SET THEORETIC SOLUTIONS OF THE YANG-BAXTER EQUATION, GRAPHS AND COMPUTATIONS

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ABSTRACT. We extend our recent work on set-theoretic solutions of the Yang-Baxter or braid relations with new results about their automorphism groups, strong twisted unions of solutions and multipermutation solutions. We introduce and study graphs of solutions and use our graphical methods for the computation of solutions of finite order and their automorphisms. Results include a detailed study of solutions of multipermutation level 2.

1. INTRODUCTION

It is well-known that certain matrix solutions of the braid or Yang-Baxter equations lead to braided categories, knot invariants, quantum groups and other important constructions, see [M02] for an introduction. However, these equations are also very interesting at the level of set maps $r : X \times X \to X \times X$ where X is a set and r is a bijection, a line of study proposed in [Dri, Wei]. Solutions extend linearly to very special linear solutions but also lead to a great deal of combinatorics and to algebras with very nice homological properties including those relating to the existence of noncommutative Groebner bases. We start with the definitions, using conventions from recent works [GI, GIM].

Definition 1.1. Let X be a nonempty set and let $r : X \times X \longrightarrow X \times X$ be a bijective map. We use notation (X, r) and refer to it as a *quadratic set*. The image of (x, y) under r is presented as

$$(1.1) \qquad r(x,y) = ({}^x y, x^y).$$

The formula (1.1) defines a "left action" $\mathcal{L} : X \times X \longrightarrow X$, and a "right action" $\mathcal{R} : X \times X \longrightarrow X$, on X as:

$$(1.2) \qquad \mathcal{L}_x(y) = {}^x y, \quad \mathcal{R}_y(x) = x^y,$$

for all $x, y \in X$. The map r is *nondegenerate*, if the maps $\mathcal{R}_x$ and $\mathcal{L}_x$ are bijective for each $x \in X$. In this paper we shall always be interested in the case where r is nondegenerate, as will be indicated. As a notational tool, we shall often identify the sets $X \times X$ and X^2, the set of all monomials of length two in the free semigroup $\langle X \rangle$.

Definition 1.2.   (1) r is *square-free* if $r(x,x) = (x,x)$ for all $x \in X$.

   (2) r is *a set-theoretic solution of the Yang-Baxter equation* or, shortly *a solution* (YBE) if the braid relation
$$r^{12}r^{23}r^{12} = r^{23}r^{12}r^{23}$$
holds in $X \times X \times X$, where the two bijective maps $r^{ii+1} : X^3 \longrightarrow X^3$, $1 \leq i \leq 2$ are defined as $r^{12} = r \times Id_X$, and $r^{23} = Id_X \times r$. In this case some authors also call (X, r) *a braided set*. If in addition r is involutive (X, r) is called *a symmetric set*.

To each quadratic set (X, r) we associate canonical algebraic objects generated by X and with quadratic defining relations $\Re = \Re(r)$ defined by

(1.3) $\qquad\qquad xy = zt \in \Re(r), \quad$ whenever $\quad r(x, y) = (z, t)$.

Definition 1.3. Let (X, r) be a quadratic set.

(i) The semigroup
$$S = S(X, r) = \langle X; \Re(r) \rangle,$$
with a set of generators X and a set of defining relations $\Re(r)$, is called *the semigroup associated with* (X, r).

(ii) The *group* $G = G(X, r)$ *associated with* (X, r) is defined as
$$G = G(X, r) = {}_{gr}\langle X; \Re(r) \rangle.$$

(iii) For arbitrary fixed field k, *the k-algebra associated with* (X, r) is defined as

(1.4) $\qquad\qquad \mathcal{A} = \mathcal{A}(k, X, r) = k\langle X \rangle / (\Re(r)).$

If (X, r) is a solution, then $S(X, r)$, resp. $G(X, r)$, resp. $\mathcal{A}(k, X, r)$ is called the *Yang-Baxter semigroup*, resp. the *Yang-Baxter group*, resp. the *Yang-Baxter algebra* associated to (X, r).

Example 1.4. For arbitrary nonempty set X, *the trivial solution* (X, r) is defined as $r(x, y) = (y, x)$, for all $x, y \in X$. It is clear that (X, r) is the trivial solution *iff* ${}^x y = y$, and $x^y = x$, for all $x, y \in X$, or equivalently $\mathcal{L}_x = id_X = \mathcal{R}_x$ for all $x \in X$. In this case $S(X, r)$ is the free abelian monoid, $G(X, r)$ is the free abelian group, $\mathcal{A}(k, X, r)$ the algebra of commutative polynomials in X.

The algebras $\mathcal{A}(X, r)$ provided new classes of Noetherian rings [GI94, GI96-1], Gorentstein (Artin-Schelter regular) rings [GI96-2, GI00, GI04] and so forth. Artin-Schelter regular rings were introduced in [AS] and are of particular interest. The algebras $\mathcal{A}(X, r)$ are similar in spirit to the quadratic algebras associated to linear solutions, particularly studied in [M], but have their own remarkable properties in the set theoretic case. The semigroups $S(X, r)$ were studied particularly in [GIM] with a systematic theory of 'exponentiation' from the set to the semigroup by means of the 'actions' $\mathcal{L}_x, \mathcal{R}_x$ (which in the process become a matched pair of semigroup actions) somewhat in analogy with the Lie theoretic exponentiation in [M90]. Also in [GIM] are first results about construction of extensions of solutions that we shall take further. There are many other works on set-theoretic solutions and related structures, of which a relevant selection for the interested reader is [ESS, GB, GJO, JO, LYZ, R, T]. Of particular interest in the present paper, we define:

Definition 1.5. When (X, r) is nondegenerate, $\mathcal{L} : G(X, r) \longrightarrow \mathrm{Sym}(X)$ is a group homomorphism defined via the left action. We denote by $\mathcal{G} = \mathcal{G}(X, r)$ the subgroup $\mathcal{L}(G(X, r))$ of $\mathrm{Sym}(X)$.

For example, $\mathcal{G}(X, r) = \{id_X\}$ in the case of the trivial solution.

Remark 1.6. As is well known, see for example [GIM], a quadratic set (X, r) is a braided set (i.e. r obeys the YBE) *iff* the following conditions hold

$$\mathbf{l1}: \quad {}^x({}^y z) = {}^{x}y({}^{x^y} z), \qquad \mathbf{r1}: \quad (x^y)^z = (x^{y^z})^{y^z},$$

$$\mathbf{lr3}: \quad ({}^x y)^{({}^{x^y}(z))} = {}^{(x^{y^z})}(y^z),$$

for all $x, y, z \in X$. Clearly, conditions **l1**, respectively, **r1** imply that the group $G(X, r)$ acts on the left, respectively on the right, on the set X. This is needed for the definition of $\mathcal{G}(X, r)$ above to make sense.

We also have some technical conditions and special cases of interest, particularly certain 'cyclicity' conditions. Systematic treatments were provided in [GI, GIM] and we recall some essentials needed for the paper in the remainder of this introduction.

Definition 1.7. [GIM] Given (X, r) we extend the actions ${}^x\bullet$ and $\bullet^x$ on X to left and right actions on $X \times X$ as follows. For $x, y, z \in X$ we define:

$$^x(y, z) := ({}^x y, {}^{x^y} z), \quad \text{and} \quad (x, y)^z := (x^{y^z}, y^z).$$

We say that r is respectively left and right invariant if

$$\mathbf{l2}: \quad r(^x(y, z)) = {}^x(r(y, z)), \quad \mathbf{r2}: \quad r((x, y)^z) = (r(x, y))^z$$

hold for all $x, y, z \in Z$.

Definition 1.8. [GIM], A quadratic set (X, r) is called *cyclic* if the following conditions are satisfied

$$\mathbf{cl1}: \quad {}^{y^x} x = {}^y x \quad \text{for all } x, y \in X; \qquad \mathbf{cr1}: \quad x^{{}^x y} = x^y, \quad \text{for all } x, y \in X;$$

$$\mathbf{cl2}: \quad {}^{x^y} x = {}^y x, \quad \text{for all } x, y \in X; \qquad \mathbf{cr2}: \quad x^{y^x} = x^y \quad \text{for all } x, y \in X.$$

We refer to these conditions as *cyclic conditions*.

Theorem 1.13 given below shows that every square-free solution (X, r) is cyclic, furthermore it satisfies condition **lri** and is a *left and a right cycle set*, see the definitions below.

Definition 1.9. [GIM] Let (X, r) be a quadratic set. We define the condition condition

$$\mathbf{lri:} \qquad ({}^x y)^x = y = {}^x(y^x) \text{ for all } x, y \in X.$$

In other words **lri** holds if and only if (X, r) is nondegenerate and $\mathcal{R}_x = \mathcal{L}_x^{-1}$ and $\mathcal{L}_x = \mathcal{R}_x^{-1}$

Remark 1.10. By contrast with this involutiveness does not entail nondegeneracy. For example *the identity solution* (X, r_X), with $r_X = id_{X \times X}$ is obviously involutive, but for each X of order ≥ 2 it is not nondegenerate.

Definition 1.11. [R, GIM] Let (X, r) be nondegenerate. (X, r) is called *a left cycle set* if

$$\mathbf{csl:} \qquad {}^{({}^y t)}({}^y x) = {}^{({}^t y)}({}^t x) \text{ for all } x, y, t \in X.$$

The definition of *a right cycle set* is analogous, see [GIM].

Proposition 1.12. [GIM, Proposition 2.24] *Let (X, r) be a quadratic set. Then any two of the following conditions imply the remaining third condition.*

(1) (X, r) *is involutive.*
(2) (X, r) *is nondegenerate and cyclic, see Definition 1.8.*
(3) **lri** *holds.*

Theorem 1.13. [GIM, Theorem 2.34] *Suppose (X, r) is nondegenerate, involutive and square-free quadratic set. Then the following conditions are equivalent:*

(1) (X, r) *is a set-theoretic solution of the Yang-Baxter equation.*
(2) (X, r) *satisfies* **l1**.
(3) (X, r) *satisfies* **l2**.
(4) (X, r) *satisfies* **r1**.
(5) (X, r) *satisfies* **r2**.
(6) (X, r) *satisfies* **lr3**
(7) (X, r) *satisfies* **csl**.

In this case (X, r) is cyclic and satisfies **lri**.

Corollary 1.14. *Every nondegenerate involutive square-free solution (X, r) is uniquely determined by the left action $\mathcal{L} : X \times X \longrightarrow X$, more precisely,*

$$r(x, y) = (\mathcal{L}_x(y), \mathcal{L}_y^{-1}(x)).$$

Furthermore it is cyclic.

2. Homomorphisms of solutions, automorphisms

Definition 2.1. Let (X, r_X) and (Y, r_Y) be arbitrary solutions (braided sets). A map $\varphi : X \longrightarrow Y$ is a *homomorphism of solutions*, if it satisfies the equality

$$(\varphi \times \varphi) \circ r_X = r_Y \circ (\varphi \times \varphi).$$

A bijective homomorphism of solutions is called (as usual) *an isomorphism* of solution. An isomorphism of the solution (X, r) onto itself is *an r-automorphism.*

We denote by $Hom((X, r_X), (Y, r_Y))$ the set of all homomorphisms of solutions $\varphi : X \longrightarrow Y$. The group of r-automorphisms of (X, r) will be denoted by $\mathrm{Aut}(X, r)$. Clearly, $Aut(X, r)$ is a subgroup of $Sym(X)$.

Remark 2.2. Let $(X, r_X), (Y, r_Y)$ be finite symmetric sets with **lri**. Every homomorphism of solutions $\varphi : (X, r_X) \longrightarrow (Y, r_Y)$ induces canonically a homomorphism of their graphs (see Definition 5.1):

$$\varphi_\Gamma : \Gamma(X, r_X) \longrightarrow \Gamma(Y, r_Y).$$

Furthermore there is a one-to one correspondence between $Aut(X, r)$ and $Aut(\Gamma(X, r))$, the group of *automorphisms* of the multigraph $\Gamma(X, r)$.

The next lemma is straightforward from the definition.

