The Maximum Principle for Elliptic Inequalities on Stratified Sets *

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In this paper we present some new results about the strong maximum principle for elliptic inequalities on stratified sets (the exact definitions of the basic notions will be presented later). Our proof of the strong maximum principle is based on a special necessary extremum's condition. In order to make the main idea clearer we illustrate it in a classical case. The following lemma is almost obvious in this case.

Lemma. Let $u : \Omega_0 \subset \mathbb{R}^{n+1} \to \mathbb{R}$ be a sufficiently smooth function in domain Ω_0. If $X \in \Omega_0$ is a point of nontrivial local maximum then there exists $r > 0$ which is small enough so that

$$\int\limits_{S_r^n(X)} \frac{\partial u}{\partial \nu} < 0, \tag{1}$$

where $S_r^n(X)$ is a usual sphere and $\vec{\nu}$ is a unit exterior normal to the sphere. A point X is said to be a point of nontrivial local maximum of function u if $u(Y) \leq u(X)$ for all Y sufficiently close to X and u is not a constant function in any neighborhood of X.

Proof of the lemma is the immediate consequence of the formula

$$\frac{d}{dr}\left(\frac{1}{r^n}\int\limits_{S_r^n(X)} u\right) = \frac{1}{r^n}\int\limits_{S_r^n(X)} \frac{\partial u}{\partial \nu}. \tag{2}$$

Now we can easily obtain proof of the strong maximum principle for solutions of the inequality $\Delta u \geq 0$, because this inequality contradicts (2). One can see that a standard formulation of the strong maximum is equivalent to the following one.

Theorem 1 *A solution of inequality $\Delta u \geq 0$ couldn't have a point of nontrivial maximum on Ω_0.*

We formulate the maximum principle in this form, because the standard formulation cannot be extended without any changes to the stratified sets.

Now we are ready to start on the main subject.

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1 Definitions and Auxiliary Assertions

A connected subset $\Omega \subset \mathbb{R}^n$ is sad to be stratified if it is presented as a union of a finite number of smooth submanifolds σ_{kj} (strata) which are attached to each other similar to the cells of CW-complex. In this work we confine ourselves to the case when all the strata are open convex polyhedrons of different dimensions. More general definitions are presented in [1] or [2].

Set Ω is assumed to be divided into two parts Ω_0 and $\partial\Omega_0$. The first one is an open connected part of Ω (in the topology, induced on Ω from $\mathbb{R}^n$ taken with its standard topology), consisting of the above mentioned strata in Ω. Besides, we assume $\overline{\Omega}_0 = \Omega$.

We define the so-called stratified measure of subset $\omega \subset \Omega$ by means of formula

$$\mu(\omega) = \sum_{\sigma_{kj}} \mu_k(\omega \cap \sigma_{kj}), \tag{3}$$

where μ_k is a standard Lebesgues measure on σ_{kj}. It is supposed that $\mu_0(\sigma_{0j}) = 1$. So, we have concentrated the unit measure on 0-dimensional strata. Set ω is said to be measurable if the sum (3) is finite. Using the last notion we can define a class of measurable functions. One can easily prove that for a measurable function $f : \Omega \to \mathbb{R}$ its Lebesgues integral reduces to the sum

$$\int\limits_{\Omega} f\, d\mu = \sum_{\sigma_{kj}} \int\limits_{\sigma_{kj}} f\, d\mu_k$$

of the Lebesgues integrals of the restrictions of f onto σ_{kj}.

Let $X \in \Omega$. If $r > 0$ does not exceed the distance between X and all the strata which closures do not contain X (such r will be called admissible), then set

$$S_r(X) = \{Y \in \Omega : \ \|Y - X\| = r\}$$

will be called a stratified sphere (or, simply, a sphere). We denote it by $S_r(X)$. Set $S_r(X)$ may be considered as an intersection of the usual sphere $\hat{S}_r(X) \subset \mathbb{R}^n$ with Ω. The corresponding open ball will be denoted by $B_r(X)$. By $S_r^m(X)$ will be denoted an intersection of $S_r(X)$ with the union of all $(m+1)$-dimensional strata. We call it the m-dimensional region of the sphere $S_r(X)$.

Set $S_r(X)$ may be considered as stratified, if we take connected components of $S_r^m(X)$ $(m = 1, 2 \ldots,)$ as its m-dimensional strata. This stratification generates measure μ on $S_r(X)$ as it was described earlier. When we integrate a function on the sphere we have this measure in mind.

