An index theory for uniformly locally finite graphs

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Abstract: An index theory for uniformly locally finite (ULF) graphs is developed based on the adjacency operator $\mathcal{A}$ acting on the space of bounded sequences defined on the vertices. It turns out that the characterization by upper and lower nonnegative eigenvectors is an appropriate tool to overcome the difficulties imposed by the ℓ^∞–setting. A distinctive property of the spectral radius $r_\infty(\mathcal{A})$ in ℓ^∞ is the identity

$$r_\infty = \sup\{\lambda \geq 0 | \exists x \in \ell^\infty(\Gamma),\, x > 0 \,:\, \mathcal{A}x \geq \lambda x\} =: \mathrm{I},$$

while the ℓ^2–spectral radius r_2 of the adjacency operator satisfies

$$r_2 = \inf\{\lambda \geq 0 | \exists x \in \ell^\infty(\Gamma),\, x > 0 \,:\, \mathcal{A}x \leq \lambda x\}.$$

The index I, as well as other order indices, can serve in classifying ULF graphs and enables connections with various graph invariants. E.g., the chromatic number can be estimated from above by $1 + r_\infty$. Moreover, results on the index I in the periodic case, the regular one and for graphs having only finitely many essential ramification nodes are presented.

Keywords: Uniformly locally finite graphs, adjacency operator, graph spectra, spectral theory of positive operators.

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1 Introduction

The index of a finite graph, i.e. the spectral radius of its adjacency matrix, plays an important role within graph theory, cf. e.g. [1, 7, 10, 11]. It is closely related to the chromatic number, the complexity, and other combinatorial invariants of the graph. For infinite graphs, especially in the context of random walks on graphs, cf. e.g. [16, 20], the spectral radius r_2 of the adjacency operator $\mathcal{A}$ in the ℓ^2–setting has been considered in the uniformly locally finite (ULF) case, where $\mathcal{A}$ becomes a selfadjoint operator in $\ell^2(\Gamma)$. Nevertheless, the point spectrum of the adjacency operator in the ℓ^2–setting can be rather poor when compared to the ℓ^∞–case, realizing that, in general, the determination of the eigenvalues can be quite delicate.

In the present context, we define the index of a ULF graph by the spectral radius r_∞ of the adjacency operator $\mathcal{A}$ acting in the space $\ell^\infty(\Gamma)$ of bounded sequences defined on the vertex set $V(\Gamma)$, using the order properties of the adjacency operator as a positive operator in the Banach lattice $\ell^\infty(\Gamma)$. One of the key tools is the characterization of the spectral radius r_∞ by means of nonnegative upper eigenvectors:

$$r_\infty = \sup\{\lambda \geq 0 \,|\, \exists x \in \ell^\infty(\Gamma),\, x > 0 \,:\, \mathcal{A}x \geq \lambda x\} =: \mathrm{I}. \tag{1}$$

Moreover, its lower counterpart I_∞^- is shown to satisfy

$$r_2 = \inf\{\lambda \geq 0 \,|\, \exists x \in \ell^\infty(\Gamma),\, x > 0 \,:\, \mathcal{A}x \leq \lambda x\} =: \mathrm{I}_\infty^-, \tag{2}$$

see Theorems 4.9 and 4.12. Both entities reflect certain combinatorial properties of the graph, but are different in general. Here we mention only the fact that, if Γ is regular of valency d, then $d = \mathrm{I}(\Gamma)$, see Corollary 4.15, while $r_2 < d$ is possible in that case, e.g. for the 3–regular tree.

The present paper is organized as follows: After Section 2 with some notations, basic assumptions and preliminaries from graph theory and from operator theory, some general spectral properties of the adjacency operator $\mathcal{A}$ in the spaces $\ell^p(\Gamma)$ are investigated in Section 3. In Section 4 the order indices $\mathrm{I}_p^\pm$ are introduced and compared to the corresponding spectral radii of $\mathcal{A}$ in $\ell^p(\Gamma)$. The identity (2) is part of the following general result, see Theorem 4.9:

1. $\forall p \in [1,\infty] \,:\, \mathrm{I}_p^- \leq r_p \quad \& \quad \mathrm{I}_p^+ \leq r_p.$
2. $\forall p \in [1,2] \,:\, \mathrm{I}_1^+ = \mathrm{I}_p^+ = r_2 \leq \mathrm{I}_p^- \leq \mathrm{I}_1^-.$
3. $\forall p \in [2,\infty] \,:\, \mathrm{I}_\infty^- = \mathrm{I}_p^- = r_2 \leq \mathrm{I}_p^+ \leq \mathrm{I}_\infty^+ \quad \& \quad \mathrm{I}_p^+ \leq \mathrm{I}_q^-$ for $\frac{1}{p} + \frac{1}{q} = 1$.

The identity (1) is based in addition on the fact that r_∞ belongs to the approximate spectrum and on the positivity properties of $\mathcal{A}$, see Theorem 4.12. For periodic graphs or generalized lattices it is shown in Section 5 that the periodic index introduced in [2, 9] coincides with I and I_∞^-. In Section 6 Wilf's theorem on the chromatic number is extended to ULF graphs, see Theorem 6.2 while a characterization of vertex bipartition being valid for finite and periodic graphs is shown not to hold in the ULF case. Section 7 is devoted to the characterization of the regular case by means of the index I. Finally, in Section 8 conditions are presented under which $\mathrm{I}_\infty^-(\Gamma) = \mathrm{I}(\Gamma)$. Beyond the finite and periodic case, infinite graphs with finitely many essential ramification nodes are shown to fulfill this identity, see Theorem 8.2. Moreover, the index of these graphs is shown to satisfy a characteristic equation of the form

$$\det\left(\mathcal{A}(S) + \left(\frac{\rho}{2} - \sqrt{\frac{\rho^2}{4} - 1}\right)\begin{pmatrix} \mathbf{I}_t & 0 \\ 0 & 0 \end{pmatrix} - \rho\,\mathbf{I}_n\right) = 0,$$

where S is a finite subgraph containing all essential ramification nodes of the graph.

2 Preliminaries

For any graph $\Gamma = (V, E, \in)$, the vertex set is denoted by $V = V(\Gamma)$, the edge set by $E = E(\Gamma)$ and the incidence relation by $\in \subset V \times E$. The valency of each vertex v is denoted by $\gamma(v) = \text{card}\{e \in E \,|\, v \in e\}$. We distinguish the *boundary vertices* $V_b = \{v \in V \,|\, \gamma(v) = 1\}$ from the *ramification nodes* $V_r = \{v \in V \,|\, \gamma(v) \geq 2\}$, especially, we define the *essential ramification nodes* by $V_{\text{ess}} = \{v \in V \,|\, \gamma(v) \geq 3\}$. Set

$$\gamma_{\min} = \min\{\gamma(v) \,|\, v \in V\}.$$

For a subgraph $\Delta \leq \Gamma$ let $\overline{\Delta} = \overline{V(\Delta)} = \big(V(\Delta), K(\overline{\Delta}), \in\big)$ denote the subgraph of Γ spanned by the vertices in Δ with $E(\overline{\Delta}) = \{e \,|\, e \in E(\Gamma),\ e \cap V(\Gamma) \subset V(\Delta)\}$. The subgraph Δ is called *induced* if $\overline{\Delta} = \Delta$. The *distance* between two vertices v_1 and v_2 is defined as the minimal number of edges of all paths joining v_1 and v_2.

Unless otherwise stated, all graphs considered in this paper are assumed to be *nonempty*, i.e. $V \neq \emptyset$, *simple*, i.e. Γ contains no loops, and at most one edge can join two vertices in Γ, *countable*, i.e. $V(\Gamma)$ is countable, and *uniformly locally finite (ULF)*, i.e.

$$\max_{v \in V(\Gamma)} \gamma(v) =: \gamma_{\max} < \infty. \tag{3}$$

For further graph theoretical terminology we refer to [21], for the algebraic graph theory to [7] and [10], and for the theory of positive operators to [17].

A vector, a sequence or a matrix x is called *positive* ($x \gg 0$) if all its entries satisfy $x_i > 0$, and *nonnegative* ($x \geq 0$) if all $x_i \geq 0$. Moreover, $x > 0$ denotes $x \geq 0$ and $x \neq 0$. Sequences or vectors with constant entries equal to 1 are denoted by $\mathbf{e}$, while $e_k := (\delta_{hk})_{h \in J}$ for $k \in J$. With respect to the above order, the *positive part* and the *negative part* of x are defined as $x^+ = \sup\{x, 0\}$ and $x^- = \sup\{-x, 0\}$ respectively such that $x = x^+ - x^-$. Throughout we shall use the following notations.

Definition 2.1

$$\begin{aligned}
\ell^p(\Gamma) &= \ell^p(V(\Gamma)) \text{ for } p \in [1, \infty] \\
|\cdot|_p &= \ell^p\text{–norm} \\
|T|_p &= \text{operator norm of an endomorphism } T : \ell^p \to \ell^p \\
\sigma(T, B) &= \text{spectrum of the endomorphism } T \text{ in the Banach space } B \\
\sigma_{\text{pt}}(T, B) &= \text{point spectrum of the endomorphism } T \text{ in } B \\
\sigma_{\text{apt}}(T, B) &= \text{approximate point spectrum of the endomorphism } T \text{ in } B \\
r(T, B) &= \sup\{|\lambda| \,|\, \lambda \in \sigma(T, B)\} = \text{spectral radius of } T \text{ in } B \\
r(A) &= \text{spectral radius of a finite matrix } A \\
\mathbf{I} &= \text{identity matrix} \\
\mathbf{I}_n &= n \times n\text{–identity matrix}
\end{aligned}$$

3 The adjacency operator

For a given numbering of the vertices $V(\Gamma) = \{v_i | i \in J\}$ with $J \subset \mathbb{N}$ set $\gamma_i = \gamma(v_i)$ and define the *adjacency matrix* or *adjacency operator* by

$$\mathcal{A}(\Gamma) = (e_{ih})_{i,h\in J} : \mathbb{R}^{V(\Gamma)} \longrightarrow \mathbb{R}^{V(\Gamma)} \tag{4}$$

where

$$e_{ih} = \begin{cases} 1 & \text{if } v_i \text{ and } v_h \text{ are adjacent in } \Gamma \\ 0 & \text{else} \end{cases}$$

The operator $\mathcal{A}(\Gamma)$ is symmetric with respect to the usual ℓ^2–scalar product. Due to a result by Mohar [15], for an arbitrary locally finite graph Γ, the closure in $\ell^2(\Gamma)$ of $\mathcal{A}(\Gamma)$, defined on the sequences of finite support, is bounded iff Γ is ULF, and then, of course, $\mathcal{D}(\mathcal{A}) = \ell^2(\Gamma)$ and $\mathcal{A}(\Gamma)$ is self–adjoint. Moreover, $\mathcal{A} : \ell^2(\Gamma) \to \ell^2(\Gamma)$ is compact iff Γ is finite. For ULF graphs Γ, we mention the following properties of the adjacency operator.