Lemma 2.3. *Let (X, r_X) and (Y, r_Y) be solutions.*

(1) *A map $\varphi : X \longrightarrow Y$ is a homomorphism of solutions iff:*

$$\varphi \circ \mathcal{L}_x = \mathcal{L}_{\varphi(x)} \circ \varphi \quad \text{and} \quad \varphi \circ \mathcal{R}_x = \mathcal{R}_{\varphi(x)} \circ \varphi \text{ for all } x \in X.$$

(2) *If both (X, r_X) and (Y, r_Y) satisfy **lri**, then φ is a homomorphism of solutions* iff

$$\varphi \circ \mathcal{L}_x = \mathcal{L}_{\varphi(x)} \circ \varphi, \text{ for all } x \in X.$$

(3) *If (X, r) obeys **lri**, then $\tau \in Sym(X)$ is an automorphism of (X, r_X)* iff

(2.1) $$\tau \circ \mathcal{L}_x \circ \tau^{-1} = \mathcal{L}_{\tau(x)}, \text{ for all } x \in X.$$

For example, if (X, r) is the trivial solution then, clearly, $\mathrm{Aut}(X, r) = Sym(X)$. This is because each $\mathcal{L}_x = id_X$. More generally, (2.1) clearly implies:

Corollary 2.4. *The group $\mathrm{Aut}(X, r)$ is a subgroup of $\mathrm{Nor}_{Sym(X)}\mathcal{G}(X, r)$, the normalizer of $\mathcal{G}(X, r)$ in $Sym(X)$.*

We now turn to two basic examples which will be used throughout the paper. The first is a solution (X, r) of order 6, with $\mathcal{G}(X, r) \subsetneq Aut(X, r)$ for which $\mathrm{Aut}(X, r) \subsetneq \mathrm{Nor}_{Sym(X)}(\mathcal{G})$. We shall show in general in Section 4 that $\mathcal{G}(X, r)$ is a subgroup of $\mathrm{Aut}(X, r)$ *iff* (X, r) is a multipermutation solution of level 2, see Theorem 5.24.

Example 2.5. Let (X, r) be the nondegenerate involutive square-free solution with

$$X = \{x_1, x_2, x_3, x_4, b, c\}, \quad \mathcal{L}_b = (x_1 x_2)(x_3 x_4), \quad \mathcal{L}_c = (x_1 x_3)(x_2 x_4)$$

and the remaining $\mathcal{L}_{x_i} = id_X$. Here and elsewhere we use Corolary 1.14. Then $\mathcal{G}(X, r) = \langle \mathcal{L}_b \rangle \times \langle \mathcal{L}_c \rangle$, so it is isomorphic to the *Klein's group* $\mathbb{Z}_2 \times \mathbb{Z}_2$. Direct computations show that the set of automorphisms consists of the following eight elements:

$$\begin{array}{llll} id_X, & \tau_1 = (bc)(x_2 x_3), & \tau_2 = (bc)(x_1 x_2 x_4 x_3), \\ \tau_3 = (bc)(x_1 x_3 x_4 x_2), & \tau_4 = (bc)(x_1 x_4), & \mathcal{L}_b, & \mathcal{L}_c, \quad \mathcal{L}_b \circ \mathcal{L}_c. \end{array}$$

Furthermore, one has

$$\tau_1^2 = 1, \quad \tau_2^4 = 1, \quad \tau_1 \tau_2 \tau_1^{-1} = \tau_2^3 = \tau_3, \quad \tau_1 \circ \tau_2 = \mathcal{L}_c, \quad \tau_2 \circ \tau_1 = \mathcal{L}_b, \quad \tau_1 \circ \tau_2^2 = \tau_4.$$

It is easy to see that

$$Aut(X, r) = {}_{gr}\langle \tau_1, \tau_2 \mid \tau_1^2 = 1, \tau_2^4 = 1, \tau_1 \tau_2 \tau_1^{-1} = \tau_2^3 \rangle \approx D_4.$$

Next we show that $Aut(X, r) \subsetneq \mathrm{Nor}_{Sym(X)}(\mathcal{G})$. Consider $\sigma = (x_1 x_3)(x_2 x_4)(bc) \in Sym(X)$. One has:

(2.2) $$\sigma \circ \mathcal{L}_b \circ \sigma^{-1} = \mathcal{L}_b, \quad \sigma \circ \mathcal{L}_c \circ \sigma^{-1} = \mathcal{L}_c,$$

so $\sigma \in \mathrm{Nor}_{Sym(X)}\mathcal{G}$. On the other hand σ does not satisfy the necessary condition (2.1) for being an automorphism, so $\mathrm{Aut}(X, r)$ is a proper subgroup of $\mathrm{Nor}_{Sym(X)}(\mathcal{G})$. The graph $\Gamma(X, r)$ and its automorphism group is given on figure 1.

The second example will contain the above solution (X, r) an r-invariant subset. (In fact (X, r) itself contains $X_0 = \{x_1, x_2, x_3, x_4\}$ as an r-invariant subset.)

Definition 2.6. Let (Z, r) be a quadratic set, $\emptyset \neq X \subseteq Z$. X is *r-invariant* if $r(X \times X) \subseteq X \times X$. In this case we consider the restriction $r_X = r_{|X \times X}$. Clearly, if (Z, r) obeys YBE then (X, r_X) is also a solution, and inherits all "good" properties of (Z, r) like being non degenerate, involutive, square-free, satisfying **lri** , the cyclic conditions, etc.
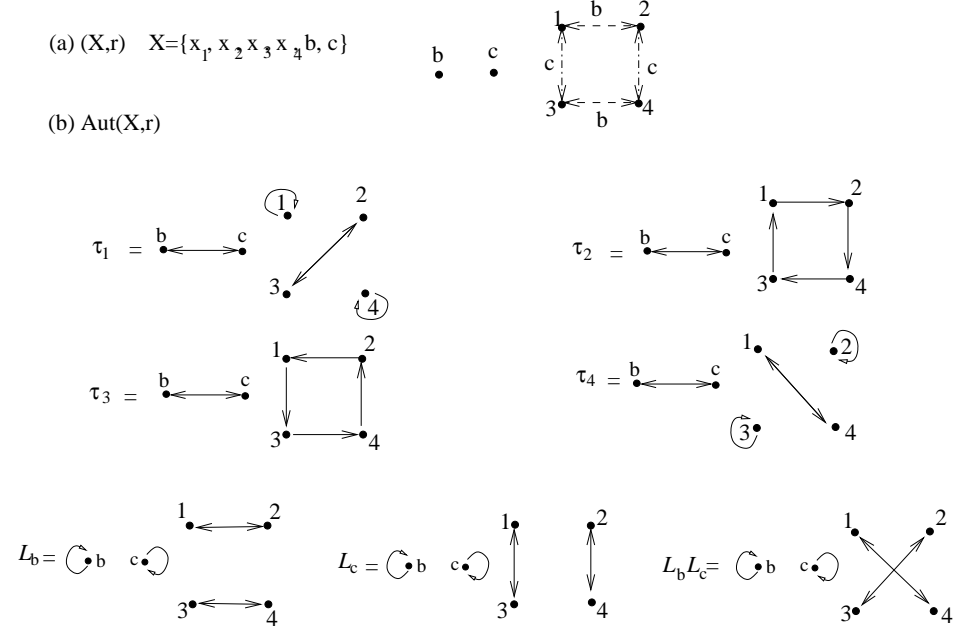
(a) (X,r)   X={x_1, x_2 x_3 x_4 b, c}

(b) Aut(X,r)

FIGURE 1. (a) Graph of the solution of Example 2.5 and (b) its automorphism group.

Lemma 2.7. *Let* (Z, r) *be a nondegenerate solution, with* **lri** *and* $\alpha \in Z$. *Suppose* X *is an* r-*invariant subset of* Z *and let* r_X *be the restriction of* r *on* $X \times X$. *Then*

1) $\mathcal{L}_{\alpha|X} \in \mathrm{Aut}(X, r_X)$ *iff*

$$^{\alpha^y}x = {}^{\alpha}x \quad \text{for all} \quad x, y, \in X.$$

In particular, $\mathcal{L}_{\alpha} \in \mathrm{Aut}(Z, r)$ *iff the displayed condition holds for all* $x, y \in Z$.

2) Furthermore, suppose that r *is involutive, then* $\mathcal{L}_{\alpha|X} \in \mathrm{Aut}(X, r_X)$ *iff*

$$^{y}\alpha x = {}^{\alpha}x \quad \text{for all} \quad x, y, \in X.$$

Proof. Let $\alpha \in Z$. By (2.1) $\mathcal{L}_{\alpha|X}$ is an automorphism of (X, r) *iff*

$$\mathcal{L}_{\alpha|X} \circ \mathcal{L}_y = \mathcal{L}_{\alpha y} \circ \mathcal{L}_{\alpha|X}, \text{ for all } y \in X$$

or equivalently

(2.3) $$^{\alpha}(^y x) = {}^{\alpha}y(^{\alpha}x), \text{ for all } x, y \in X$$

By condition **l1** on (Z, r) one has

$$^{\alpha}(^y x) = {}^{\alpha}y(^{\alpha^y}x), \text{ for all } x, y \in X,$$

which together with (2.3) implies

(2.4) $$^{\alpha}y(^{\alpha}x) = {}^{\alpha}y(^{\alpha^y}x), \text{ for all } x, y \in X$$

By the non degeneracy of (Z, r) (2.4) holds *iff* $^{\alpha^y}x = {}^{\alpha}x$. This proves part (1).

Assume now that r is involutive. Then (Z, r) is a symmetric set so its retraction $([Z], r_{[Z]})$ is well defined, see Definition 3.1. By Lemma 3.4 $([Z], r_{[Z]})$ inherits **lri**. We write (2.3) in the notation of retractions and obtain the implications

$$(2.3) \Longleftrightarrow [\alpha]^{[y]} = [\alpha^y] = [\alpha] \Longleftrightarrow^{lri} [\alpha] = {}^{[y]}[\alpha]$$

$$[\alpha] = {}^{[y]}[\alpha] \Longleftrightarrow^{Def.3.1} {}^{\alpha}x = {}^{y}{}^{\alpha}x \quad \text{for all} \quad x \in X.$$

These equations together with part (1) imply part (2) of the lemma. $\qquad\square$

For instance, we have seen that $\mathcal{L}_b, \mathcal{L}_c$ in our example above are automorphisms. For our next example they are joined by $\mathcal{L}_a = \mathcal{L}_b \circ \mathcal{L}_c$.

Example 2.8. Let (Z, r_Z) be the nondegenerate involutive square-free solution given by

$$Z = \{x_1, x_2, x_3, x_4, a, b, c\}, \ \mathcal{L}_a = (x_1 x_4)(x_2 x_3), \ \mathcal{L}_b = (x_1 x_2)(x_3 x_4), \ \mathcal{L}_c = (x_1 x_3)(x_2 x_4).$$

(Z, r_Z) is an extension of the preceding solution (X, r) by the trivial solution on the one element set $\{a\}$. Furthermore, $\mathcal{L}_a = \mathcal{L}_b \circ \mathcal{L}_c$, so $\mathcal{G}(Z, r_Z) = \mathcal{G}(X, r) \approx \mathbb{Z}_2 \times \mathbb{Z}_2$. More sophisticated arguments in Section 5 will show that the group $Aut(Z, r_Z)$ is isomorphic to S_4, the symmetric group on 4 elements. Furthermore, each automorphism $\tau \in Aut(X, r)$ can be extended uniquely to an automorphism in $Aut(Z, r_Z)$, by $\tau(a) = a$. These are the automorphisms denoted $\tau_i, \mathcal{L}_a, \mathcal{L}_b, \mathcal{L}_c$ shown in Figure 2. The remaining automorphisms and the graph of the solution are also shown.

We conclude the section with some straightforward generalities.

