CONVERGENCE OF A TWO-GRID ALGORITHM FOR THE CONTROL OF THE WAVE EQUATION

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ABSTRACT. We analyze the problem of boundary observability of the finite-difference space semi-discretizations of the 2-d wave equation in the square. We prove the uniform (with respect to the mesh size) boundary observability for the solutions obtained by the two-grid preconditioner introduced by Glowinski [6]. Our method uses previously known uniform observability inequalities for low frequency solutions and a dyadic spectral time decomposition. As a consequence we prove the convergence of the two-grid algorithm for computing the boundary controls for the wave equation. The method can be applied in any space dimension, for more general domains and other discretization schemes.

1. INTRODUCTION

Let us consider the wave equation

$$\begin{cases} y'' - \Delta y = 0 & \text{in } \Omega \times (0,T), \\ y = v \mathbf{1}_{\Gamma_0}(x) & \text{on } \Gamma \times (0,T), \\ y(0,x) = y^0(x),\ y'(0,x) = y^1(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is the unit square $\Omega = (0,1) \times (0,1)$ of $\mathbb{R}^2$ and its boundary Γ is decomposed as $\Gamma = \Gamma_0 \cup \Gamma_1$ with

$$\begin{cases} \Gamma_0 = \{(x_1,1) :\ x_1 \in (0,1)\} \cup \{(1,x_2) :\ x_2 \in (0,1)\}, \\ \Gamma_1 = \{(x_1,0) :\ x_1 \in (0,1)\} \cup \{(0,x_2) :\ x_2 \in (0,1)\}. \end{cases}$$

In equation (1.1), $y = y(t,x)$ is the state, $'$ is the time derivative and v is a control function which acts on the system through the boundary Γ_0. Classical results of existence and uniqueness for solutions of nonhomogeneous evolution equations (see for instance [16]) show that for any $v \in L^2((0,T) \times \Gamma_0)$ and $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ equation (1.1) has a unique weak solution $(y, y') \in C([0,T], L^2(\Omega) \times H^{-1}(\Omega))$.

Concerning the controllability of the above system the following exact controllability result is well known (see Lions [15]): *Given $T > 2\sqrt{2}$ and $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ there exists a control function $v \in L^2((0,T) \times \Gamma_0)$ such that the solution $y = y(t,x)$ of (1.1) satisfies*

$$y(T, \cdot) = y'(T, \cdot) = 0. \tag{1.2}$$

In fact, given $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ a control function v of minimal $L^2((0,T) \times \Gamma_0)$-norm may be obtained by the so-called Hilbert Uniqueness Method (HUM, see [15]). It reduces the exact controllability problem to an equivalent *observability* property for the adjoint problem:

$$\begin{cases} u'' - \Delta u = 0 & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \Gamma \times (0,T), \\ u(0,x) = u^0(x),\ u_t(0,x) = u^1(x) & \text{in } \Omega. \end{cases} \tag{1.3}$$

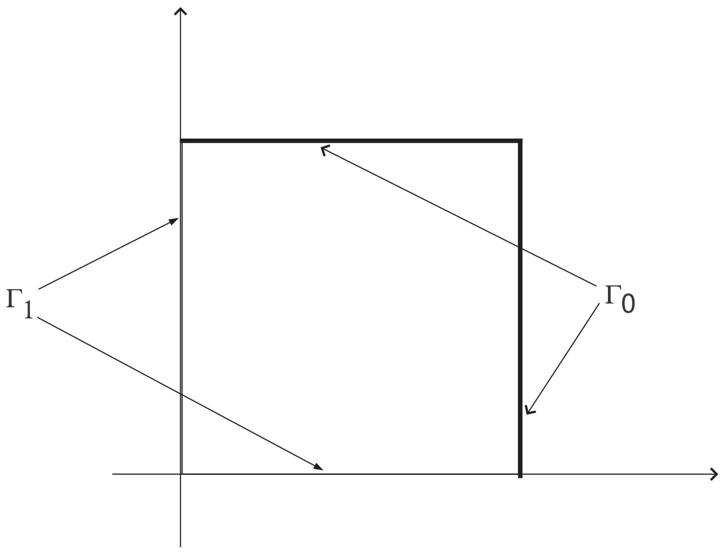

FIGURE 1. Boundary of the domain Ω under consideration. Γ_0 is the subset where the control acts while Γ_1 is the one that remains uncontrolled.

More precisely, the equivalent observability property is the following: *For any $T > 2\sqrt{2}$ there exists $C(T) > 0$ such that*

$$\|(u^0, u^1)\|^2_{H^1_0(\Omega)\times L^2(\Omega)} \le C(T) \int_0^T \int_{\Gamma_0} \left|\frac{\partial u}{\partial n}\right|^2 d\sigma dt \tag{1.4}$$

for any $(u^0, u^1) \in H^1_0(\Omega) \times L^2(\Omega)$ where u is the solution of (1.3) *with initial data (u^0, u^1).* Note that, rigorously speaking, the adjoint system should take the initial data at $t = T$. But, the wave equation being time reversible, this is irrelevant in what concerns the observability inequality (1.4).

The lower bound $2\sqrt{2}$ on the observability time T is due to the fact that, in this model, the velocity of propagation of waves is one and then, in order for (1.4) to be true, any perturbation of the initial data needs some time to reach the observation zone. The minimal time for this geometric configuration, $2\sqrt{2}$, is twice the diameter of the domain, which is the largest travel time along the diagonal that needs a wave to get into the control region after bouncing on the opposite vertex.

The main objective of this paper is to prove the convergence of a numerical approximation algorithm for computing the control function v of equation (1.1). This issue has been the object of intensive research in the past years. It is by now well known that the control of a stable numerical approximation scheme for (1.1) may diverge when the mesh-size tends to zero. This is due to the unstabilizing effect of the high frequency numerical solutions [29]. Several techniques have been introduced as possible remedies to the instabilities produced by the high frequency spurious oscillations: Tychonoff regularization [6], filtering of the high frequencies [11], [28], [29], mixed finite elements [7], [4], [5] and the two-grid algorithm [22], [17].

Possibly the one which is more systematic and convenient for practical implementations is the two-grid algorithm proposed by Glowinski in [6]. The method consists in relaxing the controllability requirement on numerical solutions by considering only its projection over a coarser grid. In what concerns the observability inequality (1.4) the method consists in analyzing the discrete or semidiscrete version of (1.4) for the solutions of the numerical approximation scheme, but only for initial data obtained through a two-grid preconditioning. To be more precise, the two-grid method consists in using a coarse and a fine grid, and interpolating the initial data for the numerical approximation of (1.3) from the coarse G_c grid to the fine one G_f. This method attenuates the short wave-length components of the initial data, which are responsable for the spurious high frequency oscillations.

The main goal of the paper is to rigorously prove the convergence of this algorithm in the context of the semidiscrete finite-difference approximation scheme for the wave equation in the square. The key ingredient of the proof is the obtention of an inequality similar to (1.4) at the semidiscrete level, independent of the mesh-size, for the two-grid data mentioned above.

Through the paper we deal with the two-dimensional case but all the arguments we present here work in any space dimension and can be also applied to other numerical schemes both semi-discrete and fully discrete ones.

Our main contribution is to develop a dyadic decomposition argument that allows reducing the problem to considering classes of solutions in which the high frequency components have been filtered, a situation that was already dealt with in the literature.

To fix the ideas let us consider the finite-difference semi-discretization of (1.3). Given $N \in \mathbb{N}$ we set $h = 1/(N+1)$, $\Omega_h = \Omega \cap h\mathbb{Z}^2$ and $\Gamma_h = \Gamma \cap h\mathbb{Z}^2$. In the same manner we define Γ_{0h} and Γ_{1h}. The finite-difference semi-discretization of system (1.1) is as follows:

$$(1.5) \qquad \begin{cases} y_h'' - \Delta_h y_h = 0 & \text{in } \Omega_h \times (0,T), \\ y_h = v_h \mathbf{1}_{\Gamma_{0h}} & \text{on } \Gamma_h \times (0,T), \\ y_h(0) = y_h^0,\ y_h'(0) = y_h^1 & \text{in } \Omega_h, \end{cases}$$

where the initial data (y_h^0, y_h^1) are approximations of (y^0, y^1) and Δ_h is the five-point approximation of the laplacian:

$$(\Delta_h u)_{j,k} = \frac{u_{j-1,k} - 2u_{j,k} + u_{j+1,k}}{h^2} + \frac{u_{j,k-1} - 2u_{j,k} + u_{j,k+1}}{h^2}.$$

For the homogeneous wave equation (1.3) we consider the following numerical scheme:

$$(1.6) \qquad \begin{cases} u_h'' - \Delta_h u_h = 0 & \text{in } \Omega_h \times [0,T], \\ u_h = 0, & \text{on } \Gamma_h \times (0,T) \\ u_h(0) = u_h^0,\ u_h'(0) = u_h^1 & \text{in } \Omega_h. \end{cases}$$

To simplify the presentation, whenever it is not strictly necessary, we will avoid the subscript h in the notation of the solution u_h.

Let us now introduce the *discrete energy* associated with system (1.6):

$$(1.7) \qquad \mathcal{E}_h(t) = \frac{h^2}{2} \sum_{j,k=0}^{N} \left[|u_{j,k}'(t)|^2 + \left| \frac{u_{j+1,k}(t) - u_{j,k}(t)}{h} \right|^2 + \left| \frac{u_{j,k+1}(t) - u_{j,k}(t)}{h} \right|^2 \right].$$

It is easy to see that the energy remains constant in time, i.e.

$$(1.8) \qquad \mathcal{E}_h(t) = \mathcal{E}_h(0),\ \forall\ 0 < t < T$$

for every solution of (1.6).

Following [1], the discrete version of the energy observed on the boundary Γ_0 is given by:

$$\int_0^T \int_{\Gamma_0} \left|\frac{\partial u}{\partial n}\right|^2 d\sigma dt \simeq \int_0^T \left[h\sum_{j=1}^N \left|\frac{u_{j,N}}{h}\right|^2 + h\sum_{k=1}^N \left|\frac{u_{N,k}}{h}\right|^2 \right] dt. \tag{1.9}$$

In the following for any $j = 1, ..., N$ and $k = 1, ..., N$, we denote

$$(\partial_n^h u)_{j,N+1} := -\frac{u_{j,N}}{h},\ (\partial_n^h u)_{N+1,k} := -\frac{u_{N,k}}{h},\ (\partial_n^h u)_{j,0} := -\frac{u_{j,1}}{h},\ (\partial_n^h u)_{0,k} := -\frac{u_{1,k}}{h}.$$

Also, in order to simplify the presentation, we shall use integrals to denote discrete sums, i.e.

$$\int_{\Omega_h} u d\Omega_h = h^2 \sum_{\mathbf{j}h\in\Omega_h} u_{\mathbf{j}},\ \int_{\Gamma_h} u d\Gamma_h := h \sum_{\mathbf{j}h\in\Gamma_h} u_{\mathbf{j}}.$$

and

$$\int_{\Gamma_{0h}} |\partial_n^h u|^2 d\Gamma_{0h} := h\sum_{j=1}^N \left|\frac{u_{j,N}}{h}\right|^2 + h\sum_{k=1}^N \left|\frac{u_{N,k}}{h}\right|^2. \tag{1.10}$$

The discrete version of (1.4) is then an inequality of the form

$$\mathcal{E}_h(0) \le C_h(T) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h u|^2 d\Gamma_{0h} dt. \tag{1.11}$$

System (1.6) being finite dimensional, it is easy to see that the so-called Kalman rank condition is satisfied and, consequently, for all $T > 0$ and $h > 0$ there exists a constant $C_h(T)$ such that inequality (1.11) holds for all the solutions of equation (1.3). But, as it was proved in [28], for all $T > 0$ the best constant $C_h(T)$ necessarily blows-up as $h \to 0$. The blow-up of the observability constant is due to two main reasons. First, the discrete version of the normal derivative in (1.9) is too weak to capture the energy of the high frequency monochromatic waves. This fact could be compensated by making stronger boundary measurements, but this would not suffice due to the fact that the numerical scheme develops high frequency wave packets whose group velocity is of the order of h. These high frequency solutions are such that the energy concentrated on the boundary Γ_{0h} is asymptotically smaller than the total one. This phenomenon was already observed by R. Glowinski et al. in [6], [8] and [9]. Using a wave-packet construction is can be shown that the observability constant $C_h(T)$ blows-up exponentially as $h \to 0$. We refer to Micu [19] for a detailed proof in the 1-d case based on explicit estimates of biorthogonal families to the complex exponentials entering in the Fourier development of solutions.

As proved in [28], inequality (1.11) holds uniformly in a class of *low frequency* solutions (initial data where the spurious high frequency modes have been filtered) provided the time T is large enough depending on the frequencies under consideration. In Section 2 we will make this concept precise and recall this result. The main result of this paper, stated in Section 2, guarantees that, once (1.11) holds uniformly for a class of *low frequency* solutions, it also holds for all solutions in an extended class of initial data whose energy is controlled by their projection on the previous *low frequency* components. As we shall see, the class of initial data for (1.6) obtained through the two-grid approach fulfills these requirements. Accordingly, we shall deduce that for $T > 0$ large enough inequality (1.11) holds uniformly (i.e. with a constant $C_h(T)$ which is independent of h) in this class of two-grid data. As a consequence of this, we will conclude that system (1.5) is uniformly controllable in the sense that the projections of the states onto the coarse grid are controllable with controls that remain bounded as $h \to 0$. Furthermore these controls converge to those of (1.1) as the mesh size $h \to 0$.

In the one-dimensional case, the two-grid method was analyzed by Negreanu and Zuazua in [22] with a discrete multiplier approach. The authors considered two meshes with quotient $1/2$ and proved the convergence of the method as $h \to 0$ for $T > 4$. The same two-grid method has been considered in a more recent work by Loreti and Mehrenberer [17], where the authors use a fine extension of Ingham's inequality to obtain a sharp time of uniform observability, namely $T > 2\sqrt{2}$. However, as far as we know, there is no proof of the uniform observability in the two-dimensional case. The main goal of this paper is to give the first complete proof of convergence of the method in the multidimensional setting.

In contrast with the strategy adopted in [22] we choose two grids with the quotient of their sizes to be $1/3$. This is done for merely technical reasons, that we shall describe in the last section, and one may expect the same result to hold when the ratio of the grids is $1/2$. The problem is open in the multidimensional case for the mesh-ratio $1/2$.

Our method, which consists in using the already well known observability inequality for a class of *low frequency* data and a dyadic time spectral decomposition of the solutions, works in any space dimension and for other discretization schemes.

The two-grid method has also been used in other contexts to filter the unwanted effect of high-frequency numerical solutions. For instance, in [10], it was employed with two meshes of mesh-ratio $1/4$ when proving dispersive estimates for conservative semi-discrete approximation schemes of the Schrödinger equation. There, using the mesh-ratio $1/4$ was necessary. Here, as mentioned above, the result might well hold for $1/2$ as in 1-d but here, for technical reasons, we prove it only for $1/3$. Our techniques allow also showing the convergence of the method for meshes with mutual ratio $1/p$ for any $p \geq 3$. We present here the case $1/3$ since it is the one in which the amount of filtering is minimal.

Indeed, when diminishing the ratio between grids, the attenuation that the two-grid algorithm introduce on the high frequency component of the solutions is enhanced and the energy is then concentrated on lower frequencies for which the velocity of propagation becomes closer to that of the continuous wave equation. It is therefore natural to expect that proving the uniform observability will be easier for smaller grid ratios. When doing that one may also expect that the time of control will get closer to the optimal one of the continuous wave equation. Both facts will be explicitly established through our analysis.

The rest of the paper is organized as follows. In section 2 we introduce the spaces $K_h^M(\gamma)$ consisting in all the discrete functions (φ, ψ) such that their norm is controlled by the one of its projection on a suitable low frequency component and state the core result of this paper: the uniform observability inequality for data that belong to these spaces. In Section 3 we will introduce the space V^h of functions defined on the fine grid G^h as linear interpolation of functions defined on the coarse one G^{3h}. We prove that (1.11) holds uniformly for all $T > 4\sqrt{2}$, in the class of two-grid initial data $V^h \times V^h$. Section 4 is devoted to the proof of the main result of this paper, namely Theorem 2.1, using the dyadic decomposition argument. The last sections are devoted to prove the convergence of controls. More precisely, in Section 5 we construct semi-discrete control functions v_h for (1.5) that approximate the control function v in (1.1). Section 6 contains convergence results for the uncontrolled problem that will be used in Section 7 to prove the convergence in $L^2((0,T) \times \Gamma_0)$ of functions v_h, constructed before, towards the continuous one v. In the last section we comment on the main result of the paper, about how it can be used or improved, what are its limitations and we also formulate a number of open problems. The paper also has two appendices containing some technical lemmas and the Fourier analysis of the discrete functions obtained by a two-grid algorithm.