Theorem 3.1 *Suppose $p \in [1, \infty]$. Then $\mathcal{A}(\Gamma) : \ell^p(\Gamma) \longrightarrow \ell^p(\Gamma)$ is a continuous endomorphism, more precisely,*

$$\forall x \in \ell^p(\Gamma) : |\mathcal{A}x|_p \le \gamma_{\max}|x|_p, \tag{5}$$

$$\gamma_{\max}(\Gamma) = |\mathcal{A}(\Gamma)|_\infty = |\mathcal{A}(\Gamma)|_1 \tag{6}$$

and

$$\gamma_{\max}(\Gamma)^{1/p} \le |\mathcal{A}(\Gamma)|_p \le \gamma_{\max}(\Gamma) \quad \textit{for all} \quad p \in (1, \infty). \tag{7}$$

Moreover

(a) $|\sigma(\mathcal{A}; \ell^p(\Gamma))| \subset [0, \gamma_{\max}]$

(b) Γ is connected iff $\mathcal{A}(\Gamma)$ is indecomposable, i.e. $\mathcal{A}(\Gamma)$ possesses no closed invariant order ideal other than the zero ideal and $\ell^p(\Gamma)$.

(c) $\mathcal{A}(\Gamma) \ge 0$, i.e. $x \le y \Longrightarrow \mathcal{A}(\Gamma)x \le \mathcal{A}(\Gamma)y$, and $\mathcal{A}(\Gamma) > 0$ if $E(\Gamma) \ne \emptyset$.

(d) The ih–th element of the matrix $\mathcal{A}(\Gamma)^k$ is equal to the number of walks of length k between v_i and v_h in Γ.

Proof. Inequality (5) is plain for $p = 1$ and $p = \infty$, while for $p \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{p'} = 1$, the finite Hölder inequality applies at each vertex v_i and yields

$$\begin{aligned} |\mathcal{A}x|_p^p &\le \sum_{i\in J} \left(\sum_{h\in J} e_{ih}|x_h| \right)^p \le \sum_{i\in J} \gamma_i^{\frac{p}{p'}} \sum_{h\in J} e_{ih}|x_h|^p \\ &\le \gamma_{\max}^{\frac{p}{p'}} \sum_{h\in J} \gamma_h |x_h|^p \le \gamma_{\max}^{1+\frac{p}{p'}} |x|_p^p. \end{aligned}$$

This shows also $|\mathcal{A}(\Gamma)|_p \le \gamma_{\max}(\Gamma)$. The remaining inequalities in (6) and (7) follow easily by applying $\mathcal{A}$ to the vectors e_k.

Assertion (a) follows immediately from (5), while the remaining assertions are easily derived as in the finite case. ■

Definition 3.2 $\quad r_p = r_p(\Gamma) = r(\mathcal{A}(\Gamma); \ell^p(\Gamma)) = \lim_{k\to\infty} \sqrt[k]{|\mathcal{A}(\Gamma)^k|_p}$

It is well–known by the theory of positive linear operators that r_p belongs to the spectrum of $\mathcal{A}$, see e.g. [17], more precisely

$$r_p(\Gamma) \in \sigma_{\mathrm{apt}}\left(\mathcal{A}(\Gamma); \ell^p(\Gamma)\right), \tag{8}$$

since the boundary of the spectrum belongs to the approximate point spectrum. Moreover, since the adjoint of $\mathcal{A}(\Gamma) : \ell^p(\Gamma) \longrightarrow \ell^p(\Gamma)$ is just $\mathcal{A}(\Gamma) : \ell^{p'}(\Gamma) \longrightarrow \ell^{p'}(\Gamma)$, we conclude that

$$r_p(\Gamma) = r_{p'}(\Gamma) \quad \text{for} \quad p \in [1, \infty) \ \text{and} \ \frac{1}{p} + \frac{1}{p'} = 1. \tag{9}$$

But, in general, $r_p(\Gamma)$ is not an eigenvalue as will be shown by Example 4.17. As in the finite case,

$$\forall p \in [1, \infty] \, : \, r_p(\Gamma) \leq \gamma_{\max}, \tag{10}$$

and

$$\gamma_{\min} \leq r_\infty(\Gamma). \tag{11}$$

The latter inequality follows from Theorem 4.12 or, more directly, from the fact that ih–th element $e_{ih}^{(k)}$ of the matrix $\mathcal{A}(\Gamma)^k$ is bounded from below by $\gamma_{\min}^k$ and that the operator norm $|\mathcal{A}(\Gamma)^k|_\infty$ is bounded from below by $\sup\{e_{ih}^{(k)} \,|\, i, h \in J\}$. Especially

$$\Gamma \text{ regular of valency } d \implies r_\infty(\Gamma) = d = |\mathcal{A}(\Gamma)|_\infty. \tag{12}$$

But, in general, (11) does not hold for finite p.

In fact, the eigenvalues of the adjacency operator are real.

Lemma 3.3 $\quad \sigma_{\mathrm{pt}}\left(\mathcal{A}(\Gamma), \ell^\infty(\Gamma)\right) \subset \mathbb{R}$.

Proof. We can follow the idea of the proof of [6, Lemma 5.2], but for the reader's convenience we repeat the arguments here. Without restriction, assume that Γ is connected. Choose some node v_0 and introduce for $k \in \mathbb{N}$

$$B_k = \{v \in V \,|\, \mathrm{dist}(v_0, v) \leq k\} \quad \text{and} \quad S_k = \{v \in V \,|\, \mathrm{dist}(v_0, v) = k\}.$$

Let $\varphi \in \ell^\infty$ be an eigensequence belonging to the eigenvalue $\mu \in \mathbb{C}$ of $\mathcal{A}$: $\mathcal{A}\varphi = \mu\varphi$ and set

$$s_k = \sum_{v_i \in S_k} |\varphi_i|^2, \qquad b_k = \sum_{v_i \in B_k} |\varphi_i|^2.$$

Then

$$\begin{aligned}
\mu b_k =& \sum_{v_i \in B_k} \overline{\varphi_i} \left(\sum_{l=0}^{\infty} e_{il}\varphi_l \right) = \sum_{v_i \in B_k} \sum_{l=0}^{\infty} e_{il}\varphi_l\overline{\varphi_i} = \sum_{l=0}^{\infty} \varphi_l \left(\sum_{v_i \in B_k} \overline{e_{li}\varphi_l} \right) \\
=& \overline{\mu} b_k - \sum_{v_i \in S_k} \sum_{v_h \in S_{k+1}} e_{ih}\varphi_i\overline{\varphi_h},
\end{aligned}$$

and by the eigenvalue relation

$$-\sum_{v_i\in S_k}\sum_{v_h\in S_{k+1}} e_{ih}\varphi_i\overline{\varphi_h} = -\overline{\mu}s_k + \sum_{v_i\in S_k}\sum_{v_h\in B_k} e_{ih}\varphi_i\overline{\varphi_h}.$$

Thus, for any $k\in\mathbb{N}$,

$$2i\Im(\mu) = \frac{1}{b_k}\left(-\overline{\mu}s_k + \sum_{v_i\in S_k}\sum_{v_h\in B_k} e_{ih}\varphi_i\overline{\varphi_h}\right) \tag{13}$$

and by Young's Inequality, we obtain

$$\forall k\in\mathbb{N}:\quad 2\,|\Im(\mu)| \le (|\mu| + |\mathcal{A}|_\infty)\,\frac{s_k}{b_k} + \frac{1}{2}|\mathcal{A}|_\infty\frac{s_{k-1}}{b_{k-1}}. \tag{14}$$

If $\varphi\in\ell^2$, then clearly $\mu(\varphi,\overline{\varphi})_{\ell^2} = (\mathcal{A}\varphi,\overline{\varphi})_{\ell^2} = (\varphi,\mathcal{A}\;\overline{\varphi})_{\ell^2} = \overline{\mu}(\varphi,\overline{\varphi})_{\ell^2}$ and $\mu\in\mathbb{R}$. Thus we can assume that

$$\lim_{k\to\infty} b_k = \infty.$$

If the sequence $(s_k)_{k\in\mathbb{N}}$ contains a bounded subsequence$\big(s_{\alpha(k)}\big)_{k\in\mathbb{N}}$ with injection $\alpha:\mathbb{N}\to\mathbb{N}$, then

$$\lim_{k\to\infty}\frac{s_{\alpha(k)}}{b_{\alpha(k)}} = 0$$

implying that $\Im(\mu)=0$. Thus we can assume that

$$\liminf_{k\to\infty} s_k = \infty.$$

If $b_k = b_{k+1} = b_{k+2}$ for some $k\in\mathbb{N}$, then φ would vanish on two consecutive spheres S_{k+1} and S_{k+2} and, thereby, it would vanish everywhere by connectedness of Γ. Thus, at most two consecutive values b_k and b_{k+1} can be identical, and the sequence $(b_k)_{k\in\mathbb{N}}$ contains a strictly increasing subsequence that is again denoted by $(b_k)_{k\in\mathbb{N}}$.
If $\lim_{k\to\infty}\frac{s_{k-1}}{s_k} = \sigma\in\mathbb{R}$ exists, then by Stolz's Theorem $\left(\frac{s_k}{b_k}\right)_{k\in\mathbb{N}}$ is convergent to

$$\lim_{k\to\infty}\frac{s_k}{b_k} = \lim_{k\to\infty}\frac{s_k - s_{k-1}}{b_k - b_{k-1}} = 1-\sigma. \tag{15}$$

Observe that $\sigma>1$ is excluded and that $\sigma=1$ yields $\lim_{k\to\infty}\frac{s_k}{b_k}=0$. The case $\sigma<1$ is impossible, since on the one side $(s_k)_{k\in\mathbb{N}}$ converges to 0 and thereby $\lim_{k\to\infty}\frac{s_k}{b_k}=0$, while on the other the latter limit is positive by (15). Thus $\Im(\mu)=0$ if $\lim_{k\to\infty}\frac{s_{k-1}}{s_k}=\sigma\in\mathbb{R}$ exists. The same argument is valid if $\left(\frac{s_{k-1}}{s_k}\right)_{k\in\mathbb{N}}$ contains a convergent subsequence.

It only remains the case in which $\left(\frac{s_{k-1}}{s_k}\right)_{k\in\mathbb{N}}$ tends to infinity. But then again Stolz's Theorem would imply that $\lim_{k\to\infty}\frac{s_k}{b_k}<0$ which is absurd. ∎

Corollary 3.4 $\forall p \in [1,\infty] : \sigma_{\mathrm{pt}}\left(\mathcal{A}(\Gamma), \ell^p(\Gamma)\right) \subset \mathbb{R}$.