Remark 2.9. Let $(X, r_X), (Y, r_Y)$ be solutions. By definition every homomorphism of solutions $\varphi : (X, r_X) \to (Y, r_Y)$ agrees with the defining relations $\Re(X, r_X)$ and $\Re(Y, r_Y)$ of the related algebraic objects, so it can be extended to

   (1) a semigroup homomorphism $\varphi_S : S(X, r_X) \to S(Y, r_Y)$ of their Yang-Baxter semigroups;
   (2) a group homomorphism $\varphi_G : G(X, r_X) \to G(Y, r_Y)$ of their Yang-Baxter groups.
   (3) an algebra homomorphism $\varphi_{\mathcal{A}} : \mathcal{A}(k, X, r_X) \to \mathcal{A}(k, Y, r_Y)$ of their Yang-Baxter algebras.

If furthermore, X is embedded in $G(X, r)$ (respectively in $S(X, r)$), each r-automorphism τ of the solution (X, r) can be extendeded to a group automorphism $\tau_G : G(X, r) \to G(X, r)$, (respectively, to a semigroup automorphism $\tau_S : S(X, r) \to S(X, r)$. In this case, we have an embedding $\mathrm{Aut}(X, r) \hookrightarrow \mathrm{Aut}(G(X, r))$, (respectively, an embedding $\mathrm{Aut}(X, r) \hookrightarrow \mathrm{Aut}(S(X, r))$.
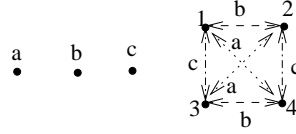
3. Multipermutation solutions

In this section we shall consider only nondegenerate symmetric sets (X, r).

Definition 3.1. [ESS], 3.2. Let (X, r) be a nondegenerate symmetric set. An equivalence relation $\sim$ is defined on X as

$$x \sim y \quad \text{iff} \quad \mathcal{L}x = \mathcal{L}y.$$

In this case we also have $\mathcal{R}_x = \mathcal{R}_y$, see [ESS].

(a) $Z = X \natural \{a\} = \{x_1, x_2, x_3, x_4, a, b, c\}$
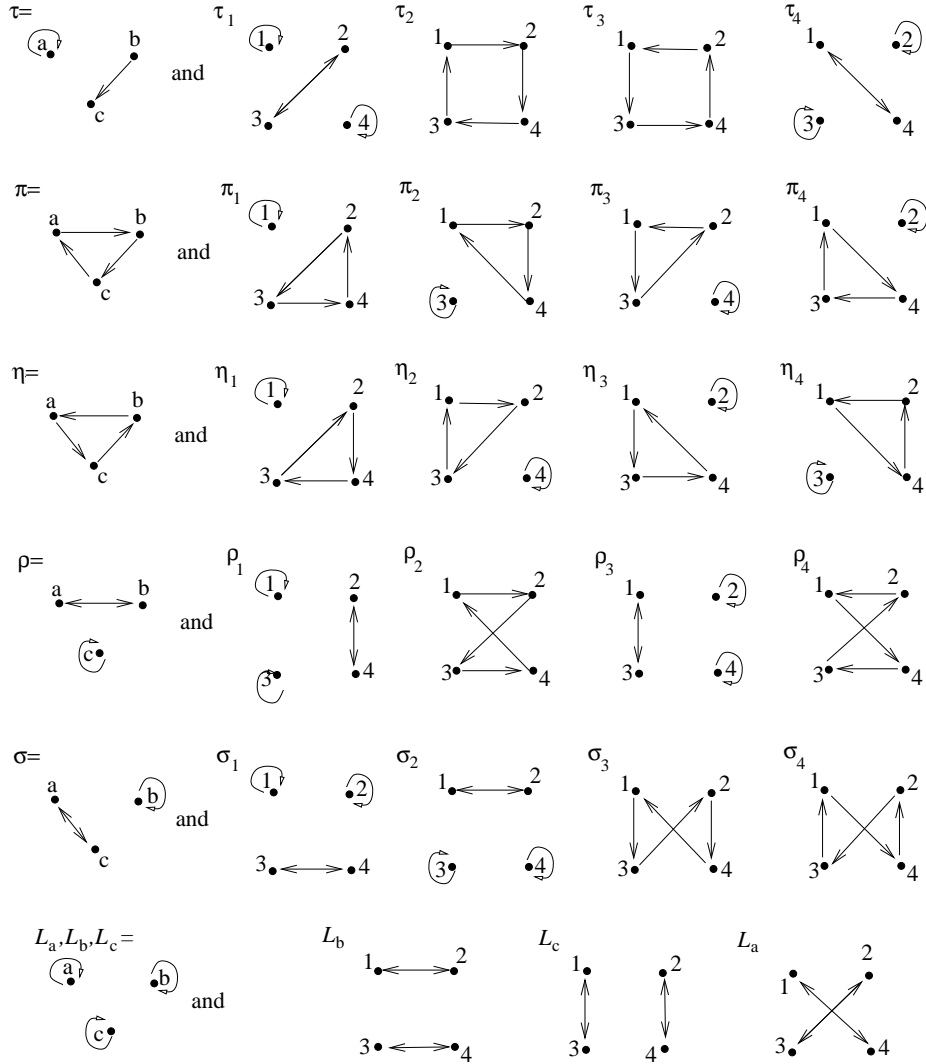

(b) $\mathrm{Aut}(Z, r_Z)$

FIGURE 2. (a) Graph of the solution of Example 2.8 and (b) its automorphism group.

We denote by $[x]$ the equivalence class of $x \in X$, $[X] = X/\sim$ is the set of equivalence classes.

It is shown in [ESS] that the solution r induces a canonical map

$$r_{[X]} : [X] \times [X] \longrightarrow [X] \times [X],$$

which makes $([X], r_{[X]})$ a nondegenerate symmetric set, called the *retraction of* (X, r), and denoted $Ret(X, r)$.

For our purposes we need concrete expressions of the left and the right actions on $[X]$ canonically induced by the left and right actions on X.

Definition 3.2. Let (X, r) be a nondegenerate symmetric set. Then the left and the right actions of X onto itself induce naturally left and right actions on the retraction $[X]$, via

$$(3.1) \qquad\qquad {}^{[\alpha]}[x] := [{}^{\alpha}x] \quad [\alpha]^{[x]} := [\alpha^x], \text{ for all } \alpha, x \in X.$$

Note that in the following there is no need to assume **lri**, since $\mathcal{L}_x = \mathcal{L}_y$ iff $\mathcal{R}_x = \mathcal{R}_y$.

Lemma 3.3. *Suppose (X, r) is a nondegenerate symmetric set. Then the left and the right actions (3.1) are well defined.*

Proof. We need to show that ${}^{[\alpha]}[x]$ does not depend on the representatives of $[\alpha]$ and $[x]$. It will be enough to show that

$$\mathcal{L}_y = \mathcal{L}_x \Longrightarrow [{}^{\alpha}y] = [{}^{\alpha}x], \quad \text{for all } \alpha \in X.$$

So fix $\alpha, x \in X$, and assume $y \in [x]$. Let $z \in X$. We have to verify:

$$(3.2) \qquad\qquad {}^{({}^{\alpha}y)}z = {}^{({}^{\alpha}x)}z.$$

By hypothesis (X, r) is nondegenerate, so there exists a $t \in X$, with

$$(3.3) \qquad\qquad z = {}^{\alpha^x}t = {}^{\alpha^y}t$$

(The right hand side equality comes from $\mathcal{L}_y = \mathcal{L}_x$, which implies $\alpha^x = \alpha^y$). Then we have

$$ {}^{({}^{\alpha}y)}z = {}^{\alpha}y({}^{\alpha^y}t) =^{\mathbf{l1}} {}^{\alpha}({}^{y}t) =^{[x]=[y]} {}^{\alpha}({}^{x}t) = {}^{\alpha^x}({}^{\alpha^x}t) =^{(3.3)} {}^{({}^{\alpha}x)}z.$$

$\square$

Lemma 3.4. *Suppose (X, r) is a nondegenerate symmetric set. Then*

(1) *The left and the right actions (3.1) define (as usual) a canonical map*

$$r_{[X]} : [X] \times [X] \longrightarrow [X] \times [X],$$

which makes $([X], r_{[X]})$ a nondegenerate symmetric set.
(2) *(X, r) cyclic $\Longrightarrow ([X], r_{[X]})$ cyclic.*
(3) *(X, r) is **lri** $\Longrightarrow ([X], r_{[X]})$ is **lri**.*
(4) *(X, r) square-free $\Longrightarrow ([X], r_{[X]})$ square-free.*

Proof. It is easy to see that the left and the right actions (3.1) on [X] inherit conditions **l1, r1, lr3**. Therefore $r_{[X]}$ obeys YBE. The implication in (3) follows easily from (3.1). We leave conditions (2), (4) (and so forth) to the reader. $\square$

Definition 3.5. [ESS]. The solution $([X], [r])$ is called *retraction of* (X, r) and is also denoted $Ret(X, r)$. For all integers $m \geq 1$ $Ret^m(X, r)$ is defined recursively as $Ret^m(X, r) = Ret(Ret^{m-1}(X, r))$. A nondegenerate symmetric set (X, r) is *a multipermutation solution of level* m if m is the minimal number such that $Ret^m(X, r)$ is finite of order 1, this will be denoted by $mpl(X, r) = m$. By definition (X, r) is *a multipermutation solution of level* 0 *iff* X is a one element set.

Example 3.6. An involutive *permutation solution* (X, r) is defined [Dri] (attributed to Lyubashenko) as

$$r(x, y) = (\sigma(y), \sigma^{-1}(x)),$$

where σ is a fixed permutation in $Sym(X)$. In this case (X, r) is a nondegenerate symmetric set with **lri**. Clearly, $\mathcal{L}_x = \sigma$, for all $x \in X$, thus $Ret(X, r)$ is a one element set and $mpl(X, r) = 1$. The converse also holds [ESS], so $mpl(X, r) = 1$ *iff* (X, r) is a permutation solution.

In particular, the trivial solution (X, r) with X of order ≥ 2 has $mpl(X, r) = 1$.

Remark 3.7. Let (X, r) be a nondegenerate symmetric set. Then there is a surjective homomorphism of solutions

$$\mu : (X, r) \longrightarrow ([X], [r]); \quad \mu(x) = [x],$$

Each finite symmetric set (X, r) with **lri** has a well defined oriented graph $\Gamma(X, r_X)$, see [GIM] and 5.1. In this case μ induces a homomorphism of graphs

$$\mu_\Gamma : \Gamma(X, r_X) \longrightarrow \Gamma([X], r_{[X]})$$

The graph $\Gamma([X], r_{[X]})$ is *a retraction* of $\Gamma(X, r_X)$.

The following is straightforward from Definition 3.5

Lemma 3.8. *Suppose* (X, r) *is a multipermutation solution. The following implications hold.*

$$mpl(X, r) = m \implies mpl(Ret^k(X, r)) = m - k, \quad \text{for all} \quad 1 \leq k \leq m - 1.$$

Conversely, if $k \geq 0$ *is an integer, then*

$$mpl(Ret^k(X, r)) = s \implies mpl(X, r) = s + k.$$

Lemma 3.9. *Let* (X, r) *be a nondegenerate square-free symmetric set. Then* $[x] = [y], x, y \in X \implies r(x, y) = (y, x)$.

Proof. Indeed, $[x] = [y]$ implies ${}^y x = {}^x x = x$, from which by **lri**, one has $x^y = x$ and by an analogous argument with x, y swapped one has $y^x = y$. $\square$

Corollary 3.10. *Suppose* (X, r) *is a nondegenerate square-free symmetric set (of order* ≥ 2*). Then the following conditions are equivalent.*

(1) $mpl(X, r) = 1$.
(2) (X, r) *is the trivial solution.*
(3) ${}^x y = y$, *for all* $x, y \in X$.
(4) $S(X, r)$ *is the free abelian monoid generated by* X.
(5) $G(X, r)$ *is the free abelian group generated by* X.
(6) $\mathcal{G}(X, r) = \{id_X\}$.

The kth retract orbit of an element was introduced first in the case of square-free solutions, see [GI].

Definition 3.11. [GI] Let (X, r) be a nondegenerate symmetric set. We denote by $[x]_k$ the image of x in $Ret^k(X, r)$. The set

$$\mathcal{O}(x, k) := \{\xi \in X \mid [\xi]_k = [x]_k\}$$

is called *the kth retract orbit of x*.

Suppose X is an r-invariant subset of the solution (Z, r), and $x \in X$. Then $[x]_{k,X}$ will denote the k-th retract of x in X.