When r_1 and r_2 are admissible, then the spheres $S_{r_1}(X)$, $S_{r_2}(X)$ and their m-dimensional regions $S_{r_1}^m(X)$, $S_{r_2}^m(X)$ are homothetic for each m. As a consequence, we obtain

$$\frac{d}{dr}\left(\frac{1}{r^m}\int\limits_{S_r^m(X)} u\right) = \frac{1}{r^m}\int\limits_{S_r^m(X)} \frac{\partial u}{\partial \nu}, \tag{4}$$

where ν is an exterior normal to the sphere $\hat{S}_r(X)$ attached to points of sphere $S_r(X)$. It is assumed here that $X \in \Omega_0$ and function u is continuous for the whole Ω_0 (the last condition will be needed in the following sections, not here), differentiable in the interior of each stratum $\sigma_{kj} \subset \Omega_0$ and such, that the integrals on the right hand sides of (4) are all convergent. A set of such functions will be denoted by $C^1(\Omega_0)$.

Multiplying both sides of (4) by r^m and summing through m we obtain

$$\int\limits_{S_r(X)} \frac{\partial u}{\partial \nu} = \sum_{m=0}^{d} r^m \frac{d}{dr}\left(\frac{1}{r^m}\int\limits_{S_r^m(X)} u\right). \tag{5}$$

Some m-dimensional regions of the sphere may be empty. It is assumed by convention that the integrals, corresponding to these regions are equal to zero.

2 Necessary Condition of the Extremum

This section contains our main result.

Theorem 2 *Let $X \in \Omega_0$ be a point of a local nontrivial maximum of the function $u \in C^1(\Omega_0)$. Then there exists an admissible $r > 0$ small enough so that*

$$\int\limits_{S_r(X)} \frac{\partial u}{\partial \nu} < 0. \tag{6}$$

Let us recall that we refer to X as a point of a nontrivial maximum of function u, if an inequality $u(Y) \le u(X)$ holds for each Y close to X and u is not a constant function in each neighborhood of X.

Our proof is based on the following two lemmas.

Lemma 1 *Assume f_0, f_1 are continuous functions on $[0;a]$, which are continuously differentiable on $(0;a]$. Let also $f_0(0) = f_1(0) = 0$. Then the nonpositivity of f_0 and inequality*

$$r f_1'(r) + f_0'(r) \ge 0 \tag{7}$$

lead to nonnegativity of function f_1.

Proof. Integrating (7) on the segment $[\epsilon; r]$ $(0 < \epsilon < r \le a)$ we obtain after obvious transformations

$$r f_1(r) - \epsilon f_1(\epsilon) - \int\limits_{\epsilon}^{r} f_1(\rho) d\rho + f_0(r) - f_0(\epsilon) \ge 0.$$

Taking $\epsilon \to 0$ we are getting

$$r f_1(r) - \int\limits_{0}^{r} f_1(\rho) d\rho + f_0(r) \ge 0,$$

and taking into account a positivity of f_0 we obtain

$$\frac{1}{r}\int\limits_{0}^{r} f_1(\rho) d\rho \le f_1(r).$$

According to the mean-value theorem the last inequality may be rewritten in the form $f_1(\xi) \le f_1(r)$ for some $\xi \in [0; r)$. Denoting by $\xi^\star$ a greatest lower bound of all ξ, for which the last inequality still holds, let us show that $\xi^\star = 0$ Indeed, if it is not so, than we can take $\xi^\star$ instead of r and derive

$$\frac{1}{\xi^\star}\int\limits_0^{\xi^\star} f_1(\rho)d\rho \leq f_1(\xi^\star).$$

This implies, as an above, an existence of such $\xi^0 \in [0;\xi^\star)$, that $f_1(\xi^0) \leq f_1(\xi^\star) \leq f_1(r)$. But it contradicts the definition of $\xi^\star$. So, $\xi^\star = 0$ and as a consequence $0 = f_1(0) \leq f_1(r)$ for all $r \in [0;a]$. ■

Lemma 2 *Assume $f_0,\ldots,f_n$ are continuous functions on $[0;a]$ as well as continuously differentiable on $(0;a]$. Let further $f_i(0) = 0$ $(i = 0,\ldots,n)$. Then a nonpositivity of functions f_i and inequality*

$$r^n f_n'(r) + r^{n-1} f_{n-1}'(r) + \cdots + f_0'(r) \geq 0 \tag{8}$$

follow $f_i(r) \equiv 0$ for each i.

Proof. This assertion is trivial when $n = 0$. Besides, in the case $n = 1$ it is an easy consequence of the previous lemma. In the general case we shall apply the mathematical induction.