2. The observability problem

To make our statements precise, let us consider the eigenvalue problem associated to (1.6):

$$\begin{cases} -\Delta_h \varphi_h = \lambda \varphi_h & \text{in}\,\Omega_h, \\ \varphi_h = 0 & \text{on}\,\Gamma_h. \end{cases} \tag{2.1}$$

Denoting $\Lambda_N := [1,N]^2 \cap \mathbb{Z}^2$, the eigenvalues and eigenvectors of system (2.1) are

$$\lambda_{\mathbf{j}}(h) = \frac{4}{h^2}\left[\sin^2\left(\frac{j_1 \pi h}{2}\right) + \sin^2\left(\frac{j_2 \pi h}{2}\right)\right],\ \mathbf{j} = (j_1, j_2) \in \Lambda_N \tag{2.2}$$

and

$$\varphi_h^{\mathbf{j}}(\mathbf{k}) = 2\sin(j_1 k_1 \pi h)\sin(j_2 k_2 \pi h),\ \mathbf{k} = (k_1,k_2) \in [0, N+1]^2 \cap \mathbb{Z}^2,\ \mathbf{j} = (j_1, j_2) \in \Lambda_N.$$

The vectors $\{\varphi_h^{\mathbf{j}}\}_{\mathbf{j}\in\Lambda_N}$ form a basis for the discrete functions ϕ_h defined on $G^h = \Omega_h \cup \Gamma_h$ and vanishing on its boundary, allowing us to write, for any discrete function ϕ_h,

$$\phi_h = \sum_{\mathbf{j}\in\Lambda_N} \widehat{\phi}_h(\mathbf{j}) \varphi_h^{\mathbf{j}},$$

where $\widehat{\phi}(\mathbf{j}) = (\phi_h, \varphi_h^{\mathbf{j}})_h$, $(\cdot,\cdot)_h$ being the inner product in $l^2(\Omega_h)$:

$$(u,v)_h = h^2 \sum_{\mathbf{k}h\in\Omega_h} u(\mathbf{k})v(\mathbf{k}).$$

In view of this representation, for every $s \in \mathbb{R}$, we will denote by $\mathcal{H}_h^s(\Omega_h)$ the space of all functions defined on the grid G^h, endowed with the norm

$$\|\phi_h\|_{s,h} = \Big(\sum_{\mathbf{j}\in\Lambda_N} \lambda_{\mathbf{j}}^{2s}(h) |\widehat{\phi}_h(\mathbf{j})|^2\Big)^{1/2}.$$

Let us consider $\{\widehat{u}_h^0(\mathbf{j})\}_{\mathbf{j}\in\Lambda_N}$ and $\{\widehat{u}_h^1(\mathbf{j})\}_{\mathbf{j}\in\Lambda_N}$ the coefficients of the initial data (u_h^0, u_h^1) of system (1.6) in the basis $\{\varphi_h^{\mathbf{j}}\}_{\mathbf{j}\in\Lambda_N}$. Then the solution u_h is given by

$$u_h(t) = \frac{1}{2}\sum_{\mathbf{j}\in\Lambda_N}\left[e^{it\omega_{\mathbf{j}}(h)}\widehat{u}_{\mathbf{j}+}^h + e^{-it\omega_{\mathbf{j}}(h)}\widehat{u}_{\mathbf{j}-}^h\right]\varphi_h^{\mathbf{j}}, \tag{2.3}$$

where $\omega_{\mathbf{j}}(h) = \sqrt{\lambda_{\mathbf{j}}(h)}$ and

$$\widehat{u}_{\mathbf{j}\pm}^h = \widehat{u}_h^0(\mathbf{j}) \pm \frac{\widehat{u}_h^1(\mathbf{j})}{i\sqrt{\lambda_{\mathbf{j}}(h)}}.$$

Using the above notations, the energy of the system introduced in (1.7) is conserved in time and satisfies

$$\mathcal{E}_h(u_h) = \sum_{\mathbf{j}\in\Lambda_N} \omega_{\mathbf{j}}^2(h)(|\widehat{u}_{\mathbf{j}+}^h|^2 + |\widehat{u}_{\mathbf{j}-}^h|^2).$$

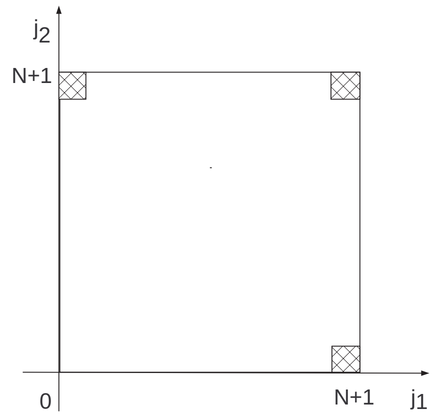

FIGURE 2. The three dashed corners contain solutions whose group velocity is of order of h

Let us introduce the class of filtered solutions of (1.6) in which the high frequencies have been truncated or filtered. More precisely, for any $0 < \gamma \leq 2\sqrt{2}$ we set

$$I_h(\gamma) = \Big\{ u_h : u_h = \sum_{\omega_{\mathbf{j}}(h) \leq \gamma/h} \widehat{u}_{\mathbf{j}}^h \varphi_h^{\mathbf{j}} \text{ with } \widehat{u}_{\mathbf{j}}^h \in \mathbb{C} \Big\}. \tag{2.4}$$

The class $I_h(\gamma)$ has been intensively used for control problems ([12], [2], [13]) and the dispersive properties of PDE's ([3]). For any solution u_h of equation (1.6) we denote by $\Pi_h^\gamma u_h$ its projection on the space $I_h(\gamma)$, which consists simply in restricting the Fourier expansion (2.3) to the class of indices entering in $I_h(\gamma)$ for which $\omega_{\mathbf{j}}(h) \leq \gamma/h$.

The uniform observability in the class $I_h(\gamma)$ has been analyzed in [28] by the multiplier technique. In that article it is shown that for any $0 < \gamma < 2$ and

$$T > T(\gamma) = \frac{8\sqrt{2}}{4 - \gamma^2} \tag{2.5}$$

there exists $C(\gamma, T) > 0$ such that

$$\mathcal{E}_h(u_h) \leq C(\gamma, T) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h u_h(t)|^2 d\Gamma_{0h} dt \tag{2.6}$$

holds for every solution u of (1.6) in the class $I_h(\gamma)$ and $h > 0$. This observability result will be systematically used along the paper. The choice of the filtering parameter $\gamma < 2$ in [28] is sharp. More precisely, for $\gamma = 2$ and any $T > 0$ it was shown that there is no constant $C(T)$ (see [28]) such that (2.6) holds for all solutions u_h of (1.6), uniformly on h:

$$\sup_{u_h \in I_h(2)} \frac{\mathcal{E}_h(u_h)}{\displaystyle\int_0^T \int_{\Gamma_{0h}} |\partial_n^h u_h(t)|^2 d\Gamma_{0h} dt} \to \infty,\ h \to 0.$$

This is a consequence of the presence of solutions which have group velocity of order h and spend a time of order $1/h$ to reach the boundary. In Figure 2 we can see the areas of the spectrum in which these solutions with group velocity of order h can occur and in Figure 3 we illustrate how, some of them, enter in the class of filtered solutions $I_h(\gamma)$ for $\gamma = 2$.

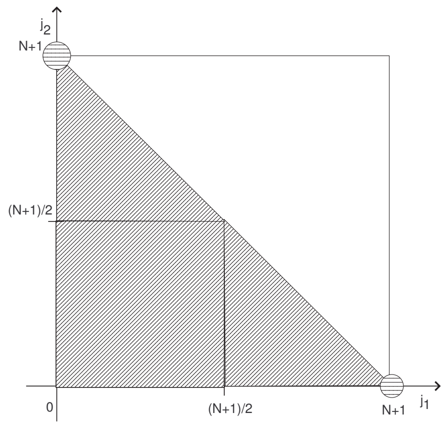

FIGURE 3. The dashed area below the diagonal of the square represents the frequencies involved in $I_h(2)$. The two circles on the corners correspond to frequencies with group velocity of order h that enter in the class $I_h(2)$ but that are exluded for filtering parameter $\gamma < 2$.

The classes $I_h(\gamma)$ make sense for all $0 < \gamma \le 2\sqrt{2}$ in view of the obvious spectral bound $\lambda_{\mathbf{j}}(h) \le 8/h^2$ that immediately holds as a consequence of the explicit expression (2.2). But, obviously, the observability estimate (2.6) fails to be uniform in $I_h(\gamma)$ for all $2 \le \gamma \le 2\sqrt{2}$ because it actually fails for $\gamma = 2$.

The main goal of this paper is to extend this uniform observability inequality to a more general class of initial data obtained through the two-grid filtering strategy. In this class the high frequency components do not vanish but a careful analysis shows that their energy is dominated by the low frequency ones.

To be more precise, let Π_h^γ be the orthogonal projection of discrete functions over the subspace $I_h(\gamma)$. Let us now fix $M > 0$. For any $0 < \gamma \le 2\sqrt{2}$ we define $K_h^M(\gamma)$ as the subspace of $\mathcal{H}_h^1(\Omega_h) \times \mathcal{H}_h^0(\Omega_h)$ consisting in all the discrete functions (φ, ψ) such that their square norm is controlled by the one of its projection on $I_h(\gamma)$ by a factor of M:

$$K_h^M(\gamma) = \{(\varphi, \psi) : \|\varphi\|_{1,h}^2 + \|\psi\|_{0,h}^2 \le M(\|\Pi_h^\gamma \varphi\|_{1,h}^2 + \|\Pi_h^\gamma \psi\|_{0,h}^2)\}. \tag{2.7}$$

We point out that the conservation of energy (1.8) guarantees that the solutions of equation (1.6) with initial data $(u_h^0, u_h^1) \in K_h^M(\gamma)$ satisfy

$$\mathcal{E}_h(u_h) \le M\mathcal{E}_h(\Pi_h^\gamma u_h). \tag{2.8}$$

Therefore $K_h^M(\gamma)$ is stable under the flow and $(u_h(t), u_h'(t)) \in K_h^M(\gamma)$ for any $t \ge 0$.

The main result of this section is given by the following theorem.

Theorem 2.1. *Let $\gamma > 0$ and $M > 0$ be given. Assume the existence of a time $T(\gamma)$ such that for all $T > T(\gamma)$ there exists a positive constant $C = C(\gamma, T)$, independent of h, such that*

$$\mathcal{E}_h(u_h) \le C \int_0^T \int_{\Gamma_{0h}} |\partial_n^h u_h(t)|^2 d\Gamma dt \tag{2.9}$$

holds for all $(u_h^0, u_h^1) \in I_h(\gamma)$*. Then for all* $T > T(\gamma)$ *there exists a positive constant* $C = C_1(\gamma, T, M)$*, such that* (2.9) *holds for all the solutions* u_h *of problem* (1.6) *with initial data* $(u_h^0, u_h^1) \in K_h^M(\gamma)$ *and* $h > 0$*.*

Remark 2.1. *According to Theorem 2.1 the uniform observability inequality can be automatically transferred from* $I_h(\gamma)$ *to* $K_h^M(\gamma)$*. Let us briefly explain the main difficulty of the proof of Theorem 2.1. Inequalities (2.8) and (2.9) show that the uniform boundary observability inequality*

$$\mathcal{E}_h(u_h) \leq C(T) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h \Pi_h^\gamma u_h|^2 d\Gamma_{0h} dt$$

holds in the class $K_h^M(\gamma)$ *as well. But, unfortunately, the right side term cannot be estimated directly in terms of the energy of the solution* u_h *measured at the boundary* Γ_{0h}*:*

$$\int_0^T \int_{\Gamma_{0h}} |\partial_n^h u_h|^2 d\Gamma_{0h} dt.$$

A careful analysis is required to show that estimate. The essential contribution of this article is to show how this may be done by means of a dyadic decomposition.

Remark 2.2. *In the proof of the above theorem we use that the so-called "direct inequality" holds. In fact it is well known that (see* [28]*) for any* $T > 0$ *there exists a constant* $C(T)$*, independent of* h*, such that*

$$\int_0^T \int_{\Gamma_{0h}} |\partial_n^h u_h|^2 d\Gamma_{0h} dt \leq C(T) \mathcal{E}_h(u_h). \tag{2.10}$$

for all solutions u *of the semi-discrete system (1.6) and for all* $h > 0$*.*

Remark 2.3. *The same result holds if the two-grid filtered initial data are taken at any time* $t_0 \in [0, T]$*. In this sense our method of proof is more robust that that in* [17] *that makes a distinction between observability in the interval* $[0, T]$ *or* $[-T/2, T/2]$ *that our arguments show is not necessary.*

Since the proof of Theorem 2.1 is quite laborious it will be postponed until Section 4.

3. A Two-grid Method

In this section we describe a two-grid method that naturally produces classes of initial data of the form $K_h^M(\gamma)$. In view of Theorem 2.1 this will allow to show immediately uniform observability estimates for these classes of two-grid data.

The two-grid algorithm we propose is the following: Let N be such that $N \equiv 2 \pmod 3$ and $h = 1/(N+1)$. We introduce a coarse grid of mesh-size $3h$:

$$G^{3h} : x_{\mathbf{j}},\ x_{\mathbf{j}} = 3\mathbf{j}h,\ \mathbf{j} \in \left[0, \frac{N+1}{3}\right]^2 \cap \mathbb{Z}^2$$

and a fine one of size h:

$$G^h : y_{\mathbf{j}},\ y_{\mathbf{j}} = \mathbf{j}h,\ \mathbf{j} \in [0, N+1]^2 \cap \mathbb{Z}^2.$$

We consider the space V^h of all functions φ defined on the fine grid G^h as a linear interpolation of the functions ψ defined on the coarse grid G^{3h}. To be more precise let us consider the spaces $\mathcal{G}_h$ and $\mathcal{G}_{3h}$ of all the functions defined on the fine and coarse grids G^h and G^{3h}

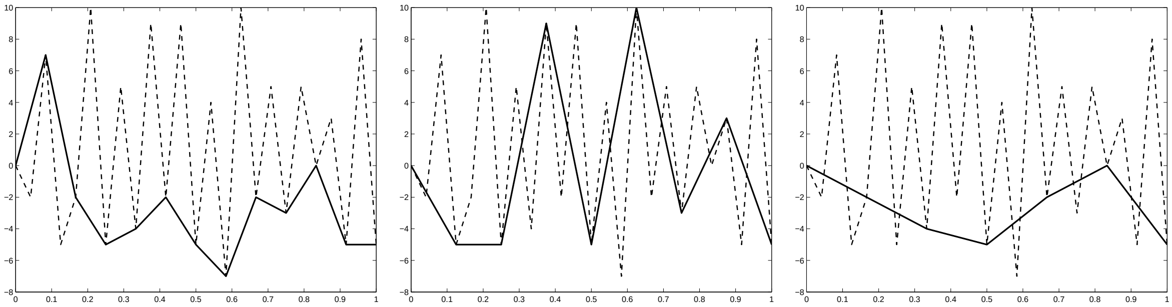

FIGURE 4. The dashed line is the original discrete function u. From left to write the new functions $\mathbf{\Pi}_h^{2h}u$, $\mathbf{\Pi}_h^{3h}u$, $\mathbf{\Pi}_h^{4h}u$ respectively.

respectively. We also introduce the extension operator $\mathbf{\Pi}_h^{3h}$ which associates to any function $\psi \in \mathcal{G}_{3h}$ a new function $\mathbf{\Pi}_h^{3h}\psi \in \mathcal{G}_h$ obtained by an interpolation process:

$$(\mathbf{\Pi}_h^{3h}\psi)_{\mathbf{j}} = (\mathbf{P}_{3h}^1\psi)(\mathbf{j}h),\ \mathbf{j} \in \mathbb{Z}^2,$$

where $\mathbf{P}_{3h}^1\psi$ is the piecewise multi-linear interpolator of $\psi \in \mathcal{G}_{3h}$. We then define $V^h = \mathbf{\Pi}_h^{3h}(\mathcal{G}_{3h})$, the image of operator $\mathbf{\Pi}_h^{3h}$. Obviously this constitutes a subspace of slowly oscillating discrete functions defined on the fine grid G^h. Examples of this interpolation process are given in Figure 4.

We define now another class of filtered functions, better adapted to the spectral analysis of the two-grid ones. In the sequel we denote for any $\mathbf{j} = (j_1, j_2) \in \mathbb{Z}^2$, its maximal component by $\|\mathbf{j}\|_\infty = \max\{j_1, j_2\}$. For any $0 < \eta \leq 1$ we set

$$J_h(\eta) = \left\{ u_h : \ u_h = \sum_{\|\mathbf{j}\|_\infty \leq \eta(N+1)} \widehat{u}_{\mathbf{j}}^h \varphi_h^{\mathbf{j}} \text{ with } \widehat{u}_{\mathbf{j}}^h \in \mathbb{C} \right\}, \tag{3.1}$$

and for any solution u_h of (1.6) we denote by $\Upsilon_h^\eta u_h$, its projection on the space $J_h(\eta)$.