Remark 3.12. cf. [GI, Lemma 8.9] Let (X, r) be a square-free symmetric set. For every positive integer $k \leq mpl(X, r)$ the kth retract orbit $\mathcal{O}(x, k)$ is r-invariant. Furthermore if we denote by $r_{x,k}$ the corresponding restriction of r, then $(\mathcal{O}(x, k), r_{x,k})$ is a multipermutation solution and

$$(3.4) \qquad\qquad mpl(\mathcal{O}(x, k), r_{x,k}) \leq k.$$

In [GI] (3.4) is actually written as an equality, but this should be corrected as the following example shows.

Example 3.13. Let $X = \{x_1, \cdots, x_k, y\}$, and r be defined via the actions $\mathcal{L}_y = (x_1 \cdots x_k), \mathcal{L}_{x_i} = id_X$. Then $\mathcal{O}(x_1, 1) = \{x_1, \cdots, x_k\}, \mathcal{O}(y, 1) = \{y\}$. Clearly, $[X] = \{[x_1], [y]\}$ is the trivial solution, so $mplX = 2$. Note that $mpl\mathcal{O}(y, 1) = 0, mpl(\mathcal{O}(x_1, 1)) = 1$.

4. Strong twisted unions of solutions

In this section we study special extensions of solutions called *strong twisted unions*. We recall first some basic facts and definitions.

The notion of *a union* of solutions and one-sided extensions were introduced in [ESS], but only for nondegenerate involutive solutions $(X, r_X), (Y, r_Y)$. In [GIM] are introduced and studied more general extensions (Z, r) of arbitrary solutions $(X, r_X), (Y, r_Y)$, and given necessary and sufficient conditions (in terms of left and right actions) so that a regular extension (Z, r) satisfies YBE.

Definition 4.1. [GIM] Let (X, r_X) and (Y, r_Y) be disjoint quadratic sets (i.e. with bijective maps $r_X : X \times X \longrightarrow X \times X$, $r_Y : Y \times Y \longrightarrow Y \times Y$). Let (Z, r) be a set with a bijection $r : Z \times Z \longrightarrow Z \times Z$. We say that (Z, r) is *a (general) extension of* $(X, r_X), (Y, r_Y)$, if $Z = X \bigcup Y$ as sets, and r extends the maps r_X and r_Y, i.e. $r_{|X^2} = r_X$, and $r_{|Y^2} = r_Y$. Clearly in this case X, Y are r-invariant subsets of Z. (Z, r) is *a YB-extension of* $(X, r_X), (Y, r_Y)$ if r obeys YBE.

Remark 4.2. In the assumption of the above definition, suppose (Z, r) is a nondegenerate extension of $(X, r_X), (Y, r_Y)$, (without any further restrictions on the solutions). Then the equalities $r(x, y) = (^xy, x^y)$, $r(y, x) = (^yx, y^x)$, and the nondegeneracy of r, r_X, r_Y imply that

$$^yx, x^y \in X, \quad ^xy, y^x \in Y, \text{ for all } x \in X, y \in Y.$$

Therefore, r induces bijective maps

$$(4.1) \qquad\qquad \rho : Y \times X \longrightarrow X \times Y, \quad \sigma : X \times Y \longrightarrow Y \times X,$$

and left and right "actions"

(4.2) $\quad\quad\quad {}^{()}\bullet : Y \times X \longrightarrow X, \quad \bullet^{()} : Y \times X \longrightarrow Y$, projected from ρ

(4.3) $\quad\quad\quad \rhd : X \times Y \longrightarrow Y, \quad \lhd : X \times Y \longrightarrow X$, projected from σ.

Clearly, the 4-tuple of maps (r_X, r_Y, ρ, σ) uniquely determine the extension r. The map r is also uniquely determined by r_X, r_Y, and the maps (4.2), (4.3).

In the present paper we restrict our attention to particular extensions called *strong twisted unions*, also introduced in [GIM]. However, in the present paper we prefer to avoid the most general form of this and focus on the case where the extension is nondegenerate as in the remark above, and involutive.

Definition 4.3. [GIM] In the notation of Remark 4.2 a nondegenerate involutive extension (Z, r) is a *strong twisted union* of the quadratic sets (X, r_X) and (Y, r_Y) if

  (1) The assignment $\alpha \longrightarrow {}^{\alpha}\bullet$ extends to a left action of the associated group $G(Y, r_Y)$ (and the associated semigroup $S(Y, r_Y)$) on X, and the assignment $x \longrightarrow \bullet^{x}$ extends to a right action of the associated group of $G(X, r_X)$ (and the associated semigroup $S(X, r_X)$) on Y;
  (2) The pair of ground actions satisfy

$$\mathbf{stu}: \quad {}^{\alpha^{y}}x = {}^{\alpha}x; \quad \alpha^{{}^{\beta}x} = \alpha^{x}, \quad \text{for all} \quad x, y \in X, \alpha, \beta \in Y$$

We shall use notation $(Z, r) = (X, r_X) \natural (Y, r_Y)$ (or shortly, $(Z, r) = X \natural Y$) for a strong twisted union. A strong twisted union (Z, r) of (X, r_X) and (Y, r_Y) is *non-trivial* if at least one of the actions in (1) is nontrivial. In the case when both actions (1) are trivial we write $(Z, r) = X \natural_0 Y$). In this case one has $r(x, \alpha) = (\alpha, x)$ for all $x \in X, \alpha \in Y$.

The following example is extracted from [ESS, Definition 3.3, Proposition 3.9],

Example 4.4. Let $(X, r_X), (Y, r_Y)$ be nondegenerate symmetric sets and $\sigma \in Aut(X, r), \rho \in Aut(Y, r)$. Define the nondegenerate involutive extension (Z, r) via r_X, r_Y and the formulae

$$r(\alpha, x) = (\sigma(x), \rho^{-1}(\alpha)), \quad r(x, \alpha) = (\rho(\alpha), \sigma^{-1}(x)).$$

This is, moreover, a symmetric set (obeys the YBE) and is called a *twisted union* in [ESS]. We denote it $(Z, r) = X \natural_0 Y$ as a special case of a strong twisted union. Note that

$$\mathcal{L}_{x|Z} = \mathcal{L}_{x|X}.\rho, \quad \mathcal{L}_{\alpha|Z} = \mathcal{L}_{\alpha|Y}.\sigma, \quad \text{for all } x \in X, \alpha \in Y$$

from which it is clear that (Z, r) is nondegenerate as (X, r_X) and (Y, r_Y) are.

Clearly a trivial extension (Z, r) of $(X, r_X), (Y, r_Y)$ is a particular case of twisted union. Another easy example is:

Example 4.5. Let (X, r) be a nondegenerate symmetric set and $Y = \{a\}$ a one element set with trivial solution. A strong twisted union (in fact any regular extension) of these that obeys the YBE is necessarily a twisted union $Z = X \natural_0 Y$ given by $\mathcal{L}_a \in Aut(X, r)$ and id_Y. This will be clear from Proposition 4.10 below. Our previous Example 2.8 is of this form.

The following lemma is straightforward.

Lemma 4.6. *Suppose $(Z, r) = X \natural_0 Y$, is a solution and $mpl(X, r_X) < \infty, mpl(Y, r_Y) < \infty$. Then $mpl(Z, r) = max\{mpl(X, r_X), \ mpl(Y, r_Y)\}$.*

In [ESS, Definition 3.3] the notion of *a generalized twisted union* (Z, r) of the solutions (X, r_X) and (Y, r_Y), is introduced in the class of symmetric sets.

Definition 4.7. A symmetric set (Z, r) is *a generalized twisted union* of the disjoint symmetric sets (X, r_X), and (Y, r_Y) if it is an extension, and for every $x \in X, \alpha \in Y$ the ground action $^{\alpha^x} \bullet : Y \times X \longrightarrow X$ does not depend on x, and the ground action $\bullet^{\alpha^x} : Y \times X \longrightarrow Y$ does not depend on α.

Remark 4.8. Note that a strong twisted union (Z, r) of (X, r_X) and (Y, r_Y) does not necessarily obey YBE, in contrast with twisted unions and generalized twisted unions. It follows straightforwardly from Definition 4.7 that a strong twisted union (Z, r) which is symmetric set is a generalized twisted union of (X, r_X) and (Y, r_Y).

Furthermore, it is shown in [GI, Proposition 8.3] that an YB-extension (Z, r) of two involutive square-free solutions $(X, r_X), (Y, r_Y)$ is a generalized twisted union *iff* it is a strong twisted union. Proposition 4.9 now generalizes this result for arbitrary symmetric sets with **lri**, without necessarily assuming that the solutions are square-free.

Proposition 4.9. *Suppose the symmetric set (Z, r) has **lri** and is an extension of the solutions $(X, r_X), (Y, r_Y)$. Then (Z, r) is a generalized twisted union iff it is a strong twisted union.*

Proof. By hypothesis (Z, r) is an involutive solution with **lri**, thus Proposition 1.12 implies that (Z, r) also satisfies the cyclic conditions, see Definition 1.8. Assume (Z, r) is a generalized twisted union. Then by Definition 4.7 for every $x \in X, \alpha \in Y$ the ground action $^{\alpha^x} \bullet : Y \times X \longrightarrow X$ does not depend on x, and the ground action $\bullet^{\alpha^x} : Y \times X \longrightarrow Y$ does not depend on α. It follows then that for all $x, y \in X, \alpha, \beta \in Y$ there are equalities:

$$\alpha^x y = \alpha^y y =^{\textbf{cl1}} {}^{\alpha}y \quad \text{and} \quad \beta^{\alpha^x} = \beta^{\beta^x} =^{\textbf{cr1}} \beta^x.$$

We have shown that conditions **stu** are satisfied, it follows then that $(Z, r) = (X, r_X) \natural (Y, r_Y)$. The converse implication is straightforward, see Remark 4.8. $\square$

Let $Ext^\natural(X, Y)$ denote the set of strong twisted unions which obey the YBE

Proposition 4.10.

(1) *A strong twisted union of solutions $(Z, r) = (X, r_X) \natural (Y, r_Y)$ obeys YBE iff*
  (a) The assignment $\alpha \longrightarrow {}^{\alpha}\bullet$ extends to a a group homomorphism

$$\varphi : G(Y, r_Y) \longrightarrow Aut(X, r); \quad and$$

  (b) The assignment $x \longrightarrow \bullet^x$ extends to a a group homomorphism

$$\psi : G(X, r_X) \longrightarrow Aut(Y, r).$$

 In this case the pair of group homomorphisms (φ, ψ) is uniquely determined by the pair of the ground actions, or equivalently by r.