Example 3.5 Let $\mathbb{T}_d$ denote the regular tree of valency $d \geq 3$. Then

$$\sigma_{\mathrm{pt}}\left(\mathcal{A}(\mathbb{T}_d), \ell^\infty\right) = [-d, d],$$

and each eigenvalue λ is of infinite multiplicity in $\ell^\infty(\mathbb{T}_d)$, see [5]. This holds especially for $\lambda = d = \mathrm{I}(\mathbb{T}_d)$, showing that $\mathbb{T}_d$ is not a Liouville space, see [4]. For $d \geq 3$ the ℓ^2–spectral radius differs from the above one, namely

$$r\left(\mathcal{A}(\mathbb{T}_d); \ell^2(\mathbb{T}_d)\right) = 2\sqrt{d-1}$$

due to result by P. Cartier 1972 e.a., see e.g. [16]. This case furnishes also an example for an ULF graph satisfying $\gamma_{\min} > r_p$ for $p \in [1,\infty)$. For $d = 2$, the situation is different, the ℓ^2– and ℓ^∞–spectra coincide and all eigenvalues in $\ell^\infty(\mathbb{T}_d)$ are of multiplicity 2 or 1, see Example 4.18.

4 Order indices

We introduce the order indices $\mathrm{I}_p^+(\Gamma)$ and $\mathrm{I}_p^-(\Gamma)$ that are based on the order properties of each real Banach lattice $\ell^p(\Gamma)$, as follows: For $p \in [1,\infty]$ introduce

$$\begin{aligned} \sigma_p^+(\Gamma) &= \{\lambda \geq 0 \,|\, \exists x \in \ell^p(\Gamma),\, x > 0 \,:\, \mathcal{A}x \geq \lambda x\}, \\ \sigma_p^-(\Gamma) &= \{\lambda \geq 0 \,|\, \exists x \in \ell^p(\Gamma),\, x > 0 \,:\, \mathcal{A}x \leq \lambda x\}. \end{aligned}$$

Both sets are never empty due to the following

Lemma 4.1 $(r_p(\Gamma), \infty) \subseteq \sigma_p^-(\Gamma)$ *and* $\{r(\mathcal{A}(\Lambda)) \,|\, \Lambda \leq \Gamma,\, \Lambda \text{ finite}\} \subset \sigma_p^+(\Gamma)$.

Proof. Choose $\rho \in (r_p(\Gamma), \infty)$. Then, by a well known result for positive operators, see e.g. [17], the resolvent is positive in $\ell^p(\Gamma)$:

$$\left(\rho\, \mathbf{I}_{\ell^p(\Gamma)} - \mathcal{A}(\Gamma)\right)^{-1} \geq 0. \tag{16}$$

For any $y \in \ell^p(\Gamma)$ with $y > 0$ this yields $x := (\rho\,\mathbf{I} - \mathcal{A})^{-1} y > 0$, i.e. $(\rho\,\mathbf{I} - \mathcal{A})\, x = y > 0$.

For the second assertion, choose some finite subgraph Λ with positive eigenvector v satisfying $\mathcal{A}(\Lambda)v = r(\mathcal{A}(\Lambda))v$. Since $\mathcal{A}(\Lambda)$ is smaller or equal than a principal minor of $\mathcal{A}(\Gamma)$, the zero extension $\tilde{v}$ of v to $V(\Gamma)$ satisfies $r(\mathcal{A}(\Lambda))\tilde{v} \leq \mathcal{A}(\Lambda)\tilde{v}$ and shows $r(\mathcal{A}(\Lambda)) \in \sigma_p^+(\Gamma)$. ∎

Moreover, $\sigma_p^+(\Gamma)$ and $\sigma_p^-(\Gamma)$ are intervals of the positive real half line. Now we can define the indices

Definition 4.2

$$\begin{aligned} \mathrm{I}_p^+(\Gamma) &= \sup \sigma_p^+(\Gamma) \\ \mathrm{I}_p^-(\Gamma) &= \inf \sigma_p^-(\Gamma) \end{aligned}$$

Let us collect the basic properties of these indices in the following lemmata.

Lemma 4.3 *If Γ is finite, then* $\mathrm{I}_p^-(\Gamma) = r(\mathcal{A}(\Gamma)) = \mathrm{I}_p^+(\Gamma)$.

Proof. Without restriction we can assume that Γ is connected. Then there exists $x \gg 0$ with $\mathcal{A}x = rx$, $r = r(\mathcal{A})$ and by definition, $\mathrm{I}_p^-(\Gamma) \le r \le \mathrm{I}_p^+(\Gamma)$. Next, for $\mathcal{A}y \ge \lambda y > 0$, $y > 0$ and for $0 < \mathcal{A}z \le \mu z$, $z > 0$ we conclude $r \ge \lambda$ and $r \le \mu$ since

$$r(x,y) = (\mathcal{A}x, y) = (x, \mathcal{A}y) \ge \lambda(x,y) > 0 < r(x,z) = (\mathcal{A}x, z) = (x, \mathcal{A}z) \le \mu(x,z).$$

∎

Connectedness and adjacency imply immediately

Lemma 4.4 *If Γ is connected and $x > 0$ a node vector with $\mathcal{A}x \le \lambda x$ then $x \gg 0$.*

An important property is the monotonicity of the indices.

Lemma 4.5 *If Δ is any subgraph of Γ then* $\mathrm{I}_p^+(\Delta) \le \mathrm{I}_p^+(\Gamma)$ *and* $\mathrm{I}_p^-(\Delta) \le \mathrm{I}_p^-(\Gamma)$.

Proof. Without restriction we can assume that Γ is connected. The matrix $\mathcal{A}(\Delta)$ is smaller or equal than a principal minor of $\mathcal{A}(\Gamma)$ and its zero extension $\tilde{\mathcal{A}}(\Delta)$ to $V(\Gamma)^2$ satisfies $\tilde{\mathcal{A}}(\Delta) \le \mathcal{A}(\Gamma)$. For $\mathcal{A}(\Delta)y \ge \lambda y$ with $\ell^p(\Delta) \ni y > 0$ and its zero extension x to $V(\Gamma)$ we conclude $\mathcal{A}(\Gamma)x \ge \tilde{\mathcal{A}}(\Delta)x \ge \lambda x$.

For $\mathcal{A}(\Gamma)z \le \lambda z$ with $\ell^p(\Delta) \ni z \gg 0$ (Lemma 4.4) and its restriction v to $V(\Delta)$ we conclude $v \gg 0$ and $\mathcal{A}(\Delta)v \le \lambda v$. ∎

Corollary 4.6 $\quad R := \sup\{r(\mathcal{A}(\Lambda)) \,|\, \Lambda \le \Gamma,\ \Lambda$ *finite* $\} \le \mathrm{I}_p^-(\Gamma),\ \mathrm{I}_p^+(\Gamma)$

Lemma 4.7 $\quad \mathrm{I}_p^+(\Gamma) \le r_p(\Gamma)$

Proof. For $\lambda \in \sigma_p^+(\Gamma)$ it readily follows that $\lambda \le \sqrt[k]{|\mathcal{A}(\Gamma)^k|_p}$ for all $k \in \mathbb{N}$. ∎

By duality it follows from Definition 4.2

Lemma 4.8 $\quad \mathrm{I}_p^+(\Gamma) \le \mathrm{I}_q^-(\Gamma)$ *and* $\mathrm{I}_q^+(\Gamma) \ge \mathrm{I}_p^-(\Gamma)$ *for* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $p \ge 2$.

Clearly, I_p^+ is increasing with p, while I_p^- is decreasing in p. Combining this with Lemmata 4.1–4.8 and with a result by B. Mohar 1982 or E. Seneta 1981, see [16], that states

$$r_2(\Gamma) = R, \tag{17}$$

we can resume the relations between the different indices in the following

Theorem 4.9

(a) $\forall p \in [1,\infty] \,:\, \mathrm{I}_p^-(\Gamma) \le r_p(\Gamma)$.

(b) $\forall p \in [1,\infty] \,:\, R \le \mathrm{I}_p^+(\Gamma) \le r_p(\Gamma)$.

(c) $\forall p \in [1,2] \,:\, R = \mathrm{I}_1^+(\Gamma) = \mathrm{I}_p^+(\Gamma) = r_2(\Gamma) \le \mathrm{I}_p^-(\Gamma) \le \mathrm{I}_1^-(\Gamma)$.

(d) $\forall p \in [2,\infty] : R = \mathrm{I}^-_\infty(\Gamma) = \mathrm{I}^-_p(\Gamma) = r_2(\Gamma) \leq \mathrm{I}^+_p(\Gamma) \leq \mathrm{I}^+_\infty(\Gamma)$.

(e) $\forall p \in [2,\infty] : \mathrm{I}^+_p(\Gamma) \leq \mathrm{I}^-_q(\Gamma)$ *and* $\mathrm{I}^+_q(\Gamma) = \mathrm{I}^-_p(\Gamma)$ *for* $\frac{1}{p} + \frac{1}{q} = 1$.

Corollary 4.10 $\mathrm{I}^+_1(\Gamma) = R = r_2(\Gamma) = \mathrm{I}^-_\infty(\Gamma) \leq \mathrm{I}^+_\infty(\Gamma) \leq \mathrm{I}^-_1(\Gamma) \leq r_\infty(\Gamma) = r_1(\Gamma)$

All indices and spectral radii are bounded from above by $\gamma_{\max}(\Gamma)$, but only $\mathrm{I}^+_\infty(\Gamma)$ is bounded from below by $\gamma_{\min}(\Gamma)$ as follows from $\mathcal{A}\mathbf{e} \geq \gamma_{\min}\mathbf{e}$. In general, this is false for finite p or for $\mathrm{I}^-(\Gamma)$.

Lemma 4.11 $\gamma_{\min}(\Gamma) \leq \mathrm{I}^+_\infty(\Gamma) \leq \gamma_{\max}(\Gamma)$.

