H\mathcal{M}$ has trivial associator, the new category \mathcal{C} in general needs an associator $\Phi : (F(X) \otimes F(Y)) \otimes F(Z) \rightarrow F(X) \otimes (F(Y) \otimes F(Z))$, given by the action of the following element $\phi \in H^{\otimes 3}$:

$$\phi = (\text{id} \otimes \mathcal{F}) ((\text{id} \otimes \Delta)\mathcal{F}) ((\Delta \otimes \text{id})\mathcal{F}^{-1}) (\mathcal{F}^{-1} \otimes \text{id}) .$$

The condition that this is the identity is exactly that \mathcal{F} is a cocycle, but we shall keep to the general case for the moment.

Now we suppose that H is a $*$ -Hopf algebra, and thus that ${}_H\mathcal{M}$ becomes a bar category with the notations given earlier in 2.4. The problem is how to make ${}_H\mathcal{F}\mathcal{M}$ into a bar category. We use the following conventions for operations on ${}_H\mathcal{F}\mathcal{M}$:

$$\begin{aligned} \text{bb}(F(x)) &= \overline{\overline{F(\gamma \triangleright x)}} , \\ \text{fb}(F(x)) &= \overline{F(\varphi \triangleright x)} , \\ \Upsilon(\overline{F(x) \otimes F(y)}) &= \overline{F(\mathcal{G}^{(2)} \triangleright y) \otimes F(\mathcal{G}^{(1)} \triangleright x)} , \end{aligned} \tag{6}$$

for some $\varphi \in H$ and $\mathcal{G} \in H \otimes H$.

Theorem 3.4.1 *Suppose that H is a $*$ -Hopf algebra, and $\mathcal{F} \in H \otimes H$ is an invertible element with $(\epsilon \otimes \text{id})\mathcal{F} = (\text{id} \otimes \epsilon)\mathcal{F} = 1$. Then ${}_{H^{\mathcal{F}}}\mathcal{M}$ is a bar category such that the identity map from ${}_{H}\mathcal{M}$ to ${}_{H^{\mathcal{F}}}\mathcal{M}$ is a morphism of bar categories if and only if*

$$\gamma = \varphi^{-1} (S\varphi^{-1})^* , \quad \mathcal{G} = \varphi^{-1} (S\mathcal{F}^{(2)})^* \varphi_{(1)} \mathcal{F}^{-(1)} \otimes \varphi^{-1} (S\mathcal{F}^{(1)})^* \varphi_{(2)} \mathcal{F}^{-{(2)}} , \quad (7)$$

obey

$$\gamma \otimes \gamma = (S\tilde{\mathcal{G}}^{(2)})^* \mathcal{G}^{(1)} \mathcal{F}^{(1)} \gamma_{(1)} \mathcal{F}^{-{(1)}} \otimes (S\tilde{\mathcal{G}}^{(1)})^* \mathcal{G}^{(2)} \mathcal{F}^{(2)} \gamma_{(2)} \mathcal{F}^{-{(2)}} . \quad (8)$$

$$\begin{aligned} & (S\tilde{\phi}^{(1)})^* \tilde{\mathcal{G}}^{(2)} \mathcal{F}^{(2)} \mathcal{G}^{(2)} \mathcal{F}^{-{(2)}} \otimes (S\tilde{\phi}^{(2)})^* \tilde{\mathcal{G}}^{(1)} \mathcal{F}^{(1)} \mathcal{G}^{(1)} \mathcal{F}^{-{(1)}} \otimes (S\tilde{\phi}^{(3)})^* \mathcal{G}^{(1)} \\ = & \mathcal{G}^{(2)} \phi^{-{(3)}} \otimes \tilde{\mathcal{G}}^{(2)} \mathcal{F}^{(2)} \mathcal{G}^{(1)} \mathcal{F}^{-{(2)}} \phi^{-{(2)}} \otimes \tilde{\mathcal{G}}^{(1)} \mathcal{F}^{(1)} \mathcal{G}^{(1)} \mathcal{F}^{-{(1)}} \phi^{-{(1)}} . \end{aligned} \quad (9)$$

$$\gamma = (S\gamma)^* , \quad (10)$$

$$\epsilon(\varphi) = 1 . \quad (11)$$

Proof: We calculate bb from the following formula:

$$\begin{aligned} \text{bb}_{F(X)}(F(x)) &= \overline{\text{fb}_X^{-1} \circ \text{fb}_{\bar{X}}^{-1} \circ F(\text{bb}_X)(F(x))} \\ &= \overline{\text{fb}_X^{-1} \circ \text{fb}_{\bar{X}}^{-1}(F(\bar{x}))} \\ &= \overline{\text{fb}_X^{-1}(F(\varphi^{-1} \triangleright \bar{x}))} \\ &= \overline{\text{fb}_X^{-1}(F((S\varphi^{-1})^* \triangleright x))} \\ &= \overline{F(\varphi^{-1} (S\varphi^{-1})^* \triangleright x)} \end{aligned}$$

The diagram for Υ gives

$$\begin{aligned} \Upsilon(\overline{F(x) \otimes F(y)}) &= (\text{fb}^{-1} \otimes \text{fb}^{-1}) f^{-1} F(\Upsilon) \text{fb} \overline{f(F(x) \otimes F(y))} \\ &= (\text{fb}^{-1} \otimes \text{fb}^{-1}) f^{-1} F(\Upsilon) \text{fb} (\overline{F(\mathcal{F}^{-{(1)}} \triangleright x \otimes \mathcal{F}^{-{(2)}} \triangleright y)}) \\ &= (\text{fb}^{-1} \otimes \text{fb}^{-1}) f^{-1} F(\Upsilon) (\overline{F(\varphi_{(1)} \mathcal{F}^{-{(1)}} \triangleright x \otimes \varphi_{(2)} \mathcal{F}^{-{(2)}} \triangleright y)}) \\ &= (\text{fb}^{-1} \otimes \text{fb}^{-1}) f^{-1} (\overline{F(\varphi_{(2)} \mathcal{F}^{-{(2)}} \triangleright y \otimes \varphi_{(1)} \mathcal{F}^{-{(1)}} \triangleright x)}) \\ &= (\text{fb}^{-1} \otimes \text{fb}^{-1}) (\overline{F(\mathcal{F}^{(1)} \triangleright \varphi_{(2)} \mathcal{F}^{-{(2)}} \triangleright y) \otimes F(\mathcal{F}^{(2)} \triangleright \varphi_{(1)} \mathcal{F}^{-{(1)}} \triangleright x)}) \\ &= \overline{F(\varphi^{-1} (S\mathcal{F}^{(1)})^* \varphi_{(2)} \mathcal{F}^{-{(2)}} \triangleright y) \otimes F(\varphi^{-1} (S\mathcal{F}^{(2)})^* \varphi_{(1)} \mathcal{F}^{-{(1)}} \triangleright x)} . \end{aligned}$$

Now we look at 2.1.1 (d):

$$\begin{aligned} \Upsilon \overline{\Upsilon} \text{bb}(F(x) \otimes F(y)) &= \Upsilon \overline{\Upsilon} (\overline{F(\mathcal{F}^{(1)} \gamma_{(1)} \mathcal{F}^{-{(1)}} \triangleright x) \otimes F(\mathcal{F}^{(2)} \gamma_{(2)} \mathcal{F}^{-{(2)}} \triangleright y)}) \\ &= \Upsilon (\overline{F(\mathcal{G}^{(2)} \mathcal{F}^{(2)} \gamma_{(2)} \mathcal{F}^{-{(2)}} \triangleright y) \otimes F(\mathcal{G}^{(1)} \mathcal{F}^{(1)} \gamma_{(1)} \mathcal{F}^{-{(1)}} \triangleright x)}) \end{aligned}$$

$$= \frac{\overline{\overline{F((S\tilde{\mathcal{G}}^{(2)})^* \mathcal{G}^{(1)} \mathcal{F}^{(1)} \gamma_{(1)} \mathcal{F}^{-(1)} \triangleright x)}}}{\otimes \overline{\overline{F((S\tilde{\mathcal{G}}^{(1)})^* \mathcal{G}^{(2)} \mathcal{F}^{(2)} \gamma_{(2)} \mathcal{F}^{-(2)} \triangleright y)}}} ,$$

and as this must be $\text{bb} \otimes \text{bb}$ we have the first condition.

The condition in 2.1.1 with Υ and the associators gives the equality of the following expressions:

$$\begin{aligned} & \Phi (\Upsilon \otimes \text{id}) \overline{\overline{\Upsilon (F(x) \otimes (F(y) \otimes F(z)))}} \\ = & \Phi (\Upsilon \otimes \text{id}) \overline{\overline{(F(\mathcal{F}^{(1)} \mathcal{G}^{(2)}_{(1)} \mathcal{F}^{-(1)} \triangleright y)) \otimes F(\mathcal{F}^{(2)} \mathcal{G}^{(2)}_{(2)} \mathcal{F}^{-(2)} \triangleright z) \otimes \overline{\overline{F(\mathcal{G}^{(1)} \triangleright x)}}}} \\ = & \Phi ((\overline{\overline{F(\tilde{\mathcal{G}}^{(2)} \mathcal{F}^{(2)} \mathcal{G}^{(2)}_{(2)} \mathcal{F}^{-(2)} \triangleright z)} \otimes \overline{\overline{F(\tilde{\mathcal{G}}^{(1)} \mathcal{F}^{(1)} \mathcal{G}^{(2)}_{(1)} \mathcal{F}^{-(1)} \triangleright y)}}}) \otimes \overline{\overline{F(\mathcal{G}^{(1)} \triangleright x)}}} , \\ & (\text{id} \otimes \Upsilon) \overline{\overline{\Upsilon \Phi^{-1} (F(x) \otimes (F(y) \otimes F(z)))}} \\ = & (\text{id} \otimes \Upsilon) \overline{\overline{\Upsilon ((F(\phi^{-(1)} \triangleright x) \otimes F(\phi^{-(2)} \triangleright y)) \otimes F(\phi^{-(3)} \triangleright z))}} \\ = & (\text{id} \otimes \Upsilon) \overline{\overline{(F(\mathcal{G}^{(2)} \phi^{-(3)} \triangleright z) \otimes F(\mathcal{F}^{(1)} \mathcal{G}^{(1)}_{(1)} \mathcal{F}^{-(1)} \phi^{-(1)} \triangleright x) \otimes F(\mathcal{F}^{(2)} \mathcal{G}^{(1)}_{(2)} \mathcal{F}^{-(2)} \phi^{-(2)} \triangleright y))}} \\ = & \overline{\overline{F(\mathcal{G}^{(2)} \phi^{-(3)} \triangleright z) \otimes (F(\tilde{\mathcal{G}}^{(2)} \mathcal{F}^{(2)} \mathcal{G}^{(1)}_{(2)} \mathcal{F}^{-(2)} \phi^{-(2)} \triangleright y) \otimes F(\tilde{\mathcal{G}}^{(1)} \mathcal{F}^{(1)} \mathcal{G}^{(1)}_{(1)} \mathcal{F}^{-(1)} \phi^{-(1)} \triangleright x))}} , \end{aligned}$$

so we deduce the second condition. Then 2.1.1 (e) gives the third condition, and the conditions on l and r give the fourth. \square

Proposition 3.4.2 *Suppose that H is a cocommutative Hopf algebra and that \mathcal{F} is a cocycle (i.e. $\phi = 1 \otimes 1 \otimes 1$ in (6)) for which $(\epsilon \otimes \text{id})\mathcal{F} = (\text{id} \otimes \epsilon)\mathcal{F} = 1$ and $((S \otimes S)\mathcal{F})^{*\otimes*} = \mathcal{F}$. Then we can set $\varphi = 1$ and $\mathcal{G} = \mathcal{F}_{21} \mathcal{F}^{-1}$, and all the conditions for 3.4.1 are satisfied.*

Proof: Substituting the given conditions into (7), we find $\gamma = 1$ and $\mathcal{G} = \mathcal{F}_{21} \mathcal{F}^{-1}$. Now as $\mathcal{G}_{21} = \mathcal{G}^{-1}$ we see that (7) holds. Using the equation for the ϕ (6) and cocommutativity shows that 9 holds. \square

Now we look at this from a different point of view. Suppose that we are just given a quasi-Hopf algebra, without a description of how it was obtained by twisting (i.e. we do not know \mathcal{F} or φ in the discussion above). However we do know $\mathcal{F}\Delta$, as it is just the coproduct in the quasi-Hopf algebra. We describe a quasi $*$ -Hopf algebra via an antilinear algebra map $T : H \rightarrow H$ which obeys

$$(T \otimes T) \Delta h = \mathcal{G}_{21} (\tau \Delta(T(h))) \mathcal{G}_{21}^{-1} .$$

[For a usual $*$ -Hopf algebra, this is just $T(h) = (Sh)^*$.]

Definition 3.4.3 *A quasi $*$ -Hopf algebra is $(H, \Delta, \phi, T, \gamma, \mathcal{G})$ where*

$$\begin{aligned} \gamma \otimes \gamma &= ((T \otimes T)\mathcal{G}_{21}) \mathcal{G} \Delta \gamma , \\ \gamma &= T(\gamma) , \\ ((T \otimes T \otimes T)\phi_{321}) (1 \otimes \mathcal{G}) ((\text{id} \otimes \Delta)\mathcal{G}) &= (\mathcal{G} \otimes 1) (\Delta \otimes \text{id})\mathcal{G} \phi^{-1} . \end{aligned}$$

Proposition 3.4.4 *A quasi $*$ -Hopf algebra (see 3.4.3) has ${}_H\mathcal{M}$ which is a bar category.*

Proof: Check the equations (8), (10) and (9). \square

Remark 3.4.5 *Clearly one can factor out the antipode S of a quasi-Hopf algebra and look at the formal properties of $*$ = $T \circ S^{-1}$ in line with usual $*$ -Hopf algebras but this is not particularly convenient in the quasi-Hopf algebra case.*

We will later give a concrete example of this theory applied to the Octonions.

3.5 Example: The coset representative category

Suppose that G is a subgroup of a finite group X . The set $G \backslash X$ consists of the equivalence classes of X under left translation by elements of G , i.e. the left cosets. The subgroup G acts on $G \backslash X$ by right translation, which we write as $\triangleleft : G \backslash X \times G \rightarrow G \backslash X$ (i.e. $p \triangleleft u \in G \backslash X$ for $p \in G \backslash X$ and $u \in G$).

Definition 3.5.1 *The category $\mathcal{C}_{\triangleleft}$ consists of all finite dimensional vector spaces over a field \mathbb{K} , whose objects are right representations of the group G and possess $G \backslash X$ -gradings, i.e. an object V decomposes as a direct sum of subspaces $V = \bigoplus_{p \in G \backslash X} V_p$, with a compatibility condition between the action and the grading. If $\xi \in V_p$ for some $p \in G \backslash X$ we say that ξ is a homogenous element of V , with grade $\langle \xi \rangle = p$.*

We write the action for the representation as $\triangleright : V \times G \rightarrow V$. The compatibility condition is $\langle \xi \triangleright u \rangle = \langle \xi \rangle \triangleleft u$.

The morphisms are linear maps which preserve both the grading and the action, i.e. for a morphism $\phi : V \rightarrow W$ we have $\langle \phi(\xi) \rangle = \langle \xi \rangle$ and $\phi(\xi) \triangleright u = \theta(\xi \triangleright u)$ for all homogenous $\xi \in V$ and $u \in G$.

The category has not yet been given a tensor product operation, and to do this we need to make a choice of coset representatives. This is a subset $M \subset X$ such that for every $x \in X$ there is a unique $s \in M$ so that $x \in Gs$. We shall call the decomposition $x = us$ for $u \in G$ and $s \in M$ the unique factorisation of x . For convenience, we identify the coset representatives with the cosets. The identity in X will be denoted e , and we assume that $e \in M$.