(2) *Furthermore, there is a one-to-one correspondence between the sets $Ext^\natural(X, Y)$ and $Hom(G(Y, r_Y), Aut(X, r_X)) \times Hom(G(X, r_X), Aut(Y, r_Y))$ using (1).*

Proof. We use [GIM, Theorem 4.9] which breaks down the condition for any regular extension (Z, r) (of which a strong twisted union is an example) to obey the YBE into explicit conditions **ml1,ml2,mr1,mr2**. Particularly, as equalities in $X \times X$

$$\textbf{ml2}: \quad {}^\alpha r(x, y) = r({}^\alpha(x, y)) \quad \text{for all } x, y \in X, \alpha \in Y$$

(the condition **mr2** similar but for the right action and the roles of X, Y swapped.) We can interpret this condition by computing both sides as

$${}^\alpha r(x, y) = {}^\alpha({}^x y, x^y) = ({}^{\alpha x} y, {}^{(\alpha^x y)}(x^y)) = ({}^\alpha({}^x y), {}^\alpha(x^y)) = \mathcal{L}_\alpha \times \mathcal{L}_\alpha \circ r(x, y)$$

and

$$r({}^\alpha(x, y)) = r({}^\alpha x, {}^{\alpha^x} y) = r({}^\alpha x, {}^\alpha y) = r \circ (\mathcal{L}_\alpha \times \mathcal{L}_\alpha)(x, y).$$

using our assumption **stu**. Thus the condition has the meaning

$$\mathcal{L}_\alpha \times \mathcal{L}_\alpha \circ r = r \circ (\mathcal{L}_\alpha \times \mathcal{L}_\alpha)$$

that is $\mathcal{L}_{\alpha|X} \in Aut(X, r)$, for every $\alpha \in Y$ as in part (a). Similarly for **mr2** as in part (b). By definition every strong twisted union obeys **ml1, mr1** so the above are the only conditions for (Z, r) obeys YBE. Note that this proof works for general strong twisted unions as defined in [GIM] not only the nondegenerate involutive case in Definition 4.3, as long as the same definition is used for $Ext^\natural$. □

Lemma 4.11. *Let (Z, r) be a strong twisted union of (X, r_X) and (Y, r_Y) which is a symmetric set with **lri**. Then the retraction $([Z], r_{[Z]})$ is a strong twisted union of the retractions $([X], r_{[X]})$ and $([Y], r_{[Y]})$.*

Proof. By hypothesis Z is a disjoint union of X, Y thus $[Z]$ is a disjoint union of $[X], [Y]$. It follows from Lemma 3.4 and conditions **stu** on Z that

$$[{}^{[\alpha]^{[y]}}[x] = {}^{(3.1)} [{}^{\alpha^y} x] = {}^{\textbf{stu}} [{}^\alpha x] = {}^{(3.1)} [{}^{[\alpha]}[x].$$

This gives the left hand side part of **stu**, which together with **lri** implies the right hand side of **stu**. In view of Proposition 4.9 this could be said equally well in terms of generalised twisted unions; the present version can also be applied to general twisted unions obeying the YBE with **lri** provided the retracts make sense. □

In the following example the strong twisted union $Z = X \natural Y$ satisfies $mplZ = max(mplX, mplY)$.

Example 4.12. Let $Z = X \bigcup Y$, where (X, r_X), (Y, r_Y), are the nondegenerate involutive square-free solutions defined by follows. $X = \{x, y, z\}$, $Y = \{\alpha, \beta, \gamma\}$, with

$$\mathcal{L}_{x|X} = (yz), \ \mathcal{L}_{y|X} = \mathcal{L}_{z|X} = id_X, \ \mathcal{L}_{\alpha|Y} = (\beta\gamma), \ \mathcal{L}_{\beta|Y} = \mathcal{L}_{\gamma|Y} = id_Y.$$

It is easy to see that $mplX = mplY = 2$. Define

$$\mathcal{L}_{x|Y} = (\beta\gamma), \ \mathcal{L}_{y|Y} = \mathcal{L}_{z|Y} = id_Y, \ \mathcal{L}_{\alpha|X} = (yz), \ \mathcal{L}_{\beta|X} = \mathcal{L}_{\gamma|X} = id_X.$$

and extend these to Z (necessarily for any regular extension) by $\mathcal{L}_z = \mathcal{L}_{z|X}\mathcal{L}_{z|Y}$ where the restricted parts are considered to act trivially on the rest. So we have actions on Z :

$$\mathcal{L}_x = \mathcal{L}_\alpha = (yz)(\beta\gamma), \ \ \mathcal{L}_y = \mathcal{L}_z = \mathcal{L}_\beta = \mathcal{L}_\gamma = id_Z,$$

which define a nondegenerate involutive square-free solution (Z, r). It is a strong twisted union of (X, r_X), and (Y, r_Y). Clearly, $[x] = [\alpha], [y] = [z] = [\beta] = [\gamma]$, thus $[Z] = \{[x], [y]\}$ is a two element set, and $([Z], [r])$ is the trivial solution. This gives $mplZ = 2 = mplX = mplY$.

However, (Z, r) can also be looked at as a strong twisted union of the trivial solutions $X_0 = \{x, \alpha\}, Y_0 = \{y, z, \beta, \gamma\}$ with

$$\mathcal{L}_{x|Y_0} = \mathcal{L}_{\alpha|Y_0} = (yz)(\beta\gamma); \quad \mathcal{L}_{y|X_0} = \mathcal{L}_{z|Y_0} = \mathcal{L}_{\beta|X_0} = \mathcal{L}_{\gamma|X_0} = id_{X_0}.$$

This way we have again $Z = X_0 \natural Y_0$, but

$$mpl(Z, r) = mpl(X_0) + 1 = mpl(Y_0) + 1.$$

Remark 4.13. Theorem 5.24 in the next section shows that in the case when $mpl(Z, r) = 2$, Z can always be presented as a strong twisted union of a finite number of solutions $X_i, 1 \leq i \geq s$ where $mpl(X_i), \leq 1$, and there exists an i, such that $mpl(X_i) \geq 1$. The case $mpl(Z, r) \geq 3$ is more complicated. We show in Proposition 5.27 that every solution (Z, r) with $mpl(Z, r) = 3$ splits into r-invariant components $X_i, 1 \leq i \geq s$, where $X_i = \mathcal{V}(\Gamma_i)$, $mplX_i \leq 2$, and for each pair $i, j, 1 \leq i \leq s$, the subset $X_{ij} = X_i \bigcup X_j$ is r-invariant and has presentation as $X_{ij} = X_i \natural X_j$. However, in order to understand the nature of solutions with higher multipermutation level and for their classification we should understand under what conditions $\natural$ associates or what replaces this. We believe that it is always possible to present (Z, r) as a strong twisted union of components of strictly smaller multipermutation level.

We now give two more examples of strong twisted unions of solutions. In the first we have a strong twisted union $Z = X \natural Y$, with $mplX = 2, mplY = 1$ and $mplZ = 3$. Z is also split as a strong twisted union of three r-invariant components, each of multipermutation level 1, and the "associative law" holds. The second example, see Example 4.15, gives a solution (Z, r) as an extension (but not necessarily a strong twisted union) of two r-invariant subsets. It illustrates Remark 4.13 with respect to $\natural$ not being nonassociative in general.

Example 4.14. Let (X, r_X) be the nondegenerate involutive square-free solution defined by $X = \{a, b, c, x_1, x_2, y_1, y_2, z_1, z_2\}$ and left action:

$$(4.4) \qquad \mathcal{L}_{a|X} = \mathcal{L}_{b|X} = \mathcal{L}_{c|X} = (x_1x_2)(y_1y_2)(z_1z_2),$$

and $\mathcal{L}_{x_i|X} = \mathcal{L}_{y_i|X} = \mathcal{L}_{z_i|X} = id_X$ for $i = 1, 2$.

Let (Y, r_Y) be the trivial solution on the set $Y = \{\alpha, \beta\}$. We extend the left actions on $Z = X \bigcup Y$ as

$$\mathcal{L}_\alpha = \mathcal{L}_\beta = (abc)(x_1y_1z_1x_2y_2z_2),$$
$$\mathcal{L}_{a|Y} = \mathcal{L}_{b|Y} = \mathcal{L}_{c|Y} = \mathcal{L}_{x_i|Y} = \mathcal{L}_{y_i|Y} = \mathcal{L}_{z_i|Y} = id_Y, i = 1, 2.$$

This defines a left action of Z which we verify defines (Z, r) as a nondegenerate involutive square-free solution. Clearly, $mplY = 1$, and it is easy to see that $mpl(X, r_X) = 2$. We leave the reader to verify that $mpl(Z, r) = 3$.
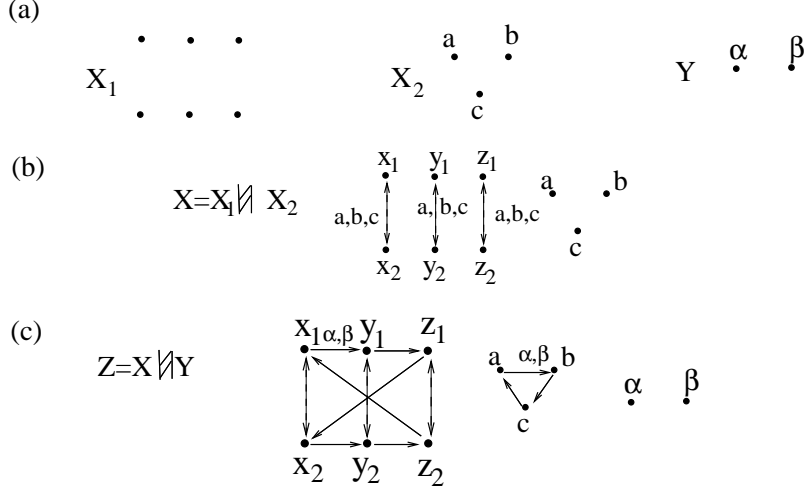
(a)

(b)

(c)

FIGURE 3. Graph of Example 4.14 of a strong twisted union

The graph $\Gamma(X, r_X)$ has three nontrivial components and three one vertex components, as shown in Figure 3 part (b). Consider the presentation of $X = X_1 \bigcup X_2$, where $X_1 = \{x_1, x_2, y_1, y_2, z_1, z_2\}$, $X_2 = \{a, b, c\}$. Both are r-invariant sets and the restrictions $r_1 = r_{|X_1 \times X_1}$, $r_2 = r_{|X_2 \times X_2}$, are the trivial solutions as shown along with Y in part (a) of the figure. Moreover, the actions given by (4.4) make X a strong twisted union $X = X_1 \natural X_2$. This way we have

$$Z = (X_1 \natural X_2) \natural Y = X_1 \natural (X_2 \natural Y) = X_2 \natural (X_1 \natural Y)$$

$$mpl X_1 = mpl X_2 = mpl Y = 1, mpl(Z) = 3.$$

The graph $\Gamma(Z, r)$ is shown in part (c) of the figure.

Example 4.15. Let $X_3 = \{a\}$ be the one element solution $(r_3 = id_{X_3 \times X_3})$, and let $(X_1, r_1), (X_2, r_2)$ be the trivial solutions on the sets

$$X_1 = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}, \quad X_2 = \{a, b, c\}$$

as depicted in Figure 4 (a). Consider the strong twisted union $(X, r_0) = X_1 \natural X_2$ with r_0 defined by the left actions

$$\mathcal{L}_b = (x_1 x_2)(x_3 x_4)(x_5 x_6)(x_7 x_8), \quad \mathcal{L}_c = (x_1 x_5)(x_2 x_6)(x_3 x_7)(x_4 x_8)$$

and the condition **lri**. As usual, in this way we find a nondegenerate involutive square-free solution, as depicted in part (b). It is easy to see that $Ret(X, r_0)$ is the trivial solution on the set $[X] = \{[x_1], [b], [c]\}$, therefore $mpl(X, r_0) = 2$.