The class of filtered solutions $I_\gamma(h)$, introduced in Section 2, is obtained through a filtering process along the level curves of $\omega_{\mathbf{j}}(h)$. The second one, leading to the space $J_h(\eta)$, consists in filtering the range of indices $\mathbf{j}$ to a square with length side $\eta(N+1)$. Observe that, in dimension one there exists a one-to-one correspondence between the two classes. In dimension two, excepting the case $\gamma = 2\sqrt{2}$, that corresponds to $\eta = 1$, there is no one-to-one correspondence. However the two classes can be easily compared with each other by analyzing the shape of the level curves of $\omega_{\mathbf{j}}(h)$. In Figure 5 we can see the support of the discrete functions in the frequency domain for the classes $J_h(1/3)$ and $I_h(\sqrt{2})$ that occur in the analysis of our two-grid method.

The second class of filtered data $J_h(\eta)$ is better adapted to analyze the two-grid discrete functions. In fact we will prove that the total energy of a solution u_h of (1.6) with initial data in the space $V^h \times V^h$ is bounded above by the energy of its projection on the space $J_h(1/3)$:

$$\mathcal{E}_h(u_h) \leq M\mathcal{E}_h(\Upsilon_h^{1/3}u_h), \tag{3.2}$$

for some positive constant M, independent of h. We point out that it is sufficient to prove this bound for $t = 0$, i.e. for the initial data, and use that the space $J_h(1/3)$ remains invariant under the semidiscrete flow to deduce that (3.2) holds for all $t > 0$. More precisely, it is

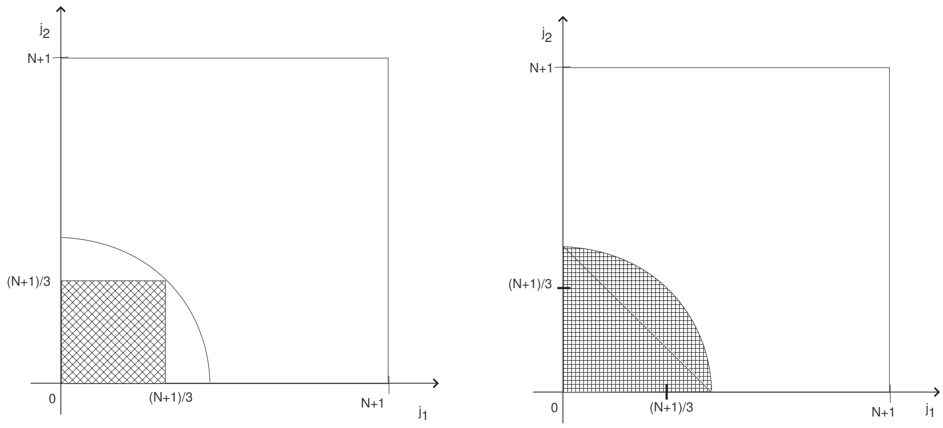

FIGURE 5. On the left, the dashed area represents the frequencies $\omega_{\mathbf{j}}(h)$, $\mathbf{j} \in \Lambda_{(N+1)/3}$; On the right, the dashed area represents the frequencies involved in $I_h(\sqrt{2})$.

sufficient to show that, for $(u_h^0, u_h^1) \in V^h \times V^h$:

$$\|u_h^0\|_{1,h}^2 \leq M \|\Upsilon_h^{1/3} u_h^0\|_{1,h}^2 \tag{3.3}$$

and

$$\|u_h^1\|_{0,h}^2 \leq M \|\Upsilon_h^{1/3} u_h^1\|_{0,h}^2. \tag{3.4}$$

Observe that any $\omega_{\mathbf{j}}(h)$ with $\|\mathbf{j}\|_\infty \leq (N+1)/3$ satisfies

$$\omega_{\mathbf{j}}(h) \leq \left(\frac{8}{h^2} \sin^2\left(\frac{\pi}{6}\right) \right)^{1/2} \leq \frac{\sqrt{2}}{h},$$

and thus, in view of (3.2), the energy of u_h is bounded above by the energy of its projection on the space $I_h(\sqrt{2})$:

$$\mathcal{E}_h(u_h) \leq M \mathcal{E}_h(\Upsilon_h^{1/3} u_h) \leq M \mathcal{E}_h(\Pi_h^{\sqrt{2}} u_h), \tag{3.5}$$

i.e. $(u_h, u_h') \in K_h^M(\gamma)$ with $\gamma = \sqrt{2}$.

The following theorem gives us the property of uniform boundary observability for the solutions u_h of system (1.6) with initial data $(u_h^0, u_h^1) \in V^h \times V^h$. This theorem is in fact a consequence of Theorem 2.1, estimate (3.5) and the well-known results for observability in classes of the form $I_h(\gamma)$ from [28] mentioned above.

Theorem 3.1. *Let $T > 4\sqrt{2}$. There exists a constant $C(T)$ such that*

$$\mathcal{E}_h(u_h) \leq C(T) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h u_h|^2 d\Gamma_{0h} dt \tag{3.6}$$

holds for all solutions u_h of (1.6) with $(u_h^0, u_h^1) \in V^h \times V^h$, uniformly on $h > 0$, V^h being the class of the two-grid data obtained with grids of mesh-size ratio 1/3.

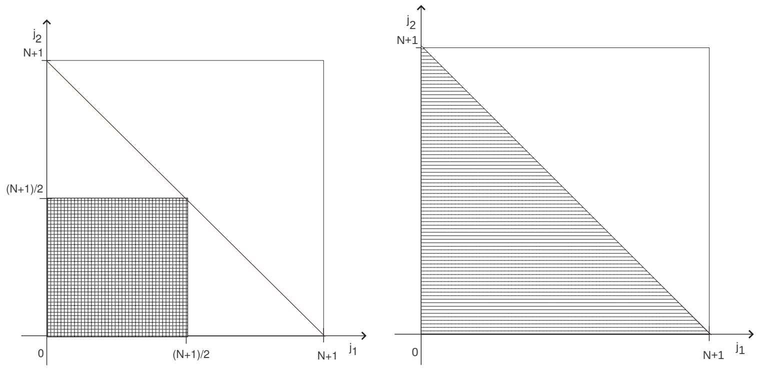

FIGURE 6. On the left, the black area represents the frequencies involved in $J_h(1/2)$; On the right the dashed area represents the the frequencies involved in $I_h(2)$.

Remark 3.1. *The time $T > 4\sqrt{2}$ is the one corresponding to the class of solutions belonging to $I_h(\sqrt{2})$, the smallest class I_h that contains $J_h(1/3)$, as obtained in* [28]*. Indeed, in view of (2.5) the known observability time for the above class of solutions is given by $T(\sqrt{2}) = 4\sqrt{2}$.*

In fact, Theorem 3.1 would hold for all $T > T^*$, T^* being the optimal time for uniform observability in the class $I_h(\sqrt{2})$. Very likely the estimate $T^* = 4\sqrt{2}$ given in [28] *is not optimal. An analysis of the velocity of propagation of the associated bicharacteristic rays shows that, according to* [29]*, the expected minimal time T^* should be*

$$T^* = \frac{2\sqrt{2}}{\cos(\pi/6)} = \frac{4\sqrt{2}}{\sqrt{3}}. \tag{3.7}$$

Although the uniform observability inequality in the class $I_h(\sqrt{2})$ for all $T > T^*$ with T^* as in (3.7) is very likely to hold, as far as we know, it has not been rigourously proved so far. Thus, improving the optimal time in Theorem 3.1 from $T > 4\sqrt{2}$ to $T > T^*$ as in (3.7) *is an open problem. This improvement would automatically lead to an improvement of the minimal time in Theorem 3.1 too.*

Remark 3.2. *We could apply the same two-grid algorithm with grids of mesh sizes ratio $1/2$, i.e. G^h and G^{2h}. In this case we would get, for some constant C, independent of h,*

$$\mathcal{E}_h(u_h) \leq C\mathcal{E}_h(\Upsilon_h^{1/2} u_h) \leq C\mathcal{E}_h(\Pi_h^2 u_h)$$

for all solutions u_h obtained by this two-grid method. Indeed, the smallest γ such that $I_h(\gamma)$ contains all the frequencies $\omega_{\mathbf{j}}(h)$ with $\|\mathbf{j}\|_\infty \leq (N+1)/2$ is $\gamma = 2$. Unfortunately, as we pointed before, inequality (2.9) does not hold in the class $I_h(2)$. This is why we have chosen the ratio between the fine and coarse grids in the two-grid method to be $1/3$. This will guarantee that the two hypotheses (2.8) and (2.9) are verified.

Remark 3.3. *The method also works for size meshes ratio $1/p$ with $p \geq 3$. In this case,*

$$J_h\Big(\frac{1}{p}\Big) \subset I_h\left(2\sqrt{2}\sin\big(\frac{\pi}{2p}\big)\right)$$

and thus the observability time given by Theorem 3.1 is

$$T\left(2\sqrt{2}\sin\left(\frac{\pi}{2p}\right)\right) = \frac{2\sqrt{2}}{\cos(\pi/p)}.$$

Remark 3.4. *The two-grid method proposed here has always a mesh-ratio of the form $1/p$. The same two-grid algorithm makes sense for ratios m/n with $m < n$. One could expect the uniform observability to hold in 1-d for any mesh-ratio $m/n < 1$, in the multidimensional case, when $m/n < 1/2$. But, by now, these are open problems. As we shall see, the only difficulty for doing that is to prove the following estimate for the functions u_h^0 belonging to $\mathbf{\Pi}_h^{n/mh}\mathcal{G}_h$:*

$$\|u_h^0\|_{s,h} \leq C(m/n,s)\|\Upsilon_h^{m/n}u_h^0\|_{s,h},\ s\in\{0,1\}.$$

Proof of Theorem 3.1. As we shall see Theorem 3.1 is an easy consequence of Theorem 2.1. Let u_h be the solution of system (1.6) with initial data $(u_h^0, u_h^1) \in V^h \times V^h$. Using that $J_h(1/3) \subset I_h(\sqrt{2})$ we obtain that

$$\mathcal{E}_h(\Upsilon_h^{1/3}u_h) \leq \mathcal{E}_h(\Pi_h^{\sqrt{2}}u_h).$$

To apply Theorem 2.1 with $\gamma = \sqrt{2}$ it remains to prove (3.2), i.e. (3.3) and (3.4).

We make use of the following lemma, which will be proved in Appendix B.

Lemma 3.1. *Let $p \geq 2$ and $V^h = \mathbf{\Pi}_h^{ph}(\mathcal{G}_{ph})$. For any $s \in [0,2]$ there exists a positive constant $C(p,s)$ such that the following*

$$\|v\|_{s,h} \leq C(p,s)\|\Upsilon_h^{1/p}v\|_{s,h},\ 0\leq s\leq 2. \tag{3.8}$$

holds for any $v \in V^h$.

Applying this Lemma with $p = 3$ to $u_h^0 \in V^h$ and $u_h^1 \in V^h$ we get the existence of a positive constant $M = \max\{C(3,0), C(3,1)\}^2$ such that

$$\|u_h^0\|_{1,h}^2 \leq M\|\Upsilon_h^{1/3}u_h^0\|_{1,h}^2 \qquad \text{and} \qquad \|u_h^1\|_{0,h}^2 \leq M\|\Upsilon_h^{1/3}u_h^1\|_{0,h}^2.$$

This proves (3.2) and finishes the proof of Theorem 3.1.

□

4. Proof of Theorem 2.1

First of all we introduce the projectors P_k that we shall use. Let us consider a function $P \in C_c^\infty(\mathbb{R})$ and $c > 1$. For any function $f \in L^1(\mathbb{R})$ and $k \geq 0$ we define the projector $P_k f$ as follows:

$$(P_k f)(t) = \int_{\mathbb{R}_\tau}\int_{\mathbb{R}_s} P(c^{-k}\tau)f(s)e^{i(t-s)\tau}dsd\tau,\ t\in\mathbb{R}. \tag{4.1}$$

In view (2.6), for any $T > T(\gamma)$ there exist two positive constants δ and ϵ such that

$$\mathcal{E}_h(v_h) \leq C(T,\gamma,\epsilon,\delta)\int_{2\delta}^{T-2\delta}\int_{\Gamma_{0h}}|\partial_n^h v_h|^2 d\Gamma_{0h}dt \tag{4.2}$$

for all $v_h \in I_h(\gamma+\epsilon)$. More precisely, using the continuity of the map $\gamma \to T(\gamma)$ we obtain the existence of a small constant ϵ such that $T > T(\gamma+\epsilon)$. We then choose a positive δ such that $T - 4\delta > T(\gamma+\epsilon)$. Then, the invariance by time translation guarantees that (4.2) holds.

With ϵ verifying (4.2) let us choose positive constants a, b, c and μ satisfying

$$1 < c < \frac{b-\mu}{a+\mu} \text{ and } \frac{b}{a+\mu} < \frac{\gamma+\epsilon}{\gamma}. \tag{4.3}$$

Let $F \in C_c^\infty(\mathbb{R})$ be supported in (a, b), $0 \le F \le 1$ such that $F \equiv 1$ in $[a+\mu, b-\mu]$. Set $P(\tau) = F(\tau) + F(-\tau)$ and then consider P_k as in (4.1).

In view of (2.3) the Fourier transform of u_h, in the t variable, reads

$$\widehat{u}_h(\tau) = \sum_{\mathbf{j}\in\Lambda_N} \left[\delta(\tau - \omega_{\mathbf{j}}(h))\widehat{u}^h_{\mathbf{j}+} + \delta(\tau + \omega_{\mathbf{j}}(h))\widehat{u}^h_{\mathbf{j}-}\right]\varphi^{\mathbf{j}}_h.$$

Therefore, the projector $P_k u_h$ is given by

$$P_k u_h(t) = \sum_{\mathbf{j}\in\Lambda_N} F(c^{-k}\omega_{\mathbf{j}}(h))\left[e^{it\omega_{\mathbf{j}}(h)}\widehat{u}^h_{\mathbf{j}+} + e^{-it\omega_{\mathbf{j}}(h)}\widehat{u}^h_{\mathbf{j}-}\right]\varphi^{\mathbf{j}}_h \tag{4.4}$$

and its energy satisfies

$$\mathcal{E}_h(P_k u_h) = \sum_{\mathbf{j}\in\Lambda_N} F^2(c^{-k}\omega_{\mathbf{j}}(h))\omega^2_{\mathbf{j}}(h)(|\widehat{u}^h_{\mathbf{j}+}|^2 + |\widehat{u}^h_{\mathbf{j}-}|^2). \tag{4.5}$$

Conditions (4.3) guarantee the existence of an index k_h such that $\{P_k u_h\}_{k=0}^{k_h}$ covers all the frequencies occurring in the representation of $\Pi^\gamma_h u_h$ and all these projections belong to $I_h(\gamma+\epsilon)$.

Step I. Sketch of the main steps. We first give the main ideas of the proof. We choose k_h as above and $k_0 \le k_h$, k_0 independent of h, such that $\{P_k u_h\}_{k=k_0}^{k_h}$ covers, except possibly for a finite number, all the frequencies occurring in $\Pi^\gamma_h u_h$, the projection of u_h on the space $I_h(\gamma)$ defined in (2.4):

$$\Pi^\gamma_h u_h = \frac{1}{2}\sum_{\omega_{\mathbf{j}}(h)\le\gamma/h}\left[e^{it\omega_{\mathbf{j}}(h)}\widehat{u}^h_{\mathbf{j}+} + e^{-it\omega_{\mathbf{j}}(h)}\widehat{u}^h_{\mathbf{j}-}\right]\varphi^{\mathbf{j}}_h.$$

The precise value of k_0 and k_h will be specified later.

Our hypothesis on the initial data $(u^0_h, u^1_h) \in K^h_\gamma(M)$ guarantees (see (2.7) for the definition of the spaces $K^h_\gamma(M)$) that the total energy u_h is controlled by the energy of $\Pi^\gamma_h u_h$:

$$\mathcal{E}_h(u_h) \le M\mathcal{E}_h(\Pi^\gamma_h u_h). \tag{4.6}$$

Firstly we will prove that

$$\mathcal{E}_h(\Pi^\gamma_h u_h) \le \sum_{k=k_0}^{k_h}\mathcal{E}_h(P_k u_h) + LOT \tag{4.7}$$

where LOT is a lower order term involving only a fixed number of Fourier components. In particular this LOT will be compact when passing to the limit $h \to 0$.