A key result of this section is the following

Theorem 4.12 $r_\infty(\Gamma) = \mathrm{I}^+_\infty(\Gamma)$

Proof. By Lemma 4.7, it remains to show $r := r_\infty(\Gamma) \leq \mathrm{I}^+_\infty(\Gamma)$. Let $\left(x^{(k)}\right)_{k\in\mathbb{N}}$ denote a sequence of nonzero vectors in $\ell^\infty(\Gamma)$ and $(\lambda_k)_{k\in\mathbb{N}}$ a scalar sequence with

$$\lim_{k\to\infty} \lambda_k = r \quad \text{and} \quad \lim_{k\to\infty} \left|\mathcal{A}(\Gamma)x^{(k)} - \lambda_k\, x^{(k)}\right|_\infty = 0.$$

Since r is a real number, we can assume w.l.o.g. that the vectors $x^{(k)}$ real and that all

$$\lambda_k > 0 \quad \text{and} \quad \lambda_k \nearrow r \quad \text{as} \quad k \to \infty.$$

Set

$$\varepsilon^{(k)} = \mathcal{A}(\Gamma)x^{(k)} - \lambda_k\, x^{(k)}$$

and get by hypothesis that $\lim_{k\to\infty} \left|\varepsilon^{(k)}\right|_\infty = 0$. For each $k \in \mathbb{N}$ we find

$$\begin{aligned} \mathcal{A}x^{(k)+} &\geq \lambda_k\, x^{(k)+} + \varepsilon^{(k)} \geq \lambda_k\, x^{(k)+} - \varepsilon^{(k)-}, \\ -\mathcal{A}x^{(k)-} &\leq \lambda_k\, x^{(k)-} + \varepsilon^{(k)} \leq \lambda_k\, x^{(k)+} + \varepsilon^{(k)+}. \end{aligned}$$

Introduce $y^{(k)} = \left|x^{(k)}\right| > 0$ and get

$$\forall k \in \mathbb{N} : \mathcal{A}y^{(k)} \geq \lambda_k\, y^{(k)} - \left|\varepsilon^{(k)}\right|. \tag{18}$$

Fix $N \in \mathbb{N}$. Since $(\lambda_k)_{k\in\mathbb{N}}$ is increasing,

$$k \geq N \implies \mathcal{A}y^{(k)} \geq \lambda_N\, y^{(k)} - \left|\varepsilon^{(k)}\right|.$$

Thus, the vector

$$z^{(N)} := \sup_{k\geq N} y^{(k)} = \left(z_i^{(N)}\right)_{i\in J} \quad \text{with} \quad z_i^{(N)} = \sup_{k\geq N} y_i^{(k)}$$

satisfies $z^{(N)} > 0$ and for each $i \in J$,

$$\left(\mathcal{A}z^{(N)}\right)_i \geq \lambda_N\, y_i^{(k)} - \left|\varepsilon_i^{(k)}\right| \quad \text{for all} \quad k \geq N$$

and

$$\left(\mathcal{A}z^{(N)}\right)_i \geq \sup_{k\geq N} \left\{\lambda_N\, y_i^{(k)} - \left|\varepsilon_i^{(k)}\right|\right\} = \lambda_N\, z_i^{(N)}.$$

This shows $\mathcal{A}z^{(N)} \geq \lambda_N\, z^{(N)}$ and that $\lambda_N \in \sigma^+_\infty(\Gamma)$, and permits to conclude that $r = \sup\{\lambda_N \,|\, N \in \mathbb{N}\} \leq \mathrm{I}^+_\infty(\Gamma)$. ∎

In view of the above properties and results it seems reasonable to mark with distinction the ℓ^∞–case. Let us define the index of an ULF graph as follows.

Definition 4.13 The *index* $\mathrm{I}(\Gamma)$ of an ULF graph Γ is defined by

$$\mathrm{I}(\Gamma) = \mathrm{I}^+_\infty(\Gamma) = \sup\{\lambda \geq 0 \mid \exists x \in \ell^\infty(\Gamma),\, x > 0 \,:\, \mathcal{A}x \geq \lambda x\}\,.$$

Moreover we set

$$\mathrm{I}^-(\Gamma) = \mathrm{I}^-_\infty(\Gamma) = \inf\{\lambda \geq 0 \mid \exists x \in \ell^\infty(\Gamma),\, x > 0 \,:\, \mathcal{A}x \leq \lambda x\}\,.$$

Corollary 4.14 $\quad \mathrm{I}^-(\Gamma) \leq \mathrm{I}(\Gamma)$ *and*

$$\mathrm{I}(\Gamma) = \sup_{\ell^\infty(\Gamma)\ni x>0}\ \inf_{i\in J} \sum_{h\in J} e_{ih}\frac{x_h}{x_i} = r(\mathcal{A}(\Gamma);\ell^\infty(\Gamma)),$$
$$\mathrm{I}^-(\Gamma) = \inf_{\ell^\infty(\Gamma)\ni x\gg 0}\ \sup_{i\in J} \sum_{h\in J} e_{ih}\frac{x_h}{x_i} = r(\mathcal{A}(\Gamma);\ell^2(\Gamma)).$$

Corollary 4.15

(a) $\gamma_{\min}(\Gamma) \leq \mathrm{I}(\Gamma) \leq \gamma_{\max}(\Gamma)$

(b) If Γ is regular of valency d, then $d = \mathrm{I}(\Gamma)$.

(c) $\forall p \in [1,\infty] \,:\, \mathrm{I}(\Gamma) \leq |\mathcal{A}(\Gamma)|_p$

(d) If Δ is any subgraph of Γ, then $\mathrm{I}(\Delta) \leq \mathrm{I}(\Gamma)$.

Note that the above results yield also a proof of the monotonicity of the spectral radius of $\mathcal{A}$ in the corresponding ℓ^∞–spaces. Moreover, the index $\mathrm{I}^-(\Gamma)$ is always attained by a bounded nonzero nonnegative sequence:

Lemma 4.16 *If Γ is connected, then $\mathrm{I}^-(\Gamma)$ is attained by a positive sequence belonging to $\ell^\infty(\Gamma)$.*

Proof. Choose a decreasing real sequence $(\lambda_k)_{k\in\mathbb{N}}$ with $\lambda_k \searrow \mathrm{I}^-(\Gamma)$ as $k\to\infty$, and a sequence $\left(x^{(k)}\right)_{k\in\mathbb{N}}$ of positive vectors in $\ell^\infty(\Gamma)$ such that

$$\mathcal{A}(\Gamma)x^{(k)} \leq \lambda_k\, x^{(k)} \quad \text{for all} \quad k\in\mathbb{N}.$$

Since each $x^{(k)} \gg 0$, we can assume furthermore that

$$x^{(k)}_0 = 1 \quad \text{for all} \quad k\in\mathbb{N}. \tag{19}$$

For $N\in\mathbb{N}$ set

$$y^{(N)} = \sup_{k\geq N} x^{(k)} = \left(y^{(N)}_i\right)_{i\in J} \quad \text{with} \quad y^{(N)}_i = \sup_{k\geq N} x^{(k)}_i.$$

By the decreasing character of $(\lambda_k)_{k\in\mathbb{N}}$ and by (19), it follows that

$$\mathcal{A}(\Gamma)y^{(N)} \leq \lambda_N\, y^{(N)} \quad \text{and} \quad y^{(N)}_0 = 1 \quad \text{for all} \quad N\in\mathbb{N}.$$

Since $\left(y^{(k)}\right)_{k\in\mathbb{N}}$ is decreasing, for each $i \in J$ the limit

$$z_i := \lim_{N\to\infty} y_i^{(N)} = \inf_{N\in\mathbb{N}} y_i^{(N)}$$

exists. Thus, $z := \inf_{N\in\mathbb{N}} y^{(N)} = (z_i)_{i\in J}$ is a bounded sequence satisfying

$$z > 0 \quad \text{and} \quad z_0 = 1.$$

Moreover, for each $i \in J$ and for all $N \in \mathbb{N}$,

$$(\mathcal{A}z)_i \leq \lambda_N \, y_i^{(N)}$$

and, thereby,

$$(\mathcal{A}z)_i \leq \inf_{N\in\mathbb{N}} \left(\lambda_N \, y_i^{(N)}\right) = \mathrm{I}_\infty^-(\Gamma)\, z_i.$$

This shows $\mathcal{A}z \leq \mathrm{I}_\infty^-(\Gamma)\, z$ and Lemma 4.4 permits to conclude that $z \gg 0$. ■

Note that, in general, r_2 is not an eigenvalue in the ℓ^2–setting, despite Lemma 4.16. For the regular tree $\mathbb{T}_d$ of valency $d \geq 3$, the value $r_2(\mathbb{T}_d) = 2\sqrt{d-1} = \mathrm{I}^-(\mathbb{T}_d)$ possesses a positive bounded lower eigenvector that has to take arbitrarily small values in view of Lemma 8.1.

An analogous construction for the index $\mathrm{I}(\Gamma)$ fails, since here (19) cannot be fulfilled in general. In fact, $\mathrm{I}(\Gamma)$ needs not to be attained as will be well displayed by the next example.

Example 4.17 Let Γ_0 denote the one-sided unbounded path with vertex set $V(\Gamma_0) = \mathbb{N}$ and edges $\{\{i,k\} \,|\, |i-k| = 1\}$. A recurrence argument based on the eigenvector equation shows that $\sigma_{\mathrm{pt}}\left(\mathcal{A}(\Gamma_0); \ell^\infty(\Gamma_0)\right) = (-2, 2)$. Suppose that $0 < x \in \ell^\infty(\mathbb{N})$, $\lambda \geq 2$ and $\mathcal{A}(\Gamma_0)x \geq \lambda x$. Then x is increasing and $x_1 \geq \lambda x_0$ and

$$\forall k \in \mathbb{N}^* : \; x_{k+1} \geq \lambda x_k - x_{k-1} \geq (\lambda - 1)x_k.$$

This imposes $\lambda = 2$. But still the difference sequence $(x_{k+1} - x_k)_{k\in\mathbb{N}}$ is increasing, which enforces that x is constant. Since this is excluded, $\mathrm{I}(\Gamma_0) = 2$ is not attained. Moreover, $\mathrm{I}^-(\Gamma_0) = 2$ by Lemma 4.5, since the index $2\cos\frac{\pi}{n+1}$ of any finite path with n vertices is a lower bound for $\mathrm{I}^-(\Gamma_0)$. It is well attained by the constant sequence $\mathbf{e}$.

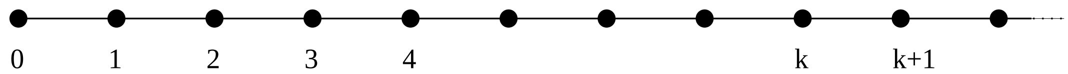

Figure 1: The one-sided unbounded path Γ_0

Example 4.18 Let Γ_1 denote the two-sided unbounded path, i.e. the connected 2 - regular graph with $V(\Gamma_1) = \mathbb{Z}$ and edges $\{\{i,k\} \,|\, |i-k| = 1\}$. The eigenvectors obey the difference equations

$$x_{k+1} = \lambda x_k - x_{k-1} \quad \text{for all } k \in \mathbb{Z},$$

that lead to the characteristic roots

$$a_1(\lambda) = \frac{\lambda}{2} + \sqrt{\frac{\lambda^2}{4} - 1}, \qquad a_2(\lambda) = \frac{\lambda}{2} - \sqrt{\frac{\lambda^2}{4} - 1}. \tag{20}$$

Then the boundedness condition implies $\sigma_{\mathrm{pt}}\left(\mathcal{A}(\Gamma_1);\ell^\infty(\Gamma_1)\right) = [-2,2]$. By Corollary 4.15, or more directly as in Example 4.17, we conclude $\mathrm{I}(\Gamma_1) = 2 = \mathrm{I}^-(\Gamma_1)$ and both of the indices are attained by the constant sequence $\mathbf{e}$.