Consider now the set $Z = X \bigcup X_3$. Let

$$\mathcal{L}_a = (bc)(x_1 x_4)(x_2 x_3)(x_5 x_8)(x_6 x_7).$$

We leave the reader to verify that $\mathcal{L}_a \in Aut(X, r_0)$. Next we define the extension $(Z, r) = X \natural X_3$, with r defined via $\mathcal{L}_a$, the condition **lri** that we are extending r_0, r_3. Clearly, (Z, r) is nondegenerate involutive square-free solution and $mpl(Z, r) = 3$. This way we have a presentation which we denote $Z = (X_1 \natural X_2) \natural X_3 = (X_2 \natural X_1) \natural X_3$ as in part (c).
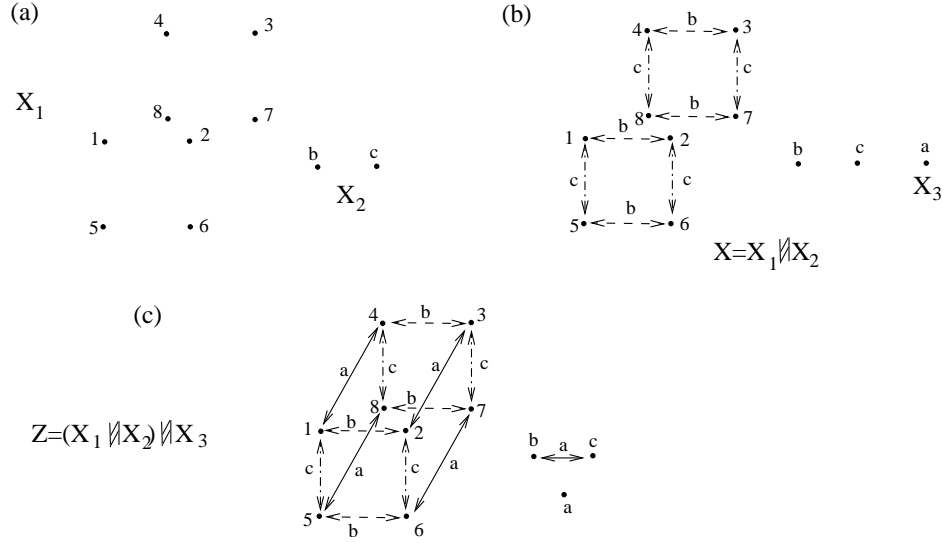
FIGURE 4. Graph of Example 4.15

Consider now the set $Y = X_1 \bigcup X_3$, which is an r-invariant subset of (Z, r). Let $r_Y = r_{|Y \times Y}$, then $(Y, r_Y) = X_1 \natural X_3$, ($r_Y$ can be defined also via the action $\mathcal{L}_{a|X_1} = (x_1 x_4)(x_2 x_3)(x_5 x_8)(x_6 x_7)$). One has $mpl(Y, r_y) = 2$. Clearly (Z, r) as an extension of (Y, r_Y) and (X_2, r_2). However, this extension is not a strong twisted union, since neither $\mathcal{L}_b$ nor $\mathcal{L}_c$ is in $Aut(Y, r_Y)$. Or equivalently, a direct verification shows that the pair Y, X_3 does not satisfy condition **stu**, since

$$ {}^{b^a}x_1 = {}^c x_1 = x_5 \neq {}^b x_1 = x_2. $$

Thus, while one also has $Z = X_1 \natural (X_3 \natural X_2)$ by similar computations, the parentheses do not exactly associate with $\natural$; we have shown that (Z, r) is a YB-extension of the nondegenerate square-free involutive solutions $Y = X_1 \natural X_3$ and X_2 as desired but this extension is not a strong twisted union.

Theorem 4.16. *Let (X, r_X) and (Y, r_Y), be square-free multipermutation solutions, with $mplX = m$, $mplY = k$. Suppose $(Z, r) = X \natural Y$ is a nondegenerate symmetric set. Then*

(4.5) $$ mpl(Z, r) \leq max\{m, k\} + 1. $$

Proof. Without loss of generality we can assume $k \leq m$. We shall prove (4.5) by induction on m. For the base for the induction, $m = 1$, we have to show that $mpl(Z, r) \leq 2$, which by Lemma 4.5 is equivalent to $mpl(Ret(Z, r)) = 1$. By hypothesis **stu** holds on Z, so

$$ {}^{\alpha^y}x = {}^\alpha x, \quad {}^{x^\alpha}\beta = {}^x \beta, \text{ for every } x, y \in X, \ \alpha, \beta \in Y, $$

or equivalently,

(4.6) $\quad \mathcal{L}_{\alpha^y|X} = \mathcal{L}_{\alpha|X} = \mathcal{L}_{{}^y\alpha|X}; \ \mathcal{L}_{x^\alpha|Y} = \mathcal{L}_{x|Y} = \mathcal{L}_{{}^\alpha x|Y}$, for all $x, y \in X, \alpha, \beta \in Y$.

By Corollary 3.10 $mplX = 1$ implies $\mathcal{L}_{x|X} = id_X$. As a set Z is a disjoint union $Z = X \bigcup Y$, hence the following equalities hold.

$$(4.7) \qquad \mathcal{L}_{x|Z} = \mathcal{L}_{x|X} \circ \mathcal{L}_{x|Y} = id_X \circ \mathcal{L}_{x|Y} = \mathcal{L}_{x|Y}, \quad \forall x \in X.$$

Analogously,

$$(4.8) \qquad \mathcal{L}_{\alpha|Z} = \mathcal{L}_{\alpha|X}, \quad \forall \alpha \in Y.$$

Now from (4.6), (4.7), and (4.8) we deduce the equalities

$$(4.9) \;\; \mathcal{L}_{\alpha^y|Z} = \mathcal{L}_{\alpha|Z} = \mathcal{L}_{y_\alpha|Z}; \quad \mathcal{L}_{x^\alpha|Z} = \mathcal{L}_{x|Z} = \mathcal{L}_{\alpha_x|Z}, \text{ for all } x, y \in X, \alpha, \beta \in Y.$$

This, together with Corollary 3.10 imply that either i) $Ret(Z, r)$ is a trivial solution of order ≥ 2, hence $mpl(Z, r) = 2$; or ii) $Ret(Z, r)$ is one element solution so $mpl(Z, r) = 1$. We have verified (4.5) for $m = 1$. Assume now (4.5) holds for all $m \leq m_0$. Suppose $mplX = m_0 + 1$, thus by Lemma 4.5, $mpl([X], r_{[X]}) = m_0$. Clearly, $mpl([Y], r_{[Y]}) = mpl(Y, r_Y) - 1 \leq m_0$. By Lemma 4.11 the retract $([Z], r_{[Z]})$ is a strong twisted union $([Z], r_{[Z]}) = [X]\natural[Y]$, hence by the inductive assumption one has $mpl([Z], r_{[Z]}) \leq m_0 + 1$. This and the equality $mpl(Z, r) = mpl([Z], r_{[Z]}) + 1$ implies $mpl(Z, r) \leq m_0 + 2$. It follows then that (4.5) holds for all $m, m \geq 1$, which proves the theorem. $\qquad\square$

The following lemma is straightforward.

Lemma 4.17. *Let* $(X_1, r_1), (X_2, r_2)$ *be disjoint solutions with* $mpl(X_1, r_1) = m_1$, $mpl(X_2, r_2) = m_2$, *let* $m = max\{m_1, m_2\}$. *Suppose* $(Z, r) = X_1 \natural X_2$ *is a symmetric set. Then* $mplZ = m + 1$ *iff for some* $i, 1 \leq i \leq 2$ *with* $m_i = m$ *there are* $x \in X_i$, *and* $\alpha \in Z \backslash X_i$ *such that the orbit* $O^{m-1}(x)$ *does not contain* $^\alpha x$.

Lemma 4.17 has a clear interpretation in terms of the graphs $\Gamma(X_i, r_i), i = 1, 2, \Gamma(Z, r)$, see Definitions 5.1, 5.3. In the case discussed by the lemma, $\mathcal{L}_{\alpha|X_i}$ acts as an automorphism of (X_i, r_i) which maps the connected component Γ_x of $\Gamma(X_i, r_i)$ onto a different isomorphic component $\Gamma_{\alpha x}$ of $\Gamma(X_i, r_i)$. Clearly this is possible only in the case when $\Gamma(X_i, r_i)$ has at least two connected components which are isomorphic as graphs. The last is a necessary but not a sufficient condition.

5. Graphs of symmetric sets with **lri**

Each finite involutive solution (X, r) with **lri** can be represented geometrically by its *graph of the left action* $\Gamma(X, r)$. It is an oriented labeled multi-graph (although we refer to it as a *graph*). It was introduced in [GI00] for square-free solutions, see also [GIM]. Here we recall the definition.

Definition 5.1. [GIM] Let (X, r) be a finite symmetric set with **lri**, we define the (complete) graph $\Gamma = \Gamma(X, r)$ as follows. It is an oriented graph, which reflects the left action of $G(X, r)$ on X. The set of vertices of Γ is exactly X. There is a labeled arrow $x \xrightarrow{a} y$, if $x, y, a \in X$ and $^a x = y$. An edge $x \xrightarrow{a} y$, with $x \neq y$ is called *a nontrivial edge.* We will often consider the simplified graph in which to avoid clutter we typically omit self-loops unless needed for clarity or contrast. Also for the same reason, we use the line type to indicate when the same type of element acts, rather than labeling every arrow. Clearly, $x \xleftrightarrow{a} y$ indicates that $^a x = y$ and $^a y = x$. (One can make such graphs for arbitrary solutions but then it should be indicated which action is considered).

Note that two solutions are isomorphic if and only if their (complete) oriented graphs are isomorphic. Various properties of a solution (X, r) are reflected in the properties of its graph $\Gamma(X, r)$, see for example the remark below, Proposition 5.6, Theorem 5.22.

Remark 5.2. Let (Z, r) be a symmetric set with **lri**, $\Gamma = \Gamma(Z, r)$.

(1) (Z, r) is a square-free solution *iff* Γ does not contain a nontrivial edge $x \xrightarrow{x} y, x \neq y$.
(2) In this case, (Z, r) is a trivial solution, or equivalently, $mpl(Z, r) = 1$ *iff* Γ does not contain nontrivial edges $x \xrightarrow{a} y, x \neq y$.

By Proposition 1.12 our assumptions that (X, r) is a symmetric set with **lri** imply the cyclic conditions (without necessarily assuming (X, r) square-free). We will need the assumption (X, r) square-free any time when we claim $^{x}x = x$, for every $x \in X$. Examples of graphs of square-free solutions were already given in Section 2, see Examples 2.5, and 2.8, as well as in Section 4.

We will find now various properties of $\Gamma = \Gamma(X, r)$.

Notation 5.3. Suppose Γ_0 is a subgraph of Γ. Denote by $\mathcal{V}(\Gamma_0)$ the set of all vertices of Γ_0. $\mathcal{E}(\Gamma_0)$ denotes the set of all labels of (nontrivial) edges that occur in Γ_0, i.e.
$$\mathcal{E}(\Gamma_0) = \{a \in X \mid \exists \text{ an edge } x \xrightarrow{a} y \subset \Gamma_0, x \neq y\}.$$
Clearly, each $x \in X$ determines uniquely a connected component of Γ which contains x as a vertex, we shall denote it by Γ_x.

Let $a, x \in X$. Suppose $^{a}x \neq x$. Then the orbit of x under the left action of the cyclic group $_{gr}\langle a \rangle$ (or equivalently under the action of $\mathcal{L}_a$) on X is a cycle $(x_1 x_2 \cdots x_m)$ of length $m \geq 2$ in the symmetric group $Sym(X)$, where for symmetry we set $x_1 = x$. One has $^{a}x_i = x_{i+1}, 1 \leq i \leq m - 1, {}^{a}x_m = x_1,$. This cycle participates in the presentation of the permutation $\mathcal{L}_a \in Sym(X)$ as a product of disjoint cycles. Clearly, $x_2 \cdots x_m \in \Gamma_x$.

Notation 5.4. For x, a as above we use notation $\mathcal{L}_a^x = (x_1 \cdots x_m) = \mathcal{L}_a^{x_i}, 1 \leq i \leq m$.

Conventions 5.5. Till the end of the section we shall consider only finite square-free nondegenerate symmetric sets (X, r). We recall from Theorem 1.13 that such solutions satisfy both **lri** and the cyclic conditions. In particular, one has

(5.1) $$\mathcal{L}_{(a^x)}^x = \mathcal{L}_a^x = \mathcal{L}_{(^{x}a)}^x.$$

Moreover, it is known that for each such square-free solution (X, r) with $1 < |X| < \infty$, its YB group $G(X, r)$ acts nontransitively on X, see [GI, R], and therefore the graph $\Gamma(X, r)$ has at least two connected components.

Proposition 5.6. *Let (X, r) be a finite nondegenerate involutive square-free solution, $G = G(X, r)$, be its YB group, and $\Gamma = \Gamma(X, r)$ be its graph. Suppose $\Gamma_1, \Gamma_2, \cdots, \Gamma_s$ is the set of all connected components, with sets of vertices, respectively, $X_i = \mathcal{V}(\Gamma_i), 1 \leq i \leq s$.*

(1) *Each set $X_i, 1 \leq i \leq s$, is precisely an orbit of the left action of the group G on X.*

(2) X_i is r-invariant, so (X_i, r_i), with $r_i = r_{|X_i \times X_i}$, is a nondegenerate involutive square-free solution. Its graph $\Gamma(X_i, r_i)$ is the subgraph of Γ_i obtained by erasing the edges labeled with elements $a \in \mathcal{E}(\Gamma_1) \backslash X_i$.