Definition 3.5.2 *Given $s, t \in M$, define $\tau(s, t) \in G$ and $s \cdot t \in M$ by the unique factorisation $st = \tau(s, t)(s \cdot t)$ in X . We also define functions $\triangleright : M \times G \rightarrow G$ and $\triangleleft : M \times G \rightarrow M$ by the unique factorisation $su = (s \triangleright u)(s \triangleleft u)$ for $s, s \triangleleft u \in M$ and $u, s \triangleright u \in G$.*

The assumption that $e \in M$ gives the binary operation (M, \cdot) a 2-sided identity. Also there is a unique left inverse t^L for every $t \in M$, satisfying the equation $t^L \cdot t = e$. We will assume that there is also a unique right inverse t^R , satisfying $t \cdot t^R = e$. For the identities relating the binary operation and the ‘cochain’ τ , see [5].

Proposition 3.5.3 *Given a set of coset representatives M , we can make \mathcal{C}_{\bowtie} into a tensor category by taking $V \otimes W$ to be the usual vector space tensor product, with actions and gradings given by*

$$\langle \xi \otimes \eta \rangle = \langle \xi \rangle \cdot \langle \eta \rangle \quad \text{and} \quad (\xi \otimes \eta) \bar{\triangleleft} u = \xi \bar{\triangleleft} (\langle \eta \rangle \triangleright u) \otimes \eta \bar{\triangleleft} u .$$

For morphisms $\theta : V \rightarrow \tilde{V}$ and $\phi : W \rightarrow \tilde{W}$ we define the morphism $\theta \otimes \phi : V \otimes W \rightarrow \tilde{V} \otimes \tilde{W}$ by $(\theta \otimes \phi)(\xi \otimes \eta) = \theta(\xi) \otimes \phi(\eta)$, which is just the usual vector space formula.

The identity for the tensor operation is just the vector space \mathbb{K} with trivial G -action and grade $e \in M$. For any object V the morphisms $l_V : V \rightarrow V \otimes \mathbb{K}$ and $r_V : V \rightarrow \mathbb{K} \otimes V$ are given by the formulae $l_V(\xi) = \xi \otimes 1$ and $r_V(\xi) = 1 \otimes \xi$, where 1 is the multiplicative identity in k .

This tensor product is not consistent with the usual idea of ignoring the brackets in tensor products. The non-trivial associator $\Phi_{UVW} : (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$ is given by

$$\Phi((\xi \otimes \eta) \otimes \zeta) = \xi \bar{\triangleleft} \tau(\langle \eta \rangle, \langle \zeta \rangle) \otimes (\eta \otimes \zeta) .$$

We can make \mathcal{C}_{\bowtie} into a bar category as follows. Again we identify objects V and \bar{V} as sets. The grading and action on \bar{V} are given by

$$\langle \bar{v} \rangle = \langle v \rangle^R , \quad \bar{v} \bar{\triangleleft} u = \overline{v \bar{\triangleleft} (\langle v \rangle^R \triangleright u)} .$$

It is left to the reader to show that this formula gives a right G -action, and that the compatibility condition is satisfied. Then the morphism $\text{bb}_V : V \rightarrow \bar{\bar{V}}$ is given by

$$\text{bb}_V(v) = \overline{\overline{v \bar{\triangleleft} \tau(\langle v \rangle^L, \langle v \rangle)^{-1}}} .$$

We also have the following formula for $\Upsilon_{WV}^{-1} : \bar{V} \otimes \bar{W} \rightarrow \bar{\bar{W}} \otimes \bar{\bar{V}}$,

$$\begin{aligned} \Upsilon_{WV}^{-1}(\bar{v} \otimes \bar{w}) &= \overline{\overline{w \bar{\triangleleft} \tau(\langle v \rangle \triangleleft \tau(\langle v \rangle^R, \langle w \rangle^R), \langle v \rangle^R \cdot \langle w \rangle^R)^{-1} \otimes v \triangleleft \tau(\langle v \rangle^R, \langle w \rangle^R)}} \\ &= \overline{\overline{(w \bar{\triangleleft} \tau(\langle v \rangle, \langle v \rangle^R)^{-1} \otimes v) \triangleleft \tau(\langle v \rangle^R, \langle w \rangle^R)}} \end{aligned}$$

4 Braided bar categories

We recall that monoidal category is braided if there is a natural equivalence $\Psi : \otimes \rightarrow \otimes^{\text{op}}$ obeying two natural ‘hexagon coherence identities’ with respect to \otimes of each factor. These are sufficient to ensure the Yang-Baxter or representation of the braid relations between any three objects. There is also compatibility with l, r . We refer to [12, 11] for details and [13] for our notations.

4.1 Real and hermitian braidings

If the bar category \mathcal{C} is also braided, we can ask how the braiding Ψ behaves under the bar functor. In particular we can highlight two cases:

Definition 4.1.1 *If the braiding in a bar category \mathcal{C} fits the following diagram:*

$$\begin{array}{ccc} \overline{X} \otimes \overline{Y} & \xrightarrow{\Psi} & \overline{Y} \otimes \overline{X} \\ \Upsilon^{-1} \downarrow & \xrightarrow{\overline{\Psi^{\pm 1}}} & \downarrow \Upsilon^{-1} \\ \overline{Y} \otimes \overline{X} & \xrightarrow{\quad} & \overline{X} \otimes \overline{Y} \end{array}$$

and the sign is $+1$, we call the braiding real. If the sign is -1 , we call the braiding antireal.

Clearly the two notions coincide in the symmetric case where $\Psi^2 = \text{id}$. In this case we say that the bar category is symmetric. Clearly **Vect** is an elementary example with Ψ the usual flip on tensor products.

Example 4.1.2 *Consider modules ${}_H\mathcal{M}$ over a $*$ -Hopf algebra H (see 2.4), where H is also quasitriangular, with $\mathcal{R} \in H \otimes H$. If we write $\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$ (summation implicit), then the braiding in ${}_H\mathcal{M}$ is defined as*

$$\Psi_{V,W}(v \otimes w) = \mathcal{R}^{(2)} \triangleright w \otimes \mathcal{R}^{(1)} \triangleright v .$$

In the literature there are two cases for $\mathcal{R}^{(1)*} \otimes \mathcal{R}^{(2)*}$ singled out, and we will consider the effect on the braiding of both of these possibilities. The reader should remember that $(S \otimes S)\mathcal{R} = \mathcal{R}$.

Case 1 - the real case - the braiding is real: $\mathcal{R}^{(1)*} \otimes \mathcal{R}^{(2)*} = \mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)}$.

$$\begin{aligned} \Psi_{\overline{V},\overline{W}}(\overline{v} \otimes \overline{w}) &= \overline{\mathcal{R}^{(2)*} \triangleright w} \otimes \overline{\mathcal{R}^{(1)*} \triangleright v} \\ &= \overline{\mathcal{R}^{(1)} \triangleright w} \otimes \overline{\mathcal{R}^{(2)} \triangleright v} . \end{aligned}$$

Case 2 - the antireal case - the braiding is antireal: $\mathcal{R}^{(1)*} \otimes \mathcal{R}^{(2)*} = \mathcal{R}^{-1}$.

$$\begin{aligned} \Psi_{\overline{V},\overline{W}}(\overline{v} \otimes \overline{w}) &= \overline{\mathcal{R}^{(2)*} \triangleright w} \otimes \overline{\mathcal{R}^{(1)*} \triangleright v} \\ &= \overline{(\mathcal{R}^{-1})^{(2)} \triangleright w} \otimes \overline{(\mathcal{R}^{-1})^{(1)} \triangleright v} . \end{aligned}$$

Example 4.1.3 *For Yetter Drinfeld modules over $*$ -Hopf algebras (see 3.1), there is a braiding given by*

$$\Psi(v \otimes w) = w_{[0]} \otimes v \triangleleft w_{[1]} .$$

This is antireal, as can be seen by the following calculation:

$$\begin{aligned}
\Psi(\bar{v} \otimes \bar{w}) &= \overline{w_{[0]} \otimes \bar{v} \triangleleft w_{[1]}} \\
&= \overline{w_{[0]} \otimes \bar{v} \triangleleft w_{[1]}^*} \\
&= \overline{w_{[0]} \otimes v \triangleleft S^{-1}(w_{[1]})} , \\
\Upsilon^{-1}\Psi(\bar{v} \otimes \bar{w}) &= \overline{v \triangleleft S^{-1}(w_{[1]}) \otimes w_{[0]}} \\
&= \overline{\Psi^{-1}(w \otimes v)} \\
&= \overline{\Psi^{-1}\Upsilon^{-1}(\bar{v} \otimes \bar{w})}
\end{aligned}$$

Example 4.1.4 For the example of $u_q(SU(2))$, where q is a primitive l th root of unity (for odd integer $l > 1$) (see 3.3.6) the braiding defined by

$$\Psi_{V,W}(v \otimes w) = \mathcal{R}^{(2)} \triangleright w \otimes \mathcal{R}^{(1)} \triangleright v$$

is antireal. Note that Υ is non-trivial in this case.

4.2 Real forms of bar categories

Definition 4.2.1 A star bar category is a bar category \mathcal{C} such that

- There is a natural transformation \star from the identity to the bar functor, making every object into a star object (see 2.6). (For the identity object this is the same as the \star in the definition of bar category.)
- $\star_{\bar{X}} = \overline{\star_X} : \bar{X} \rightarrow \overline{\bar{X}}$.
- There is an invertible natural transformation Γ between the functors \otimes and $\otimes \circ \text{flip}$ from $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ so that the following diagram commutes:

$$\begin{array}{ccc}
X \otimes Y & \xrightarrow{\star \otimes \star} & \bar{X} \otimes \bar{Y} \\
\Gamma_{Y,X} \uparrow & & \downarrow \Gamma^{-1} \\
Y \otimes X & \xrightarrow{\star} & \overline{\bar{X} \otimes \bar{Y}}
\end{array}$$

- The following diagram commutes:

$$\begin{array}{ccc}
(X \otimes Y) \otimes Z & \xrightarrow{\star_{(X \otimes Y) \otimes Z}} & \overline{(X \otimes Y) \otimes Z} \\
\Phi_{X,Y,Z} \downarrow & & \downarrow \overline{\Phi_{X,Y,Z}} \\
X \otimes (Y \otimes Z) & \xrightarrow{\star_{X \otimes (Y \otimes Z)}} & \overline{X \otimes (Y \otimes Z)}
\end{array}$$

Let us pause to unpack this definition. Firstly, the choice of the morphisms \star is natural in the sense that, for any morphism $\phi : X \rightarrow Y$ we have $\bar{\phi} \circ \star_X = \star_Y \circ \phi : X \rightarrow \bar{Y}$. Secondly,

we have two obvious morphisms from \bar{X} to $\overline{\bar{X}}$, $\star_{\bar{X}}$ and $\overline{\star_{\bar{X}}}$, and we insist that these are the same. Thirdly, we have two obvious morphisms from X to $\overline{\bar{X}}$, bb_X and $\star_{\bar{X}}\star_X$, and we insist that these are the same. In the example of the bimodule category, we could write $\star_E : E \rightarrow \bar{E}$ as $e \mapsto \overline{e^*}$, and we would then have $e^{**} = e$, which is a rather more familiar way of writing an involution. There are some immediate deductions which we can make from the definition.

Proposition 4.2.2 *Given the rest of the definition, the following condition is implied by 4.2.1 part (b): $\overline{\Gamma} \Upsilon^{-1} = \Upsilon^{-1} \Gamma^{-1} : \bar{X} \otimes \bar{Y} \rightarrow \overline{\bar{X} \otimes \bar{Y}}$.*

Proof: We can rewrite the equation $\overline{\star_{X \otimes Y}} \star_{X \otimes Y} = \text{bb}_{X \otimes Y}$, using the definition of Γ on one side and the definition of bar category on the other, to give

$$\overline{\Upsilon^{-1}(\star \otimes \star)} \overline{\Gamma} \Upsilon^{-1}(\star \otimes \star) \Gamma = \overline{\Upsilon^{-1}} \Upsilon^{-1}(\overline{\star \otimes \star}) (\star \otimes \star) .$$

Yrom the naturality of Γ , $(\star \otimes \star) \Gamma = \Gamma(\star \otimes \star)$, so

$$\overline{\Upsilon^{-1}(\star \otimes \star)} \overline{\Gamma} \Upsilon^{-1} \Gamma(\star \otimes \star) = \overline{\Upsilon^{-1}} \Upsilon^{-1}(\overline{\star \otimes \star}) (\star \otimes \star) ,$$

and some cancellation gives

$$\overline{(\star \otimes \star)} \overline{\Gamma} \Upsilon^{-1} \Gamma = \Upsilon^{-1}(\overline{\star \otimes \star}) .$$

Yrom the naturality of Υ , $\Upsilon^{-1}(\overline{\star \otimes \star}) = \overline{(\star \otimes \star)} \Upsilon^{-1}$, and using this we get the result. \square

Proposition 4.2.3 *Given the rest of the definition, the following condition is equivalent to 4.2.1 part (d):*

$$\Phi(\Gamma_{Y,Z} \otimes \text{id}) \Gamma_{X,Y \otimes Z} = (\text{id} \otimes \Gamma_{X,Y}) \Gamma_{X \otimes Y,Z} \Phi^{-1} : X \otimes (Y \otimes Z) \rightarrow Z \otimes (Y \otimes X) .$$

Proof: Yirst consider $\star_{(X \otimes Y) \otimes Z}$. Yrom the definition of Γ we can rewrite this as

$$\star_{(X \otimes Y) \otimes Z} = \Upsilon_{X \otimes Y,Z}^{-1} (\star_Z \otimes \star_{X \otimes Y}) \Gamma_{X \otimes Y,Z} ,$$

and applying this again,

$$\star_{(X \otimes Y) \otimes Z} = \Upsilon_{X \otimes Y,Z}^{-1} (\text{id} \otimes \Upsilon_{X,Y}^{-1}) (\star_Z \otimes (\star_Y \otimes \star_X)) (\text{id} \otimes \Gamma_{X,Y}) \Gamma_{X \otimes Y,Z} .$$

Now we rewrite $\star_{X \otimes (Y \otimes Z)}$ using Γ , as

$$\star_{X \otimes (Y \otimes Z)} = \Upsilon_{X,Y \otimes Z}^{-1} (\Upsilon_{Y,Z}^{-1} \otimes \text{id}) ((\star_Z \otimes \star_Y) \otimes \star_X) (\Gamma_{Y,Z} \otimes \text{id}) \Gamma_{X,Y \otimes Z}$$

Yrom the definition of bar category, we have

$$\overline{\Phi_{X,Y,Z}} \star_{(X \otimes Y) \otimes Z} = \Upsilon_{X,Y \otimes Z}^{-1} (\Upsilon_{Y,Z}^{-1} \otimes \text{id}) \Phi_{\bar{Z},\bar{Y},\bar{X}}^{-1} (\star_Z \otimes (\star_Y \otimes \star_X)) (\text{id} \otimes \Gamma_{X,Y}) \Gamma_{X \otimes Y,Z} ,$$

and by naturality of Φ ,

$$\overline{\Phi_{X,Y,Z} \star_{(X \otimes Y) \otimes Z}} = \Upsilon_{X,Y \otimes Z}^{-1} (\Upsilon_{Y,Z}^{-1} \otimes \text{id}) ((\star_Z \otimes \star_Y) \otimes \star_X) \Phi_{Z,Y,X}^{-1} (\text{id} \otimes \Gamma_{X,Y}) \Gamma_{X \otimes Y, Z} .$$

If we substitute this into the condition in 4.2.1 part (d), we can cancel terms to give the equivalence. \square

The following result is motivated by connections on bimodules in noncommutative Riemannian geometry:

Proposition 4.2.4 *Suppose that \mathcal{C} is a bar category, and a braided category, in which the braiding Ψ is antireal (see 2.6.1). Also suppose that*

a) *There is a natural transformation \star from the identity to the bar functor, making every object into a star object (see 2.6).*

b) $\star_{\overline{X}} = \overline{\star_X} : \overline{X} \rightarrow \overline{\overline{X}}$.

c) *For all objects X and Y , where Ψ is the braiding,*

$$\star_{Y \otimes X} = \overline{\Psi_{X,Y}} \Upsilon_{X,Y}^{-1} (\star_Y \otimes \star_X) = \Upsilon_{Y,X}^{-1} (\star_X \otimes \star_Y) \Psi_{X,Y}^{-1} .$$

[The last two expressions are equal as the braiding is antireal.]