Let us define a family of functions $\phi_0,\ldots,\phi_{n-1}$ by recurrence, taking for $k \geq 1$

$$\phi_k = r\phi_{k-1} + f_{n-k} - \int\limits_0^r \phi_{k-1}(\rho)d\rho, \tag{9}$$

and lying $\phi_0 = f_n$. Then the inequality (8) may be reduced to $r\phi_{n-1}'(r) + f_0'(r) \geq 0$. Indeed,

$$\begin{aligned} r^n f_n' + r^{n-1} f_{n-1}' + \cdots + f_0' &= r(r(\ldots r(rf_n' + f_{n-1}') + f_{n-2}') + \cdots + f_1') + f_0' = \\ = r(r(\ldots(r\phi_1' + f_{n-2}') + \cdots + f_1') + f_0' &= r(r(\ldots(r\phi_2' + f_{n-3}') + \cdots + f_1') + f_0' = \\ &= \cdots = r\phi_{n-1}' + f_0'. \end{aligned}$$

Here we have used (9) repeatedly. Using this formula again we can see, that $\phi_i(0) = 0$ for all i. Using nonpositivity of f_0 and lemma 1 we obtain $\phi_{n-1}(r) \geq 0$ or

$$r\phi_{n-2}(r) + f_1(r) - \int\limits_0^r \phi_{n-2}(\rho)d\rho \geq 0,$$

and as a consequence

$$\phi_{n-2}(r) \geq \frac{1}{r}\int\limits_0^r \phi_{n-2}(\rho)d\rho.$$

Arguing as in lemma 1 we obtain $\phi_{n-2}(r) \geq 0$. Continuing these constructions further we derive $\phi_0(r) = f_n(r) \geq 0$. Comparing with the assumption $f_n(r) \leq 0$ we obtain $f_n(r) \equiv 0$, and inequality (8) reduces to

$$r^{n-1} f_{n-1}'(r) + \cdots + f_0'(r) \geq 0.$$

So, one inductive step was done. ■

Proof of the theorem 2 is an easy consequence of the last lemma. Indeed, if we assume, by contradiction, that it is not so, then there exists a positive a, so that the integral on the left hand side of (6) is nonnegative when $r \in (0; a]$. But taking into account (5), we obtain inequality $r^d f_d'(r) + r^{d-1} f_{d-1}'(r) + \cdots + f_0'(r) \geq 0$ for $r \in (0; a]$, where we have denoted

$$f_m(r) = \frac{1}{r^m} \int\limits_{S_r^m(X)} u.$$

Without the loss of generality we can assume $u(X) = 0$ at the point of maximum. Then f_m $(m = 1, dots)$ appears to be nonpositive and may be extended at $r = 0$ by continuity taking $f_m(0) = 0$. So, we are under the conditions of the lemma 2. As a consequence $f_m(r) \equiv 0$ for all m. From this we have immediately $u \equiv 0$ in $B_a(X)$, which contradicts to nontriviality of the maximum at X.

Theorem (6) plays an important role in the proof of the strong maximum principle for elliptic inequalities on the stratified sets. The remainder of this paper will be devoted to this subject.

3 Divergence and Laplacian on Stratified Set

Vector field $\vec{F}$ on Ω_0 will be called tangent to Ω_0, if for each stratum $\sigma_{kj} \subset \Omega_0$ and each point $X \in \sigma_{kj}$ vector $\vec{F}(X)$ lies in tangent space $T_X\sigma_{kj}$ attached to σ_{kj} at point X. It is natural to assume $\vec{F} = 0$ in 0-dimensional strata.

The divergence of vector field $\vec{F}$ at the arbitrary $X \in \Omega_0$ will be defined as

$$(\nabla \vec{F})(X) = \lim_{S \to X} \frac{\Phi_{\vec{F}}(S)}{\mu(B)}, \tag{10}$$

where $\Phi_{\vec{F}}(S)$ is a flux of vector field $\vec{F}$ through the "stratified" surface S. This surface is an intersection of Ω with a smooth (or piecewise smooth) closed surface $\hat{S} \subset \mathbb{R}^n$ and B is part of Ω cut out by $\hat{S}$. Surface S is supposed to be situated in the interior of ball $B_r(X)$ of admissible radius. Normal vector to $\hat{S}$ at point $X \in \sigma_{kj} \cap S$ is assumed to be lying in a tangent space to σ_{kj} for all k, j.

Flux $\Phi_{\vec{F}}(S)$ consists of fluxes through m-dimensional regions S_m $(m = 1, \ldots, d)$ of surface S. Summing these fluxes we have

$$\Phi_{\vec{F}}(S) = \int\limits_S \vec{F} \cdot \vec{\nu}\, d\mu,$$

where $\vec{\nu}$ is an exterior normal to S, defined as it was described earlier and μ is a stratified measure on surface S, which is being considered as a stratified set. The method of stratification was described in case $S = S_r(X)$ in section 1.