(3) Furthermore, if $mpl(X, r) = m$, then $mpl(X_i, r_i) \leq m - 1$, for each $i, 1 \leq i \leq s$. More precisely, in this case, $x \in X_i$ implies $X_i \subseteq \mathcal{O}(x, m-1)$.

Proof. Sketch of the proof. Condition (1) follows straightforwardly from the definition of the graph $\Gamma(X, r)$. The definition of $\Gamma(X, r)$ also implies that the set of vertices X_i of each connected component is invariant under the left action. By **lri** it is also invariant with respect to the right action, and therefore it is r-invariant. We shall prove (3). By 3.8 one has $mpl(Ret^{m-1}(X, r)) = 1$. $Ret^{m-1}(X, r)$ inherits the property of being square-free, so by Lemma 3.10, $Ret^{m-1}(X, r)$ is a trivial solution. In terms of the left action this is equivalent to

$$^y x \in \mathcal{O}(x, m-1) \quad \text{for all} \quad x, y \in X.$$

It follows then that the orbit of x, $X_i = \mathcal{O}_G(x)$ is contained in the $(m-1)$st retract orbit $\mathcal{O}(x, m-1)$, and Remark 3.12, implies $mpl(X_i) \leq m - 1$. $\qquad \square$

Definition 5.7. Let $a \in X$, and let Γ_1 be a connected component of Γ. We shall use notation $\mathcal{L}_{a|\Gamma_1}$ for the restriction of the action $\mathcal{L}_a$ on the set of vertices $\mathcal{V}(\Gamma_1)$. We say that *the (left) actions of a and b commute on Γ_1* if the following equalities hold:

$$(5.2) \qquad \mathcal{L}_{(^b a)|\Gamma_1} = \mathcal{L}_{a|\Gamma_1}, \quad \mathcal{L}_{(^a b)|\Gamma_1} = \mathcal{L}_{b|\Gamma_1}.$$

In this case we also say that each two edges of Γ_1 labeled by a and b commute.

The graph $\Gamma(X, r)$ and the combinatorial properties of the solution (X, r) can be used as a "mini-computer" to deduce missing relation of (X, r). For example, see the proofs of Lemmas 5.8, 5.11, 5.13 and Example 5.9.

The following straightforward lemma describes the "cells" from which we build $\Gamma(X, r)$ in the general case of (X, r) with **lri**.

Lemma 5.8. *Let (X, r) be a symmetric set with* **lri**.

(1) *For each $x_1, a, b \in X, a \neq b$, there exists a uniquely determined subgraph Γ_0 of Γ:*

$$(5.3) \qquad \begin{array}{ccc} x_1 & \xrightarrow{\ a\ } & x_2 \\ {\scriptstyle b}\big\downarrow & & \big\downarrow{\scriptstyle ^a b} \\ x_3 & \xrightarrow[\ ^b a\ ]{} & x_4 \end{array} \quad ,$$

where the vertices $x_1, x_2, x_3, x_4 \in X$ are not necessarily pairwise distinct.

(2) *Given a, b, each vertex x_i, of Γ_0 in (5.3) determines the remaining three verteces $x_j, 1 \leq j \leq 4, j \neq i$.*

(3) *In the notation of (1) let Γ_{x_1} be the connected component which contains x_1 as a vertex. Suppose that the left actions of a and b commute on Γ_{x_1},*
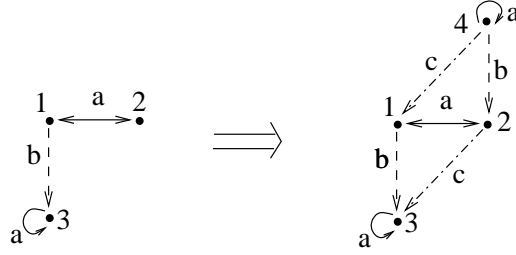
FIGURE 5. Diagram for Example 5.9

see Definition 5.7. Then the subgraph Γ_0 has the shape

(5.4)
$$
\begin{array}{ccc}
x_1 & \xrightarrow{\ a\ } & x_2 \\
{\scriptstyle b}\downarrow & & \downarrow{\scriptstyle b} \\
x_3 & \xrightarrow[\ a\ ]{} & x_4
\end{array}
\ ,
$$

and the following implications hold:

(5.5)
$$
\begin{aligned}
x_1 \neq x_2 &\iff x_3 \neq x_4 \\
x_1 \neq x_3 &\iff x_2 \neq x_4.
\end{aligned}
$$

Proof. (1) is clear. Note that given $\xi, a \in X$ there exists unique $x \in X$, such that $^a x = \xi$. Indeed, by **lri**, $x = \xi^a$. One uses this and **lri** again to deduce (2). Now we prove the first implication of (3). It follows from the diagram (5.4) that one has $x_3 = {}^b x_1, x_4 = {}^{ab} x_2 = {}^b x_2$, so by **lri** $x_3{}^b = x_1$, $x_4{}^b = x_2$. Clearly this implies

$$
x_1 = x_2 \iff x_3 = x_4.
$$

Analogous argument proves the second implication. $\qquad\qquad\square$

Note that the diagram (5.3) is just a formal graphic expression of condition **l1**, so it always presents a subgraph of $\Gamma(X, r)$, although in the case when some vertices coincide it is deformed to a segment or to a single vertex with loops. It is convenient (and mathematically correct) for computational purposes to use sometimes diagrams of the shape of squares (as in (5.3)) even when some vertices coincide. See for example Corollary 5.12.

Example 5.9. Let (Z, r) be a nondegenerate involutive square-free solution with at least 7 elements. Suppose where $a, b, c, x_1, x_2, x_3 \in Z$, and

(5.6)
$$
\begin{aligned}
&r(a, b) = (c, a), &&r(a, c) = (b, a), &&r(a, x_1) = (x_2, a) \\
&r(a, x_2) = (x_1, a), &&r(a, x_3) = (x_3, a), &&r(b, c) = (c, b), &&r(b, x_1) = (x_3, b)
\end{aligned}
$$

Then $r(c, x_2) = (x_3, c)$ and there exists some $x_4 \in X$, distinct from a, b, c, x_1, x_2, x_3, such that

(5.7)
$$
r(c, x_1) = (x_4, c), \quad r(b, x_2) = (x_4, b), \quad r(a, x_4) = (x_4, a).
$$

This is indicated in terms of the graph in Figure 5. Furthermore, if (Z, r) has exactly 7 elements then (5.6) determines uniquely the solution (Z, r). It is not

difficult to see that in addition to (5.6), (5.7), (Z, r) satisfies $r(x_i, x_j) = (x_j, x_i)$, for all $1 \le i, j \le 4$. For the left actions we have:

$$\mathcal{L}_{x_i} = id_Z, 1 \le i \le 4, \quad \mathcal{L}_b = (x_1 x_3)(x_2 x_4), \quad \mathcal{L}_c = (x_1 x_4)(x_2 x_3), \quad \mathcal{L}_a = (bc)(x_1 x_2).$$

So the retracts of (Z, r) are:

$$\begin{aligned}
Ret(Z, r) &= \{[x_1], [a], [b], [c]\}, \quad &\mathcal{L}_{[a]} &= ([b][c]), \quad &\mathcal{L}_{[b]} &= \mathcal{L}_{[c]} = \mathcal{L}_{[x_1]} = id_{[Z]} \\
Ret^2(Z, r) &= \{[a]^{(2)}, [b]^{(2)}\}, \quad &\text{the trivial solution } Ret^3(Z, r) &= \{[a]^{(3)}\}.
\end{aligned}$$

It follows then that $mpl(Z, r) = 3$.

Example 5.10. The solution from the previous example can be constructed as a strong twisted union of disjoint solutions of lower multipermutation levels. Consider the trivial solutions $(X_1, r_1), (X_2, r_2)$, where

$$X_1 = \{x_1, x_2, x_3, x_4\}, \quad X_2 = \{b, c\}.$$

Let $(X, r_X) = X_1 \natural X_2$ be the strong twisted union defined via the actions

$$\mathcal{L}_b = (x_1 x_3)(x_2 x_4), \quad \mathcal{L}_c = (x_1 x_4)(x_2 x_3), \quad \mathcal{L}_{x_i} = id_X.$$

We leave the reader to verify that (X, r_X) is a solution. The graph $\Gamma(X, r_X)$ has three connected components: two one vertex components, b and c and Γ_1, with $\mathcal{V}(\Gamma_1) = X_1$, $\mathcal{E}(\Gamma_1) = X_2$. Moreover, Γ_1 is a 'graph of first type', see Definition 5.17 below. One has $mpl(X, r_X) = 2$.

Let (Y, r_Y) be the one element solution, with $Y = \{a\}$. We will build an extension $(Z, r) = X \natural Y$. Since Y is one element solution, the left action on Y on X has to be via an automorphism. Consider the permutation $\tau = (bc)(x_1 x_2) \in Sym(X)$. It is easy to see that is an automorphism of (X, r_X). Now one defines strong twisted union $(Z, r) = X \natural Y$ via the action $\mathcal{L}_a = (bc)(x_1 x_2)$. We obtain again the solution of Example 5.10. Now one has $mpl(Z, r) = 3 = max\{mpl(X), mpl(Y)\} + 1$.

Lemma 5.11. Let $x, a, b \in X$, $\mathcal{L}_a^x = (x_1 x_2 \cdots x_k)$, where $x_1 = x, k \ge 2$, and let $^b x = x_{21} \ne x$. Suppose that the left actions of a and b commute on Γ_x. Then

(1) $\mathcal{L}_a^{(^b x)} = (x_{21} x_{22} \cdots x_{2k})$ is a cycle of length exactly k.
(2) If $^b x = x_{m+1}$ for some $m, 1 \le m \le k-1$, then $\mathcal{L}_a^{^b x} = \mathcal{L}_a^x$, and $\mathcal{L}_b^x = (\mathcal{L}_a^x)^m$.
(3) If part (2) does not apply then $\mathcal{L}_a^x$ and $\mathcal{L}_a^{^b x}$ are two disjoint cycles.

Proof. We apply Lemma 5.8 to the commuting edges

$$\begin{array}{c}
x_1 \xrightarrow{\ a\ } x_2 \\
\Big\downarrow{\scriptstyle b} \\
x_{21}
\end{array}$$

and obtain the "cell" subgraph of Γ:

(5.8)
$$\begin{array}{ccc}
x_1 & \xrightarrow{\ a\ } & x_2 \\
\Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle b} \\
x_{21} & \xrightarrow{\ a\ } & x_{22}.
\end{array}$$

Now we use this method to recursively build the diagram

(5.9)
$$x_1 \xrightarrow{a} x_2 \xrightarrow{a} \cdots \xrightarrow{a} x_k \xrightarrow{a} x_1 \xrightarrow{a} \cdots$$
$$\downarrow b \quad\quad \downarrow b \quad\quad\quad\quad \downarrow b \quad\quad \downarrow b$$
$$x_{21} \xrightarrow{a} x_{22} \xrightarrow{a} \cdots \xrightarrow{a} x_{2k} \xrightarrow{a} x_{21} \xrightarrow{a} \cdots$$
$$\downarrow b \quad\quad \downarrow b \quad\quad\quad\quad \downarrow b \quad\quad \downarrow b$$
$$\vdots \quad\quad\quad \vdots \quad\quad \cdots \quad\quad \vdots \quad\quad\quad \vdots \quad\quad \cdots$$

It is clear from the first two rows of the diagram that the cycle $\mathcal{L}_a^{x_{21}}$ (represented by the second row) has length at most k. Assume that its length q is strictly less than k then we obtain from (5.9)

(5.10)
$$x_1 \xrightarrow{a} x_2 \xrightarrow{a} \cdots \xrightarrow{a} x_q \xrightarrow{a} x_{q+1} \xrightarrow{a} \cdots$$
$$\downarrow b \quad\quad \downarrow b \quad\quad\quad\quad \downarrow b \quad\quad \downarrow b$$
$$x_{21} \xrightarrow{a} x_{22} \xrightarrow{a} \cdots \xrightarrow{a} x_{2q} \xrightarrow{a} x_{21} \xrightarrow{a} \cdots$$
$$\downarrow b \quad\quad \downarrow b \quad\quad\quad\quad \downarrow b \quad\quad \downarrow b$$
$$\vdots \quad\quad\quad \vdots \quad\quad \cdots \quad\quad \vdots \quad\quad\quad \vdots \quad\quad \cdots$$

which implies ${}^b x_1 = x_{21}$ and ${}^b x_{q+1} = x_{21}$. The last is impossible in view of the nondegeneracy of (X, r) and $x_1 \neq x_{q+1}$. We have shown that $\mathcal{L}_a^{({}^b x)}$ is a cycle of length exactly k.