We close this section with some general lower bounds for both of the indices.

Corollary 4.19 *Let Γ be an infinite and connected graph. Then*

(a) $\mathrm{I}^-(\Gamma) \geq 2$,

(b) $\mathrm{I}^-(\Gamma) > \sqrt{2+\sqrt{5}} = 2.058171..$, *if Γ is not a tree.*

(c) $\mathrm{I}^-(\Gamma) > \frac{3}{\sqrt{2}} = 2.121320..$, *if Γ contains a periodic graph of rank at least 2.*

Proof. By hypothesis, Γ contains a subgraph isomorphic to Γ_0. Thus, (a) follows by Lemma 4.5. It applies also for (b) in connection with the corresponding result for finite graphs containing proper circuits due to A. J. Hoffman [12] and for (c) in connection with the lower bound found in [2, Theorem 8.1]. ■

5 Periodic graphs

A *periodic graph*, see [2], [9] or *generalized lattice*, see [20] is a ULF graph whose automorphism group contains a transitive subgroup G isomorphic to some $\mathbb{Z}^m$. In detail:

Definition 5.1 A ULF graph Γ is called *periodic of rank m* with *translation group* $G = \oplus_{i=1}^m \mathbb{Z}b_i \leq \mathrm{Aut}(\Gamma)$, with *kernel* N and with *cell* F, if the following conditions hold:

(a) Γ is connected.

(b) N and F are finite connected subgraphs of Γ.

(c) $V(N)^G = V(\Gamma)$,

(d) $F = \overline{N \cup \bigcup_{i=1}^m N^{b_i}}$ and $E(F)^G = E(\Gamma)$,

(e) $\forall g,h \in G: \quad g \neq h \quad \Longrightarrow \quad V(N^g) \cap V(N^h) = \emptyset$,

(f) rank $G = m$.

Classical examples are given by the graphs of the Keplerian plane tilings, as e.g. the tiling with regular triangles and dodecagons in Fig. 2, where a kernel is given by any pair of adjacent triangles. To the periodic graph Γ we associate the finite *nuclear matrix* $\mathbb{A}(N,\Gamma)$ as follows. The vertices of the kernel $P_1,\ldots,P_n$ decompose $V(\Gamma)$ into n disjoint classes $P_1,\ldots,P_n$. Then $\mathbb{A}(N,\Gamma) = (a_{ik})_{n\times n}$ is the weighted adjacency matrix between these equivalence classes, i.e. a_{ik} is the number of vertices in Γ of the class P_k that are adjacent to any vertex of class P_i. The conditions 5.1 ensure that $\mathbb{A}(N,\Gamma)$ is a symmetric, nonnegative and indecomposable matrix. Moreover, the spectral radii of all nuclear matrices of Γ coincide and, thereby, the *"periodic"* index of Γ is well-defined:

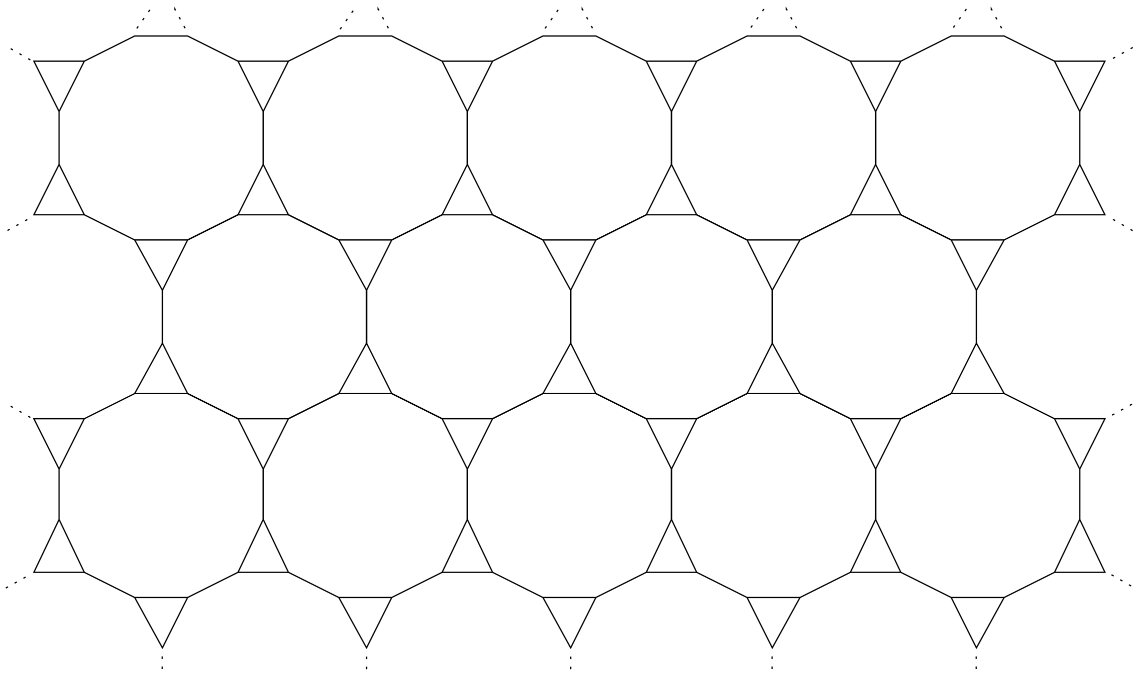

Figure 2: Kepler's plane tiling with regular triangles and dodecagons

Definition 5.2 $\quad \mathrm{I}_{per}(\Gamma) = r(\mathbb{A}(N, \Gamma))$

For this and more details we refer to [2]. In fact, the periodic index coincides with the indices from Definition 4.13.

Theorem 5.3 *If Γ is a periodic graph, then $\mathrm{I}_{per}(\Gamma) = \mathrm{I}^-(\Gamma) = \mathrm{I}(\Gamma)$.*

Proof. Set $\rho = \mathrm{I}_{per}(\Gamma)$. The periodic graph Γ possesses a periodic vector $\mathbf{p} \gg 0$ such that $\mathcal{A}(\Gamma)\mathbf{p} = \rho\mathbf{p}$, which shows $\mathrm{I}^-(\Gamma) \leq \rho \leq \mathrm{I}(\Gamma)$.

Suppose $\mathcal{A}(\Gamma)y \geq \lambda y$ with $\ell^\infty(\Gamma) \ni y > 0$. Find a scalar $\alpha > 0$ with $\alpha\mathbf{p} \geq y$ and $\alpha p_j = y_j > 0$ for some j. Then $\rho\alpha\mathbf{p} = \mathcal{A}(\Gamma)\alpha\mathbf{p} \geq \mathcal{A}(\Gamma)y \geq \lambda y$ and $0 \leq (\rho - \lambda)\alpha p_j$, which implies $\lambda \leq \rho$. Thus $\rho \geq \mathrm{I}(\Gamma)$.

In order to show $\rho \leq \mathrm{I}^-(\Gamma)$, observe that for any $0 \ll x \in \ell^\infty(\Gamma)$ with $\mathcal{A}x \leq \lambda x$ and for any kernel N, the restriction z_x of x to $V(N)$ satisfies $\mathbb{A}(N;\Gamma)z \leq \lambda z$. Thus

$$\{\lambda \,|\, \exists x > 0 \,:\, \mathcal{A}x \leq \lambda x\} \subset \{\lambda \,|\, \exists z > 0 \,:\, \mathbb{A}(N;\Gamma)z \leq \lambda z\}\,.$$

Since for the finite matrix $\mathbb{A}(N, \Gamma)$ the infimum of the r.h.s. amounts to ρ, the assertion is shown. ■

Combining this result with Theorem 4.9 we are lead to the

Corollary 5.4 *For a periodic graph Γ all the indices $I_p^\pm$ and corresponding spectral radii coincide:*

$$\forall p \in [1, \infty] \,:\, \mathrm{I}_{per}(\Gamma) = R = \mathrm{I}_p^-(\Gamma) = \mathrm{I}_p^+(\Gamma) = r_p(\Gamma).$$

The last result generalizises the characterization of the periodic index in [3] and applies especially to the $2m$-regular infinite m-dimensional grid Γ_m in $\mathbb{R}^m$ with $V(\Gamma_m) = \mathbb{Z}^m$ and the edges generated by the adjacency

$$e_{zw} = 1 \iff \sum_{j=1}^{m} |z_j - w_j| = 1.$$

It includes as a special case $m = 2$ the regular graph $\mathbf{K}_1$ belonging to the Keplerian plane tiling with squares. Thus, $r_p(\mathbb{Z}^m) = 2m$ for all $p \in [1, \infty]$. It has been shown in [2] that

$$\sigma\left(\mathcal{A}(\mathbb{Z}^m); \ell^\infty(\mathbb{Z}^m)\right) = \sigma_{\mathrm{pt}}\left(\mathcal{A}(\mathbb{Z}^m); \ell^\infty(\mathbb{Z}^m)\right) = [-2m, 2m]$$

with the aid of the mapping $(z_1, \ldots, z_m) \mapsto x_{z_1+\cdots+z_m}$ that induces a *spectral embedding* $\lambda \mapsto m\lambda$ from $\sigma_{\mathrm{pt}}\left(\mathcal{A}(\Gamma_1); \ell^\infty(\Gamma_1)\right)$ onto $\sigma\left(\mathcal{A}(\mathbb{Z}^m); \ell^\infty(\mathbb{Z}^m)\right)$ using eigenvectors x on Γ_1. The same technique applies to the corresponding resolvent sets and yields

$$\forall p \in [1, \infty] \, : \, \sigma\left(\mathcal{A}(\mathbb{Z}^m); \ell^p(\mathbb{Z}^m)\right) = [-2m, 2m].$$

Note that in general the ℓ^∞–point spectrum is not connected. The graph in Fig. 2 bears that property, see [5, 3, 14], a more simple example is given by the infinite comb Z_1 from [5, Example 8.2] with $\sigma_{\mathrm{pt}}\left(\mathcal{A}(Z_1); \ell^\infty(Z_1)\right) = [-\sqrt{2} - 1, 1 - \sqrt{2}] \cup [\sqrt{2} - 1, 1 + \sqrt{2}]$.

6 Vertex colourings

Recall that the *chromatic number* $\nu(\Gamma)$ of the graph Γ is defined as the minimal number of colours necessary to partition $V(\Gamma)$ such that no class contains two adjacent vertices. A classical result by H. W. Wilf [19] relates $\nu(\Gamma)$ to the graph index in the finite case as follows.