Next we use that each projection $P_k u_h$, $k_0 \le k \le k_h$ belongs to the class $I_h(\gamma+\epsilon)$ and, consequently, according to (4.2), satisfies the observability inequality:

$$\mathcal{E}_h(P_k u_h) \le C(T,\gamma,\delta,\epsilon)\int_{2\delta}^{T-2\delta}\int_{\Gamma_{0h}}|\partial^h_n P_k u_h|^2 d\Gamma_{0h}dt. \tag{4.8}$$

Thus, combining (4.7) and (4.8) we obtain the following estimate:

$$\mathcal{E}_h(\Pi_h^\gamma u_h) \leq C(T,\gamma,\delta,\epsilon) \sum_{k=k_0}^{k_h} \int_{2\delta}^{T-2\delta} \int_{\Gamma_{0h}} |\partial_n^h P_k u_h|^2 d\Gamma_{0h} dt + LOT. \tag{4.9}$$

Using ideas previously developed in [12] and [2] the right hand side sum can be estimated in terms of the energy of u_h measured on Γ_{0h}. More precisely, we will prove the existence of constants $C(P,c)$ and $C(\epsilon,\delta,T)$ such that

$$\sum_{k\geq k_0} \int_{2\delta}^{T-2\delta} \int_{\Gamma_h} |\partial_n^h P_k u_h|^2 d\Gamma_h dt \leq C(P,c) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h u_h|^2 d\Gamma_h dt + \frac{C(\epsilon,\delta,T)}{c^{2k_0}} \mathcal{E}_h(u_h) \tag{4.10}$$

holds for any $k_0 \geq 0$ and u_h solution of (1.6), uniformly on $h > 0$. Then combining (4.6), (4.9) and (4.10) the following holds:

$$\mathcal{E}_h(u_h) \leq C(T,P,\gamma,\delta,\epsilon,c) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h u_h|^2 d\Gamma_{0h} dt + \frac{C(\epsilon,\delta,T)}{c^{2k_0}} \mathcal{E}_h(u_h) + LOT. \tag{4.11}$$

Choosing h small and k_0 sufficiently large, but still independent of h, the energy term from the right side may be absorbed and then we obtain

$$\mathcal{E}_h(u_h) \leq C(T,P,\gamma,\delta,\epsilon,c) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h u_h|^2 d\Gamma_{0h} dt + LOT. \tag{4.12}$$

Finally, classical arguments of compactness-uniqueness allow us to get rid of the lower order term. For a complete development of this argument we refer to [28].

In the following we give the details of the proofs of the above steps.

Step II. Upper bounds of $\mathcal{E}_h(\Pi_h^\gamma u_h)$ in terms of $\{\mathcal{E}_h(P_k u_h)\}_{k\geq 0}$.
Let us choose a positive integer k_h such that

$$c^{k_h}(a+\mu) \leq \gamma/h < c^{k_h+1}(a+\mu). \tag{4.13}$$

The choice of k_h is always possible for small enough h. Also let us fix a positive integer $k_0 \leq k_h$ independent of h. Its precise value will be chosen later on in the proof. Using that $c < (b-\mu)/(a+\mu)$ (see (4.3)) we obtain that the following inequality holds:

$$c^{k_h}(a+\mu) \leq \gamma/h \leq c^{k_h+1}(a+\mu) \leq c^{k_h}(b-\mu).$$

Then any frequency $\omega_{\mathbf{j}}(h)$ belonging to $[(a+\mu)c^{k_0}, \gamma/h]$ is contained in at least one interval of the form $[c^k(a+\mu), c^k(b-\mu)]$ with $k_0 \leq k \leq k_h$ where the function $F(c^{-k}\cdot) \equiv 1$. Thus for any frequency $\omega_{\mathbf{j}}(h) \in [(a+\mu)c^{k_0}, \gamma/h]$ we have

$$1 \leq \sum_{k=k_0}^{k_h} F(c^{-k}\omega_{\mathbf{j}}(h))^2. \tag{4.14}$$

In view of (4.5) and (4.14) the energy of $\Pi_h^\gamma u_h$ excepting a lower order term involving a finite number of Fourier components only, can be bounded from above by the energy of all the projections $(P_k u_h)_{k=k_0}^{k_h}$:

$$\begin{aligned}(4.15) \qquad \mathcal{E}_h(\Pi_h^\gamma u_h) &\leq c^{2k_0}(a+\mu)^2 \sum_{\omega_{\mathbf{j}}(h)<(a+\mu)c^{k_0}} \Big(|\widehat{u}_{\mathbf{j}+}^h|^2 + |\widehat{u}_{\mathbf{j}-}^h|^2\Big) \\ &\quad + \sum_{k=k_0}^{k_h}\sum_{\mathbf{j}\in\Lambda_N} F^2(c^{-k}\omega_{\mathbf{j}}(h))\omega_{\mathbf{j}}^2(h)\Big(|\widehat{u}_{\mathbf{j}+}^h|^2 + |\widehat{u}_{\mathbf{j}-}^h|^2\Big) \\ &= C(a,k_0,\mu) \sum_{\omega_{\mathbf{j}}(h)<(a+\mu)c^{k_0}} \Big(|\widehat{u}_{\mathbf{j}+}^h|^2 + |\widehat{u}_{\mathbf{j}-}^h|^2\Big) + \sum_{k=k_0}^{k_h} \mathcal{E}_h(P_k u_h).\end{aligned}$$

Step III. Observability inequalities for the projections $P_k u_h, k \leq k_h$.
The next step is to apply the observability inequality (4.2) to each projection $P_k u_h$, $k \leq k_h$. We show that each of them belongs to the class $I_h(\gamma+\epsilon)$ where (4.2) holds. We remark that the projector $P_k u_h$ contains only the frequencies $\omega_{\mathbf{j}}(h) \in (c^k a, c^k b)$. In view of (4.13) any frequency $\omega_{\mathbf{j}}(h)$ involved in the decomposition of $P_k u_h$, $k \leq k_h$, satisfies

$$\omega_{\mathbf{j}}(h) < c^{k_h} b \leq \frac{\gamma b}{h(a+\mu)} < \frac{\gamma+\epsilon}{h},$$

which shows that $P_k u_h \in I_h(\gamma+\epsilon)$. Then for any $k \leq k_h$ the following holds:

$$(4.16) \qquad \mathcal{E}_h(P_k u_h) \leq C(T,\delta,\epsilon,\gamma)\int_{2\delta}^{T-2\delta}\int_{\Gamma_{0h}} |\partial_n^h(P_k u_h)|^2 d\Gamma_{0h} dt.$$

Using (4.15) and the above inequalities we obtain that

$$\begin{aligned}(4.17) \qquad \mathcal{E}_h(\Pi_h^\gamma u_h) &\leq C(T,\gamma,\delta,\epsilon)\sum_{k=k_0}^{k_h}\int_{2\delta}^{T-2\delta}\int_{\Gamma_{0h}} |\partial_n^h(P_k u_h)|^2 d\Gamma_{0h} dt \\ &\quad + C(a,k_0,\mu)\sum_{\omega_{\mathbf{j}}(h)<(a+\mu)c^{k_0}} \Big[|\widehat{u}_{\mathbf{j}+}^h|^2 + |\widehat{u}_{\mathbf{j}-}^h|^2\Big].\end{aligned}$$

It remains to prove (4.10). Once this inequality holds then (4.11) and (4.12) hold as well, which finishes the proof.

The key point is the following lemma which will be proved in Appendix A.

Lemma 4.1. *Let μ be a Borel measure, Ω a μ-measurable set such that $\mu(\Omega) < \infty$, $P \in C_c^\infty(\mathbb{R})$, $c > 1$ and $1 \leq p \leq \infty$. We set $X = L^p(\Omega, d\mu)$ and P_k as in (4.1). For any positive T and $\delta < T/4$ there are positive constants $C(P,c)$ and $C(\delta,T,P)$ such that the following holds*

$$(4.18) \qquad \sum_{k\geq k_0}\int_{2\delta}^{T-2\delta} \|P_k w\|_X^2 dt \leq C(P,c)\int_0^T \|w\|_X^2 dt + \frac{C(\delta,T,P)}{c^{2k_0}} \sup_{l\in\mathbb{Z}} \|w\|^2_{L^2((lT,(l+1)T),X)}$$

for all positive integer k_0 and $w \in L^2_{loc}(\mathbb{R}, X)$.

We now apply Lemma 4.1 with $X = l^2(\Gamma_{0h})$ and $w = \partial_n^h u_h$. Using that $P_k(\partial_n^h u_h) = \partial_n^h(P_k u_h)$, we obtain the existence of a constant $C(\delta,T,P)$ such that

$$\sum_{k\geq k_0}\int_{2\delta}^{T-2\delta}\int_{\Gamma_{0h}}|\partial_n^h P_k u_h(t)|^2 d\Gamma_{0h}dt \;\leq\; C(P,c)\int_0^T\int_{\Gamma_{0h}}|\partial_n^h u_h(t)|^2 d\Gamma_{0h}dt$$
$$+\frac{C(\delta,T,P)}{c^{2k_0}}\sup_{l\in\mathbb{Z}}\int_{lT}^{(l+1)T}\int_{\Gamma_{0h}}|\partial_n^h u_h(t)|^2 d\Gamma_{0h}dt.$$

At this point we apply the so-called *"direct inequality"* (2.10), which holds for all solutions u_h of system (1.6). Thus, a translation in time in (2.10) together with the conservation of energy shows that

$$\sup_{l\in\mathbb{Z}}\int_{lT}^{(l+1)T}\int_{\Gamma_{0h}}|\partial_n^h u_h(t)|^2 d\Gamma_{0h}dt \leq C(T)\mathcal{E}_h(u_h). \tag{4.19}$$

and then (4.10) holds.

5. Construction of the Control

In this section we introduce a numerical approximation for the HUM control v of the continuous wave equation (1.1) based on the two-grid method.

First, we define a restriction operator which carries any function of $\mathcal{G}_h$ to $\mathcal{G}_{3h}$. The most natural way is to define it as the formal adjoint of the $\mathbf{\Pi}_h^{3h}$ operator:

$$(\psi,\mathbf{\Pi}_h^{3h}\phi)_h=(\mathbf{\Pi}_h^{3h,*}\psi,\phi)_{3h},\ \forall\ \phi\in\mathcal{G}_{3h}.$$

To obtain the control v_h in (1.5) that is intended to approximate the control of (1.1), it would be rather natural to approximate the initial data (y^0,y^1) by (y_h^0,y_h^1) and take the corresponding controls v_h. But this has to be done carefully taking into account the high frequency pathologies. In fact not all the approximation of the initial data has to be done carefully but also the final requirement (1.2) has to be relaxed conveniently. To do this we shall consider controls v_h for which $\mathbf{\Pi}_h^{3h,*}y_h$, the projection of solutions over the coarse grid G^{4h}, vanishes at the time $t=T$.

The following holds:

Theorem 5.1. *Let be $T>4\sqrt{2}$. There exists a constant $C(T)$ such that for any $h>0$ and (y_h^0,y_h^1), there exists a function v_h satisfying*

$$\|v_h\|^2_{L^2((0,T)\times\Gamma_{0h})}\leq C(T)(\|y_h^0\|^2_{0,h}+\|y_h^1\|^2_{-1,h}) \tag{5.1}$$

such that the solution u_h of system (1.5) *with (y_h^0,y_h^1) as initial data and v^h acting as control satisfies:*

$$\mathbf{\Pi}_h^{3h,*}y_h(T)=\mathbf{\Pi}_h^{3h,*}y_h'(T)=0. \tag{5.2}$$

In order to construct the function v_h we need some notations and preliminary results. We define the duality product between $L^2(\Omega)\times H^{-1}(\Omega)$ and $H_0^1(\Omega)\times L^2(\Omega)$ by

$$\langle(\varphi^0,\varphi^1),(u^0,u^1)\rangle=(\varphi^1,u^0)_{-1,1}-(\varphi^0,u^1).$$

Also for the discrete spaces $\mathcal{H}_h^0(\Omega_h)\times\mathcal{H}_h^{-1}(\Omega_h)$ and $\mathcal{H}_h^1(\Omega_h)\times\mathcal{H}_h^0(\Omega_h)$ we introduce a similar duality product

$$\langle(\varphi^0,\varphi^1),(u^0,u^1)\rangle_h=(\varphi^1,u^0)_h-(\varphi^0,u^1)_h.$$

Let us introduce the adjoint discrete problem:

$$\begin{cases} u_h'' - \Delta_h u_h = 0 & \text{in } \Omega_h \times (0,T), \\ u_h(t) = 0 & \text{on } \Gamma_h \times (0,T), \\ u_h(T) = u_h^0,\ \partial_t u_h(T) = u_h^1 & \text{in } \Omega_h. \end{cases} \tag{5.3}$$

Note that the system (5.3) can be transformed into (1.6) by reversing the sense of time ($t \to T - t$). Thus, all the previous estimates on (1.6) apply to (5.3) too.

Following the same steps as in the continuous case, i.e. multiplying the control problem (1.5) by solutions of the adjoint problem (5.3) and integrating (summing) by parts we obtain the following result for the solutions of system (1.5):

Lemma 5.1. *Let y_h be a solution of system (1.5). Then*

$$\int_0^T \int_{\Gamma_{0h}} v_h(t) \partial_n^h u_h(t) d\Gamma_{1h} dt + \langle (y_h, y_h'), (u_h, u_h') \rangle_h \Big|_0^T = 0 \tag{5.4}$$

for all solutions u_h of the adjoint problem (5.3).

Proof of Lemma 5.1. Multiplying (1.5) and (5.3) by u_h, respectively y_h, integrating on $[0,T]$ and summing on Ω_h yields

$$\int_0^T \int_{\Omega_h} (y_h'' u_h - u_h'' y_h) d\Omega_h dt = \int_0^T \int_{\Omega_h} [(\Delta_h y_h) u_h - (\Delta_h u_h) y_h] d\Omega_h dt. \tag{5.5}$$

Integration by parts in the left hand side term gives us

$$\int_{\Omega_h} \int_0^T (y_h'' u_h - u_h'' y_h) dt d\Omega_h = \int_{\Omega_h} \Big(y_h' u_h \Big|_0^T - u_h' y_h \Big|_0^T \Big) d\Omega_h = \langle (y_h, y_h'), (u_h, u_h') \rangle_h \Big|_0^T. \tag{5.6}$$

For the second term of (5.5) we have:

$$\begin{aligned} &\int_0^T \int_{\Omega_h} [(\Delta_h y_h) u_h - (\Delta_h u_h) y_h] d\Omega_h dt \\ &= \sum_{i,j=1}^N \big[(y_{i-1,j} + y_{i+1,j}) u_{i,j} - (u_{i-1,j} + u_{i+1,j}) y_{i,j} \big] \\ &\quad + \sum_{i,j=1}^N \big[(y_{i,j-1} + y_{i,j+1}) u_{i,j} - (u_{i,j-1} + u_{i,j+1}) y_{i,j} \big] \\ &= \sum_{j=1}^N (y_{0,j} u_{1,j} + y_{N+1,j} u_{N,j}) + \sum_{i=1}^N (y_{i,0} u_{i,1} + y_{i,N+1} u_{i,N}) \\ &= \sum_{j=1}^N y_{N+1,j} u_{N,j} + \sum_{i=1}^N y_{i,N+1} u_{i,N} = -\int_0^T \int_{\Gamma_{0h}} v_h(t) \partial_n^h u_h(t) dt d\Gamma_{1h}. \end{aligned} \tag{5.7}$$

Identities (5.6) and (5.7) prove (5.4). □

Proof of Theorem 5.1. **Step I. Construction of v_h.** First, using variational methods we will prove the existence of a function v_h such that

$$\int_0^T \int_{\Gamma_{0h}} v_h(t) \partial_n^h u_h(t) d\Gamma_{0h} dt + \langle (y_h^0, y_h^1), (u_h(0), u_h'(0)) \rangle_h = 0 \tag{5.8}$$

for all solutions u_h of the adjoint problem (5.3) with final state $(u_h^0, u_h^1) \in V^h \times V^h$. This is equivalent to (5.2) in view of (5.4).

To do this we consider the space $\mathcal{F}_h = V^h \times V^h$ endowed with the norm

$$\|(u_h^0, u_h^1)\|_{\mathcal{F}_h} = \left(\|u_h^0\|_{1,h}^2 + \|u_h^1\|_{0,h}^2\right)^{1/2}$$

and the functional $\mathcal{J}_h : \mathcal{F}_h \to \mathbb{R}$ defined by

$$\mathcal{J}_h((u_h^0, u_h^1)) = \frac{1}{2}\int_0^T \int_{\Gamma_{0h}} |\partial_n^h u_h|^2 d\Gamma_{0h} dt + \langle (y_h^0, y_h^1), (u_h(0), u_h'(0))\rangle_h \tag{5.9}$$

where u_h is the solution of the adjoint problem (5.3) with final state (u_h^0, u_h^1). To construct the control v_h satisfying the relaxed controllability condition (5.8) for all $(u_h^0, u_h^1) \in V^h \times V^h$ it is sufficient to minimize $\mathcal{J}_h$ over $\mathcal{F}_h$.