The set of all tangent vector fields, having a uniformly continuous divergence on each stratum in Ω_0 will be denoted by $\vec{C}^1(\Omega_0)$. Inclusion $\vec{C}^1(\Omega_0)$ does not suppose a continuity on the whole Ω_0. In other words, vector field $\vec{F}\vec{C}^1(\Omega_0)$ will be considered as a collection of independent fields on a separate strata. One can prove that (see [2]), that for $X \in \sigma_{k-1i}$ we have

$$(\nabla \vec{F})(X) = (\nabla_{k-1}\vec{F})(X) + \sum_{\sigma_{kj} \succ \sigma_{k-1i}} \vec{\nu} \cdot \vec{F}\Big|_{\overline{kj}}(X), \tag{11}$$

where ∇_{k-1} is a classical divergence on σ_{k-1i}, and notation $\sigma_{kj} \succ \sigma_{k-1i}$ means joining the stratum σ_{k-1i} to σ_{kj}. Vector $\vec{\nu}$ is a unit normal to σ_{k-1i} at point X, directed at the interior of σ_{kj}. Here and further a notation of type $u\Big|_{\overline{kj}}(X)$ $(X \in \sigma_{k-1i} \succ \sigma_{kj})$ means an extension by continuity to point X of restriction $u\Big|_{\sigma_{kj}}$ to σ_{kj} of function u. It is assumed, of course, that such extensions exist. If u is not necessarily continuous on the whole Ω_0, then $u\Big|_{\overline{kj}}(X)$ is not necessarily equal to $u(X)$.

Let $u : \Omega_0 \to \mathbb{R}$ be a differentiable in the interior of each strata in Ω_0. Then we can consider vector field ∇u consisting of gradient fields $\nabla_k u$ of function u on separate strata $\sigma_{ki} \subset \Omega_0$. Let us note that the presence of any connections between the restrictions of u on different strata was not assumed. So, at the moment gradient ∇u is a collection of independent gradients $\nabla_k u$ on the separate strata. But in applications it is naturally for u to be continuous. So we will denote by $C^2(\Omega_0)$ a set of the continuous functions on Ω so that $\nabla u \in \vec{C}^1(\Omega_0)$. On this class we can define an analogue of the Laplace operator $\Delta u = \nabla(\nabla u)$. Here symbol ∇ has been used in two different senses. Exterior symbol ∇ is a divergence, whereas the interior one is a gradient.

Together with Δ we shall consider operator Δ_p, which is defined by the formula $\Delta_p u = \nabla(p\nabla u)$, where p is a so-called stratified constant. It means that its restrictions to the strata are different constants. In this paper p is supposed to be taken only two values, zero or one. All these operators may be considered as analogues of the Laplace operator. The simplest case will be obtained when $p = 1$ only on the so-called free strata (the strata which are not lying on the boundaries of other strata). The corresponding Laplacian will be called "soft" in contrast to "hard", which corresponds to $p \equiv 1$.

We can also consider the case when $p = 0$ on some free strata, but it is rather meaningless.

4 The Strong Maximum Principle

First of all, it should be noted that in contrast to the classical case the solution of inequality $\Delta u \geq 0$ on stratified set admits nonconstant solutions with local maximums. Nevertheless, we have the following exact analogue of the strong maximum principle.

Theorem 3 *Let $u \in C^2(\Omega_0)$ be a solution of inequality $\Delta u \geq 0$ on Ω_0. Then u could not have in Ω_0 the points of nontrivial maximum.*

In fact, inequality $\Delta u \geq 0$ follows a nonnegativity of the integrals of the normal derivative on the spheres of admissible radius. But such integrals may be presented in the form of $r^n f_n'(r) + r^{n-1} f_{n-1}'(r) + \cdots + f_0'(r)$. As a consequence, inequality $\Delta u \geq 0$ follows (8). But the last one contradicts (if X is a point of nontrivial maximum) theorem 2. A proof of the fact, that inequality $\Delta u \geq 0$ implies a nonnegativity of the integrals of the normal derivatives may be deduced from the following analogue of Green's formula.

Theorem 4 *Let $u, v \in C^2(\Omega_0)$. Then for each ball of admissible radius a following formula takes place*

$$\int\limits_{B_r(X)} (u\Delta v - v\Delta u)\, d\mu = \int\limits_{S_r(X)} (u(\nabla v)_\nu - v(\nabla u)_\nu)\, d\mu.$$

This assertion is an easy consequence of a more general assertion from the book [2].

It should be noted that in case $\partial\Omega_0 = \emptyset$ inequality $\Delta u \geq 0$ admits only constant solutions. (see [2]), so theorem 3 is trivial in this case.

As a consequence of the theorem, we obtain proof of impossibility for the solutions of the inequality to have a point of nontrivial positive maximums, if q is nonnegative. Besides, we assume q to be continuous on each strata Ω_0.

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