Then \mathcal{C} is a star bar category, with $\Gamma = \Psi^{-1}$.

Proof: We just have to verify 4.2.1 parts (c) and (d). These follow from 4.2.2 (using the antireal property) and 4.2.3 (using the usual properties of braidings). \square

5 \star -Algebras and \star -Hopf algebras in (braided) bar categories

We already introduced the notions of algebras and coalgebras in bar categories. In this section we give some nontrivial examples and, in the braided case, bring the two notions together to the study of Hopf algebras in braided bar categories. We recall that a Hopf algebra in a braided category means an object B which is both a unital algebra, a counital algebra, Δ is multiplicative for the braided tensor product algebra on $B \otimes B$, ϵ is multiplicative for which there exists an antipode $S : B \rightarrow B$ defined as for Hopf algebras. We refer to [13] for details.

5.1 Braided \star -Hopf algebras

Definition 5.1.1 *A braided Hopf algebra B in a braided category \mathcal{C} which is also a star category with antireal braiding is a braided \star -Hopf algebra if:*

1) *B is a star algebra in \mathcal{C} .*

2) *The coproduct satisfies*

$$\overline{\Delta} \star = \overline{\Psi} \Upsilon^{-1} (\star \otimes \star) \Delta : B \rightarrow \overline{B \otimes B} .$$

Proposition 5.1.2 *If B is a braided \star -Hopf algebra, then for the antipode S , $S = \star_B^{-1} \overline{S^{-1}} \star_B$.*

Proof: We use the uniqueness of the antipode. First, from the definition of star algebra,

$$\begin{aligned} \mu(\star_B^{-1} \overline{S^{-1}} \star_B \otimes \text{id})\Delta &= \star_B^{-1} \overline{\mu \Upsilon^{-1}(\overline{S^{-1}} \star_B \otimes \star_B)}\Delta \\ &= \star_B^{-1} \overline{\mu(\text{id} \otimes S^{-1})\Upsilon^{-1}(\star_B \otimes \star_B)}\Delta . \end{aligned}$$

Now, from the coproduct rule in 5.1.1,

$$\mu(\star_B^{-1} \overline{S^{-1}} \star_B \otimes \text{id})\Delta = \star_B^{-1} \overline{\mu(\text{id} \otimes S^{-1})\Psi^{-1}\Delta} \star_B .$$

Now, in a purely braided Hopf algebra calculation,

$$\begin{aligned} S\mu(\text{id} \otimes S^{-1})\Psi^{-1}\Delta &= \mu\Psi(S \otimes S)(\text{id} \otimes S^{-1})\Psi^{-1}\Delta \\ &= \mu\Psi(S \otimes \text{id})\Psi^{-1}\Delta \\ &= \mu(\text{id} \otimes S)\Delta = 1_B \cdot \epsilon . \end{aligned}$$

From this,

$$\begin{aligned} \mu(\star_B^{-1} \overline{S^{-1}} \star_B \otimes \text{id})\Delta &= \star_B^{-1} \overline{1_B \cdot \epsilon} \star_B \\ &= 1_B \cdot \epsilon \star_B^{-1} \star_B = 1_B \cdot \epsilon , \end{aligned}$$

where we have used the fact that \star_B is a natural transformation and that $1_B \cdot \epsilon$ is a morphism from B to B . \square

We shall give some examples of this definition. For the first we refer back to 3.3.6, the example of $u_q(SU(2))$, where q is a primitive l th root of unity (for odd integer $l = 2k + 1 > 1$). We shall look at \mathbb{C}_q^2 in the category of left $u_q(SU(2))$ modules where the quantum group is viewed as a quasi- \star -Hopf algebra.

Lemma 5.1.3 *In 3.3 a \star -algebra in the category of modules of a quasi- \star Hopf algebra means a module algebra A and a map $\star : A \rightarrow \overline{A}$ (written $a \mapsto \overline{a^\star}$) such that*

$$(h \triangleright a)^\star = (Sh)^\star \triangleright a^\star, \quad a^{\star\star} = \gamma \triangleright a, \quad (ab)^\star = (\mathcal{F}^{-(1)} \triangleright b^\star)(\mathcal{F}^{-(2)} \triangleright a^\star)$$

for all $a, b \in A$.

Proposition 5.1.4 *The algebra \mathbb{C}_q^2 with generators x, y with relations $yx = qxy$ is a \star -algebra in the category of $u_q(SU(2))$ modules, where*

$$(x^m y^n)^\star = q^{-(m+n)(m+n-1)(k+1)/2} q^{(n-m)/2} q^{nm} x^n y^m .$$

Proof: The algebra \mathbb{C}_q^2 has generators x, y with relations $yx = qxy$. We define a left action of $u_q(SU(2))$ by

$$K \triangleright x = qx, \quad K \triangleright y = q^{-1}y, \quad E \triangleright x = 0 = F \triangleright y, \quad E \triangleright y = q^{-1/2}x, \quad F \triangleright x = q^{1/2}y.$$

The braiding in the category ${}_G\mathcal{M}$ is given by $\Psi(v \otimes w) = \mathcal{R}^{(2)} \triangleright w \otimes \mathcal{R}^{(1)} \triangleright v$, which in this case gives (using $1 \div 2 = k + 1$ modulo l)

$$\begin{aligned} \Psi(x \otimes x) &= q^{k+1} x \otimes x, \quad \Psi(y \otimes y) = q^{k+1} y \otimes y, \quad \Psi(x \otimes y) = q^k y \otimes x, \\ \Psi(y \otimes x) &= q^k x \otimes y + q^{k+1}(1 - q^{-2})y \otimes x. \end{aligned} \quad (12)$$

We can make a star object from \mathbb{C}_q^2 by

$$x^* = z q^{-1/2} y, \quad y^* = z q^{1/2} x,$$

where z is a complex parameter with $|z| = 1$. We immediately set $z = 1$ for simplicity! Of course, this only defines star on the generators, we need to define it on all monomials $x^m y^n$. The hint for this is that the cocycle giving Υ is the same as \mathcal{R} which gives the braiding, so the condition for a star algebra can be rewritten as

$$(ab)^* = \mu \Psi^{-1}(a^* \otimes b^*).$$

This is combined with $\mu \Psi^{-1}(a \otimes b) = q^{-|a||b|(k+1)} a.b$, where $|a|$ and $|b|$ are the orders of a and b respectively. Then, supposing that we can extend to a star algebra, we get the rule on a normal ordered basis:

$$(x^m y^n)^* = q^{-(m+n)(m+n-1)(k+1)/2} q^{(n-m)/2} q^{nm} x^n y^m.$$

A quick induction on order shows that this is consistent with the rule $(g \triangleright v)^* = (Sg)^* \triangleright v^*$, in that if it works for $v = a$ and $v = b$, then it works for $v = ab$. Thus we get a genuine star operation and star algebra. \square

A much more interesting question is whether \mathbb{C}_q^2 can be made into a braided Hopf algebra, with some sort of star structure. The answer without the star structure has been known for some time [13]. There is a coproduct

$$\Delta(x^m y^n) = \sum \begin{bmatrix} m \\ r \end{bmatrix}_q \begin{bmatrix} n \\ s \end{bmatrix}_q q^{s(m-r)} x^r y^s \otimes x^{m-r} y^{n-s},$$

where the q -binomial coefficients are defined in terms of q -factorials by

$$\begin{bmatrix} m \\ r \end{bmatrix}_q = \frac{[m]_q!}{[r]_q! [m-r]_q!},$$

and the q -factorials are recursively defined in terms of q -integers $[m]_q = (q^{2m} - 1)/(q^2 - 1)$ by

$$[0]_q! = 1, \quad [n+1]_q! = [n+1]_q [n]_q! .$$

If we normalise the quasitriangular structure to $\tilde{\mathcal{R}}$, giving braiding

$$\begin{aligned} \tilde{\Psi}(x \otimes x) &= q^2 x \otimes x, \quad \tilde{\Psi}(y \otimes y) = q^2 y \otimes y, \quad \tilde{\Psi}(x \otimes y) = q y \otimes x, \\ \tilde{\Psi}(y \otimes x) &= q x \otimes y + q^2(1 - q^{-2}) y \otimes x, \end{aligned} \quad (13)$$

then this gives a braided Hopf algebra. First we note that for q a primitive cube root of unity ($k = 1$), no renormalisation is necessary. For the other roots, the best way to carry out this renormalisation is to introduce the central extension of $u_q(SU(2))$ described in 3.3.7. We suppose that \tilde{K} acts as q^β on both x and y . Then, as explained in 3.3.5, the extra factor $\mathcal{R}_{\tilde{K}}$ introduces an extra factor of $q^{\beta^2 \div 2}$ into (12), so to get (13) we need $\beta^2 = 3 \pmod{l}$. This is obviously satisfied by $\beta = 0$ if $l = 3$, so as already noted no extension is necessary. For $l > 3$ the existence of a β with $\beta^2 = 3 \pmod{l}$ is well studied in number theory. If there is such a β , then 3 is said to be a quadratic residue for l . For l prime, this is summarised in the Legendre symbol:

$$\left(\frac{3}{l}\right) = \begin{cases} +1 & \exists \beta : \beta^2 = 3 \pmod{l} \\ -1 & \nexists \beta : \beta^2 = 3 \pmod{l} \end{cases}$$

The list of those primes in the first hundred prime numbers for which 3 is a quadratic residue is:

11, 13, 23, 37, 47, 59, 61, 71, 73, 83, 97, 107, 109, 131, 157, 167, 179, 181, 191, 193
227, 229, 239, 241, 251, 263, 277, 311, 313, 337, 347, 349, 359, 373, 383, 397, 409
419, 421, 431, 433, 443, 457, 467, 479, 491, 503, 541 .

For example, in the first case on the list, $5^2 = 3 \pmod{11}$.

Theorem 5.1.5 *In the following cases the coproduct*

$$\Delta(x^m y^n) = \sum \begin{bmatrix} m \\ r \end{bmatrix}_q \begin{bmatrix} n \\ s \end{bmatrix}_q q^{s(m-r)} x^r y^s \otimes x^{m-r} y^{n-s}$$

makes \mathbb{C}_q^2 into a \star -braided Hopf algebra, with braiding

$$\begin{aligned} \tilde{\Psi}(x \otimes x) &= q^2 x \otimes x, \quad \tilde{\Psi}(y \otimes y) = q^2 y \otimes y, \quad \tilde{\Psi}(x \otimes y) = q y \otimes x, \\ \tilde{\Psi}(y \otimes x) &= q x \otimes y + q^2(1 - q^{-2}) y \otimes x. \end{aligned} \quad (14)$$

and the star operation

$$(x^m y^n)^* = q^{-(m+n)(m+n-1)} q^{(n-m)/2} q^{nm} x^n y^m .$$

Case 1: $l = 3$ and we use the usual $u_q(SU(2))$.

Case 2: We use the centrally extended $u_q(\widetilde{SU(2)})$, and restrict to the case where 3 is a quadratic residue mod l . The central generator \bar{K} acts as q^β on x and y , where $\beta^2 = 3 \pmod{l}$.

Proof: Explicit calculation, using examples 3.3.6 and 3.3.7. \square

5.2 Transmutation of Hopf algebras

We recall that the transmutation of a quasitriangular Hopf algebra H is a braided Hopf algebra in the braided category ${}_H\mathcal{M}$ of left H -modules. It is defined by BH being H as a left H module (under the left adjoint action $h \triangleright b = h_{(1)}bS(h_{(2)})$), and has the same algebra, unit and counit as H . For the braided coalgebra, we have

$$\underline{\Delta}b = b_{(1)}S(\mathcal{R}^{(2)}) \otimes \mathcal{R}^{(1)} \triangleright b_{(2)}, \quad \underline{S}(b) = \mathcal{R}^{(2)}S(\mathcal{R}^{(1)} \triangleright b).$$

We shall see that, under some quite common restrictions, the transmutation of a Hopf algebra with star operation gives a braided $*$ -Hopf algebra.

Theorem 5.2.1 *Suppose that H is a quasitriangular $*$ -Hopf algebra, and that $\mathcal{R}^{*\otimes*} = \mathcal{R}^{-1}$. Then if we define $\star : BH \rightarrow \overline{BH}$ by $b \mapsto \bar{b}^*$ (using the star operation from H), then BH is a braided $*$ -Hopf algebra.*

Proof: The reader should remember that in this case $*$ preserves the order of coproduct and switches S and S^{-1} . First check that $\star : BH \rightarrow \overline{BH}$ is a morphism in ${}_H\mathcal{M}$:

$$\begin{aligned} \star(h \triangleright b) &= \overline{(h \triangleright b)^*} = \overline{(h_{(1)}bS(h_{(2)}))^*} \\ &= \overline{S(h_{(2)})^* b^* h_{(1)}^*}, \\ h \triangleright (\star b) &= \overline{S(h)^* \triangleright b^*} = \overline{S(h)_{(1)}^* b^* S(S(h)_{(2)}^*)} \\ &= \overline{S(h_{(2)})^* b^* S(S(h_{(1)}^*))} \\ &= \overline{S(h_{(2)})^* b^* S^{-1}(S(h_{(1)}))^*}. \end{aligned}$$

Next for the coproduct,

$$\begin{aligned} (\star \otimes \star) \underline{\Delta}b &= \overline{S(\mathcal{R}^{(2)})^* b_{(1)}^*} \otimes \overline{(\mathcal{R}^{(1)} \triangleright b_{(2)})^*} \\ &= \overline{S(\mathcal{R}^{(2)})^* b_{(1)}^*} \otimes \overline{S(\mathcal{R}^{(1)})^* \triangleright b_{(2)}^*} \\ &= \overline{\mathcal{R}^{(2)*} b_{(1)}^*} \otimes \overline{\mathcal{R}^{(1)*} \triangleright b_{(2)}^*} \\ \Upsilon^{-1}(\star \otimes \star) \underline{\Delta}b &= \overline{\mathcal{R}^{(1)*} \triangleright b_{(2)}^*} \otimes \overline{\mathcal{R}^{(2)*} b_{(1)}^*} \\ &= \overline{\mathcal{R}^{-(1)} \triangleright b_{(2)}^*} \otimes \overline{\mathcal{R}^{-(2)} b_{(1)}^*} \\ \overline{\Psi} \Upsilon^{-1}(\star \otimes \star) \underline{\Delta}b &= \overline{\mathcal{R}^{(2)} \triangleright (\mathcal{R}^{-(2)} b_{(1)}^*)} \otimes \overline{\mathcal{R}^{(1)} \mathcal{R}^{-(1)} \triangleright b_{(2)}^*} \end{aligned}$$

$$\begin{aligned}
&= \overline{\mathcal{R}^{(2)} \triangleright (\mathcal{R}^{-(2)} b_{(1)}^*) \otimes \mathcal{R}^{(1)} \mathcal{R}^{-(1)} \triangleright b_{(2)}^*} \\
&= \overline{\mathcal{R}^{(2)}_{(1)} \mathcal{R}^{-(2)} b_{(1)}^* S(\mathcal{R}^{(2)}_{(2)}) \otimes \mathcal{R}^{(1)} \mathcal{R}^{-(1)} \triangleright b_{(2)}^*} .
\end{aligned}$$