In order to apply the fundamental theorem of the calculus of variations, guaranteeing the existence of a minimizer for $\mathcal{J}_h$, we prove that the functional $\mathcal{J}_h$ restricted to $\mathcal{F}_h$ which is convex, it is also continuous and uniformly coercive (with respect to the parameter h).

The linear term in the right side of (5.9) satisfies

$$|\langle (y_h^0, y_h^1), (u_h(0), u_h'(0))\rangle_h| \le (\|y_h^1\|_{-1,h} + \|y_h^0\|_{0,h})\|(u_h(0), u_h'(0))\|_{\mathcal{F}_h}.$$

Using the *direct inequality* (2.10) and the conservation of the energy $\mathcal{E}_h(u_h)$ we get

$$|\mathcal{J}_h((u_h^0, u_h^1))| \quad \le \quad \|(u_h^0, u_h^1)\|_{\mathcal{F}_h} \left(C(T)\|(u_h^0, u_h^1)\|_{\mathcal{F}_h} + \|y_h^1\|_{-1,h} + \|y_h^0\|_{0,h}\right)$$

which proves the continuity of the functional $\mathcal{J}_h$.

In view of the observability inequality (3.6), for any $T > 4\sqrt{2}$, the functional $\mathcal{J}_h$ is uniformly (with respect to h) coercive on $\mathcal{F}_h$:

$$|\mathcal{J}_h((u_h^0, u_h^1))| \quad \ge \quad \|(u_h^0, u_h^1)\|_{\mathcal{F}_h} \left(C(T)\|(u_h^0, u_h^1)\|_{\mathcal{F}_h} - \|y_h^1\|_{-1,h} - \|y_h^0\|_{0,h}\right),$$

for all $(u_h^0, u_h^1) \in \mathcal{F}_h$, where $C(T)$ is a constant obtained in (3.6).

Applying the fundamental theorem of the calculus of variations we obtain the existence of a minimizer $(u_h^{0,*}, u_h^{1,*}) \in \mathcal{F}_h$ such that

$$\mathcal{J}_h((u_h^{0,*}, u_h^{1,*})) = \min_{((u_h^0, u_h^1)) \in \mathcal{F}_h} \mathcal{J}_h((u_h^0, u_h^1)).$$

This implies that $\mathcal{J}_h'$, the Gateaux derivative of $\mathcal{J}_h$, satisfies

$$\mathcal{J}_h'((u_h^{0,*}, u_h^{1,*}))(u_h^0, u_h^1) = 0$$

for all $(u_h^0, u_h^1) \in \mathcal{F}_h$ and that u_h^*, solution of (5.3) with final state $(u_h^{0,*}, u_h^{1,*})$, satisfies

$$\int_0^T \int_{\Gamma_{0h}} (\partial_n^h u_h^*)\partial_n^h u(t) d\Gamma_{0h} dt + \langle (y_h^0, y_h^1), (u_h(0), u_h'(0))\rangle_h = 0$$

for all u_h solution of the adjoint problem (5.3) with final state $(u_h^0, u_h^1) \in \mathcal{F}_h$.

We set

$$v_h(t) = \partial_n^h u_h^*(t), \ \ t \in [0, T]$$

and then (5.8) holds.

Step II. Proof of property (5.2)**.** In view of Lemma 5.1, the solution y_h of system (1.5) with the above function v_h acting as control on Γ_{0h} satisfies

$$(y_h'(T), u_h^0)_h - (y_h(T), u_h^1)_h = 0$$

for all function $(u_h^0, u_h^1) \in V^h \times V^h$. Using that $V^h = \mathbf{\Pi}_h^{3h}(\mathcal{G}^{3h})$ we obtain

$$(y_h(T), \mathbf{\Pi}_h^{3h} w)_h = (y_h'(T), \mathbf{\Pi}_h^{3h} w)_h = 0$$

for all functions $w \in \mathcal{G}^{3h}$. Then

$$(\mathbf{\Pi}_h^{3h,*} y_h(T), w)_{3h} = (\mathbf{\Pi}_h^{3h,*} y_h'(T), w)_{3h} = 0$$

for all $w \in \mathcal{G}^{3h}$ and obviously (5.2) holds.

Step III. Proof of estimate (5.1)**.** Using that $(u_h^{0,*}, \phi_h^{1,*})$ is a minimizer of $\mathcal{J}_h$ we have $\mathcal{J}_h((u_h^{0,*}, u_h^{1,*})) \le \mathcal{J}_h((0_h, 0_h))$, where 0_h is the function vanishing identically on the mesh G_h. Consequently

$$\int_0^T \int_{\Gamma_{0h}} |\partial_n^h u_h^*|^2 d\Gamma_{0h} dt \le (\|y_h^1\|_{-1,h} + \|y_h^0\|_{0,h})(\|u_h^{0,*}\|_{1,h} + \|u_h^{1,*}\|_{0,h}).$$

Applying the observability inequality (3.6) to the solution u_h^* we get

$$\|u_h^{0,*}\|_{1,h}^2 + \|u_h^{1,*}\|_{0,h}^2 \le C(T) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h u_h^*|^2 d\Gamma_{0h} dt.$$

We then find that

$$\|v_h\|_{L^2((0,T)\times\Gamma_{0h})}^2 = \int_0^T \int_{\Gamma_{0h}} |\partial_n^h u_h^*|^2 d\Gamma_{0h} dt \le C(T)(\|y_h^0\|_{-1,h} + \|y_h^1\|_{0,h})^2$$

where the constant $C(T)$ is independent of h.

The proof is now complete. □

6. Convergence of the uncontrolled problem

In this section, for the sake of completness, we prove the convergence of the solutions of the uncontrolled problem (1.6). We also analyze the convergence of their normal derivatives towards the continuous one. First we introduce the interpolators needed in our analysis.

6.1. **Interpolators.** We denote by $\mathbf{P}_h^1$ the piecewise multi-linear and continuous interpolator on Ω. We also consider the operators $\mathbf{P}_h^s$ defined for any $u \in \mathcal{H}_h^s(\Omega)$ by

$$\mathbf{P}_h^s u_h = (-\Delta)^{-(s-1)/2}(\mathbf{P}_h^1(-\Delta_h)^{(s-1)/2} u_h), \tag{6.1}$$

that, for any $s \in \mathbb{R}$, they continuously map $\mathcal{H}_h^s(\Omega_h)$ to $H^s(\Omega)$.

In the sequel we will denote by ∇_h^+ the following operator

$$(\nabla_h^+ u)_{j,k} = (\frac{u_{j+1,k} - u_{j,k}}{h}, \frac{u_{j,k+1} - u_{j,k}}{h}).$$

The representation of the operator $\mathbf{P}_h^0$ in the Fourier space shows that this operator is exactly the piecewise constant interpolator:

$$\mathbf{P}_h^0 u_h(x) = u_{jk},\ x \in ((j-1/2)h, (j+1/2)h) \times ((k-1/2)h, (k+1/2)h).$$

Concerning the operator $\mathbf{P}_h^{-1}$, it satisfies

$$\begin{aligned} \|\mathbf{P}_h^{-1} u_h\|_{H^{-1}(\Omega)} &= \|\mathbf{P}_h^1(-\Delta_h)^{-1} u_h\|_{H_0^1(\Omega)} = \|\nabla_h^+(-\Delta_h)^{-1} u_h\|_{\mathcal{H}_h^0(\Omega_h)} \\ &= \|u_h\|_{\mathcal{H}_h^{-1}(\Omega_h)}. \end{aligned}$$

Also for any pair of functions u_h and w_h defined on G^h and vanishing on Γ_h the following holds:

$$\begin{aligned}\int_\Omega \mathbf{P}_h^0 u_h \mathbf{P}_h^0 w_h &= \int_{\Omega_h} u_h w_h d\Omega_h = \int_{\Omega_h} (-\Delta_h)(-\Delta_h)^{-1} u_h w_h d\Omega_h \\ &= \int_{\Omega_h} \nabla_h((-\Delta_h)^{-1} u_h) \cdot \nabla_h w_h d\Omega = \int_\Omega \nabla(\mathbf{P}_h^1(-\Delta_h)^{-1} u_h) \cdot \nabla(\mathbf{P}_h^1 w_h) \\ &= \langle \mathbf{P}_h^{-1} u_h, \mathbf{P}_h^1 w_h \rangle_{-1,1}.\end{aligned}$$

Lemma 6.1. *The following holds for all $h > 0$ and all sequences u_h:*

$$\|\mathbf{P}_h^{-1} u_h - \mathbf{P}_h^0 u_h\|_{H^{-1}(\Omega)} \le h \|u_h\|_{0,h}. \tag{6.2}$$

Proof. By the definition of the operators $\mathbf{P}_h^{-1}$ and $\mathbf{P}_h^0$ we get

$$(-\Delta)^{-1/2} \mathbf{P}_h^0 u_h = \mathbf{P}_h^1 (-\Delta_h)^{-1/2} u_h$$

and

$$\mathbf{P}_h^{-1} u_h = (-\Delta)^{1/2} \mathbf{P}_h^0 (-\Delta)^{-1/2} u_h.$$

Thus we have

$$\begin{aligned}\|\mathbf{P}_h^{-1} u_h - \mathbf{P}_h^0 u_h\|_{H^{-1}(\Omega)} &= \|(-\Delta)^{1/2} \mathbf{P}_h^0 (-\Delta_h)^{-1/2} u_h - \mathbf{P}_h^0 u_h\|_{H^{-1}(\Omega)} \\ &= \|\mathbf{P}_h^0 (-\Delta_h)^{-1/2} u_h - (-\Delta)^{-1/2} \mathbf{P}_h^0 u_h\|_{L^2(\Omega)} \\ &= \|\mathbf{P}_h^0 (-\Delta_h)^{-1/2} u_h - \mathbf{P}_h^1 (-\Delta_h)^{-1/2} u_h\|_{L^2(\Omega)}.\end{aligned}$$

Using that the two interpolators $\mathbf{P}_h^0$ and $\mathbf{P}_h^1$ satisfy (see [24], Th. 3.4.1, p. 88)

$$\|\mathbf{P}_h^0 u_h - \mathbf{P}_h^1 u_h\|_{L^2(\Omega)} \le h \|u_h\|_{\mathcal{H}_h^1(\Omega)}$$

we obtain

$$\|\mathbf{P}_h^{-1} u_h - \mathbf{P}_h^0 u_h\|_{H^{-1}(\Omega)} \le h \|(-\Delta_h)^{-1/2} u_h\|_{\mathcal{H}_h^1(\Omega_h)} = h \|u_h\|_{l^2(\Omega_h)},$$

which finishes the proof. □

6.2. **Convergence of the solutions.** The following propositions describe how a uniformly bounded family of solutions of (1.6) weakly converges (up to a subsequence) as $h \to 0$ to a solution of finite energy of the continuous wave equation (1.3).

Let us consider the family $\{u_h\}_{h>0}$ of solutions of (1.6) and let us denote by $\mathbf{P}_h^1 u_h$ their piecewise linear interpolator, that belongs to $H_0^1(\Omega)$ for all $0 \le t \le T$ as the solution of the continuous problem does.

Proposition 6.1. *Let $\{u_h\}_{h>0}$ be a family of solutions of (5.3) depending on the parameter $h \to 0$, whose energies are uniformly bounded, i.e.*

$$E_h(0) \le C, \forall\, h > 0. \tag{6.3}$$

Then there exists a solution $u \in C([0,T], H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega))$ of problem (1.3) such that, by extracting a suitable subsequence $h \to 0$, we may guarantee that

$$\mathbf{P}_h^1 u_h \rightharpoonup u \text{ in } L^2([0,T], H_0^1(\Omega)), \tag{6.4}$$

$$\mathbf{P}_h^0 u_h' \rightharpoonup u' \text{ in } L^2([0,T], L^2(\Omega)). \tag{6.5}$$

Moreover, if the family $\{u_h\}_{h>0}$ *is such that* $\mathbf{P}_h^1 u_h(0) \to u^0$ *in* $H_0^1(\Omega)$ *and* $\mathbf{P}_h^0 u_h'(0) \to u^1$ *in* $L^2(\Omega)$ *for some* $(u^0, u^1) \in H^1(\Omega) \times L^2(\Omega)$ *then all the above convergences hold in the corresponding strong topologies.*

Proof of Proposition 6.1. **Step I. Weak convergence.** In view of the uniform bound (6.3) and the conservation of energy we deduce that

$$\begin{cases} \mathbf{P}_h^1 u_h \text{ is uniformly bounded in } W^{1,\infty}((0,T), L^2(\Omega)), \\ \mathbf{P}_h^0 u_h \text{ is uniformly bounded in } W^{1,\infty}((0,T), L^2(\Omega)). \end{cases} \tag{6.6}$$

Using that

$$\|\mathbf{P}_h^1 u_h - \mathbf{P}_h^0 u_h\|_{L^2((0,T),\, L^2(\Omega))} \leq h \|u_h\|_{L^2((0,T),\, \mathcal{H}_h^1(\Omega_h))} \overset{h\to 0}{\to} 0$$

we obtain the existence of a function $u \in W^{1,\infty}((0,T),\ L^2(\Omega))$ such that, up to subsequences,

$$\begin{cases} \mathbf{P}_h^1 u_h \rightharpoonup u \text{ in } H^1((0,T), L^2(\Omega)), \\ \mathbf{P}_h^0 u_h' \rightharpoonup u' \text{ in } L^2((0,T), L^2(\Omega)). \end{cases} \tag{6.7}$$

Also, by (6.3) we get that $\{\mathbf{P}_h^1 u_h\}_h$ is uniformly bounded in $C([0,T], H_0^1(\Omega))$. Using the classical Aubin-Lions's compactness result (see for instance [27]) we deduce that $\{\mathbf{P}_h^1 u_h\}_h$ is relatively compact in $C([0,T], L^2(\Omega))$. Thus we obtain that

$$\mathbf{P}_h^1 u_h \rightharpoonup u \text{ in } H^1((0,T), L^2(\Omega)) \cap L^2((0,T), H_0^1(\Omega)), \tag{6.8}$$

and

$$\mathbf{P}_h^1 u_h \to u \text{ in } C([0,T], L^2(\Omega)). \tag{6.9}$$

Also we prove that

$$(\mathbf{P}_h^0 u_h'') \text{ is uniformly bounded in } L^2((0,T),\ H^{-1}(\Omega). \tag{6.10}$$

For any function function $\varphi \in L^2((0,T),\ H_0^1(\Omega))$ and $t \in (0,T)$ we have

$$\begin{aligned} \langle \mathbf{P}_h^0 u_h''(t), \varphi \rangle_{-1,1} &= \int_\Omega \mathbf{P}_h^0 u_h''(t)\varphi = \sum_{j,k=1}^N \int_{jh-h/2}^{jh+h/2} \int_{kh-h/2}^{kh+h/2} (\Delta_h u_h)_{jk}(t)\varphi \\ &= \sum_{j,k=1}^N (\Delta_h u_h)_{jk}(t) \int_{jh-h/2}^{jh+h/2} \int_{kh-h/2}^{kh+h/2} \varphi := \sum_{j,k=1}^N (\Delta_h u_h)_{jk}(t) \widetilde{\varphi}_{jk}^h \\ &= -h^2 \sum_{j,k=0}^N (\nabla_h^+ u_h)_{jk}(t) (\nabla_h^+ \widetilde{\varphi})_{jk}^h = \int_\Omega \nabla(\mathbf{P}_h^1 u_h)(t) \nabla(\mathbf{P}_h^1 \widetilde{\varphi}^h) \\ &\lesssim \|\mathbf{P}_h^1 u_h(t)\|_{H_0^1(\Omega)} \|\varphi\|_{H_0^1(\Omega)}. \end{aligned}$$

Thus we obtain (6.10). Using (6.10), (6.7) and the compactness result mentioned above we deduce that

$$\mathbf{P}_h^0 u_h' \to u' \text{ in } C([0,T], L^2(\Omega)). \tag{6.11}$$

Observe that, according to the bounds (6.6), the subsequences may be extracted so that

$$\mathbf{P}_h^1 u_h(0) \rightharpoonup u^0 \text{ in } H_0^1(\Omega) \text{ and } \mathbf{P}_h^0 u_h'(0) \rightharpoonup u^1 \text{ in } L^2(\Omega)$$

for some $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. Note that, in view of (6.9) and (6.11), $u(0) = u^0$ and $u'(0) = u^1$.

Step II. Equation solved by the limit. We prove that u solves the wave equation (1.3) with initial data (u_0, u_1).