Theorem 6.1 *If Γ is a finite graph then $\nu(\Gamma) \leq 1 + r(\mathcal{A}(\Gamma))$.*

The same formula holds for ULF graphs.

Theorem 6.2 $\nu(\Gamma) \leq 1 + \mathrm{I}^-(\Gamma) \leq 1 + \mathrm{I}(\Gamma)$.

Proof. A fundamental result by N. G. de Bruijn and P. Erdös [8] states that, if for an arbitrary infinite graph G there exists $N \in \mathbb{N}$ such that for all finite subgraphs $\Delta \leq G$ their chromatic numbers satisfy $\nu(\Delta) \leq N$, then $\nu(G)$ is bounded from above by N too. In the ULF case, we find a sufficiently large finite subgraph $\Delta \leq \Gamma$ with $\nu(\Delta) = \nu(\Gamma)$. Then

$$\nu(\Gamma) = \nu(\Delta) \leq 1 + r(\mathcal{A}(\Delta)) \leq 1 + \mathrm{I}^-(\Gamma)$$

by Wilf's Theorem 6.1 and Lemma 4.5. ■

By Theorem 4.9 the chromatic number is also a lower bound for all $\mathrm{I}_p^-(\Gamma)$, $\mathrm{I}_p^+(\Gamma)$ and $r_p(\Gamma)$. The periodic example given in [2, Example 5.5] shows that the estimate in Theorem 6.2 is optimal. In the bipartite case, the ℓ^2–spectrum is symmetric with respect to the origin, see [15]. This remains true in $\ell^\infty(\Gamma)$:

Lemma 6.3 *Suppose Γ is bipartite. Then $\sigma(\mathcal{A}(\Gamma); \ell^\infty(\Gamma))$ is symmetric with respect to the origin and*

$$\dim_{\mathbb{R}} \ker(\lambda \mathbf{I} - \mathcal{A}(\Gamma)) = \dim_{\mathbb{R}} \ker(-\lambda \mathbf{I} - \mathcal{A}(\Gamma)).$$

Proof. Choose a 2–colouring $c : V(\Gamma) \to \{0, 1\}$ and define the diagonal matrix $\mathcal{C} = (c_{ih})_{i,h \in J}$ by $c_{ii} = (-1)^{c(v_i)}$. Then

$$\mathcal{C}\mathcal{A}(\Gamma) = -\mathcal{A}(\Gamma)\mathcal{C}$$

and by putting $y = \mathcal{C}x$,

$$\rho x - \mathcal{A}(\Gamma)x = b \iff -\rho y - \mathcal{A}(\Gamma)y = -\mathcal{C}b.$$

This shows that $\rho - \mathcal{A}$ is a bounded automorphism of $\ell^\infty(\Gamma))$ with bounded inverse iff $-\rho - \mathcal{A}$ has the same properties. Thus the resolvent set is symmetric with respect to the origin and so does $\sigma(\mathcal{A}(\Gamma); \ell^\infty(\Gamma))$. Moreover, the multiplicity formula follows readily by $\mathcal{C}$ being an automorphism of $\ell^\infty(\Gamma)$. ■

In the finite or periodic case, index and bipartite character are closely related by the

Theorem 6.4 *([1],[2]) Suppose that Γ is a finite or periodic graph. Then Γ is bipartite iff*

$$-\mathrm{I}(\Gamma) \in \sigma(\Gamma).$$

For general infinite ULF graphs, it has been shown in [18] that, if $r_2(\Gamma)$ is an eigenvalue with eigenvector belonging to $\ell^2(\Gamma)$, then Γ is bipartite iff $-\mathrm{I}^-(\Gamma) = -r_2(\Gamma) \in \sigma_{\mathrm{pt}}(\mathcal{A}(\Gamma); \ell^2(\Gamma))$. But for the index $\mathrm{I}(\Gamma)$, this is no longer true as the following example shows.

Example 6.5 Take a 3–regular tree in which one node is replaced by a triangle such that the resulting graph Θ remains 3–regular, see Fig. 3 Then Θ is not bipartite, but $-\mathrm{I}(\Gamma)$ is an eigenvalue of $\mathcal{A}(\Gamma)$ in $\ell^\infty(\Gamma)$ of infinite multiplicity, as follows readily with the constructions in [4, Section 5] or [5, Section 8].

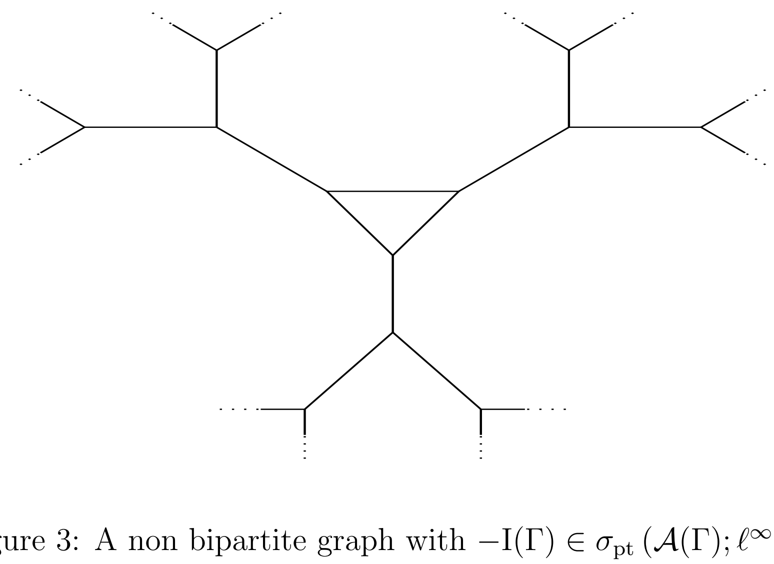

Figure 3: A non bipartite graph with $-\mathrm{I}(\Gamma) \in \sigma_{\mathrm{pt}}(\mathcal{A}(\Gamma); \ell^\infty(\Gamma))$

7 Maximal regular subgraphs

For finite or periodic graphs all conditions in Theorem 7.1 are equivalent. But in the ULF case the situation is more complicated. Let us first note the implications in this context that are plain.

Theorem 7.1 *The following implications hold*

$$(a) \Longleftarrow (c) \Longleftrightarrow (d) \Longleftrightarrow (e) \Longleftrightarrow (f) \Longrightarrow (b),$$

where

(a) $\gamma_{\max} = \mathrm{I}(\Gamma)$,

(b) $\gamma_{\min} = \mathrm{I}(\Gamma)$,

(c) Γ *is regular of valency* $r(\Gamma)$,

(d) $\mathbf{e}$ *is eigenvector of* Γ,

(e) $\mathcal{A}\mathbf{e} \geq \gamma_{\max}\,\mathbf{e}$,

(f) $\mathcal{A}\mathbf{e} \leq \gamma_{\min}\,\mathbf{e}$.

In general, $\gamma_{\max} = \mathrm{I}(\Gamma)$ does not imply that Γ is regular. Any easy example is furnished by the one-sided unbounded path Γ_0 in Example 4.17. The invalidity of $(b) \Longrightarrow (c)$ in the general ULF case is more complicated and will be shown by Example 7.4. Let us first note the following

Lemma 7.2 *If* Γ *is infinite and satisfies* $\gamma_{\min} = \mathrm{I}(\Gamma) =: d$*, then* Γ *does not contain any finite subgraph* Λ *such that* $\gamma_{\min}(\Lambda) = \gamma_{\min}(\Gamma)$*, and any* d*–regular subgraph is infinite. Moreover,* Γ *contains a maximal* d*–regular infinite subgraph* Δ.

Proof. Without restriction we can assume that Γ is connected. By hypothesis, $d = \mathrm{I}^-(\Gamma) = \mathrm{I}(\Gamma)$ and Γ contains d–regular subgraphs. If a connected finite subgraph $\Lambda \leq \Gamma$ would satisfy $\gamma_{\min}(\Lambda) = \gamma_{\min}(\Gamma)$, then there would exist a connected finite subgraph $\tilde{\Lambda} \leq \Gamma$ containing Λ properly and satisfying $d = r(\Lambda) < r(\tilde{\Lambda}) \leq r(\Gamma) = d$ by the strict monotonicity of the spectral radius in the class of indecomposable nonnegative finite matrices and by Lemma 4.5. Thus $\gamma_{\min}(\Lambda) < \gamma_{\min}(\Gamma)$. Especially, Γ cannot contain d–regular finite subgraphs, and d–regular subgraphs must be infinite. Since any two of those are vertex and edge disjoint, the union of all of them is maximal with the regularity property and defines Δ as desired. ∎

At least for $d = 2$, $\Delta = \Gamma$, as will be shown next.

Corollary 7.3 *If* Γ *is infinite with* $2 = \gamma_{\min} = \mathrm{I}(\Gamma)$*, then* Γ *is* 2*–regular.*

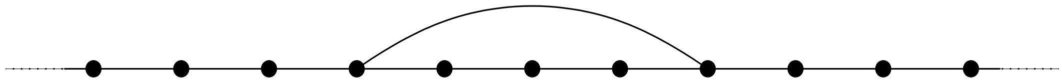

Figure 4: Ξ

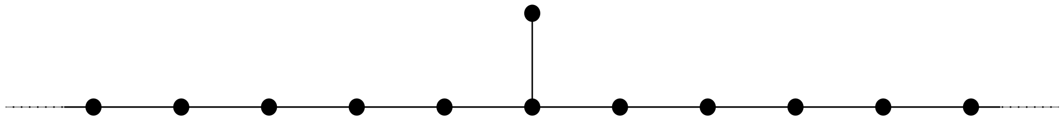

Figure 5: T

Proof. Suppose that Γ is not 2–regular. Then Γ contains either an infinite subgraph Ξ as depicted in Fig. 4 in the case $V(\Delta) = V(\Gamma)$, or an infinite subgraph T as depicted in Fig. 5 in the case $V(\Delta) \neq V(\Gamma)$. In the first case Corollary 4.19 yields $\mathrm{I}(\Xi) > \sqrt{2+\sqrt{5}}$, while in the second one $\sqrt{2+\sqrt{5}} \leq \mathrm{I}(T)$, as will be shown next. Thus, both cases are impossible, since by Lemma 4.5, $2 = \mathrm{I}(\Gamma)$ dominates the subgraph indices.