Now from the definition of quasitriangular structure use $(\text{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}$ to get

$$\overline{\Psi\Upsilon^{-1}(\star \otimes \star)\underline{\Delta}b} = \overline{b_{(1)}^* S(\mathcal{R}^{(2)}) \otimes \mathcal{R}^{(1)} \triangleright b_{(2)}^*} .$$

Finally we check the antipode,

$$\begin{aligned}
\underline{S}(b) &= \mathcal{R}^{(2)} S(\mathcal{R}^{(1)} \triangleright b) \\
&= \mathcal{R}^{(2)} S^2(\mathcal{R}^{(1)}_{(2)}) S(b) S(\mathcal{R}^{(1)}_{(1)}) ,
\end{aligned}$$

and using $(\Delta \otimes \text{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}$,

$$\begin{aligned}
\underline{S}(b) &= \tilde{\mathcal{R}}^{(2)} \mathcal{R}^{(2)} S^2(\mathcal{R}^{(1)}) S(b) S(\tilde{\mathcal{R}}^{(1)}) \\
&= \tilde{\mathcal{R}}^{(2)} u^{-1} S(b) S(\tilde{\mathcal{R}}^{(1)}) \\
&= \mathcal{R}^{-(2)} u^{-1} S(b) \mathcal{R}^{-(1)} , \\
(\underline{S}b)^* &= \mathcal{R}^{(1)} S(b)^* u^{-*} \mathcal{R}^{(2)} , \\
\underline{S}((\underline{S}b)^*) &= \tilde{\mathcal{R}}^{(2)} S^2(\tilde{\mathcal{R}}^{(1)}_{(2)}) S(\mathcal{R}^{(1)} S(b)^* u^{-*} \mathcal{R}^{(2)}) S(\tilde{\mathcal{R}}^{(1)}_{(1)}) \\
&= \tilde{\mathcal{R}}^{(2)} S^2(\tilde{\mathcal{R}}^{(1)}_{(2)}) S(\mathcal{R}^{(2)}) S(u^{-*}) b^* S(\mathcal{R}^{(1)}) S(\tilde{\mathcal{R}}^{(1)}_{(1)}) \\
&= \tilde{\mathcal{R}}^{(2)} \hat{\mathcal{R}}^{(2)} S^2(\tilde{\mathcal{R}}^{(1)}) S(\mathcal{R}^{(2)}) S(u^{-*}) b^* S(\mathcal{R}^{(1)}) S(\tilde{\mathcal{R}}^{(1)}) \\
&= \tilde{\mathcal{R}}^{(2)} u^{-1} S(\mathcal{R}^{(2)}) S(u^{-*}) b^* S(\mathcal{R}^{(1)}) S(\tilde{\mathcal{R}}^{(1)}) \\
&= \tilde{\mathcal{R}}^{(2)} S^{-1}(\mathcal{R}^{(2)}) u^{-1} S(u^{-*}) b^* S(\mathcal{R}^{(1)}) S(\tilde{\mathcal{R}}^{(1)}) \\
&= \tilde{\mathcal{R}}^{(2)} S^{-1}(\mathcal{R}^{(2)}) u^{-1} S(u^{-*}) b^* S(\tilde{\mathcal{R}}^{(1)} \mathcal{R}^{(1)}) \\
&= \tilde{\mathcal{R}}^{(2)} \mathcal{R}^{-(2)} u^{-1} S(u^{-*}) b^* S(\tilde{\mathcal{R}}^{(1)} \mathcal{R}^{-(1)}) \\
&= u^{-1} S(u^{-*}) b^* . \quad \square
\end{aligned}$$

Theorem 5.2.2 *Suppose that H is a quasitriangular flip- $*$ -Hopf algebra, and that $\mathcal{R}^{*\otimes*} = \mathcal{R}_{21}^{-1}$. Then if we define $\star : BH \rightarrow \overline{BH}$ by*

$$\star b = \overline{\mathcal{R}^{-(1)} b^* S^{-1}(\mathcal{R}^{-(2)}) S^{-1}(u)} .$$

(using the star operation from H), then BH is a braided $$ -Hopf algebra.*

Proof: The reader should remember that in this case $*$ flips the order of the coproduct and $S(h^*) = S(h)^*$. First we check that \star is a morphism:

$$\begin{aligned}
h \triangleright \star b &= \overline{S(h)^*_{(1)} \mathcal{R}^{-(1)} b^* S^{-1}(\mathcal{R}^{-(2)}) S^{-1}(u) S(S(h)^*_{(2)})} \\
&= \overline{S(h^*_{(2)}) \mathcal{R}^{-(1)} b^* S^{-1}(\mathcal{R}^{-(2)}) S^{-1}(u) S^2(h^*_{(1)})} \\
&= \overline{S(S^{-1}(\mathcal{R}^{-(1)}) h^*_{(2)}) b^* S^{-1}(\mathcal{R}^{-(2)}) h^*_{(1)} S^{-1}(u)}
\end{aligned}$$

$$\begin{aligned}
&= \overline{S(\mathcal{R}^{-(1)}h^*_{(2)})b^*\mathcal{R}^{-(2)}h^*_{(1)}S^{-1}(u)} \\
&= \overline{S(h^*_{(1)}\mathcal{R}^{-(1)})b^*h^*_{(2)}\mathcal{R}^{-(2)}S^{-1}(u)} \\
&= \overline{S(\mathcal{R}^{-(1)})S(h^*_{(1)})b^*h^*_{(2)}S^{-1}(S(\mathcal{R}^{-(2)}))S^{-1}(u)} \\
&= \overline{\mathcal{R}^{-(1)}S(h^*_{(1)})b^*h^*_{(2)}S^{-1}(\mathcal{R}^{-(2)})S^{-1}(u)} \\
&= \overline{\mathcal{R}^{-(1)}S(h_{(2)}^*)b^*h_{(1)}^*S^{-1}(\mathcal{R}^{-(2)})S^{-1}(u)} \\
&= \overline{\mathcal{R}^{-(1)}(h \triangleright b)^*S^{-1}(\mathcal{R}^{-(2)})S^{-1}(u)} ,
\end{aligned}$$

where we have used $(S \otimes S)\mathcal{R} = \mathcal{R}$.

Next, we do \star twice:

$$\begin{aligned}
\star \star b &= \overline{\star(\overline{\mathcal{R}^{-(1)}b^*S^{-1}(\mathcal{R}^{-(2)})S^{-1}(u)})} \\
&= \overline{\overline{\tilde{\mathcal{R}}^{-(1)}S^{-1}(u^*)S^{-1}(\mathcal{R}^{-(2)*})b\mathcal{R}^{-(1)*}S^{-1}(\tilde{\mathcal{R}}^{-(2)})S^{-1}(u)}} \\
&= \overline{\overline{\tilde{\mathcal{R}}^{-(1)}S^{-1}(u^{-1})S^{-1}(\mathcal{R}^{(1)})b\mathcal{R}^{(2)}S^{-1}(\tilde{\mathcal{R}}^{-(2)})S^{-1}(u)}} \\
&= \overline{\overline{S^{-1}(u^{-1})S^{-2}(\tilde{\mathcal{R}}^{-(1)})S^{-1}(\mathcal{R}^{(1)})b\mathcal{R}^{(2)}S^{-1}(\tilde{\mathcal{R}}^{-(2)})S^{-1}(u)}} \\
&= \overline{\overline{S^{-1}(u^{-1})S^{-1}(\mathcal{R}^{(1)}S^{-1}(\tilde{\mathcal{R}}^{-(1)}))b\mathcal{R}^{(2)}S^{-1}(\tilde{\mathcal{R}}^{-(2)})S^{-1}(u)}} \\
&= \overline{\overline{S^{-1}(u^{-1})S^{-1}(\mathcal{R}^{(1)}\tilde{\mathcal{R}}^{-(1)})b\mathcal{R}^{(2)}\tilde{\mathcal{R}}^{-(2)}S^{-1}(u)}} \\
&= \overline{S^{-1}(u^{-1})bS^{-1}(u)}
\end{aligned}$$

using $u^* = u^{-1}$ and $(S \otimes S)\mathcal{R} = \mathcal{R}$. As $\gamma = v^{-1}\nu$ is grouplike, we get

$$\begin{aligned}
\gamma \triangleright b &= \gamma b \gamma^{-1} \\
&= v^{-1}\nu b \nu^{-1}v = v^{-1}bv ,
\end{aligned}$$

using ν central. Now use $S^{-1}(u) = v$.

Now check that we get a star algebra. The following expression should be $\star(ab)$:

$$\begin{aligned}
&\overline{\overline{\bar{\mu}\Upsilon^{-1}(\tilde{\mathcal{R}}^{-(1)}a^*S^{-1}(\tilde{\mathcal{R}}^{-(2)})S^{-1}(u) \otimes \tilde{\mathcal{R}}^{-(1)}b^*S^{-1}(\tilde{\mathcal{R}}^{-(2)})S^{-1}(u))}} \\
&= \overline{\overline{(R^{-(1)}\triangleright(\tilde{\mathcal{R}}^{-(1)}b^*S^{-1}(\tilde{\mathcal{R}}^{-(2)})S^{-1}(u))) (R^{-(2)}\triangleright(\tilde{\mathcal{R}}^{-(1)}a^*S^{-1}(\tilde{\mathcal{R}}^{-(2)})S^{-1}(u)))}} \\
&= \overline{\overline{R^{-(1)}_{(1)}\tilde{\mathcal{R}}^{-(1)}b^*S^{-1}(\tilde{\mathcal{R}}^{-(2)})S^{-1}(u)S(R^{-(1)}_{(2)})R^{-2}_{(1)}\hat{\mathcal{R}}^{-(1)}a^*S^{-1}(\hat{\mathcal{R}}^{-(2)})S^{-1}(u)S(\mathcal{R}^{-2}_{(2)})}} \\
&= \overline{\overline{R^{-(1)}_{(1)}\tilde{\mathcal{R}}^{-(1)}b^*S^{-1}(\tilde{\mathcal{R}}^{-(2)})S^{-1}(u)S(R^{-(1)}_{(2)})R^{-2}_{(1)}\hat{\mathcal{R}}^{-(1)}a^*S^{-1}(\mathcal{R}^{-2}_{(2)})\hat{\mathcal{R}}^{-(2)}S^{-1}(u)}}
\end{aligned}$$

From the usual rules for the quasitriangular structure we get

$$(\Delta \otimes \Delta)\mathcal{R} = \mathcal{R}_{14}\mathcal{R}_{13}\mathcal{R}_{24}\mathcal{R}_{23} ,$$

so

$$\mathcal{R}^{-(1)}_{(1)} \otimes S^{-1}(u)S(\mathcal{R}^{-(1)}_{(2)})\mathcal{R}^{-2}_{(1)} \otimes \mathcal{R}^{-2}_{(2)}$$

$$\begin{aligned}
&= \check{\mathcal{R}}^{-(1)}\mathcal{R}^{-(1)} \otimes S^{-1}(u)S(\check{\mathcal{R}}^{-(1)}\check{\mathcal{R}}^{-(1)})\check{\mathcal{R}}^{-{(2)}\check{\mathcal{R}}^{-{(2)}} \otimes \check{\mathcal{R}}^{-{(2)}\mathcal{R}^{-{(2)}} \\
&= \check{\mathcal{R}}^{-(1)}\mathcal{R}^{-(1)} \otimes S^{-1}(u)S(\check{\mathcal{R}}^{-(1)})S^{-1}(u^{-1})\check{\mathcal{R}}^{-{(2)}} \otimes \check{\mathcal{R}}^{-{(2)}\mathcal{R}^{-{(2)}} \\
&= \check{\mathcal{R}}^{-(1)}\mathcal{R}^{-(1)} \otimes S^{-1}(\check{\mathcal{R}}^{-(1)})\check{\mathcal{R}}^{-{(2)}} \otimes \check{\mathcal{R}}^{-{(2)}\mathcal{R}^{-{(2)}} \\
&= \check{\mathcal{R}}^{-(1)}\mathcal{R}^{-(1)} \otimes S^{-1}(S(\check{\mathcal{R}}^{-{(2)}})\check{\mathcal{R}}^{-{(1)}}) \otimes \check{\mathcal{R}}^{-{(2)}\mathcal{R}^{-{(2)}} \\
&= \check{\mathcal{R}}^{(1)}\mathcal{R}^{-(1)} \otimes S^{-1}(\check{\mathcal{R}}^{(2)}\check{\mathcal{R}}^{-{(1)}}) \otimes \check{\mathcal{R}}^{-{(2)}\mathcal{R}^{-{(2)}}
\end{aligned}$$

Now, substitute this back into the expression we hope is $\star(ab)$ to get:

$$\begin{aligned}
&\overline{\check{\mathcal{R}}^{(1)}\mathcal{R}^{-(1)}\check{\mathcal{R}}^{-{(1)}}b^*S^{-1}(\check{\mathcal{R}}^{-{(2)}})S^{-1}(\check{\mathcal{R}}^{(2)}\check{\mathcal{R}}^{-{(1)})\hat{\mathcal{R}}^{-{(1)}}a^*S^{-1}(\check{\mathcal{R}}^{-{(2)}\mathcal{R}^{-{(2)})\hat{\mathcal{R}}^{-{(2)}}}S^{-1}(u)} \\
&= \overline{\check{\mathcal{R}}^{(1)}\mathcal{R}^{-(1)}\check{\mathcal{R}}^{-{(1)}}b^*S^{-1}(\check{\mathcal{R}}^{(2)}\check{\mathcal{R}}^{-{(1)})\check{\mathcal{R}}^{-{(2)})\hat{\mathcal{R}}^{-{(1)}}a^*S^{-1}(\check{\mathcal{R}}^{-{(2)}\mathcal{R}^{-{(2)})\hat{\mathcal{R}}^{-{(2)}}}S^{-1}(u)} .
\end{aligned}$$

From this we have, using the braid relations,

$$\check{\mathcal{R}}_{12}\check{\mathcal{R}}_{23}^-\mathcal{R}_{13}^-\check{\mathcal{R}}_{12}^- = \mathcal{R}_{13}^-\check{\mathcal{R}}_{23}^-$$

which gives, as required,

$$\begin{aligned}
&\overline{\mathcal{R}^{-{(1)}}b^*S^{-1}(\check{\mathcal{R}}^{-{(1)})\hat{\mathcal{R}}^{-{(1)}}a^*S^{-1}(\mathcal{R}^{-{(2)}\check{\mathcal{R}}^{-{(2)})\hat{\mathcal{R}}^{-{(2)}}}S^{-1}(u)} \\
&= \overline{\mathcal{R}^{-{(1)}}b^*\check{\mathcal{R}}^{(1)}\hat{\mathcal{R}}^{-{(1)}}a^*S^{-1}(\mathcal{R}^{-{(2)}\check{\mathcal{R}}^{(2)}\hat{\mathcal{R}}^{-{(2)})}S^{-1}(u)} \\
&= \overline{\mathcal{R}^{-{(1)}}b^*a^*S^{-1}(\mathcal{R}^{-{(2)})}S^{-1}(u)} \\
&= \star(ab) .
\end{aligned}$$