Let us choose $w \in C^2([0,T], C_c^3(\Omega))$. Using the following identity

$$\int_0^T \int_{\Omega_h} (\Delta_h u_h) w d\Omega_h = -\int_0^T h^2 \sum_{j,k=0}^{N} \nabla_h^+ u_h \cdot \nabla_h^+ w,$$

integrating (1.5) on $[0,T]$ and summing on Ω_h we get

$$\int_0^T h^2 \sum_{\mathbf{j}h\in\Omega_h} (u_h)_{\mathbf{j}} w''_{\mathbf{j}} dt + \int_0^T h^2 \sum_{j,k=0}^{N} \nabla_h^+ u_h \cdot \nabla_h^+ w = \langle (u_h, u'_h), (w, w')\rangle \Big|_0^T.$$

Thus

$$\int_0^T \int_\Omega \mathbf{P}_h^0 u_h \mathbf{P}_h^0 w'' + \int_0^T \int_\Omega \nabla(\mathbf{P}_h^1 u_h) \cdot \nabla(\mathbf{P}_h^1 w) = \langle (\mathbf{P}_h^0 u_h, \mathbf{P}_h^0 u'_h), (\mathbf{P}_h^0 w, \mathbf{P}_h^0 w')\rangle \Big|_0^T. \tag{6.12}$$

Using that

$$\begin{cases} \mathbf{P}_h^0 w'' \to w'' & \text{in } L^2((0,T), L^2(\Omega)), \\ \nabla(\mathbf{P}_h^1 w) \to \nabla w & \text{in } L^2((0,T), L^2(\Omega^2)), \\ (\mathbf{P}_h^0 w, \mathbf{P}_h^0 w')(0) \to (w(0), w'(0)) & \text{in } L^2(\Omega) \times L^2(\Omega), \\ (\mathbf{P}_h^0 w, \mathbf{P}_h^0 w')(T) \to (w(T), w'(T)) & \text{in } L^2(\Omega) \times L^2(\Omega) \end{cases}$$

and

$$\begin{cases} \mathbf{P}_h^0 u_h \rightharpoonup u & \text{in } L^2((0,T), L^2(\Omega)), \\ \nabla(\mathbf{P}_h^1 u_h) \rightharpoonup \nabla u & \text{in } L^2((0,T), L^2(\Omega^2)), \\ (\mathbf{P}_h^0 u_h, \mathbf{P}_h^0 u'_h)(0) \rightharpoonup (u(0), u'(0)) & \text{in } L^2(\Omega) \times L^2(\Omega), \\ (\mathbf{P}_h^0 u_h, \mathbf{P}_h^0 u'_h)(T) \rightharpoonup (u(T), u'(T)) & \text{in } L^2(\Omega) \times L^2(\Omega) \end{cases}$$

we obtain that the limit u satisfies

$$\int_0^T \int_\Omega u w'' + \int_0^T \int_\Omega \nabla u \cdot \nabla w = \langle (u, u'), (w, w')\rangle \Big|_0^T \tag{6.13}$$

for any function $w \in C^2([0,T],\, H_0^1(\Omega))$. This shows that u is a solution of the homogenous wave equation on Ω.

Under the assumption of strong convergence of the initial data (u_h^0, u_h^1), this together with the conservation of the energy, gives us that

$$\int_0^T \Big[\|\mathbf{P}_h^1 u_h(t)\|^2_{H_0^1(\Omega)} + \|\mathbf{P}_h^0 u'_h(t)\|^2_{L^2(\Omega)}\Big] dt \to \int_0^T \Big[\|u(t)\|^2_{H_0^1(\Omega)} + \|u'(t)\|^2_{L^2(\Omega)}\Big] dt.$$

Thus all the above weak convergences hold in the strong topology as well. □

6.3. **Convergence of the normal derivatives.** In this subsection we prove that the interpolated discrete normal derivatives $\mathbf{P}^h_{0,\Gamma}(\partial^h_n u_h)$ converge to the continuous one $\partial_n u$, where $\mathbf{P}^h_{0,\Gamma}$ is the piecewise constant interpolator on the boundary Γ_h.

Proposition 6.2. *Let $\{u_h(t)\}_h$ be a family of solutions of (5.3) satisfying (6.3). Let u be any solution of (1.6) obtained as limit when $h\to 0$ as in the statement of Proposition 6.1. Then*

$$\mathbf{P}^h_{0,\Gamma}(\partial^h_n u_h) \rightharpoonup \partial_n u \text{ weakly in } L^2((0,T)\times\Gamma). \tag{6.14}$$

Moreover, if the family $\{u_h\}_{h>0}$ is such that $\mathbf{P}^1_h u_h(0)\to u^0$ in $H^1_0(\Omega)$ and $\mathbf{P}^0_h u_h'(0)\to u^1$ in $L^2(\Omega)$ for some $(u^0,u^1)\in H^1(\Omega)\times L^2(\Omega)$ then the above convergences hold in the strong topologies.

Proof. For any functions $u\in\mathcal{G}^h$ such that $u_{|\Gamma_h}=0$ and $w\in\mathcal{G}^h$, explicit computations give us

$$\begin{aligned}\int_{\Omega_h}(\Delta_h u)wd\Omega_h + h^2\sum_{j,k=0}^N(\nabla^+_h u)_{jk}(\nabla^+_h w)_{jk} &= \\ &= h^2\sum_{j,k=1}^N(\Delta_h u)_{jk}w_{jk} + h^2\sum_{j,k=0}^N(\nabla^+_h u)_{jk}(\nabla^+_h w)_{jk} = \\ &= -\sum_{k=1}^N(u_{N,k}w_{N+1,k}+y_{1,k}w_{0,k}) - \sum_{j=1}^N(u_{j,N}w_{j,N+1}+u_{j,1}w_{j,0}) \\ &= \int_{\Gamma_h}(\partial^h_n u)wd\Gamma_h.\end{aligned} \tag{6.15}$$

Let us choose $w\in C^2([0,T]\times\Omega)$. Applying identity (6.15) to the solution u_h of equation (5.3) and $w|_{G_h}$ we find that

$$\int_0^T\int_{\Omega_h}u_h w'' d\Omega_h dt + h^2\sum_{j,k=0}^N(\nabla^+_h u_h)_{jk}(\nabla^+_h w)_{jk} = \langle(u_h,u_h'),(w,w')\rangle\Big|_0^T + \int_{\Gamma_h}(\partial^h_n u_h)wd\Gamma_h.$$

Rewriting the above identity in terms of the interpolators $\mathbf{P}^0_h$ and $\mathbf{P}^1_h$ we get

$$\begin{aligned}\int_0^T\int_\Omega \mathbf{P}^0_h u_h \mathbf{P}^0_h w'' + \int_0^T\int_\Omega\nabla(\mathbf{P}^1_h u_h)\cdot\nabla(\mathbf{P}^1_h w) &= \langle(\mathbf{P}^0_h u_h,\mathbf{P}^0_h u_h'),(\mathbf{P}^0_h w,\mathbf{P}^0_h w')\rangle\Big|_0^T \\ &+ \int_0^T\int_\Gamma \mathbf{P}^h_{0,\Gamma}(\partial^h_n u_h)\mathbf{P}^h_{0,\Gamma}wd\Gamma dt.\end{aligned}$$

Using that solution u of problem (1.3) satisfies

$$\int_0^T\int_\Omega uw''dxdt + \int_0^T\int_\Omega\nabla u\cdot\nabla w = \langle(u,u'),(w,w')\rangle\Big|_0^T + \int_0^T\int_\Gamma\partial_n uwd\Gamma dt$$

for all $w\in L^2((0,T),H^1(\Omega))$ with $w''\in L^2((0,T),L^2(\Omega))$, and the convergences for $\mathbf{P}^0_h u_h$, $\mathbf{P}^1_h u_h$ and $\mathbf{P}^0_h u_h'$ given by Proposition 6.1, we obtain that

$$\int_0^T\int_\Gamma\mathbf{P}^h_{0,\Gamma}(\partial^h_n u_h)wd\Gamma dt \to \int_0^T\int_\Gamma\partial_n uwd\Gamma dt. \tag{6.16}$$

This shows that

$$\mathbf{P}^h_{0,\Gamma}(\partial^h_n u_h) \rightharpoonup \partial_n u \text{ weakly on } L^2((0,T)\times\Gamma).$$

The proof of the strong convergence is more subtle. For any $\epsilon > 0$ we can choose smooth functions $(\tilde{u}^0, \tilde{u}^1) \in H^2(\Omega)\times H^1(\Omega))$ such that $\|\tilde{u}^0 - u^0\|_{H^1(\Omega)} \leq \epsilon$ and $\|\tilde{u}^1 - u^1\|_{L^2(\Omega)} \leq \epsilon$.

We denote by $(\tilde{u}^0_h, \tilde{u}^1_h)$ the approximations of $(\tilde{u}^0, \tilde{u}^1)$. In this case the discrete solutions $(\tilde{u}_h, \tilde{u}'_h)$ of equation (5.3) are smooth enough to guarantee that $\mathbf{P}^h_{0,\Gamma}(\partial^h_n \tilde{u}_h)$ is compact in $L^2((0,T)\times\Gamma)$, and thus

$$\mathbf{P}^h_{0,\Gamma}(\partial^h_n \tilde{u}_h) \to \partial_n \tilde{u} \text{ in } L^2((0,T)\times\Gamma). \tag{6.17}$$

Denoting $\tilde{\tilde{u}} = u - \tilde{u}$, $\tilde{\tilde{u}}_h = u_h - \tilde{u}_h$ and using that the energy on the boundary is controlled by the total energy both in the discrete and continuous setting we have

$$\|\mathbf{P}_{0,\Gamma}(\partial^h_n \tilde{\tilde{u}}_h)\|_{L^2((0,T)\times\Gamma)} \leq C(T)\mathcal{E}(\tilde{\tilde{u}}_h) \leq C(T)\epsilon \tag{6.18}$$

and

$$\|\partial_n \tilde{\tilde{u}}\|_{L^2((0,T)\times\Gamma)} \leq C(T)\mathcal{E}(\tilde{\tilde{u}}) \leq C(T)\epsilon. \tag{6.19}$$

Using now (6.17), (6.18) and (6.19) we obtain the strong convergence of $\mathbf{P}_{0,\Gamma}(\partial^h_n u_h)$ towards $\partial_n u$ in $L^2((0,T)\times\Gamma)$. □

7. Convergence of the controlled problem

Concerning the convergence of the semidiscrete control of (1.5) we prove the following result.

Theorem 7.1. *Let $(y^0, y^1) \in L^2(\Omega)\times H^{-1}(\Omega)$ and (y^0_h, y^1_h) be such that*

$$\mathbf{P}^0_h y^0_h \rightharpoonup y^0 \text{ in } L^2(\Omega), \quad \mathbf{P}^{-1}_h y^1_h \rightharpoonup y^1 \text{ in } H^{-1}(\Omega). \tag{7.1}$$

Then for any $T > 4$ the solution (y_h, y'_h) and its partial controls v_h given by Theorem 5.1 satisfy

$$\mathbf{P}^0_h y_h \overset{*}{\rightharpoonup} y \text{ in } L^\infty([0,T], L^2(\Omega)),\ (\mathbf{P}^0_h y_h)' \overset{*}{\rightharpoonup} y' \text{ in } L^\infty([0,T], H^{-1}(\Omega))$$

and

$$\mathbf{P}^h_{0,\Gamma} v_h \rightharpoonup v \text{ in } L^2([0,T], L^2(\Gamma_0)),$$

where (y, y_t) solves (1.1), with the limit control v, and satisfies (1.2)*. The limit control v is given by*

$$v = \partial_n u^* \text{ on } \Gamma_0,$$

where u^* is solution of the adjoint system

$$\begin{cases} u'' - \Delta u = 0 & \text{in } \Omega\times(0,T), \\ u = 0 & \text{on } \Gamma\times(0,T), \\ u(T,x) = u^0(x),\ u_t(T,x) = u^1(x) & \text{in } \Omega, \end{cases} \tag{7.2}$$

with data $(u^{0,*}, u^{1,*}) \in H^1_0(\Omega)\times L^2(\Omega)$ minimizing the functional

$$J((u^0, u^1)) = \frac{1}{2}\int_0^T \int_{\Gamma_0} |\partial_n u|^2 dt + \langle (y^0, y^1), (u^0, u^1)\rangle \tag{7.3}$$

in $H^1_0(\Omega)\times L^2(\Omega)$.

Proof. **Step I. Weak convergence of v_h.** Theorem 5.1 gives us the function $v_h = \partial_n^h u_h^*(t)$, that depends on (y_h^0, y_h^1) and satisfies (5.1). Recall that u_h^* solves (5.3) with final state $(u_h^{0,*}, u_h^{1,*}) \in V^h \times V^h$ minimizing the function J_h.

Moreover, as a consequence of the observability inequality (3.6), we have

$$\|u_h^{0,*}\|_{1,h} + \|u_h^{1,*}\|_{0,h} \leq C(T)\|\partial_n^h u_h^*\|_{L^2((0,T)\times\Gamma_{0h})} \leq C(T)(\|y_h^1\|_{0,h} + \|y_h^0\|_{-1,h}) \leq C(T).$$

In these conditions, Proposition 6.1 guarantees the existence of a function u^* that solves (1.3) and, in addition,

$$\mathbf{P}_{0,\Gamma}^h v_h(t) = \mathbf{P}_{0,\Gamma}^h(\partial_n^h u_h^*) \rightharpoonup \partial_n u^* \text{ weakly in } L^2((0,T)\times\Gamma_0) \text{ as } h \to 0.$$

Step II. Weak convergence of y_h. Let us now consider equation (1.5) with initial data (y_h^0, y_h^1) and v_h as above. Then for any solution u_h of the adjoint problem (5.3), the following holds for all $0 < s < T$:

$$\int_0^s \int_{\Gamma_{0h}} v_h(t)\partial_n^h u_h(t) d\Gamma_{0h} dt + \langle (y_h, y_h'), (u_h, u_h')\rangle_h \Big|_0^s = 0. \tag{7.4}$$

Thus, in view of the *direct inequality* (2.10) and the conservation of the energy applied to u_h, we get, for any $s < T$, that

$$\begin{aligned} |\langle (y_h(s), y_h'(s)), (u_h^0, u_h^1)\rangle_h| &\leq |\langle (y_h^0, y_h^1), (u_h(0), u_h'(0))\rangle_h| \\ &\quad + \|v_h\|_{L^2((0,T)\times\Gamma_{0h})}\|\partial_n^h u_h\|_{L^2((0,T)\times\Gamma_{0h})} \\ &\leq C(T)(\|y_h^0\|_{0,h} + \|y_h^1\|_{-1,h})(\|u_h^0\|_{1,h} + \|u_h^1\|_{0,h}). \end{aligned}$$

This means that for any $0 \leq s \leq T$ the following holds:

$$\|y_h(s)\|_{0,h} + \|y_h'(s)\|_{-1,h} \leq C. \tag{7.5}$$

Using this estimate we claim the existence of a positive constant such that

$$\begin{cases} \|\mathbf{P}_h^0 y_h\|_{L^\infty([0,T],\, L^2(\Omega))} \leq C, \\ \\ \|\mathbf{P}_h^0 y_h'\|_{L^\infty([0,T],\, H^{-1}(\Omega))} \leq C \end{cases} \tag{7.6}$$

and

$$\begin{cases} \|\mathbf{P}_h^{-1} y_h\|_{L^\infty([0,T],\, L^2(\Omega))} \leq C, \\ \\ \|\mathbf{P}_h^{-1} y_h'\|_{L^\infty([0,T],\, H^{-1}(\Omega))} \leq C, \\ \\ \|\mathbf{P}_h^{-1} y_h''\|_{L^2([0,T],\, H^{-2}(\Omega))} \leq C. \end{cases} \tag{7.7}$$

The first four properties follow by the definition of the interpolators and property (7.5). The last estimate follows by using that y_h solves the discrete wave equation:

$$\begin{aligned} \|\mathbf{P}_h^{-1} y_h''\|_{L^\infty([0,T],\, H^{-2}(\Omega))} &:= \|(-\Delta)\mathbf{P}_h^1(-\Delta_h)^{-1} y_h''\|_{L^2([0,T],\, H^{-2}(\Omega))} \\ &\leq \|(-\Delta_h)^{-1} y_h''\|_{L^2([0,T],\, L^2(\Omega))} \\ &\leq \|y_h\|_{L^2([0,T],\, L^2(\Omega))} + \|y_h\|_{L^2([0,T],\, L^2(\Gamma_h))} \\ &\leq C + \|v_h\|_{L^2([0,T],\, L^2(\Gamma_{0h}))} \leq 2C. \end{aligned}$$

Lemma 6.1 gives us that

$$\|\mathbf{P}_h^0 y_h - \mathbf{P}_h^{-1} y_h\|_{L^2([0,T],\, H^{-1}(\Omega))} \leq hT\|y_h\|_{0,h} \to 0$$

as $h \to 0$. Using estimates (7.6) and (7.7) we obtain the existence of a function $y \in W^{1,\infty}((0,T), H^{-1}(\Omega))$ such that, up to a subsequence,

$$\text{(7.8)} \qquad \begin{cases} \mathbf{P}_h^0 y_h \rightharpoonup y & \text{in} \quad H^1(0,T), H^{-1}(\Omega)), \\ \\ \mathbf{P}_h^{-1} y_h' \rightharpoonup y' & \text{in} \quad L^2(0,T), H^{-1}(\Omega)). \end{cases}$$

Estimates (7.6) show that (see [27], Corollary 1), up to a subsequence, $\mathbf{P}_h^0 y_h \rightharpoonup y$ in $C([0,T],\, H^{-1}(\Omega))$. In particular $\mathbf{P}_h^0 y_h(0) \to y(0)$ in $H^{-1}(\Omega)$. Using that $\mathbf{P}_h^0 y_h(0)$ is uniformly bounded in $L^2(\Omega)$ we get $\mathbf{P}_h^0 y_h(0) \rightharpoonup y(0)$ in $L^2(\Omega)$ and, by (7.1), we obtain $y(0) = y^0$.