It remains to estimate $\mathrm{I}(T)$. Set $\lambda_0 = \sqrt{2+\sqrt{5}}$. Let v_0 denote the vertex of valency 3 and b the boundary vertex. Number the remaining vertices of T by $\mathbb{Z}$ correspondingly. Then, using the function $a_2(\lambda)$ from (20), $\mathbf{p} \in \ell^1(T)$ with

$$p_k = \begin{cases} 1 & \text{if } k = 0 \\ a_2(\lambda_0)^{|k|} & \text{if } 0 \neq k \in \mathbb{Z} \\ \frac{1}{\lambda_0} & \text{at } b \end{cases}$$

satisfies $\mathcal{A}(T)\mathbf{p} = \lambda_0 \mathbf{p}$. Thus, $\mathrm{I}(T) \geq \lambda_0$. ■

In fact, it can be shown that $\mathrm{I}(T) = \lambda_0$ using the functions in (20) and comparison arguments.

The arguments in both proofs show also that if the maximal d–regular infinite subgraph Δ is induced and such that adding an adjacent vertex v_1 outside Δ increases strictly the index of the graph $(V(\Delta) \cup \{v_1\}, K(\Delta) \cup \{v_0, v_1\}, \in)$, then Γ is regular. But, in general, $\gamma_{\min} = \mathrm{I}(\Gamma) =: d$ does not imply that Γ is d–regular, as will be illustrated by the

Example 7.4 Let T_0 denote the backwards genealogical tree as depicted in Fig. 6. Connect four copies of T_0 by identifying their four boundary vertices with one ramification node v_0, and get the tree T as displayed in Fig 7 that is not regular. All the vertices are of valency 3 except the vertex v_0 of valency 4.

Claim : $$\gamma_{\min}(T) = \mathrm{I}(T) = 3 < \gamma_{\max}(T) = 4.$$

Proof. First, observe that $\mathbf{p} = (p_k)_{k \in \mathbb{N}}$ with

$$p_k = \frac{1}{2} + \frac{1}{2^{k+1}}, \quad k \in \mathbb{N}$$

satisfies $0 \ll \mathbf{p} \in \ell^\infty(T)$ and $\mathcal{A}(T)\mathbf{p} = 3\mathbf{p}$.

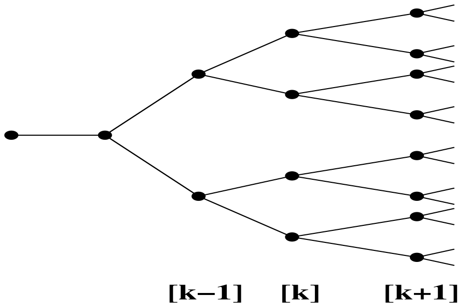

Figure 6: The tree T_0

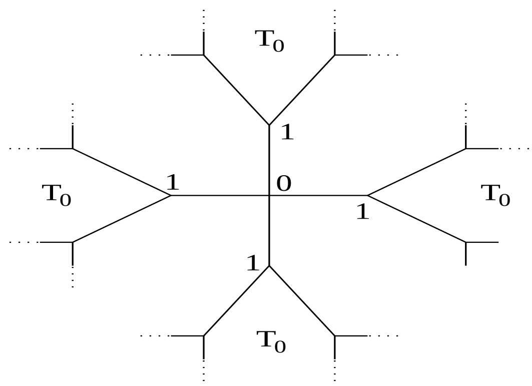

Figure 7: The non regular tree satisfying $\gamma_{\min} = \mathrm{I}(\Gamma)$ in 7.4

Next, suppose $(x_k)_{k\in\mathbb{N}} = x \in \ell^\infty(T)$ is an eigenvector with $\mathcal{A}(T)x = \lambda x$. If the sum of the four neighboring values of v_0 vanishes, then either the value x_0 at v_0 vanishes or $\lambda = 0$. In both cases, $\lambda \in \sigma_{\mathrm{p}}\left(\mathcal{A}(T_0); \ell^\infty(T_0)\right)$ and $|\lambda| \leq 3$. Thus we can assume that $x_0 = 1$, and, by symmetry, that in the k–th generation $[k]$ of all of the four T_0 the values of x coincide, still denoted by x_k. Then, for the reduced vector x, the initial condition and the recursion read

$$\lambda x_0 = \lambda = 4x_1, \quad 2x_{k+1} = \lambda x_k - x_{k-1} \text{ for } k \geq 1.$$

For $\lambda > 3$, the characteristic equation and the boundedness requirement lead to the solution

$$x_k = \left(\frac{\lambda - \sqrt{\lambda^2 - 8}}{4}\right)^k,$$

especially $x_1 = \frac{\lambda}{4} = \frac{\lambda - \sqrt{\lambda^2-8}}{4}$, which is absurd. Thus $\sigma_{\mathrm{pt}}\left(\mathcal{A}(T); \ell^\infty(T)\right) \cap (3, \infty) = \emptyset$, and the bipartite character of T guarantees that $\sigma_{\mathrm{pt}}\left(\mathcal{A}(T); \ell^\infty(T)\right) \cap (-\infty, -3) = \emptyset$. ∎

Question 7.5 *Let Γ be an infinite graph satisfying $\gamma_{\min} = \mathrm{I}(\Gamma) =: d$. Let Δ denote the maximal infinite d–regular subgraph of Γ from Lemma 7.2. Do the conditions $d \geq 3$ and $V(\Delta) = V(\Gamma)$ imply that Γ is d–regular?*

It is clear that a counterexample must have minimal valency at least 3. The following example seems to confirm the implication in Question 7.5.

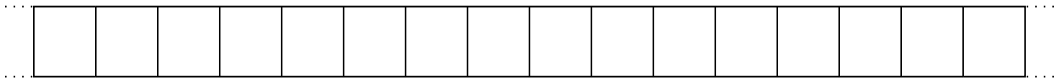

Figure 8: The periodic band Δ

Figure 9: The graph Π_k for $k = 3$

Example 7.6 Let Δ denote the 3–regular periodic band depicted in Fig. 8. Add periodically diagonal edges to each m–th square while leaving out $k = m - 1$ squares and get the graph Π_k depicted in Fig. 9. Then

$$\mathrm{I}(\Pi_k) \geq 3 + \frac{1}{k+1} > \mathrm{I}(\Delta) = 3 = \gamma_{\min}(\Pi_k) \quad \text{and} \quad V(\Delta) = V(\Pi_k).$$

The index remains strictly greater than 3 if we add only one edge as depicted in Fig. 10. The resulting graph Γ satisfies $V(\Delta) = V(\Gamma)$ and contains the finite subgraph with 30 vertices depicted in Fig. 11 that has an index bounded from below by 3.053[1]. Thus

$$\mathrm{I}(\Delta) = 3 = \gamma_{\min} < 3.053 < \mathrm{I}(\Gamma).$$

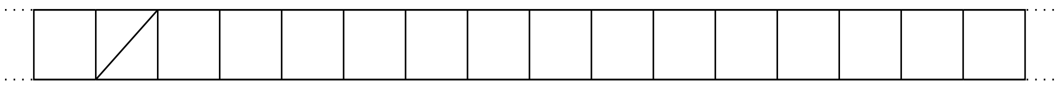

Figure 10: The graph Γ

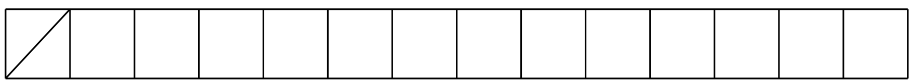

Figure 11: A finite subgraph with $\mathrm{I} > 3.053$

8 When do $\mathrm{I}^-(\Gamma)$ and $\mathrm{I}(\Gamma)$ coincide?

As pointed out already above, in general, $\mathrm{I}^-(\Gamma) < \mathrm{I}(\Gamma)$, but for finite or periodic graphs $\mathrm{I}^-(\Gamma)$ and $\mathrm{I}(\Gamma)$ coincide. For each connected graph in both of these classes, there exists a positive eigensequence belonging to $\mathrm{I}(\Gamma)$. Moreover, if Γ possesses a ℓ^1–Perron–vector $\mathbf{p}$, i.e. a positive sequence $\mathbf{p}$ belonging to $\ell^1(\Gamma)$ and satisfying $\mathcal{A}(\Gamma)\mathbf{p} = \mathrm{I}(\Gamma)\,\mathbf{p}$, then $\mathrm{I}^-(\Gamma) = r_2(\Gamma) = \mathrm{I}(\Gamma)$. This suggests the following generalizations.

[1]This value has been calculated with the aid of mathematica. The author is indebted to his colleague Shalom Eliahou for the help in calculating this value.

Lemma 8.1 *Each of the following conditions implies that* $\mathrm{I}^-(\Gamma) = \mathrm{I}(\Gamma)$*:*

(a) $\exists \mathbf{p} \in \ell^1(\Gamma) \,:\, \mathbf{p} \gg 0 \quad \& \quad \mathrm{I}(\Gamma)\,\mathbf{p} \leq \mathcal{A}(\Gamma)\mathbf{p}$.

(b) $\exists \mathbf{p} \in \ell^1(\Gamma) \,:\, \mathbf{p} \gg 0 \quad \& \quad \mathcal{A}(\Gamma)\mathbf{p} \leq \mathrm{I}^-(\Gamma)\,\mathbf{p}$.

(c) $\exists \mathbf{p} \in \ell^\infty(\Gamma) \,:\, \mathbf{p} \geq \mathbf{e} \quad \& \quad \mathcal{A}(\Gamma)\mathbf{p} \leq \mathrm{I}^-(\Gamma)\,\mathbf{p}$.

Proof. In order to show the first case, suppose $x \in \ell^\infty(\Gamma)$, $x > 0$ and $\mathcal{A}x \leq \lambda x$. Then $(p_i x_i)_{i\in J} \in \ell^1(\Gamma)$ and, by symmetry and boundedness of $\mathcal{A}$,

$$\mathrm{I}(\Gamma)(x, \mathbf{p})_{\ell^2} \leq (x, \mathcal{A}\mathbf{p})_{\ell^2} = (\mathcal{A}x, \mathbf{p})_{\ell^2} \leq \lambda(x, \mathbf{p})_{\ell^2} \text{ and } \mathrm{I}(\Gamma) \leq \lambda. \tag{21}$$

The case (b) is shown analogously. Under (c) we conclude for $x \in \ell^\infty(V)$, $x > 0$ and $\mathcal{A}\,x \geq \lambda x$, that there exists $\alpha > 0$ such that $\mathbf{p} \geq \alpha x$ and $p_{i_0} = \alpha x_{i_0}$ for some $i_0 \in J$ or $\lim_{\nu\to\infty}(p_{i_\nu} - \alpha x_{i_\nu}) = 0$ for some subsequence $(i_\nu)_{\nu\in\mathbb{N}}$. In both cases $\mathrm{I}^-(\Gamma)\,\mathbf{p} \geq \mathcal{A}\,\mathbf{p} \geq \mathcal{A}\,\alpha x \geq \lambda\alpha x$ leads to $\mathrm{I}^-(\Gamma) \geq \lambda$. ∎

In the next result we construct a ℓ^1–Perron–vector in a special class of infinite graphs.