Now we check the rule for the coproduct: Using $Q = \mathcal{R}_{21}\mathcal{R}$,

$$\begin{aligned}
\Upsilon\overline{\Psi^{-1}\underline{\Delta}}\star a &= \overline{\mathcal{R}^{-{(1)}}_{(1)}a^*_{(1)}S^{-1}(\mathcal{R}^{-{(2)})_{(1)}}S^{-1}(u)_{(1)}S(\check{\mathcal{R}}^{(2)})} \\
&\quad \otimes \overline{\check{\mathcal{R}}^{(1)}_{\triangleright}(\mathcal{R}^{-{(1)})_{(2)}}a^*_{(2)}S^{-1}(\mathcal{R}^{-{(2)})_{(2)}}S^{-1}(u)_{(2)}} \\
&= \overline{\mathcal{R}^{-{(1)}}_{(1)}a^*_{(1)}S^{-1}(\mathcal{R}^{-{(2)})_{(2)}}S^{-1}(Q^{-{(2)})}S^{-1}(u)S(\check{\mathcal{R}}^{(2)})} \\
&\quad \otimes \overline{\check{\mathcal{R}}^{(1)}_{(1)}\mathcal{R}^{-{(1)}}_{(2)}a^*_{(2)}S^{-1}(\mathcal{R}^{-{(2)})_{(1)}}S^{-1}(Q^{-{(1)})}S^{-1}(u)S(\check{\mathcal{R}}^{(1)})_{(2)}} \\
&= \overline{\mathcal{R}^{-{(1)}}_{(1)}a^*_{(1)}S^{-1}(\mathcal{R}^{-{(2)})_{(2)}}S^{-1}(Q^{-{(2)})}S^{-1}(\check{\mathcal{R}}^{(2)})S^{-1}(u)} \\
&\quad \otimes \overline{\check{\mathcal{R}}^{(1)}_{(1)}\mathcal{R}^{-{(1)}}_{(2)}a^*_{(2)}S^{-1}(\mathcal{R}^{-{(2)})_{(1)}}S^{-1}(Q^{-{(1)})}S^{-1}(\check{\mathcal{R}}^{(1)})_{(2)}S^{-1}(u)} \\
&= \overline{\mathcal{R}^{-{(1)}}_{(1)}a^*_{(1)}S^{-1}(\check{\mathcal{R}}^{(2)}Q^{-{(2)}\mathcal{R}^{-{(2)})_{(2)}}}S^{-1}(u)} \\
&\quad \otimes \overline{\check{\mathcal{R}}^{(1)}_{(1)}\mathcal{R}^{-{(1)}}_{(2)}a^*_{(2)}S^{-1}(\check{\mathcal{R}}^{(1)}_{(2)}Q^{-{(1)}\mathcal{R}^{-{(2)})_{(1)}}}S^{-1}(u)} \\
&= \overline{\mathcal{R}^{-{(1)}}_{(1)}a^*_{(1)}S^{-1}(\check{\mathcal{R}}^{(2)}\mathcal{R}^{-{(2)})_{(2)}}Q^{-{(2)}}}S^{-1}(u) \\
&\quad \otimes \overline{\check{\mathcal{R}}^{(1)}_{(1)}\mathcal{R}^{-{(1)}}_{(2)}a^*_{(2)}S^{-1}(\check{\mathcal{R}}^{(1)}_{(2)}\mathcal{R}^{-{(2)})_{(1)}}Q^{-{(1)}}}S^{-1}(u) ,
\end{aligned}$$

where we have used the fact that Q commutes with coproducts. Now we use the braid relations again:

$$(1 \otimes (\Delta \otimes \text{id})\mathcal{R})((\Delta \otimes \Delta)\mathcal{R}^{-1})(1 \otimes 1 \otimes Q^{-1}) = \mathcal{R}_{24}\mathcal{R}_{34}\mathcal{R}_{23}^-\mathcal{R}_{24}^-\mathcal{R}_{13}^-\mathcal{R}_{14}^-\mathcal{R}_{34}^-\mathcal{R}_{43}^-$$

$$\begin{aligned}
&= \mathcal{R}_{23}^- \mathcal{R}_{34}^- \mathcal{R}_{13}^- \mathcal{R}_{14}^- \mathcal{R}_{34}^- \mathcal{R}_{43}^- \\
&= \tilde{\mathcal{R}}_{23}^- \tilde{\mathcal{R}}_{14}^- \hat{\mathcal{R}}_{13}^- \tilde{\mathcal{R}}_{43}^- .
\end{aligned}$$

On substitution we get

$$\begin{aligned}
\Upsilon \overline{\Psi^{-1} \underline{\Delta}} \star a &= \overline{\mathcal{R}^{-(1)} \hat{\mathcal{R}}^{-(1)} a_{(1)}^* S^{-1}(\mathcal{R}^{-(2)} \check{\mathcal{R}}^{-(1)}) S^{-1}(u)} \\
&\quad \otimes \overline{\tilde{\mathcal{R}}^{-(1)} a_{(2)}^* S^{-1}(\tilde{\mathcal{R}}^{-(2)} \hat{\mathcal{R}}^{-(2)} \check{\mathcal{R}}^{-(2)}) S^{-1}(u)} \\
&= \overline{\mathcal{R}^{-(1)} \hat{\mathcal{R}}^{-(1)} a_{(1)}^* \check{\mathcal{R}}^{-(1)} S^{-1}(\mathcal{R}^{-(2)}) S^{-1}(u)} \\
&\quad \otimes \overline{\tilde{\mathcal{R}}^{-(1)} a_{(2)}^* \tilde{\mathcal{R}}^{-(2)} S^{-1}(\hat{\mathcal{R}}^{-(2)}) S^{-1}(\tilde{\mathcal{R}}^{-(2)}) S^{-1}(u)} \\
&= \overline{\mathcal{R}^{-(1)} \hat{\mathcal{R}}^{-(1)} \check{\mathcal{R}}^{-(1)} a_{(2)}^* S^{-1}(\mathcal{R}^{-(2)}) S^{-1}(u)} \\
&\quad \otimes \overline{\tilde{\mathcal{R}}^{-(1)} \check{\mathcal{R}}^{-(2)} a_{(1)}^* S^{-1}(\hat{\mathcal{R}}^{-(2)}) S^{-1}(\tilde{\mathcal{R}}^{-(2)}) S^{-1}(u)} .
\end{aligned}$$

Now apply the inverse of \star :

$$\begin{aligned}
(\star^{-1} \otimes \star^{-1}) \Upsilon \overline{\Psi^{-1} \underline{\Delta}} \star a &= (\hat{\mathcal{R}}^{-(1)} \check{\mathcal{R}}^{-(1)} a_{(2)}^*)^* \otimes (\check{\mathcal{R}}^{-(2)} a_{(1)}^* S^{-1}(\hat{\mathcal{R}}^{-(2)}))^* \\
&= a_{(1)} \check{\mathcal{R}}^{(2)} \hat{\mathcal{R}}^{(2)} \otimes S^{-1}(\hat{\mathcal{R}}^{(1)}) a_{(2)} \check{\mathcal{R}}^{(1)} \\
&= a_{(1)} S(\check{\mathcal{R}}^{(2)}) S(\hat{\mathcal{R}}^{(2)}) \otimes \hat{\mathcal{R}}^{(1)} a_{(2)} S(\check{\mathcal{R}}^{(1)}) \\
&= a_{(1)} S(\hat{\mathcal{R}}^{(2)} \check{\mathcal{R}}^{(2)}) \otimes \hat{\mathcal{R}}^{(1)} a_{(2)} S(\check{\mathcal{R}}^{(1)}) \\
&= a_{(1)} S(\mathcal{R}^{(2)}) \otimes \mathcal{R}^{(1)}_{(1)} a_{(2)} S(\mathcal{R}^{(1)}_{(1)}) \\
&= \underline{\Delta} a . \quad \square
\end{aligned}$$

5.3 Example: The octonians

We construct these as a \star algebra using the setting of Theorem 3.4.1 and the following immediate corollary:

Proposition 5.3.1 *Given the conditions in theorem 3.4.1, if there is an star algebra A in the category ${}_H\mathcal{M}$, then $F(A)$ is a star algebra A in the category ${}_H\mathcal{F}\mathcal{M}$, where we set $\star : F(a) \mapsto \overline{F(\varphi^{-1} \triangleright a^*)}$.*

Proof: Using Theorem 3.4.1, we apply the twisting functor the any commutative diagram in ${}_H\mathcal{M}$ to obtain one in ${}_H\mathcal{F}\mathcal{M}$. \square

In particular, consider $H = \mathbb{C}(G)$, the functions on a finite group. This has basis of delta-functions $\{\delta_a\}$ labelled by $a \in G$ and coproduct $\Delta \delta_a = \sum_{bc=a} \delta_b \otimes \delta_c$, counit $\epsilon \delta_a = \delta_{a,e}$ and antipode $S \delta_a = \delta_{a^{-1}}$. Here e is the group identity. A cochain on H is a suitable $\mathcal{F} \in H \otimes H$ i.e. a nowhere vanishing 2-argument function $\mathcal{F}(a, b)$ on the group with value 1 when either argument is the group identity e . Then

$$\Phi(a, b, c) = \frac{\mathcal{F}(b, c) \mathcal{F}(a, bc)}{\mathcal{F}(ab, c) \mathcal{F}(a, b)}$$

is the usual group-cohomology coboundary of F and is a group 3-cocycle. Then $H^{\mathcal{F}}$ is the same algebra and coalgebra as H but is viewed as a quasi-Hopf algebra with this Φ .

We next take $A = \mathbb{C}G$, the group algebra of G . This has basis $\{e_a\}$ labelled again by group elements. The product is just the product of G , so $e_a e_b = e_{ab}$. This is covariant under $\mathbb{C}(G)$ with action

$$\delta_{a \triangleright} e_b = \delta_{a,b} e_b.$$

Applying the twisting functor gives the twisted group algebra $A_{\mathcal{F}}$ with the new product

$$e_a \bullet e_b = F^{-1}(a, b) e_{ab}$$

An example is $G = \mathbb{Z}_2^3$ which we write additively as 3-vectors \vec{a} with entries in \mathbb{Z}_2 . We take

$$\mathcal{F}(\vec{a}, \vec{b}) = (-1)^{\vec{a}^T \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \vec{b} + a_1 b_2 b_3 + b_1 a_2 b_3 + b_1 b_2 a_3}, \quad \Phi(\vec{a}, \vec{b}, \vec{c}) = (-1)^{\vec{a} \cdot (\vec{b} \times \vec{c})}$$

The new product

$$e_{\vec{a}} \bullet e_{\vec{b}} = \mathcal{F}(\vec{a}, \vec{b}) e_{\vec{a} + \vec{b}}$$

is that of the octonions \mathbb{O} as explained in [1]. This is the by-now well-known construction of the octonions as a quasi-algebra obtained by twisting.

Proposition 5.3.2 *The cochain \mathcal{F} obeys the conditions of Proposition 3.4.2 and hence the category above of \mathbb{Z}_2^3 -graded spaces is a bar category with*

$$\text{bb}_V(v) = \bar{v}, \quad \Upsilon_{V,W}(\overline{v \otimes w}) = \bar{v} \otimes \bar{w} \begin{cases} -1 & 0 \neq |v| \neq |w| \neq 0 \\ 1 & \text{else} \end{cases}$$

on elements of homogeneous degree in objects V, W . Moreover, the Octonions \mathbb{O} are a \star -algebra in this category with $e_{\vec{a}}^* = e_{-\vec{a}}$ in the above basis.

Proof: The element $\mathcal{G} = \mathcal{F}_{21}\mathcal{F}$ is also the element that makes the category symmetric and was computed in [1] to give the above. The φ is trivial hence the \star operation is the same as that on $\mathbb{C}\mathbb{Z}_2^3$.

5.4 A nontrivially associated C^* algebra

There is an algebra A_{\bowtie} in the category \mathcal{C}_{\bowtie} defined as follows. It has vector space basis $\delta_s \otimes u$ for all $s \in M$ and $u \in G$. The grade is defined to be the unique solution to $s \cdot \langle \delta_s \otimes u \rangle = s \triangleleft u$, and if we set $a = \langle \delta_s \otimes u \rangle$, the action is given by

$$(\delta_s \otimes u) \triangleleft v = \delta_{s \triangleleft (a \triangleright v)} \otimes (a \triangleright v)^{-1} uv. \quad (15)$$

The formula for the product μ for A_{\bowtie} consistent with the action above is

$$(\delta_s \otimes u)(\delta_t \otimes v) = \delta_{t, s \triangleleft u} \delta_{s \triangleleft \tau(a, b)} \otimes \tau(a, b)^{-1} uv ,$$

where $a = \langle \delta_s \otimes u \rangle$ and $b = \langle \delta_t \otimes v \rangle$. There is a unit and counit defined by

$$I = \sum_t \delta_t \otimes e , \quad \epsilon(\delta_s \otimes u) = \delta_{s, e} .$$

The reader should refer to [5] for the details of the construction

Proposition 5.4.1 *The following morphism $\star : A_{\bowtie} \rightarrow \overline{A_{\bowtie}}$ makes A_{\bowtie} into a star object in \mathcal{C}_{\bowtie} , where $a = \langle \delta_s \otimes u \rangle$:*

$$\star(\delta_s \otimes u) = \overline{(\delta_s \otimes u)^*} = \overline{(\delta_{s \triangleleft u} \otimes u^{-1} \tau(a, a^R)) \overline{\tau(a, a^R)^{-1}}} . \quad (16)$$

Proof: Begin with

$$(s \triangleleft u) \cdot a^R = (s \cdot a) \cdot a^R = s \triangleleft \tau(a, a^R) \cdot (a \cdot a^R) = s \triangleleft \tau(a, a^R) .$$

This means that $\delta_{s \triangleleft u} \otimes u^{-1} \tau(a, a^R)$ has grade a^R . Now we use the formula $a^L = a^R \triangleleft \tau(a, a^R)^{-1}$ to see that the following element, which we define to be $(\delta_s \otimes u)^*$, has grade a^L :

$$\begin{aligned} (\delta_s \otimes u)^* &= (\delta_{s \triangleleft u} \otimes u^{-1} \tau(a, a^R)) \triangleleft \tau(a, a^R)^{-1} \\ &= \delta_{s \triangleleft u \tau(a^L, a)^{-1}} \otimes \tau(a^L, a) u^{-1} . \end{aligned}$$

To get the last equality we use $a^r \triangleright \tau(a, a^R)^{-1} = \tau(a^L, a)^{-1}$. Now we define a map $\star : A_{\bowtie} \rightarrow \overline{A_{\bowtie}}$ by

$$\star(\delta_s \otimes u) = \overline{(\delta_s \otimes u)^*} = \overline{(\delta_{s \triangleleft u} \otimes u^{-1} \tau(a, a^R)) \overline{\tau(a, a^R)^{-1}}} . \quad (17)$$

By construction this preserves grade (since $(a^L)^R = a$), and now we show that it also preserves the G -action, and is thus a morphism in \mathcal{C}_{\bowtie} .

$$\begin{aligned} \star((\delta_s \otimes u) \overline{v}) &= \star(\delta_{s \triangleleft (a \triangleright v)} \otimes (a \triangleright v)^{-1} uv) \\ &= \overline{(\delta_{s \triangleleft uv} \otimes v^{-1} u^{-1} (a \triangleright v) \tau(a \triangleleft v, (a \triangleleft v)^R)) \overline{\tau(a \triangleleft v, (a \triangleleft v)^R)^{-1}}} \\ &= \overline{(\delta_{s \triangleleft uv} \otimes v^{-1} u^{-1} a v (a \triangleleft v)^R) \overline{\tau(a \triangleleft v, (a \triangleleft v)^R)^{-1}}} , \\ \star((\delta_s \otimes u)) \overline{v} &= \overline{((\delta_{s \triangleleft u} \otimes u^{-1} \tau(a, a^R)) \overline{\tau(a, a^R)^{-1}}) \overline{v}} \\ &= \overline{(\delta_{s \triangleleft u} \otimes u^{-1} \tau(a, a^R)) \overline{\tau(a, a^R)^{-1} (a \triangleright v)}} . \end{aligned}$$

To show that these are the same, we first calculate

$$\begin{aligned} \tau(a \triangleleft v, (a \triangleleft v)^R)^{-1} (a \triangleright v)^{-1} \tau(a, a^R) &= (a \triangleleft v)^{R-1} (a \triangleleft v)^{-1} (a \triangleright v)^{-1} a a^R \\ &= (a \triangleleft v)^{R-1} v^{-1} a^R , \end{aligned}$$

so we have to show the following equality:

$$\delta_{s\triangleleft uv} \otimes v^{-1}u^{-1}av(a\triangleleft v)^R = (\delta_{s\triangleleft u} \otimes u^{-1}\tau(a, a^R))\bar{\triangleleft}a^{R-1}v(a\triangleleft v)^R. \quad (18)$$

Now we use the factorisation (remembering that $a^{R-1}v(a\triangleleft v)^R \in G$)

$$a^R a^{R-1}v(a\triangleleft v)^R = v(a\triangleleft v)^R = (a^R \triangleright a^{R-1}v(a\triangleleft v)^R)(a^R \triangleleft a^{R-1}v(a\triangleleft v)^R)$$

to apply the formula for the action on $A_{\triangleright\triangleleft}$,

$$(\delta_{s\triangleleft u} \otimes u^{-1}\tau(a, a^R))\bar{\triangleleft}a^{R-1}v(a\triangleleft v)^R = \delta_{s\triangleleft uv} \otimes v^{-1}u^{-1}\tau(a, a^R)a^{R-1}v(a\triangleleft v)^R.$$

This verifies (18), and completes the proof. \square

Proposition 5.4.2 $A_{\triangleright\triangleleft}$ is a star algebra in the category $\mathcal{C}_{\triangleright\triangleleft}$.