The last two estimates of (7.7) show that (see [27], Corollary 1), up to a subsequence, $\mathbf{P}_h^{-1} y_h' \to y'$ strongly in $C([0,T],\, H^{-2}(\Omega))$. In particular $\mathbf{P}_h^{-1} y_h'(0) \to y'(0)$ in $H^{-2}(\Omega))$. Using that $\mathbf{P}_h^{-1} y_h'(0)$ is uniformly bounded in $H^{-1}(\Omega)$ we get $\mathbf{P}_h^{-1} y_h'(0) \rightharpoonup y'(0)$ in $H^{-1}(\Omega)$ and, by (7.1), we obtain $y'(0) = y^1$.

Let us choose $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ as final state in the adjoint equation (7.2). We choose (u_h^0, u_h^1) in the adjoint discrete system (5.3) such that $\mathbf{P}_h^1 u_h^0 \to u^0$ in $H_0^1(\Omega)$ and $\mathbf{P}_h^0 u_h^1 \to u^1$ in $L^2(\Omega)$. In view of Proposition 6.1 we have the following strong convergence properties

$$\text{(7.9)} \qquad \begin{cases} \mathbf{P}_h^1 u_h \to u & \text{in} \quad L^2([0,T],\, H_0^1(\Omega)), \\ \mathbf{P}_h^0 u_h' \to u' & \text{in} \quad L^2([0,T],\, L^2(\Omega)), \end{cases}$$

where u is the solution of equation (1.3) with final states (u^0, u^1).

We write (7.4) as follows

$$\int_0^s \int_{\Gamma_0} \mathbf{P}_{0,\Gamma}^h v_h \mathbf{P}_{0,\Gamma}^h (\partial_n^h u_h) d\sigma dt + \langle (\mathbf{P}_h^0 y_h, \mathbf{P}_h^{-1} y_h'), (\mathbf{P}_h^1 u_h, \mathbf{P}_h^0 u_h') \rangle \Big|_0^s = 0.$$

Using that $\mathbf{P}_h^0 y_h \rightharpoonup y$ weakly in $L^2((0,T),\, L^2(\Omega))$ and $\mathbf{P}_h^{-1} y_h' \rightharpoonup y'$ weakly in $L^2((0,T),\, H^{-1}(\Omega))$ and letting $h \to 0$ we obtain

$$\int_0^s \int_{\Gamma_0} \partial_n u^* \partial_n u d\sigma dt + \langle (y, y'), (u, u') \rangle_h \Big|_0^s = 0,\ \forall\, s < T,$$

where u is solution of problem (7.2) with final state (u^0, u^1). Thus y is a solution by transposition of (1.1) with control $v = \partial_n u^*$.

Step III. Final time control requirement. We prove that (1.2) holds. We consider the case of $y(T)$ the other case being similar. Since $(y_h(T), w_h)_h = 0$ for all functions $w_h \in V^h$ we obtain that

$$\int_\Omega \mathbf{P}_h^0 y_h(T) \mathbf{P}_h^0 w_h dx = 0 \text{ for all } h > 0.$$

Using that $\mathbf{P}_h^0 y_h(T) \to y(T)$ strongly in $L^2(\Omega)$ and that $\mathbf{P}_h^0(V^h)$ is dense in $L^2(\Omega)$ we get

$$\int_\Omega y(T) w dx = 0$$

for all functions $w \in L^2(\Omega)$. Thus $y(T) \equiv 0$.

Finally, using the uniqueness results for problem (1.1) we obtain that the control v obtained before satisfies $v = \partial_n u^*$ where u^* is the solution of problem (7.2) with final state $(u^{*,0}, u^{*,1})$ minimizing functional (7.3).

□

8. Concluding remarks

In this article we have developed a quite systematic approach to prove the convergence of the controls obtained by two-grid methods. The key ingredient is to use the dyadic decomposition argument to reduce the problem of observability to that of dealing with *low frequency* solutions in which the high frequency components have been filtered. The method we propose works on regular meshes, both for finite differences and finite elements and can be also adapted to fully discrete approximation schemes.

In the following we comment about how it can be used or improved and what are its limitations.

- **Other models.** The proof of the observability for filtered solutions has been the object of several works, not only for the wave equation [11], [28], but also for Schrödinger equations [18] and beam equations [14], among others. The method we have used here can be applied to these conservative systems too.
- **Other control mechanisms.** This article has been devoted to the problem of boundary observability. But, in fact, the method we develop applies in a much more general context, and, in particular, in the problem of internal observability for which the measurement on solutions is done in an open subset ω of the domain Ω.
- **Control of nonlinear wave equations.** In the case of nonlinear problems in dimension one, in [30] the convergence of the two-grid algorithm was proved for semilinear wave equations with globally Lipschitz nonlinearities. The combination of the methods of this paper and [30] yield the same result in the multidimensional case too.
- **Fully discrete schemes.** In this article we have analyzed semidiscrete models but the same analysis of uniform observability could be performed on fully discrete discretizations of the wave equation. In the one dimensional case these models have been studied in [21] under a special assumption on the time and space steps $\Delta t = \Delta x$ and in [23] for general meshes using discrete versions of Ingham's inequalities. Combining our techniques with those of [23] the same results of the convergence of two-grid methods can be proved for fully discrete approximation schemes.
- **More general meshes.** However, the method presented here has its limitations. We used intensively Fourier analysis techniques, which is not available for irregular meshes, that require of further developments.
- **Meshes with ratio m/n.** The two-grid method we proposed here had a mesh-ratio of the form $1/p$. One could expect the uniform observability to hold in 1-d for any mesh-ratio $m/n < 1$, in the multidimensional case, when $m/n < 1/2$. The only difficulty for doing that is to prove a result similar to Lemma 3.1 for all the functions in the image of $\mathbf{\Pi}_h^{n/mh}\mathcal{G}_h$. As far as we know, these are open problems.
- **Other boundary conditions.** In the proof we use the so-called *direct inequality* whose analogue fails for other closely related problems, as the boundary control of the wave equation with Neumann boundary conditions.
- **Spectral conditions for observability.** In recent works Tucsnak [25], Miller [20] (see also Russell and Weiss [26]), the authors give a spectral condition which guarantees the observability for infinite dimensional conservative systems. This type of condition generalize the Hautus test for finite dimensional systems to infinitely dimensional ones. It would be interesting to see if these spectral methods can be adapted in order to guarantee uniform observability results for numerical methods based on the

two-grid method. The main difficulty in applying these results is due to the fact that the space V^h of two-grid data is not invariant under the semidiscrete wave flow.

Appendix A. Proof of Lemma 4.1

In this Appendix we prove Lemma 4.1. The main ingredient is the following lemma inspired in ideas of [12], [2] and adapted to our context. In the sequel X denotes the space $L^p(\Omega, d\mu)$, where μ is a Borel measure and $\mu(\Omega) < \infty$.

Lemma A.1. *Let be $c > 1$, $T > 0$, $P \in C_c^\infty(\mathbb{R})$ and $(P_k)_{k\geq 0}$ as in (4.1). Also let $\varphi \in C_0^\infty(0,T)$ and $\psi \in L^\infty(\mathbb{R})$ be satisfying $\psi \equiv 1$ on $(0,T)$. There exists a positive constant $C = C(T,\varphi,\psi,P)$ such that*

$$\int_{\mathbb{R}} \|\varphi(t)P_k(w)(t)\|_X^2 dt \leq 2\int_{\mathbb{R}} \|\varphi(t)P_k(\psi w)(t)\|_X^2 dt + Cc^{-2k} \sup_{l\in\mathbb{Z}} \|w\|^2_{L^2((lT,(l+1)T),\, X)} \tag{A.1}$$

holds for all $w \in L^2_{loc}(\mathbb{R},\, X)$ and for all $k \geq 0$.

Proof of Lemma A.1. We denote $I_l = [lT,(l+1)T)$ and $w_l = 1_{I_l} w$. We claim the existence of a positive constant $C(P)$ such that for all $\varphi \in C_0^\infty(\mathbb{R})$ and $l \in \mathbb{Z}$ with $\mathrm{dist}(I_l, \mathrm{supp}(\varphi)) > 0$ the following holds:

$$\sup_{t\in[0,T]} \|\varphi(t)P_k(w_l)\|_X \leq \frac{C(P)T^{1/2}c^{-k}}{\mathrm{dist}(I_l,\mathrm{supp}(\varphi))^2}\|\varphi\|_{L^\infty(\mathbb{R})} \sup_{l\in\mathbb{Z}} \|w_l\|_{L^2(\mathbb{R},X)}, \tag{A.2}$$

uniformly for all $k \geq 0$.

Using estimate (A.2) we will prove the existence of a positive constant $C = C(T,\varphi,\psi,P)$ such that

$$\sup_{t\in[0,T]} \|\varphi(t)(P_k(w) - P_k(\psi w))(t)\|_X \leq Cc^{-k} \sup_{l\in\mathbb{Z}} \|w_l\|_{L^2(\mathbb{R},\, X)}. \tag{A.3}$$

Then, (A.1) will be a consequence of Minkowsky's and Cauchy's inequality:

$$\begin{aligned}\int_{\mathbb{R}} \|\varphi(t)P_k(w)(t)\|_X^2 dt &\leq 2\int_{\mathbb{R}} \|\varphi(t)P_k(\psi w)(t)\|_X^2 dt + 2\int_{\mathbb{R}} \|\varphi(t)P_k(w-\psi w)(t)\|_X^2 dt \\ &\leq 2\int_{\mathbb{R}} \|\varphi(t)P_k(\psi w)(t)\|_X^2 dt + 2T \sup_{t\in[0,T]} \|\varphi(t)(P_k(w-\psi w))(t)\|_X^2 \\ &\leq 2\int_{\mathbb{R}} \|\varphi(t)P_k(\psi w)(t)\|_X^2 dt + Cc^{-k} \sup_{l\in\mathbb{Z}} \|w_l\|^2_{L^2(\mathbb{R},\, X)}.\end{aligned}$$

Step I. Proof of (A.2). The definition of the projector P_k and integration by parts give us

$$\begin{aligned}\varphi(t)P_k(w_l)(t) &= \int_{\mathbb{R}_\tau}\int_{\mathbb{R}_s} e^{i\tau(t-s)} P(c^{-k}\tau)\varphi(t)w_l(s)\,ds\,d\tau \\ &= \int_{\mathbb{R}_\tau}\int_{\mathbb{R}_s} e^{i\tau(t-s)} i^2\partial_\tau^2[P(c^{-k}\tau)]\frac{\varphi(t)w_l(s)}{(t-s)^2}\,ds\,d\tau.\end{aligned}$$

Thus, for any t in the support of φ we have $\mathrm{dist}(\mathrm{supp}(\varphi), I_l) > 0$ and by Minkowsky's inequality yields

$$\begin{aligned}
\|\varphi(t)P_k(w_l)(t)\|_X &\le c^{-2k}\|\varphi\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}_\tau}|(\partial_\tau^2 P)(c^{-k}\tau)|d\tau\int_{I_l}\frac{\|w_l(s)\|_X}{(t-s)^2}ds\\
&\le \frac{c^{-k}\|\varphi\|_{L^\infty(\mathbb{R})}}{(\mathrm{dist}(\mathrm{supp}(\varphi),I_l))^2}\int_{\mathbb{R}_\tau}|(\partial_\tau^2 P)(\tau)|d\tau\int_{I_l}\|w_l(s)\|_X ds.\\
&\le \frac{T^{1/2}c^{-k}\|\varphi\|_{L^\infty(\mathbb{R})}}{(\mathrm{dist}(\mathrm{supp}(\varphi),I_l))^2}\int_{\mathbb{R}_\tau}|(\partial_\tau^2 P)(\tau)|d\tau\left(\int_{I_l}\|w_l(s)\|_X^2 ds\right)^{1/2}.
\end{aligned}$$

Step II. Proof of (A.3). Observe that on I_0, $w \equiv w\psi$. This yields the following decomposition of the difference $P_k(w) - P_k(\psi w)$:

$$P_k(w) - P_k(\psi w) = \sum_{|l|\ge 1} P_k(w_l - (\psi w)_l) = \sum_{|l|\ge 1} P_k(b_l), \tag{A.4}$$

with $b_l = w_l - (\psi w)_l$. Let us choose $\delta > 0$ such that φ is supported on $(\delta, T-\delta)$. Thus for all $|l| \ge 2$, the function b_l satisfies $\mathrm{dist}(\mathrm{supp}(\varphi), I_l) \ge T(|l|-1)$. Also, for $|l| = 1$: $\mathrm{dist}(\mathrm{supp}(\varphi), I_l) \ge \delta$. By (A.2) we obtain

$$\sup_{t\in\mathbb{R}}\|\varphi(t)P_k(b_l)(t)\|_X \le C(P)T^{1/2}c^{-k}\|\varphi\|_{L^\infty(\mathbb{R})}\sup_{l\in\mathbb{Z}}\|b_l\|_{L^2(\mathbb{R},X)}\begin{cases}\frac{1}{T^2(|l|-1)^2}, & |l|\ge 2,\\[2mm] \frac{1}{\delta^2}, & |l| = 1.\end{cases} \tag{A.5}$$

By (A.4) and (A.5) we obtain the existence of a constant $C = C(T, \varphi, \psi, P)$ such that for any $t \in [0, T]$ the following holds

$$\begin{aligned}
\|\varphi(t)[P_k(w) - P_k(\psi w)]\|_X &\le \sum_{|l|\ge 1}\|\varphi(t)P_k(b_l)\|_X \le Cc^{-k}\sup_{l\in\mathbb{Z}}\|b_l\|_{L^2(\mathbb{R},X)}\\
&\le Cc^{-k}\sup_{l\in\mathbb{Z}}\|w\|_{L^2(\mathbb{R},X)}.
\end{aligned}$$

The proof is now complete. □

Proof of Lemma 4.1. Let us choose a function $\varphi \in C_0^\infty(0, T)$ such that $|\varphi| \le 1$ and $\varphi \equiv 1$ on $[2\delta, T-2\delta]$. Applying Lemma A.1 to the function w and $\psi = \mathbf{1}_{(0,T)}$, we obtain the existence of a positive constant $C(\delta, T, P)$ such that

$$\begin{aligned}
\int_{2\delta}^{T-2\delta}\|P_k w\|_X^2 dt &\le \int_{\mathbb{R}}\varphi^2\|P_k(w)\|_X^2 dt\\
&\le 2\int_{\mathbb{R}}\varphi^2\|P_k(\psi w)\|_X^2 dt + \frac{C(\delta,T,P)}{c^{2k}}\sup_{l\in\mathbb{Z}}\|w\|^2_{L^2((lT,(l+1)T),X)}.
\end{aligned}$$

Summing all these inequalities we get

$$\sum_{k\ge k_0}\int_{2\delta}^{T-2\delta}\|P_k w\|_X^2 dt \le 2\sum_{k\ge k_0}\int_{\mathbb{R}}\varphi^2\|P_k(\psi w)\|_X^2 dt + \frac{C(\delta,T,P)}{c^{2k_0}}\sup_{l\in\mathbb{Z}}\|w\|^2_{L^2((lT,(l+1)T),X)}.$$

In the following we prove the existence of a positive constant $C(P, c)$ such that

$$\sum_{k\ge 0}\int_{\mathbb{R}}\varphi^2\|P_k(\psi w)\|_X^2 dt \le C(P,c)\int_0^T\|w(t)\|_X^2 dt.$$

Observe that any real number τ belongs either to a finite number of intervals of the form $(\pm ac^k, \pm bc^k)$ or to none of them. Then there is a positive constant $C(P,c)$ such that

$$\sup_{\tau\in\mathbb{R}} \sum_{k\geq 0} P^2(c^{-k}\tau) \leq C(P,c). \tag{A.6}$$

Applying Plancherel's identity in the time variable we obtain

$$\begin{aligned}
\sum_{k\geq 0} \int_{\mathbb{R}} \varphi^2(t)\|P_k(\psi w)(t)\|_X^2 dt &\leq \|\varphi\|^2_{L^\infty(\mathbb{R})} \sum_{k\geq 0}\int_{\mathbb{R}} \|P_k(\psi w)(t)\|_X^2 dt \\
&= \|\varphi\|^2_{L^\infty(\mathbb{R})} \sum_{k\geq 0}\int_{\mathbb{R}} P^2(c^{-k}\tau)\|\widehat{\psi w}(\tau)\|_X^2 d\tau \\
&\leq \|\varphi\|^2_{L^\infty(\mathbb{R})} \sup_{\tau\in\mathbb{R}}\sum_{k\geq 0} P^2(c^{-k}\tau)\int_{\mathbb{R}} \|\widehat{\psi w}(\tau)\|_X^2 d\tau \\
&\leq C(P,c)\|\varphi\|^2_{L^\infty(\mathbb{R})}\int_{\mathbb{R}} \|(\psi w)(t)\|_X^2 dt = C(P,c)\|\varphi\|^2_{L^\infty(\mathbb{R})}\int_0^T \|w(t)\|_X^2 dt.
\end{aligned}$$

□

Appendix B. Spectral analysis of V^h-functions

In this Section we analyze the $\mathcal{H}^s_h(\Omega_h)$-norms of the functions belonging to V^h, i.e. the space of functions defined on the fine grid as a linear interpolation of the functions defined on the coarse one, and we prove Lemma 3.1. We will consider periodic discrete functions defined on the grid $x_0 = 0, x_1 = h, \cdots = x_{2N+1} = (2N+1)h = 2$ instead of vanishing at the boundary, but all the results also apply to this case.