Theorem 8.2 *If* Γ *is an infinite connected graph with finitely many essential ramification nodes, then*

$$\mathrm{I}^-(\Gamma) = \mathrm{I}(\Gamma).$$

More precisely: By hypothesis, Γ *consists in a finite connected graph* S *with* n *vertices, among them* t *distinguished vertices* $v_1, \ldots, v_t$*, and in* t *one–sided disjoint unbounded paths* $\Pi_1, \ldots, \Pi_t$*, each isomorphic to* Γ_0*. Each of the paths* Π_i *is linked with* S *at the node* v_i *such that* $\gamma(v_i, \Gamma) \geq 2$ *and* $\gamma(v_i, \Pi_i) = 1$*. All the other vertices of each* Π_i *have valency* 2 *in* Γ*. Then*

$$\mathrm{I}(\Gamma) \geq \max\{2, r(S)\},$$

and if $\Gamma \neq \Gamma_0$*,* $\mathrm{I}(\Gamma)$ *is attained and is equal to the maximal zero* ρ *of the equation*

$$\delta(\rho) := \det\left(\mathcal{A}(S) + \left(\frac{\rho}{2} - \sqrt{\frac{\rho^2}{4} - 1}\right)\begin{pmatrix} \mathbf{I}_t & 0 \\ 0 & 0 \end{pmatrix} - \rho\,\mathbf{I}_n\right) = 0, \tag{22}$$

especially

$$\mathrm{I}(\Gamma) \leq r(S) + \frac{1}{r(S)} \tag{23}$$

with equality iff $\Gamma = \Gamma_0$ *or* $\Gamma = \Gamma_1$*. Moreover,* $\mathrm{I}(\Gamma) = 2$ *iff* $\Gamma = \Gamma_0$*,* $\Gamma = \Gamma_1$ *or* S *is a path of length* 2 *and* $t = 1$*.*

Proof. By Lemma 4.5, $\mathrm{I}(\Gamma) \geq 2$. On each Π_i, each positive vector $\mathbf{p}$ satisfying $\mathcal{A}\mathbf{p} \leq \mu\mathbf{p}$ decays geometrically and leads to a sequence belonging to $\ell^1(\Pi_i)$. Then the conclusion (21) remains valid for any $\mu > 0$ with $\mu x \leq \mathcal{A}x$ instead of I. Thus, $\mathrm{I}^-(\Gamma) = \mathrm{I}(\Gamma)$.

Next suppose that $\Gamma \neq \Gamma_0$ and $r(S) \geq 2$, and choose $\rho > \mathrm{I}(S)$. In fact, we shall determine a ℓ^1–Perron–vector $\mathbf{p}$ belonging to $\mathrm{I}(\Gamma)$. Set $\mathbf{A} = \mathcal{A}(S)$ and denote the vertices of each Π_i

in the canonical order, i.e. by $v_k^{(i)}$ with $k \in \mathbb{N}$ endowed with the adjacency $e_{ih} = \delta_{1,|i-h|}$. For each vector $b = \sum_{i=1}^{t} b_i\, e_0^{(i)}$ with $b_1, \ldots, b_t > 0$ we obtain on S

$$x := (\rho\, \mathbf{I}_n - \mathbf{A})^{-1}\, b > 0 \quad \text{and} \quad \rho x = \mathbf{A}x + b.$$

On each Π_i any eigensequence belonging to $\ell^\infty(\Gamma)$ satisfies necessarily

$$x_k^{(i)} = x_0^{(i)}\, \alpha_2(\rho)^k \quad \text{for} \quad k \geq 1 \tag{24}$$

where we have used the function α_2 from (20). Thus the vector x defined on S can be extended to an eigenvector on the whole graph if on each Π_i (24) is satisfied and if

$$\alpha_2(\rho)\, x_0^{(i)} = b_i \quad \text{for} \quad t = 1, \ldots, t.$$

With this choice we are lead to the equation

$$\rho x = \mathbf{A}x + \alpha_2(\rho) \begin{pmatrix} \mathbf{I}_t & 0 \\ 0 & 0 \end{pmatrix} x$$

for x defined on S. As $x > 0$, the characteristic equation (22) is shown, since δ always has zeros in $[2, \infty)$ for $r(S) \geq 2$: The function

$$f(\rho) = r\left(\mathbf{A} + \alpha_2(\rho) \begin{pmatrix} \mathbf{I}_t & 0 \\ 0 & 0 \end{pmatrix}\right)$$

decays strictly on $[2, \infty)$ to $r(\mathbf{A})$. But at $\rho = 2$, f takes a value that is strictly greater than $r(\mathbf{A})$, in particular, $f(2) > 2$. Thus f possesses a unique fixed point $\rho_0 \in (2, \infty)$ that is the principal eigenvalue of the matrix $\mathbf{A} + \alpha_2(\rho_0) \begin{pmatrix} \mathbf{I}_t & 0 \\ 0 & 0 \end{pmatrix}$ and, thereby, a zero of δ. We note in passing, that this argument cannot hold for $\Gamma = \Gamma_0$, since here $f(\rho) \leq \frac{1+\sqrt{5}}{2}$ uniformly and $\delta > 0$ in $[2, \infty)$. Moreover, since

$$r\left(\mathcal{A}(S) + \alpha_2(\rho) \begin{pmatrix} \mathbf{I}_t & 0 \\ 0 & 0 \end{pmatrix}\right) \leq r\left(\mathcal{A}(S) + \alpha_2(\rho)\mathbf{I}_n\right),$$

and since the latter spectral radius is determined by the equation $\mathrm{I}(S) = \alpha_1(\rho)$ and amounts to the r.h.s. in (23), Inequality (23) is shown.

If $\Gamma = \Gamma_0$, then $\mathrm{I}(\Gamma) = 2$ without being attained according to Example 4.17. Thus, 2 cannot be a zero of δ in this case. If $\Gamma = \Gamma_1$, then $\mathrm{I}(\Gamma) = 2$ with eigenvector $\mathbf{e}$ according to Example 4.18, but 2 is also the maximal zero of δ, where S is the single edge graph. If $t \geq 2$ and $\Gamma \neq \Gamma_1$, then Γ contains a subgraph T as in the proof of Corollary 7.3 and as in Fig. 5 with $\mathrm{I}(T) > 2$, which enforces $\mathrm{I}(\Gamma) > 2$ by Lemma 4.5. If $\mathrm{I}(\Gamma) = 2$, then $t = 1$, and S is a path, since a tree with at least 2 essential ramification nodes contains a finite subgraph of index 2. But then S can have at most 3 vertices, since the graph Ψ depicted in Fig. 12 possesses a positive eigenvector $x \in \ell^1(\Psi)$ associated to $\rho = 2 + \frac{1}{30}$ defined by $x_{-3} = 1$, $x_{-2} = 2\; x_{-1} = \frac{3}{2}$, $x_0 = 3$ and $x_k = 3\left(\frac{5}{6}\right)^k$ for $k \geq 1$. If S is a path with 3 vertices, and if Γ has one vertex of valency 3, then $\mathrm{I}(\Gamma) = 2$ with associated positive eigenvector $\mathbf{p} \in \ell^\infty(\Gamma)$ coinciding with $\mathbf{e}$ on Π_1 and being $\frac{1}{2}$ at the two boundary vertices. Moreover, in this case, 2 is the only positive zero of δ, which accomplishes the proof. ∎

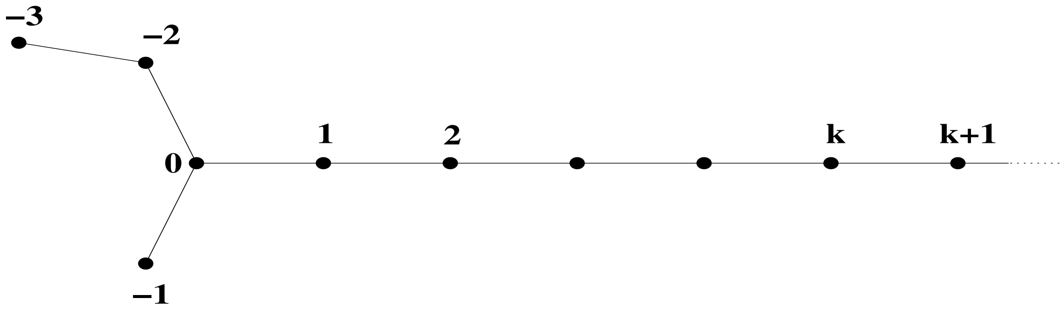

Figure 12: The graph Ψ

Note that it is no restriction to assume that a vertex v_i is incident only with one Π_i, since otherwise we can enlarge S in order to achieve this situation. Moreover, the estimate (23) is optimal: If each vertex of S is adjacent to an infinite path, i.e. $t = n$, then $\rho = \mathrm{I}(\Gamma)$ is governed by the equation $r(S) = \alpha_1(\rho)$ that leads to

$$\rho = r(S) + \frac{1}{r(S)}.$$

Example 8.3 Let A_t denote the infinite star graph with essential ramification node v_0 of valency $t \geq 3$ and with one–sided infinite paths $\Pi_1, \ldots, \Pi_t$ with v_0 as boundary vertex. Then Theorem 8.2 permits to conclude that

$$\mathrm{I}^-(A_t) = \mathrm{I}(A_t) = \frac{t}{\sqrt{t-1}} =: r$$

by calculating the zeros of $\delta(\rho) = (\alpha_2(\rho) - \rho)^{t-1}(\rho^2 - \rho\alpha_2(\rho) - t)$ according to (22). Note further that A_t admits the eigenvector $\mathbf{p} \in \ell^1(\Gamma)$ associated to r and satisfying on each Π_i

$$p_k = p_k^i = \left(\frac{r}{2} - \sqrt{\frac{r^2}{4} - 1}\right)^k \quad \text{for} \quad k \in \mathbb{N}.$$

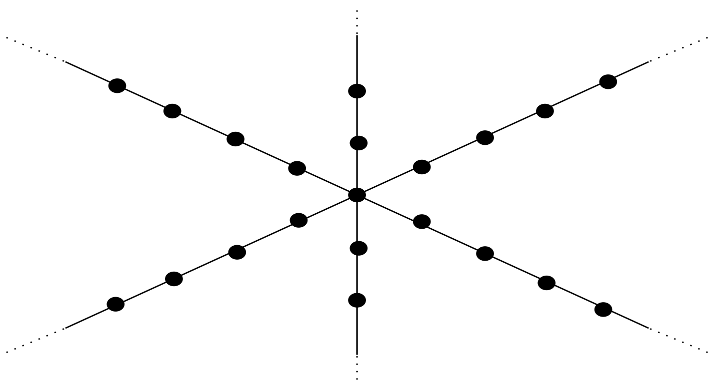

Figure 13: The infinite star graph

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