Proof: Set $x = \delta_s \otimes u$ and $y = \delta_t \otimes v$, and set $\langle x \rangle = a$ and $\langle y \rangle = b$. Now

$$\begin{aligned} \bar{\mu} \Upsilon^{-1}(\bar{x}^* \otimes \bar{y}^*) &= \overline{\mu(y^* \triangleleft \tau(a^L, a)^{-1} \otimes x^*) \bar{\triangleleft} \tau(a, b)} \\ &= \overline{((\delta_{t\triangleleft v\tau(b^L, b)^{-1}} \otimes \tau(b^L, b)v^{-1}) \bar{\triangleleft} \tau(a^L, a)^{-1})} \\ &\quad \overline{(\delta_{s\triangleleft u\tau(a^L, a)^{-1}} \otimes \tau(a^L, a)u^{-1}) \bar{\triangleleft} \tau(a, b)} \\ &= \overline{(\delta_{t\triangleleft v\tau(b^L, b)^{-1}(b^L \triangleright \tau(a^L, a)^{-1})} \otimes (b^L \triangleright \tau(a^L, a)^{-1})^{-1} \tau(b^L, b)v^{-1} \tau(a^L, a)^{-1})} \\ &\quad \overline{(\delta_{s\triangleleft u\tau(a^L, a)^{-1}} \otimes \tau(a^L, a)u^{-1}) \bar{\triangleleft} \tau(a, b)}. \end{aligned} \quad (19)$$

The grade of the first factor in this product we label as $c = b^L \triangleleft \tau(a^L, a)^{-1}$. Then we continue with the product:

$$\begin{aligned} \bar{\mu} \Upsilon^{-1}(\bar{x}^* \otimes \bar{y}^*) &= \frac{\delta_{s\triangleleft u\tau(a^L, a)^{-1}, t\triangleleft \tau(a^L, a)^{-1}} \overline{(\delta_{t\triangleleft v\tau(b^L, b)^{-1}(b^L \triangleright \tau(a^L, a)^{-1})\tau(c, a^L)} \otimes \tau(c, a^L)^{-1}(b^L \triangleright \tau(a^L, a)^{-1})^{-1} \tau(b^L, b)v^{-1}u^{-1}) \bar{\triangleleft} \tau(a, b)}}{\delta_{s\triangleleft u, t} \overline{\delta_{t\triangleleft v\tau(b^L, b)^{-1}(b^L \triangleright \tau(a^L, a)^{-1})\tau(c, a^L)w} \otimes w^{-1} \tau(c, a^L)^{-1}(b^L \triangleright \tau(a^L, a)^{-1})^{-1} \tau(b^L, b)v^{-1}u^{-1} \tau(a, b)}}, \end{aligned}$$

where we have used $w = (c \cdot a^L) \triangleright \tau(a, b)$. Next

$$\begin{aligned} &\tau(b^L, b)^{-1}(b^L \triangleright \tau(a^L, a)^{-1})\tau(c, a^L)w((c \cdot a^L) \triangleleft \tau(a, b)) \\ &= \tau(b^L, b)^{-1}(b^L \triangleright \tau(a^L, a)^{-1})ca^L\tau(a, b) \\ &= \tau(b^L, b)^{-1}b^L\tau(a^L, a)^{-1}a^L\tau(a, b) \\ &= b^{-1}a^{-1}\tau(a, b) \\ &= (\tau(a, b)(a \cdot b))^{-1}\tau(a, b) \\ &= (a \cdot b)^{-1} \\ &= \tau((a \cdot b)^L, a \cdot b)^{-1}(a \cdot b)^L. \end{aligned}$$

This means that

$$\bar{\mu} \Upsilon^{-1}(\bar{x}^* \otimes \bar{y}^*) = \delta_{s\triangleleft u, t} \overline{\delta_{t\triangleleft v\tau((a \cdot b)^L, a \cdot b)^{-1}} \otimes \tau((a \cdot b)^L, a \cdot b)v^{-1}u^{-1}\tau(a, b)}$$

$$= \delta_{s\triangleleft u, t} \overline{\delta_{s\triangleleft uv\tau((a\cdot b)^L, a\cdot b)^{-1}} \otimes \tau((a\cdot b)^L, a\cdot b)v^{-1}u^{-1}\tau(a, b)} .$$

We compare this with

$$*\mu(x \otimes y) = \delta_{s\triangleleft u, t} * (\delta_{s\triangleleft \tau(a, b)} \otimes \tau(a, b)^{-1}uv) ,$$

which gives exactly the same result. \square

There is also an object $\mathbb{K}(M)$ with basis elements δ_s for all $s \in M$, and grade and action given by $\langle \delta_s \rangle = s$ and $\delta_s \triangleleft v = \delta_{s\triangleleft v}$. This has a star operation defined by $\star \delta_s = \overline{\delta_{s^L}}$. To check that this star is a morphism, we have

$$\overline{\delta_{s^L} \triangleleft u} = \overline{\delta_{s^L} \triangleleft (s \triangleright u)} = \overline{\delta_{s^L \triangleleft (s \triangleright u)}} = \overline{\delta_{(s \triangleleft u)^L}} .$$

Proposition 5.4.3 *There is a nondegenerate inner product given by a morphism $\langle\langle, \rangle\rangle : \mathbb{K}(M) \otimes \mathbb{K}(\overline{M}) \rightarrow 1_{\mathbb{C}}$, defined by $\langle\langle \delta_t, \overline{\delta_s} \rangle\rangle = \delta_{s, t}$. It is symmetric, in the sense that*

$$\overline{\langle\langle, \rangle\rangle} \Upsilon^{-1}(\text{bb} \otimes \text{id}) = \star \langle\langle, \rangle\rangle : \mathbb{K}(M) \otimes \mathbb{K}(\overline{M}) \rightarrow \overline{\mathbb{K}} .$$

Proof: The definition is designed to preserve grades. For the actions,

$$\langle\langle \delta_t \triangleleft (\overline{\delta_s} \triangleright u), \overline{\delta_s} \triangleleft u \rangle\rangle = \langle\langle \delta_{t \triangleleft (s^R \triangleright u)}, \overline{\delta_{s \triangleleft (s^R \triangleright u)}} \rangle\rangle = \delta_{t \triangleleft (s^R \triangleright u), s \triangleleft (s^R \triangleright u)} = \delta_{t, s} .$$

To check the symmetry of the inner product on $\mathbb{K}(M)$:

$$\begin{aligned} \Upsilon^{-1}(\text{bb}(\delta_s) \otimes \overline{\delta_t}) &= \Upsilon^{-1}(\overline{\delta_{s \triangleleft \tau(s^L, s)^{-1}}} \otimes \overline{\delta_t}) \\ &= \overline{(\delta_{t \triangleleft \tau(p, p^R)^{-1}} \otimes \delta_{s \triangleleft \tau(s^L, s)^{-1}}) \triangleleft \tau(p^R, t^R)} \end{aligned}$$

where $p = \overline{\delta_{s \triangleleft \tau(s^L, s)^{-1}}} = (s \triangleleft \tau(s^L, s)^{-1})^R = s^L$. It follows that

$$\begin{aligned} \overline{\langle\langle, \rangle\rangle} \Upsilon^{-1}(\text{bb}(\delta_s) \otimes \overline{\delta_t}) &= \overline{\langle\langle \delta_{t \triangleleft \tau(p, p^R)^{-1}}, \delta_{s \triangleleft \tau(s^L, s)^{-1}} \rangle\rangle} \\ &= \overline{\delta_{t \triangleleft \tau(s^L, s)^{-1}, s \triangleleft \tau(s^L, s)^{-1}}} \\ &= \overline{\delta_{t, s}} . \quad \square \end{aligned}$$

For any object V in \mathcal{C}_{\bowtie} , there is a right action of A_{\bowtie} on V given by

$$v \triangleleft (\delta_s \otimes u) = \delta_{\langle v \rangle, s} v \triangleleft u .$$

Further $\triangleleft : V \otimes A_{\bowtie} \rightarrow V$ is a morphism in \mathcal{C}_{\bowtie} .

Definition 5.4.4 *For H in \mathcal{C}_{\bowtie} , the action $\triangleleft : H \otimes A_{\bowtie} \rightarrow H$ is compatible with an inner product $\langle\langle, \rangle\rangle : H \otimes \overline{H} \rightarrow 1_{\mathbb{C}}$ if*

$$\langle\langle, \rangle\rangle(\triangleleft \otimes \text{id}) = \langle\langle, \rangle\rangle(\text{id} \otimes \overline{\triangleleft} \Upsilon^{-1}(\star_A \otimes \text{id}))\Phi : (H \otimes A) \otimes \overline{H} \rightarrow 1_{\mathbb{C}} .$$

Proposition 5.4.5 *The usual action $\triangleleft : \mathbb{K}(M) \otimes A_{\bowtie} \rightarrow \mathbb{K}(M)$ is compatible with the inner product.*

Proof: Let $a = \langle \delta_s \otimes u \rangle$.

$$\begin{aligned} (\overline{\triangleleft} \otimes \text{id})((\delta_p \otimes (\delta_s \otimes u)) \otimes \overline{\delta_q}) &= \delta_{p,s} \cdot \delta_{p \triangleleft u} \otimes \overline{\delta_q}, \\ (\text{id} \otimes (* \otimes \text{id}))\Phi((\delta_p \otimes (\delta_s \otimes u)) \otimes \overline{\delta_q}) &= \delta_{p \triangleleft \tau(a, q^R)} \otimes (*(\delta_s \otimes u) \otimes \overline{\delta_q}) \\ &= \delta_{p \triangleleft \tau(a, q^R)} \otimes ((\overline{\delta_{s'}} \otimes u') \otimes \overline{\delta_q}), \end{aligned}$$

where $s' = s \triangleleft u \tau(a^L, a)^{-1}$ and $u' = \tau(a^L, a) u^{-1}$. Following on from this,

$$\begin{aligned} \delta_{p \triangleleft \tau(a, q^R)} \otimes \overline{\triangleleft} \Upsilon^{-1}((\overline{\delta_{s'}} \otimes u') \otimes \overline{\delta_q}) &= \delta_{p \triangleleft \tau(a, q^R)} \otimes \\ &\quad \overline{\triangleleft}((\delta_{q \triangleleft \tau(a^L, a)^{-1}} \otimes (\delta_{s'} \otimes u')) \triangleleft \tau(a, q^R)) \\ &= \delta_{p \triangleleft \tau(a, q^R)} \otimes \\ &\quad \overline{\triangleleft}((\delta_{q \triangleleft \tau(a^L, a)^{-1}} \triangleleft (\delta_{s'} \otimes u')) \triangleleft \tau(a, q^R)) \\ &= \delta_{s', q \triangleleft \tau(a^L, a)^{-1}} \cdot \delta_{p \triangleleft \tau(a, q^R)} \otimes \\ &\quad \overline{\triangleleft}_{q \triangleleft \tau(a^L, a)^{-1} u' \triangleleft \tau(a, q^R)} \\ &= \delta_{s \triangleleft u, q} \cdot \delta_{p \triangleleft \tau(a, q^R)} \otimes \overline{\triangleleft}_{q \triangleleft u^{-1} \tau(a, q^R)} \end{aligned}$$

Now apply the inner product. \square

6 Duals

6.1 Rigid tensor categories

The left dual is a contravariant functor from \mathcal{C} to \mathcal{C} , written as $X \mapsto X'$. For every object X in \mathcal{C} there are morphisms $\text{ev}_X^L : X' \otimes X \rightarrow 1_{\mathcal{C}}$ (evaluation) and $\text{coev}_X^L : 1_{\mathcal{C}} \rightarrow X \otimes X'$ (coevaluation) so that

$$\begin{aligned} l_X^{-1}(\text{id} \otimes \text{ev}_X) \Phi(\text{coev}_X \otimes \text{id}) r_X &= \text{id}_X : X \rightarrow X, \\ r_{X'}^{-1}(\text{ev}_X \otimes \text{id}) \Phi^{-1}(\text{id} \otimes \text{coev}_X) l_{X'} &= \text{id}_{X'} : X' \rightarrow X'. \end{aligned}$$

Further ev^L and coev^L obey the following properties for every morphism $\phi : X \rightarrow Y$ (we are prevented from making a simpler statement about being a natural transformation because of the contravariant nature of dual).

$$\begin{array}{ccc} Y' \otimes X & \xrightarrow{\text{id} \otimes \phi} & Y' \otimes Y \\ \phi' \otimes \text{id} \downarrow & & \downarrow \text{ev}_Y^L \\ X' \otimes X & \xrightarrow{\text{ev}_X^L} & 1_{\mathcal{C}} \end{array}$$

$$\begin{array}{ccc}
& & \text{coev}_Y^L \longrightarrow Y \otimes Y' \\
& \text{coev}_X^L \downarrow & \downarrow \text{id} \otimes \phi' \\
1_{\mathcal{C}} & \xrightarrow{\quad} & \\
X \otimes X' & \xrightarrow{\phi \otimes \text{id}} & Y \otimes X'
\end{array}$$

We could also have a right dual, which would be a contravariant functor from \mathcal{C} to \mathcal{C} , written as $X \mapsto X^\circ$. For every object X in \mathcal{C} there are morphisms $\text{ev}_X^R : X \otimes X^\circ \rightarrow 1_{\mathcal{C}}$ (evaluation) obeying the opposite versions of the rules for the left dual.

Definition 6.1.1 *A tensor category is called left rigid if every object has a left dual, and right rigid if every object has a right dual.*

6.2 Making right duals using bar

Proposition 6.2.1 *Suppose that we have a left rigid category which is also a bar category. Then*

$$\begin{aligned}
X^\circ &= \overline{(\overline{X})'} , \\
\text{ev}^R &= \star^{-1} \text{ev}_X^L \Upsilon^{-1} (\text{bb} \otimes \text{id}) : X \otimes X^\circ \rightarrow X , \\
\text{coev}^R &= (\text{id} \otimes \text{bb}^{-1}) \Upsilon \overline{\text{coev}_X^L} \star : 1 \rightarrow X^\circ \otimes X .
\end{aligned}$$

makes the bar category right rigid.