We first obtain in the following Lemma a description of the Fourier coefficients $\widehat{v}(\mathbf{j})$ of a periodic function $v \in V^h$ and then prove Lemma 3.1.

Lemma B.1. *Let $p \geq 2$, $N, \tilde{N}$ positive integers such that $2N = p\tilde{N}$, $h = 2/(2N+1)$ and the discrete function $v(pk)$, $k \in \Lambda_{\tilde{N}}$. Then the discrete function $u(k)$, $k \in \Lambda_{2N}$, obtained from the linear interpolation of v, $u = \mathbf{P}^1_h v$, has the Fourier coefficients satisfying*

$$\widehat{u}(\mathbf{j}) = e^{i(p-1)(j_1 h+\cdots+j_l h)\pi}\prod_{l=1}^{d}\Big(p^{-1}\sum_{k=0}^{p-1} e^{ik\pi j_k h}\Big)^2 \widehat{v}(\mathbf{j}),\ \mathbf{j} = (j_1,\ldots,j_d).$$

In particular for any $\mathbf{j}$

$$|\widehat{u}(\mathbf{j})| \simeq p^{-2d}|\widehat{v}(\mathbf{j})|\prod_{r=1}^{d}\left|\frac{e^{-ip\pi j_r h}-1}{e^{-i\pi j_r h}-1}\right|^2. \tag{B.1}$$

Proof. We will analyze the one-dimensional case. Iterating the same argument in each space direction the same holds in several space dimensions. In this case, we write in an explicit manner the function u:

$$u(kp+j) = \frac{(p-j)v(kp) + jv((k+1)p)}{p},\ k = 0,\ldots,\tilde{N}-1,\ j = 0,\ldots,p-1.$$

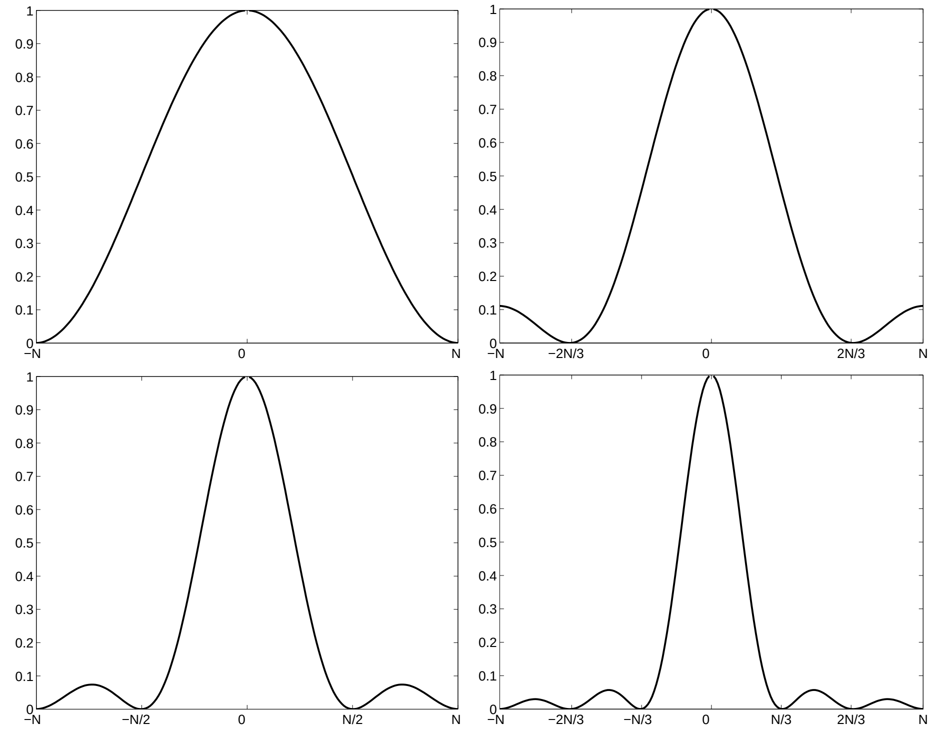

FIGURE 7. The multiplicative factors generated by the two-grid algorithm with mesh-sizes ratio 1/2, 1/3, 1/4, 1/6 respectively.

The k-th Fourier coefficient of u is given by

$$\widehat{u}(j) = h\sum_{k=0}^{2N} u_j e^{-i\pi jkh},\ k=-N,\dots,N.$$

Explicit computation give us:

$$\begin{aligned}\widehat{u}(j) &= h\sum_{k=0}^{\tilde{N}-1}\sum_{r=0}^{p-1} e^{-i\pi j(kp+r)h}u(kp+r) = h\sum_{k=0}^{\tilde{N}-1}\sum_{r=0}^{p-1} e^{-i\pi j(kp+r)h}\frac{(p-r)v(kp)+rv((k+1)p)}{p}\\ &= \frac{h}{p}\sum_{k=0}^{\tilde{N}-1} e^{-i\pi jkph}v(kp)\Big(\sum_{r=0}^{p-1} e^{-2i\pi jrh}(p-r)+\sum_{r=0}^{p-1} e^{i\pi j(p-r)h}r\Big)\\ &= \widehat{v}(j)e^{i\pi(p-1)h}\Big(p^{-1}\sum_{r=0}^{p-1} e^{-i\pi jrh}\Big)^2.\end{aligned}$$

In particular

$$|\widehat{u}(j)| \simeq p^{-2}|\widehat{v}(j)|\left|\frac{e^{-ipj\pi h}-1}{e^{-i\pi jh}-1}\right|^2.$$

$\square$

Proof of Lemma 3.1. Using that for any $\mathbf{j}$ with $\|\mathbf{j}\|_\infty \le N/p$ we have

$$p^{-2d}\prod_{r=1}^{d}\left|\frac{e^{-ip\pi j_r h}-1}{e^{-i\pi j_r h}-1}\right|^2 \simeq 1$$

we get

$$\|\Upsilon_h^{1/p}u\|^2_{\mathcal{H}_h^s} \simeq \sum_{\|\mathbf{j}\|_\infty\le N/p}\lambda_{\mathbf{j}}^{2s}|\widehat{v}(\mathbf{j})|^2.$$

We split the $\mathcal{H}_h^s$ norm of u as follows:

$$\begin{aligned}
\|u\|^2_{\mathcal{H}_h^s} &= \sum_{\|\mathbf{j}\|_\infty\le N}\lambda_{\mathbf{j}}^{2s}(h)p^{-4d}\prod_{r=1}^{d}\left|\frac{\exp(-ip\pi j_r h)-1}{\exp(-i\pi j_r h)-1}\right|^4|\widehat{v}(\mathbf{j})|^2\\
&\le p^{-4d}\sum_{\|\mathbf{j}\|_\infty\le N/p}\lambda_{\mathbf{j}}^{2s}(h)|\widehat{v}(\mathbf{j})|^2 + p^{-4d}\sum_{N/p\le\|\mathbf{j}\|_\infty\le N}\lambda_{\mathbf{j}}^{2s}(h)\prod_{r=1}^{d}\left|\frac{\exp(-ip\pi j_r h)-1}{\exp(-i\pi j_r h)-1}\right|^4|\widehat{v}(\mathbf{j})|^2\\
&\le c(p,d)\sum_{\|\mathbf{j}\|_\infty\le N/p}\lambda_{\mathbf{j}}^{2s}(h)|\widehat{v}(\mathbf{j})|^2 + c(p,d)h^{-2s}\sum_{N/p\le\|\mathbf{j}\|_\infty\le N}\prod_{r=1}^{d}\left|\frac{\exp(-ip\pi j_r h)-1}{\exp(-i\pi j_r h)-1}\right|^4|\widehat{v}(\mathbf{j})|^2\\
&\le c(p,d)(I_1+I_2).
\end{aligned}$$

We prove that for any $\mathbf{j}$ with $N/p \le \|\mathbf{j}\|_\infty \le N$ the following holds:

$$\prod_{r=1}^{d}\left|\frac{\exp(-ip\pi j_r h)-1}{\exp(-i\pi j_r h)-1}\right|^4 \le \sum_{r=1}^{d}|\exp(-ip\pi j_r h)-1|^{2s}.$$

Let us suppose that $j_1 = \|\mathbf{j}\|_\infty \ge N/p$. Thus $|\exp(-i\pi j_1 h)-1| \ge c_0 > 0$ with c_0 independent of h. Using that the following inequality

$$\left|\frac{e^{-ip\xi}-1}{e^{-i\xi}-1}\right| \le p$$

holds for any $\xi \in (-\pi,\pi)$, we obtain that

$$\begin{aligned}
\prod_{r=1}^{d}\left|\frac{\exp(-ip\pi j_r h)-1}{\exp(-i\pi j_r h)-1}\right|^4 &\le p^{d-1}\left|\frac{\exp(-ip\pi j_1 h)-1}{\exp(-i\pi j_1 h)-1}\right|^4 \le c(p,d)|\exp(-ip\pi j_1 h)-1|^4\\
&\le c(p,d,s)|\exp(-ip\pi j_1 h)-1|^{2s}
\end{aligned}$$

provided that $s \le 2$.

Then, using the periodicity of the coefficients $\widehat{v}(\mathbf{j})$ and of $\exp(-ip\pi j_r h)$, we get

$$\begin{aligned}
I_2 &\le c(p,d,s)\sum_{N/p\le\|\mathbf{j}\|_\infty\le N}|\widehat{v}(\mathbf{j})|^2\sum_{r=1}^{d}\left|\frac{\exp(-ip\pi j_r h)-1}{h}\right|^{2s}\\
&= (p^d-1)c(p,d,s)\sum_{\|\mathbf{j}\|_\infty\le N/p}|\widehat{v}(\mathbf{j})|^2\sum_{r=1}^{d}\left|\frac{\exp(-ip\pi j_r h)-1}{h}\right|^{2s}\\
&\le c(p,d,s)\sum_{\|\mathbf{j}\|_\infty\le N/p}|\widehat{v}(\mathbf{j})|^2\sum_{r=1}^{d}\left|\frac{\exp(-i\pi j_r h)-1}{h}\right|^{2s} \le c(p,d,s)\sum_{\|\mathbf{j}\|_\infty\le N/p}\lambda_{\mathbf{j}}^{2s}|\widehat{v}(\mathbf{j})|^2.
\end{aligned}$$

The proof is now complete. □

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References

1. M. Asch and G. Lebeau, *Geometrical aspects of exact boundary controllability for the wave equation–a numerical study*, ESAIM Control Optim. Calc. Var. **3** (1998), 163–212.
2. N. Burq, *Contrôlabilité exacte des ondes dans des ouverts peu réguliers*, Asymptot. Anal. **14** (1997), no. 2, 157–191.
3. N. Burq, P. Gérard, and N. Tzvetkov, *Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds*, Amer. J. Math. **126** (2004), no. 3, 569–605.
4. C. Castro and S. Micu, *Boundary controllability of a linear semi-discrete 1-D wave equation derived from a mixed finite element method*, Numer. Math. **102** (2006), no. 3, 413–462.
5. C. Castro, S. Micu, and A. Münch, *Boundary controllability of a semi-discrete 2-D wave equation with mixed finite elements*, preprint.
6. R. Glowinski, *Ensuring well-posedness by analogy; Stokes problem and boundary control for the wave equation*, J. Comput. Phys. **103** (1992), no. 2, 189–221.
7. R. Glowinski, W. Kinton, and M. F. Wheeler, *A mixed finite element formulation for the boundary controllability of the wave equation*, Internat. J. Numer. Methods Engrg. **27** (1989), no. 3, 623–635.
8. R. Glowinski, C. H. Li, and J.-L. Lions, *A numerical approach to the exact boundary controllability of the wave equation. I. Dirichlet controls: description of the numerical methods*, Japan J. Appl. Math. **7** (1990), no. 1, 1–76.
9. R. Glowinski and J.-L. Lions, *Exact and approximate controllability for distributed parameter systems*, Acta numerica, 1995, Cambridge Univ. Press, Cambridge, 1995, pp. 159–333.
10. L.I. Ignat and E. Zuazua, *A two-grid approximation scheme for nonlinear Schrödinger equations: dispersive properties and convergence*, C. R. Acad. Sci. Paris, Ser. I **341** (2005), no. 6, 381–386.
11. J.A. Infante and E. Zuazua, *Boundary observability for the space semi-discretizations of the 1-D wave equation*, M2AN Math. Model. Numer. Anal. **33** (1999), no. 2, 407–438.
12. G. Lebeau, *Contrôle de l'équation de Schrödinger*, J. Math. Pures Appl. **71** (1992), no. 3, 267–291.
13. ———, *The wave equation with oscillating density: observability at low frequency*, ESAIM Control Optim. Calc. Var. **5** (2000), 219–258.
14. L. León and E. Zuazua, *Boundary controllability of the finite-difference space semi-discretizations of the beam equation*, ESAIM Control Optim. Calc. Var. **8** (2002), 827–862.
15. J.-L. Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1*, Recherches en Mathématiques Appliquées [Research in Applied Mathematics], vol. 8, Masson, Paris, 1988.
16. J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications. Vol. 1-2*, Travaux et Recherches Mathématiques, No. 17, Dunod, Paris, 1968.
17. P. Loreti and M. Mehrenberger, *An Ingham type proof for a two-grid observability theorem*, ESAIM Control Optim. Calc. Var., to appear.
18. F. Macia, *Propagación y control de vibraciones en medios discretos y continuos, tesis de doctorado.*, Universidad Complutense de Madrid (2002).
19. S. Micu, *Uniform boundary controllability of a semi-discrete 1-D wave equation*, Numer. Math. **91** (2002), no. 4, 723–768.
20. L. Miller, *Controllability cost of conservative systems: resolvent condition and transmutation*, J. Funct. Anal. **218** (2005), no. 2, 425–444.
21. M. Negreanu and E. Zuazua, *Uniform boundary controllability of a discrete 1-D wave equation*, Systems Control Lett. **48** (2003), no. 3-4, 261–279, Optimization and control of distributed systems.

22. ———, *Convergence of a multigrid method for the controllability of a 1-D wave equation*, C. R. Math. Acad. Sci. Paris **338** (2004), no. 5, 413–418.
23. ———, *Discrete Ingham inequalities and applications*, SIAM J. Numer. Anal. **44** (2006), no. 1, 412–448 (electronic).
24. A. Quarteroni and A. Valli, *Numerical approximation of partial differential equations.*, Springer Series in Computational Mathematics. 23, 1994.
25. K. Ramdani, T. Takahashi, G. Tenenbaum, and M. Tucsnak, *A spectral approach for the exact observability of infinite-dimensional systems with skew-adjoint generator*, J. Funct. Anal. **226** (2005), no. 1, 193–229.
26. D. L. Russell and G. Weiss, *A general necessary condition for exact observability*, SIAM J. Control Optim. **32** (1994), no. 1, 1–23.
27. J. Simon, *Compact sets in the space* $L^p(0,T;B)$, Ann. Mat. Pura Appl., IV. Ser. **146** (1987), 65–96.
28. E. Zuazua, *Boundary observability for the finite-difference space semi-discretizations of the 2-D wave equation in the square*, J. Math. Pures Appl. **78** (1999), no. 5, 523–563.
29. ———, *Propagation, observation, control and numerical approximation of waves approximated by finite difference methods*, SIAM Review **2** (2005), no. 47, 197–243.
30. ———, *Control and numerical approximation of the wave and heat equations*, Proceedings of the ICM 2006, M. Sanz-Solé et. al. eds., vol. III, "Invited Lectures", European Mathematical Society Publishing Hause, 2006, pp. 1389–1417.

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