Proof: We have to see if these obey the required properties for right rigidity, beginning with the following composition being the identity:

$$X \xrightarrow{\text{id} \otimes \text{coev}_X^R} X \otimes (X^\circ \otimes X) \xrightarrow{\Phi^{-1}} (X \otimes X^\circ) \otimes X \xrightarrow{\text{ev}_X^R} X . \quad (20)$$

If we substitute in the various definitions, we have

$$\begin{aligned}
X &\xrightarrow{\text{bb}} \overline{X} \xrightarrow{l} \overline{X} \otimes 1 \xrightarrow{\text{id} \otimes \star} \overline{X} \otimes \overline{1} \xrightarrow{\text{id} \otimes \text{coev}_X^L} \overline{X} \otimes \overline{\overline{X} \otimes (\overline{X})'} \xrightarrow{\text{id} \otimes \Upsilon} \overline{X} \otimes (\overline{(\overline{X})'} \otimes \overline{X}) \\
&\xrightarrow{\Phi^{-1}} (\overline{X} \otimes \overline{(\overline{X})'}) \otimes \overline{X} \xrightarrow{\Upsilon^{-1} \otimes \text{id}} \overline{(\overline{X})'} \otimes \overline{X} \otimes \overline{X} \xrightarrow{\star^{-1} \text{ev}_X^L \otimes \text{id}} 1 \otimes \overline{X} \xrightarrow{r^{-1}} \overline{X} \xrightarrow{\text{bb}^{-1}} X .
\end{aligned}$$

Now we use the left and right identity properties in the definition of bar category to rewrite this as

$$\begin{aligned}
X &\xrightarrow{\text{bb}} \overline{X} \xrightarrow{\bar{r}} 1 \otimes \overline{X} \xrightarrow{\overline{\text{coev}_X^L} \otimes \text{id}} \overline{(\overline{X} \otimes (\overline{X})')} \otimes \overline{X} \xrightarrow{(\text{id} \otimes \Upsilon) \Upsilon} \overline{X} \otimes (\overline{(\overline{X})'} \otimes \overline{X}) \\
&\xrightarrow{\Phi^{-1}} (\overline{X} \otimes \overline{(\overline{X})'}) \otimes \overline{X} \xrightarrow{\Upsilon^{-1} (\Upsilon^{-1} \otimes \text{id})} \overline{X} \otimes (\overline{(\overline{X})'} \otimes \overline{X}) \xrightarrow{\overline{\text{id} \otimes \text{ev}_X^L}} \overline{X} \otimes 1 \xrightarrow{l^{-1}} \overline{X} \xrightarrow{\text{bb}^{-1}} X .
\end{aligned}$$

Now we can use another rule in the definition of bar category to simplify this to

$$\begin{aligned} X &\xrightarrow{\text{bb}} \overline{\overline{X}} \xrightarrow{\bar{r}} \overline{1 \otimes \overline{X}} \xrightarrow{\overline{\text{coev}_X^L \otimes \text{id}}} \overline{(\overline{X} \otimes (\overline{X})') \otimes \overline{X}} \xrightarrow{\overline{\Phi}} \overline{\overline{X} \otimes ((\overline{X})' \otimes \overline{X})} \xrightarrow{\overline{\text{id} \otimes \text{ev}_X^L}} \overline{\overline{X} \otimes 1} \\ &\xrightarrow{\bar{l}^{-1}} \overline{\overline{X}} \xrightarrow{\text{bb}^{-1}} X, \end{aligned}$$

and by right rigidity this simplifies to the identity on X .

The other required result, that the following composition is the identity, is left to the reader:

$$X^\circ \xrightarrow{\text{coev}_X^R \otimes \text{id}} (X^\circ \otimes X) \otimes X^\circ \xrightarrow{\Phi} X^\circ \otimes (X \otimes X^\circ) \xrightarrow{\text{id} \otimes \text{ev}_X^R} X^\circ.$$

Remark 6.2.2 *Given left and right rigid structures, we can define a morphism $X \rightarrow X^{o'}$ as follows:*

$$X \xrightarrow{l} X \otimes 1 \xrightarrow{\text{id} \otimes \text{coev}_{X^\circ}^L} X \otimes (X^\circ \otimes X^{o'}) \xrightarrow{\Phi^{-1}} (X \otimes X^\circ) \otimes X^{o'} \xrightarrow{\text{ev}_X^R \otimes \text{id}} 1 \otimes X^{o'} \xrightarrow{r^{-1}} X^{o'}, \quad (21)$$

and this morphism has inverse given by

$$X^{o'} \xrightarrow{l} X^{o'} \otimes 1 \xrightarrow{\text{id} \otimes \text{coev}_X^R} X^{o'} \otimes (X^\circ \otimes X) \xrightarrow{\Phi^{-1}} (X^{o'} \otimes X^\circ) \otimes X \xrightarrow{\text{ev}_X^L \otimes \text{id}} 1 \otimes X \xrightarrow{r^{-1}} X. \quad (22)$$

A Reconstruction theorem

The reconstruction theory involves a representable functor F from a braided bar category to $\mathbf{Vect}_{\mathbb{C}}$ (with its usual bar structure and braiding implemented by transposition). There are two versions of this theory, one which reconstructs a Hopf algebra via its comodules and which is technically superior, and the other via its modules which is less useful but easier for most readers. In keeping with the line taken in the paper we focus on the module case but the other has identical proofs with arrows reversed as for example in [16, 13].

This representable here means that there is a vector space H with linear maps $\text{Lin}(V, H)$ in natural 1-1 correspondence with natural transformations between the functors $V \otimes F$ and F for every vector space V . [The functor $V \otimes F$ maps X to $V \otimes F(X)$.] We begin with $\alpha_X : H \otimes F(X) \rightarrow F(X)$, which corresponds to the identity $\text{id} \in \text{Lin}(H, H)$, which may be more recognisable after reconstruction as the action of H on $F(X)$. Thus we also write α_X as $h \otimes x \mapsto h \triangleright x$. Using the bar structure, we have the following natural transformation β between $\overline{\overline{H}} \otimes F$ and F :

$$\overline{\overline{H}} \otimes F(X) \xrightarrow{\text{id} \otimes \text{bb}} \overline{\overline{H}} \otimes \overline{\overline{F(X)}} \xrightarrow{\overline{\Psi} \Upsilon^{-1}} \overline{\overline{H \otimes F(X)}} \xrightarrow{\overline{\tilde{\alpha}_X}} \overline{\overline{F(X)}} \xrightarrow{\text{bb}^{-1}} F(X),$$

where $\tilde{\alpha}_X$ is given by the composition

$$H \otimes \overline{F(X)} \xrightarrow{\text{id} \otimes \text{fb}_X} H \otimes F(\overline{X}) \xrightarrow{\alpha_X} F(\overline{X}) \xrightarrow{\text{fb}_X^{-1}} \overline{F(X)} .$$

(The reader should note that we have only used the braiding in $\mathbf{Vect}_{\mathbb{C}}$, i.e. transposition.) The natural transformation β will correspond to a map $T : \overline{H} \rightarrow H$ so that:

$$\begin{array}{ccc} \overline{H} \otimes F(X) & \xrightarrow{\beta_X} & F(X) \\ T \otimes \text{id} \downarrow & & \downarrow \text{id} \\ H \otimes F(X) & \xrightarrow{\alpha_X} & F(X) \end{array}$$

As we are operating in \mathbf{Vect} where both the braiding Ψ and Υ are transpositions (with appropriate bars in the case of Υ),

$$\tilde{\alpha}_X(h \otimes \bar{x}) = \overline{T(\bar{h}) \triangleright x} ,$$

and the definition of T simplifies to $T(\bar{h}) \triangleright$ being the following composition:

$$F(X) \xrightarrow{\text{bb}} \overline{F(X)} \xrightarrow{\overline{\text{fb}}} \overline{F(\overline{X})} \xrightarrow{\overline{h \triangleright}} \overline{F(\overline{X})} \xrightarrow{\overline{\text{fb}^{-1}}} \overline{F(X)} \xrightarrow{\text{bb}^{-1}} F(X) . \quad (23)$$

Proposition A.0.3 $T(hh') \triangleright = T(h)T(h') \triangleright$.

Proof: By applying 23 twice. \square

Proposition A.0.4 $T(\overline{T(\bar{h}) \triangleright}) = h \triangleright$.

Proof: The following composition gives $T(\overline{T(\bar{h}) \triangleright})$:

$$F(X) \xrightarrow{\text{bb}} \overline{F(X)} \xrightarrow{\overline{\text{fb}}} \overline{F(\overline{X})} \xrightarrow{\overline{T(\bar{h}) \triangleright}} \overline{F(\overline{X})} \xrightarrow{\overline{\text{fb}^{-1}}} \overline{F(X)} \xrightarrow{\text{bb}^{-1}} F(X) ,$$

and if we use the definition of T again we get

$$\begin{aligned} F(X) &\xrightarrow{\text{bb}} \overline{F(X)} \xrightarrow{\overline{\text{fb}}} \overline{F(\overline{X})} \xrightarrow{\overline{\text{bb}}} \overline{F(\overline{X})} \xrightarrow{\overline{\text{fb}}} \overline{F(\overline{X})} \xrightarrow{\overline{h \triangleright}} \overline{F(\overline{X})} \xrightarrow{\overline{\text{fb}^{-1}}} \overline{F(\overline{X})} \xrightarrow{\overline{\text{bb}^{-1}}} \overline{F(\overline{X})} \\ &\xrightarrow{\overline{\text{fb}^{-1}}} \overline{F(X)} \xrightarrow{\text{bb}^{-1}} F(X) . \end{aligned}$$

By using property (5) in the definition, this becomes

$$F(X) \xrightarrow{\text{bb}} \overline{F(X)} \xrightarrow{\overline{F(\text{bb})}} \overline{F(\overline{X})} \xrightarrow{\overline{h \triangleright}} \overline{F(\overline{X})} \xrightarrow{\overline{F(\text{bb}^{-1})}} \overline{F(X)} \xrightarrow{\text{bb}^{-1}} F(X) ,$$

and by the functorial property of $h\triangleright$ (as α is functorial), we get

$$F(X) \xrightarrow{\text{bb}} \overline{\overline{F(X)}} \xrightarrow{\overline{h\triangleright}} \overline{\overline{F(X)}} \xrightarrow{\text{bb}^{-1}} F(X) ,$$

and as bb is a natural transformation, this is just

$$F(X) \xrightarrow{h\triangleright} F(X) . \quad \square$$

The coproduct is reconstructed from the following diagram, where we have used $\Delta h\triangleright(x \otimes y) = h_{(1)\triangleright}x \otimes h_{(2)\triangleright}y$:

$$\begin{array}{ccc} F(X \otimes Y) & \xrightarrow{h\triangleright} & F(X \otimes Y) \\ c^{-1} \downarrow & & \downarrow c^{-1} \\ F(X) \otimes F(Y) & \xrightarrow{\Delta h\triangleright} & F(X) \otimes F(Y) \end{array}$$

Proposition A.0.5 $T(\overline{h})_{(1)\triangleright} \otimes T(\overline{h})_{(2)\triangleright} = T(\overline{h_{(2)}})\triangleright \otimes T(\overline{h_{(1)}})\triangleright$.

Proof: The following composition gives $T(\overline{h})\triangleright$ on $F(X \otimes Y)$:

$$F(X \otimes Y) \xrightarrow{\text{bb}} \overline{\overline{F(X \otimes Y)}} \xrightarrow{\overline{\text{fb}}} \overline{\overline{F(X \otimes Y)}} \xrightarrow{\overline{h\triangleright}} \overline{\overline{F(X \otimes Y)}} \xrightarrow{\overline{\text{fb}^{-1}}} \overline{\overline{F(X \otimes Y)}} \xrightarrow{\text{bb}^{-1}} F(X \otimes Y) .$$

By functoriality this is the same as

$$\begin{array}{c} F(X \otimes Y) \xrightarrow{\text{bb}} \overline{\overline{F(X \otimes Y)}} \xrightarrow{\overline{\text{fb}}} \overline{\overline{F(X \otimes Y)}} \xrightarrow{\overline{F(\overline{Y})}} \overline{\overline{F(\overline{Y} \otimes \overline{X})}} \xrightarrow{\overline{c^{-1}}} \overline{\overline{F(\overline{Y}) \otimes F(\overline{X})}} \xrightarrow{\overline{\Delta h\triangleright}} \\ \overline{\overline{F(\overline{Y}) \otimes F(\overline{X})}} \xrightarrow{\overline{c}} \overline{\overline{F(\overline{Y} \otimes \overline{X})}} \xrightarrow{\overline{F(\overline{Y}^{-1})}} \overline{\overline{F(X \otimes Y)}} \xrightarrow{\overline{\text{fb}^{-1}}} \overline{\overline{F(X \otimes Y)}} \xrightarrow{\text{bb}^{-1}} F(X \otimes Y) . \end{array}$$

Now we use the definition of T to get

$$\begin{array}{c} F(X \otimes Y) \xrightarrow{\text{bb}} \overline{\overline{F(X \otimes Y)}} \xrightarrow{\overline{\text{fb}}} \overline{\overline{F(X \otimes Y)}} \xrightarrow{\overline{F(\overline{Y})}} \overline{\overline{F(\overline{Y} \otimes \overline{X})}} \xrightarrow{\overline{c^{-1}}} \overline{\overline{F(\overline{Y}) \otimes F(\overline{X})}} \xrightarrow{\overline{\text{fb}^{-1}} \otimes \overline{\text{fb}^{-1}}} \\ \overline{\overline{F(\overline{Y}) \otimes F(\overline{X})}} \xrightarrow{\overline{T(\overline{h_{(1)}})\triangleright} \otimes \overline{T(\overline{h_{(2)}})\triangleright}} \overline{\overline{F(\overline{Y}) \otimes F(\overline{X})}} \xrightarrow{\overline{\text{fb} \otimes \overline{\text{fb}}}} \overline{\overline{F(\overline{Y}) \otimes F(\overline{X})}} \\ \xrightarrow{\overline{c}} \overline{\overline{F(\overline{Y} \otimes \overline{X})}} \xrightarrow{\overline{F(\overline{Y}^{-1})}} \overline{\overline{F(X \otimes Y)}} \xrightarrow{\overline{\text{fb}^{-1}}} \overline{\overline{F(X \otimes Y)}} \xrightarrow{\text{bb}^{-1}} F(X \otimes Y) . \end{array}$$

By property (4) this is the same as

$$\begin{array}{c} F(X \otimes Y) \xrightarrow{\text{bb}} \overline{\overline{F(X \otimes Y)}} \xrightarrow{\overline{c^{-1}}} \overline{\overline{F(X) \otimes F(Y)}} \xrightarrow{\overline{\Upsilon}} \overline{\overline{F(\overline{Y}) \otimes F(\overline{X})}} \\ \xrightarrow{\overline{T(\overline{h_{(1)}})\triangleright} \otimes \overline{T(\overline{h_{(2)}})\triangleright}} \overline{\overline{F(\overline{Y}) \otimes F(\overline{X})}} \xrightarrow{\overline{\Upsilon^{-1}}} \overline{\overline{F(X) \otimes F(Y)}} \xrightarrow{\overline{c}} \overline{\overline{F(X \otimes Y)}} \xrightarrow{\text{bb}^{-1}} F(X \otimes Y) . \end{array}$$

Using the functorial property of Υ gives

$$\begin{aligned} F(X \otimes Y) &\xrightarrow{\text{bb}} \overline{\overline{F(X \otimes Y)}} \xrightarrow{c^{-1}} \overline{\overline{F(X) \otimes F(Y)}} \\ &\xrightarrow{T(\overline{h(2)}) \triangleright \otimes T(\overline{h(1)}) \triangleright} \overline{\overline{F(X) \otimes F(Y)}} \xrightarrow{\bar{c}} \overline{\overline{F(X \otimes Y)}} \xrightarrow{\text{bb}^{-1}} F(X \otimes Y). \end{aligned}$$

Now as bb is a natural transformation we get

$$\begin{aligned} F(X \otimes Y) &\xrightarrow{c^{-1}} F(X) \otimes F(Y) \xrightarrow{\text{bb}} \overline{\overline{F(X) \otimes F(Y)}} \\ &\xrightarrow{T(\overline{h(2)}) \triangleright \otimes T(\overline{h(1)}) \triangleright} \overline{\overline{F(X) \otimes F(Y)}} \xrightarrow{\text{bb}^{-1}} F(X) \otimes F(Y) \xrightarrow{c} F(X \otimes Y), \end{aligned}$$

which, for the same reason, becomes

$$F(X \otimes Y) \xrightarrow{c^{-1}} F(X) \otimes F(Y) \xrightarrow{T(\overline{h(2)}) \triangleright \otimes T(\overline{h(1)}) \triangleright} F(X) \otimes F(Y) \xrightarrow{c} F(X \otimes Y). \quad \square$$

The antipode is reconstructed using the left dual structure from the following diagram for $S(h) \triangleright$:

$$\begin{aligned} F(X) &\xrightarrow{F(\text{coev}^L) \otimes \text{id}} F(X \otimes X') \otimes F(X) \xrightarrow{c^{-1} \otimes \text{id}} F(X) \otimes F(X') \otimes F(X) \xrightarrow{\text{id} \otimes h \triangleright \otimes \text{id}} \\ &F(X) \otimes F(X') \otimes F(X) \xrightarrow{\text{id} \otimes c} F(X) \otimes F(X' \otimes X) \xrightarrow{\text{id} \otimes F(\text{ev}^L)} F(X). \end{aligned} \quad (24)$$

From the right dual structure we also have the following diagram for $S^{-1}(h) \triangleright$:

$$\begin{aligned} F(X) &\xrightarrow{\text{id} \otimes F(\text{coev}^R)} F(X) \otimes F(X^\circ \otimes X) \xrightarrow{\text{id} \otimes c^{-1}} F(X) \otimes F(X^\circ) \otimes F(X) \xrightarrow{\text{id} \otimes h \triangleright \otimes \text{id}} \\ &F(X) \otimes F(X^\circ) \otimes F(X) \xrightarrow{c \otimes \text{id}} F(X \otimes X^\circ) \otimes F(X) \xrightarrow{F(\text{ev}^R) \otimes \text{id}} F(X). \end{aligned} \quad (25)$$

That these really are mutually inverse can be seen either by explicit calculation, using the functoriality of α , or by quoting the uniqueness of antipode in a Hopf algebra.

Proposition A.0.6 $S(T(\overline{h})) \triangleright$ is the same as $p \triangleright : F(\overline{X}) \rightarrow F(\overline{X})$ where $\overline{p} = T^{-1}(S^{-1}(h))$.

Proof: If we replace X by \overline{X} in (24), we get (using the functoriality of α):

$$\begin{aligned} F(\overline{X}) &\xrightarrow{c^{-1} F(\text{coev}) \otimes \text{id}} F(\overline{X}) \otimes F((\overline{X})') \otimes F(\overline{X}) \xrightarrow{\text{id} \otimes \text{fb}^{-1} F(\text{bb}) \otimes \text{id}} F(\overline{X}) \otimes \overline{F(X^\circ)} \otimes F(\overline{X}) \\ &\xrightarrow{\text{id} \otimes T(\overline{h}) \triangleright \otimes \text{id}} F(\overline{X}) \otimes \overline{F(X^\circ)} \otimes F(\overline{X}) \xrightarrow{\text{id} \otimes F(\text{bb}^{-1}) \text{fb} \otimes \text{id}} F(\overline{X}) \otimes F((\overline{X})') \otimes F(\overline{X}) \xrightarrow{\text{id} \otimes F(\text{ev})^c} F(\overline{X}), \end{aligned}$$

where we have used the fact that $\text{bb} : (\overline{X})' \rightarrow \overline{X^\circ}$. Now, using the fact that $T(\overline{T(\overline{h})}) \triangleright = h \triangleright$, we see that $S(T(\overline{h})) \triangleright$ is given by the composition

$$F(\overline{X}) \xrightarrow{c^{-1} F(\text{coev}_{\overline{X}}) \otimes \text{id}} F(\overline{X}) \otimes F((\overline{X})') \otimes F(\overline{X}) \xrightarrow{\text{id} \otimes \text{fb}^{-1} F(\text{bb}) \otimes \text{id}} F(\overline{X}) \otimes \overline{F(X^\circ)} \otimes F(\overline{X})$$

$$\text{id} \otimes \overline{h\triangleright} \otimes \text{id} \quad F(\overline{X}) \otimes \overline{F(X^\circ)} \otimes F(\overline{X}) \xrightarrow{\text{id} \otimes F(\text{bb}^{-1}) \text{fb} \otimes \text{id}} F(\overline{X}) \otimes F((\overline{X})') \otimes F(\overline{X}) \xrightarrow{\text{id} \otimes F(\text{ev}_{\overline{X}})^c} F(\overline{X}) \quad (26)$$

We can rewrite this as

$$\begin{aligned} & F(\overline{X}) \xrightarrow{r} 1 \otimes F(\overline{X}) \xrightarrow{c^{-1} F(\text{coev}_{\overline{X}}^L) \otimes \text{id}} F(\overline{X}) \otimes F((\overline{X})') \otimes F(\overline{X}) \xrightarrow{\text{fb}^{-1} \otimes \text{fb}^{-1} F(\text{bb}) \otimes \text{fb}^{-1}} \\ & \overline{F(X)} \otimes \overline{F(X^\circ)} \otimes \overline{F(X)} \xrightarrow{\text{id} \otimes \overline{h\triangleright} \otimes \text{id}} \overline{F(X)} \otimes \overline{F(X^\circ)} \otimes \overline{F(X)} \xrightarrow{\text{fb} \otimes F(\text{bb}^{-1}) \text{fb} \otimes \text{fb}} \\ & F(\overline{X}) \otimes F((\overline{X})') \otimes F(\overline{X}) \xrightarrow{\text{id} \otimes F(\text{ev}_{\overline{X}}^L)^c} F(\overline{X}) \otimes 1 \xrightarrow{l^{-1}} F(\overline{X}) \end{aligned}$$

Using the commutativity of the following diagram (from property (4) in 2.5.1) and the definition of the right dual structures,

$$\begin{array}{ccc} \overline{F(X^\circ)} \otimes \overline{F(X)} & \xrightarrow{F(\text{bb}^{-1}) \text{fb} \otimes \text{fb}} & F((\overline{X})') \otimes F(\overline{X}) \\ \Upsilon^{-1} \downarrow & & \downarrow \star F(\text{ev}_{\overline{X}}^L)^c \\ \overline{F(X)} \otimes \overline{F(X^\circ)} & \xrightarrow{F(\text{ev}_{\overline{X}}^R)^c} & \overline{1} \end{array}$$

this becomes

$$\begin{aligned} & F(\overline{X}) \xrightarrow{r} 1 \otimes F(\overline{X}) \xrightarrow{c^{-1} F(\text{coev}_{\overline{X}}^L) \otimes \text{id}} F(\overline{X}) \otimes F((\overline{X})') \otimes F(\overline{X}) \xrightarrow{\text{fb}^{-1} \otimes \text{fb}^{-1} F(\text{bb}) \otimes \text{fb}^{-1}} \\ & \overline{F(X)} \otimes \overline{F(X^\circ)} \otimes \overline{F(X)} \xrightarrow{\text{id} \otimes \overline{h\triangleright} \otimes \text{id}} \overline{F(X)} \otimes \overline{F(X^\circ)} \otimes \overline{F(X)} \xrightarrow{\text{fb} \otimes \Upsilon^{-1}} \\ & F(\overline{X}) \otimes \overline{F(X)} \otimes \overline{F(X^\circ)} \xrightarrow{\text{id} \otimes \overline{F(\text{ev}_{\overline{X}}^R)^c}} F(\overline{X}) \otimes \overline{1} \xrightarrow{\text{id} \otimes \star^{-1}} F(\overline{X}) \otimes 1 \xrightarrow{l^{-1}} F(\overline{X}) \end{aligned}$$

Using the commutativity of the following diagram (from property (4) in 2.5.1) and the definition of the right dual structures,

$$\begin{array}{ccc} 1 & \xrightarrow{c^{-1} F(\text{coev}_{\overline{X}}^L)} & F(\overline{X}) \otimes F((\overline{X})') \\ c^{-1} \overline{F(\text{coev}_{\overline{X}}^R)} \star \downarrow & & \downarrow \text{fb}^{-1} \otimes \text{fb}^{-1} F(\text{bb}) \\ \overline{F(X^\circ)} \otimes \overline{F(X)} & \xrightarrow{\Upsilon} & \overline{F(X^\circ)} \otimes \overline{F(X)} \end{array}$$

so we get

$$\begin{aligned} & F(\overline{X}) \xrightarrow{(\text{id} \otimes \star)r} \overline{1} \otimes F(\overline{X}) \xrightarrow{c^{-1} F(\text{coev}_{\overline{X}}^R) \otimes \text{id}} \overline{F(X^\circ)} \otimes \overline{F(X)} \otimes F(\overline{X}) \xrightarrow{\Upsilon \otimes \text{fb}^{-1}} \\ & \overline{F(X)} \otimes \overline{F(X^\circ)} \otimes \overline{F(X)} \xrightarrow{\text{id} \otimes \overline{h\triangleright} \otimes \text{id}} \overline{F(X)} \otimes \overline{F(X^\circ)} \otimes \overline{F(X)} \xrightarrow{\text{fb} \otimes \Upsilon^{-1}} \\ & F(\overline{X}) \otimes \overline{F(X)} \otimes \overline{F(X^\circ)} \xrightarrow{\text{id} \otimes \overline{F(\text{ev}_{\overline{X}}^R)^c}} F(\overline{X}) \otimes \overline{1} \xrightarrow{\text{id} \otimes \star^{-1}} F(\overline{X}) \otimes 1 \xrightarrow{l^{-1}} F(\overline{X}) \end{aligned}$$

From the axiom giving the behaviour of left and right identities under bar, this is

$$F(\overline{X}) \xrightarrow{\text{fb}^{-1}} \overline{F(X)} \xrightarrow{\overline{l}} \overline{F(X)} \otimes 1 \xrightarrow{\Upsilon} \overline{1} \otimes \overline{F(X)} \xrightarrow{c^{-1} F(\text{coev}_{\overline{X}}^R) \otimes \text{id}} \overline{F(X^\circ)} \otimes \overline{F(X)} \otimes \overline{F(X)}$$

$$\begin{aligned} & \overline{\Upsilon \otimes \text{id}} \overline{F(X)} \otimes \overline{F(X^\circ)} \otimes \overline{F(X)} \xrightarrow{\text{id} \otimes \overline{h_\triangleright} \otimes \text{id}} \overline{F(X)} \otimes \overline{F(X^\circ)} \otimes \overline{F(X)} \xrightarrow{\text{id} \otimes \overline{\Upsilon^{-1}}} \\ & \overline{F(X)} \otimes \overline{F(X)} \otimes \overline{F(X^\circ)} \xrightarrow{\text{id} \otimes \overline{F(\text{ev}_X^R)} c} \overline{F(X)} \otimes \overline{1} \xrightarrow{\overline{\Upsilon^{-1}}} \overline{1} \otimes \overline{F(X)} \xrightarrow{\overline{r^{-1}}} \overline{F(X)} \xrightarrow{\text{fb}} \overline{F(\overline{X})} \end{aligned}$$

The functoriality of Υ gives

$$\begin{aligned} & \overline{F(\overline{X})} \xrightarrow{\text{fb}^{-1}} \overline{F(X)} \xrightarrow{\overline{l}} \overline{F(X)} \otimes \overline{1} \xrightarrow{\text{id} \otimes c^{-1} F(\text{coev}_X^R)} \overline{F(X)} \otimes \overline{F(X^\circ)} \otimes \overline{F(X)} \\ & \xrightarrow{(\overline{\Upsilon} \otimes \text{id}) \overline{\Upsilon}} \overline{F(X)} \otimes \overline{F(X^\circ)} \otimes \overline{F(X)} \xrightarrow{\text{id} \otimes \overline{h_\triangleright} \otimes \text{id}} \overline{F(X)} \otimes \overline{F(X^\circ)} \otimes \overline{F(X)} \xrightarrow{\overline{\Upsilon^{-1}} (\text{id} \otimes \overline{\Upsilon^{-1}})} \\ & \overline{F(X)} \otimes \overline{F(X^\circ)} \otimes \overline{F(X)} \xrightarrow{F(\text{ev}_X^R) c \otimes \text{id}} \overline{1} \otimes \overline{F(X)} \xrightarrow{\overline{r^{-1}}} \overline{F(X)} \xrightarrow{\text{fb}} \overline{F(\overline{X})} \end{aligned}$$

and by more functoriality and cancelling,

$$\begin{aligned} & \overline{F(\overline{X})} \xrightarrow{\text{fb}^{-1}} \overline{F(X)} \xrightarrow{\overline{l}} \overline{F(X)} \otimes \overline{1} \xrightarrow{\text{id} \otimes c^{-1} F(\text{coev}_X^R)} \overline{F(X)} \otimes \overline{F(X^\circ)} \otimes \overline{F(X)} \\ & \xrightarrow{\text{id} \otimes (\overline{h_\triangleright}) \otimes \text{id}} \overline{F(X)} \otimes \overline{F(X^\circ)} \otimes \overline{F(X)} \xrightarrow{F(\text{ev}_X^R) c \otimes \text{id}} \overline{1} \otimes \overline{F(X)} \xrightarrow{\overline{r^{-1}}} \overline{F(X)} \xrightarrow{\text{fb}} \overline{F(\overline{X})} \end{aligned}$$

By the definition of S^{-1} , this is

$$\overline{F(\overline{X})} \xrightarrow{\text{fb}^{-1}} \overline{F(X)} \xrightarrow{\overline{S^{-1}(h_\triangleright)}} \overline{F(X)} \xrightarrow{\text{fb}} \overline{F(\overline{X})} .$$

By the definition of T , this is the same as $p\bar{p} : F(\overline{X}) \rightarrow F(\overline{X})$ where $\bar{p} = T^{-1}(S^{-1}(h))$.
□

Theorem A.0.7 *If we define $*$ by $T S^{-1}$ (allowing antilinear maps by dropping the bars), then:*

- (1) $(ab)^* = b^* a^*$
- (2) $a^{**} = a$
- (3) $\Delta(h^*) = \Delta(h)^* \otimes *$
- (4) $S(h)^* = S^{-1}(h^*)$

I.e. this $$ makes H into a $*$ -Hopf algebra.*

Proof: By the reconstruction, $h, h' \in H$ are equal if and only if h_\triangleright and h'_\triangleright are identical on all objects, so the previous propositions give these results (again, dropping the bars)

- (a) $T(h h') = T(h) T(h')$ (see A.0.3).
- (b) $T^2 = \text{id}$ (see A.0.4).
- (c) T reverses order in the coproduct (see A.0.5).
- (d) $ST = T^{-1} S^{-1}$ (see A.0.6).

As S reverses order in the product, (a) shows that $*$ reverses order in the product (1).

As S reverses order in the coproduct, (c) shows that $*$ preserves order in the product (3).

As (b) shows that $T^{-1} = T$, from (d) we get $T S^{-1} T S^{-1} = \text{id}$ (2).

This is sufficient to show that we have a $*$ -Hopf algebra, and (4) follows automatically.

□

Remark A.0.8 *The reader may wonder why the reconstruction gives a $*$ -Hopf algebra rather than some of the other variants we have mentioned. The reason goes back to 2.5.1 where we defined the properties of the functor we were to use. In inserting an ‘up to equivalence’ in some of these properties would give more freedom and allow other variants. What the reconstruction shows is that the definition of a bar category is reasonable (otherwise the reconstruction would not have worked), and that Woronowicz was right to pick out the $*$ -Hopf algebras as, in a sense, the most obvious way to add $*$ to a Hopf algebra (as it uses the fewest ‘up to equivalence’ in the definition).*

Remark A.0.9 *Similarly if F is multiplicative but not monoidal (see [16]), then we would expect to be able to reconstruct a $*$ -quasi Hopf algebra.*

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