Assume now ${}^b x = x_{21} = x_{m+1}$, for some $m \geq 1$. Then we deduce from (5.10) the following diagram:

$$x_1 \xrightarrow{a} x_2 \xrightarrow{a} \cdots \xrightarrow{a} x_{m+1} \xrightarrow{a} \cdots \xrightarrow{a} x_k \xrightarrow{a} x_1 \xrightarrow{a} \cdots$$
$$\downarrow b \quad\quad \downarrow b \quad\quad\quad\quad \downarrow b \quad\quad\quad\quad \downarrow b \quad\quad \downarrow b$$
$$x_{m+1} \xrightarrow{a} x_{m+2} \xrightarrow{a} \cdots \xrightarrow{a} x_{2m+1} \xrightarrow{a} \cdots \xrightarrow{a} x_{m+k} \xrightarrow{a} x_{m+1} \xrightarrow{a} \cdots$$
$$\downarrow b \quad\quad \downarrow b \quad\quad\quad\quad \downarrow b \quad\quad\quad\quad \downarrow b \quad\quad \downarrow b$$
$$x_{2m+1} \xrightarrow{a} x_{2m+2} \xrightarrow{a} \cdots \xrightarrow{a} x_{3m+1} \xrightarrow{a} \cdots \xrightarrow{a} x_{2m+k} \xrightarrow{a} x_{2m+1} \xrightarrow{a} \cdots$$
$$\downarrow b \quad\quad \downarrow b \quad\quad\quad\quad \downarrow b \quad\quad\quad\quad \downarrow b \quad\quad \downarrow b$$
$$\vdots \quad\quad\quad \vdots \quad\quad \cdots \quad\quad \vdots \quad\quad \cdots \quad\quad \vdots \quad\quad\quad \vdots \quad\quad \cdots$$
$$\downarrow b \quad\quad \downarrow b \quad\quad\quad\quad \downarrow b \quad\quad\quad\quad \downarrow b \quad\quad \downarrow b$$
$$x_{sm+1} \xrightarrow{a} x_{sm+2} \xrightarrow{a} \cdots \xrightarrow{a} \bullet \xrightarrow{a} \cdots \xrightarrow{a} x_{sm+k} \xrightarrow{a} x_{sm+1} \xrightarrow{a} \cdots$$
$$\downarrow b \quad\quad \downarrow b \quad\quad\quad\quad \downarrow b \quad\quad\quad\quad \downarrow b \quad\quad \downarrow b$$
$$\vdots \quad\quad\quad \vdots \quad\quad \cdots \quad\quad \vdots \quad\quad \cdots \quad\quad \vdots \quad\quad\quad \vdots \quad\quad \cdots$$

Looking at the first column of the diagram we see that $\mathcal{L}_b^x = (\mathcal{L}_a^x)^m$. It follows from (5.1) that $\mathcal{L}_a^{{}^b x} = \mathcal{L}_a^{x_{m+1}} = \mathcal{L}_a^x$. This proves (1). The rest follows from (1). $\quad\square$

Corollary 5.12. *Let $x, a, b \in X$ and suppose that the left actions of a and b commute on Γ_x.*

(1) *If $^a x = {}^b x$ then $\mathcal{L}_b^x = \mathcal{L}_a^x$.*
(2) *If $^b(^a x) = x$ then $\mathcal{L}_b^x = (\mathcal{L}_a^x)^{-1}$.*

Proof. There is nothing to prove if $^a x = x$. So suppose $^a x = x_2 \neq x$. Let $\mathcal{L}_a^x = (x_1 x_2 \cdots x_k)$, $k \geq 2$, where, as usual, we set $x_1 = x$. Suppose $^a x = {}^b x$. Then

$$x_1 \xrightarrow{\ a\ } x_2$$
$$b \downarrow$$
$$x_2$$

is a subgraph of Γ. The hypothesis of Lemma 5.11 part (2) is satisfied (here $m = 1$). Therefore $\mathcal{L}_b^x = \mathcal{L}_a^x$. This proves part (1). Assume now $^b(^a x) = x$. This implies that

$$x_1 \xrightarrow{\ a\ } x_2$$
$$\downarrow b$$
$$x_1$$

is a subgraph of Γ. It follows then by Lemma 5.8 that Γ contains also the following subgraph:

$$\begin{array}{ccc} x_1 & \xrightarrow{\ a\ } & x_2 \\ b\downarrow & & \downarrow b \\ x_k & \xrightarrow{\ a\ } & x_1. \end{array}$$

Apply Lemma 5.11 part (2) again to obtain $\mathcal{L}_b^x = (\mathcal{L}_a^x)^{k-1} = (\mathcal{L}_a^x)^{-1}$. (In the notation of the Lemma, this time $m = k - 1$). $\qquad \square$

Clearly, in the case when $\mathcal{L}_a^x = (x_1 x_2)$ the diagram in the proof of part (2) of the corollary becomes simply

$$\begin{array}{ccc} x_1 & \xrightarrow{\ a\ } & x_2 \\ b\downarrow & & \downarrow b \\ x_2 & \xrightarrow{\ a\ } & x_1. \end{array}$$

which in the real graph Γ deforms to $x_1 \xleftrightarrow{\ a,b\ } x_2$.

Lemma 5.13. *Let $a, b, x \in X$, let $\mathcal{L}_a^x = (x_1 x_2 \cdots x_k)$, $\mathcal{L}_b^x = (x_1 y_2 \cdots y_s)$, where $x = x_1$ and $s \leq k$. Suppose that the left actions of a and b commute on Γ_x and suppose that the cycles $\mathcal{L}_a^x$ and $\mathcal{L}_b^x$ have at least two common elements, i.e. $x_i = y_j$ for some $i, j \neq 1$. Then $\mathcal{L}_b^x = (\mathcal{L}_a^x)^m$, where $1 \leq m \leq k - 1$.*

Proof. We claim that $^b x \in \{x_2 \cdots x_k\}$. Assume the contrary, then we have $\mathcal{L}_b^x = (x_1 y_{21} \cdots y_{p1} x_{m+1} \cdots)$, where $p \geq 1$ and $y_{21}, \cdots, y_{p1}$ are distinct from $x_2, \cdots, x_k$.

We want to deduce the vertices of the following subgraph of Γ:

(5.11)

So starting with $i = 2$, we apply Lemma 5.11 to successively deduce the vertices of i-th row, $i = 2, 3, \cdots$. This way we obtain

(5.12)

Note first that each row of (5.12) represents a cycle of length exactly k. Compare $\mathcal{L}_a^x$ with $\mathcal{L}_b^x$ (represented by the first column of (5.12)). The hypothesis of Lemma 5.11 is satisfied but this time we have that $\mathcal{L}_a^x = (\mathcal{L}_b^x)^{qp}$, where $qp \equiv 2 (mod k)$. Thus the lengths of the two cycles k and s satisfy $k \leq s$. An equality is impossible, since by our assumption the entry y_{21} of $\mathcal{L}_b^x$ does not occur in $\mathcal{L}_a^x$. This yields $k < s$, a contradiction with the hypothesis. It follows then that ${}^b x = x_{m+1}$ for some $m \geq 1$ as we claimed. Now Lemma 5.11 again implies $\mathcal{L}_b^x = (\mathcal{L}_a^x)^m$. $\qquad \square$
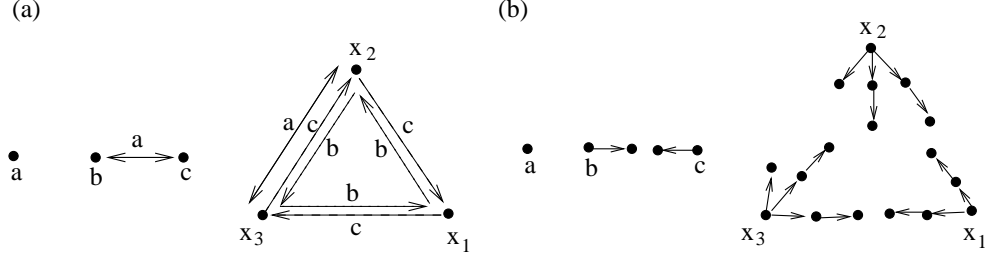
FIGURE 6. Graph of Example 5.16 illustrating *Star* at each vertex.

Definition 5.14. Let (X, r) be a nondegenerate involutive square-free solution of the YBE, and $x \in X$. The set $Star(x)$ is defined as

$$\mathrm{Star}(x) = \{\mathcal{L}_a^x \mid a \in X\}.$$

Definition 5.15. By $\Gamma(x)$ we denote the subgraph of (the complete graph) $\Gamma(X, r)$ with a set of vertices consisting of all x_j that occur in the cycles of $\mathrm{Star}(x)$, and all edges inherited from $\Gamma(X, r)$.

On the set of stars we introduce an equivalence relation $\approx$ defined as $\mathrm{Star}(x) \approx \mathrm{Star}(y)$ *iff* the complete graphs $\Gamma(x)$ and $\Gamma(y)$ are isomorphic (as labeled graphs).

As before, when we draw the graph $\Gamma(x)$ we shall often present it as a simplified graph in which if $\mathcal{L}_a^x = \mathcal{L}_b^x$, for $a \neq b \in X$, we only draw it once. In various cases is convenient to use even more schematic graph in which for each $\mathcal{L}_a^x = (x_1 x_2 \cdots x_k) \in \mathrm{Star}(x)$ with $x_1 = x$ we draw

$$\overset{x}{\bullet} \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet$$

where each variable occurring in $\mathcal{L}_a^x$ is represented by a $\bullet$ (the last corresponds to x_k).

Example 5.16. Consider the nondegenerate square-free involutive solution (X, r) defined on the set

$$X = \{x_1, x_2, x_3, a, b, c\},$$

via the left actions

$$\begin{aligned}
\mathcal{L}_{x_1} = \mathcal{L}_{x_2} = \mathcal{L}_{x_3} = id_X, && \mathcal{L}_b = (x_1 x_2 x_3), \\
\mathcal{L}_c = (x_1 x_3 x_2), && \mathcal{L}_a = (bc)(x_2 x_3),
\end{aligned}$$

The graph $\Gamma(X, r)$ is presented in figure 6 (a). The graphs of the Stars of all elements of X are represented in part (b).

We introduce now a particular type of subgraphs Γ_1 we call them *graphs of first type*.

Definition 5.17. In our usual assumption and notation, let Γ_1 be a connected component of $\Gamma = \Gamma(X, r), X_1 = \mathcal{V}(\Gamma_1)$ be its set of vertices. Γ_1 is *a graph of first type* if it contains a nontrivial edge and for each $a, b \in \mathcal{E}(\Gamma_1)$ the left actions of a and b commute on Γ_1. In other words, if

$$\mathcal{L}_{(^b a)|X_1} = \mathcal{L}_{a|X_1}, \quad \forall a, b \in \mathcal{E}(\Gamma_1).$$

Next we define

$$(5.13) \qquad \mathcal{G}_1 = \mathcal{G}(\Gamma_1) := {}_{gr}\langle \mathcal{L}_{a|X_1}; a \in \mathcal{E}(\Gamma_1)\rangle.$$

Lemma 5.18. *Let Γ_1 be a nontrivial connected component of $\Gamma(X, r)$ with a set of labels $\mathcal{E}_1 = \mathcal{E}(\Gamma_1)$. Then Γ_1 is a graph of first type* iff

$$\mathcal{L}_{(a^x)|\Gamma_1} = \mathcal{L}_{a|\Gamma_1} \quad \forall a, x \in \mathcal{E}_0.$$

Proof. The lemma follows from Definition 5.17, the cyclic conditions and **lri**.   □

Note that since (X, r) is a solution, condition **l1** hods and can be written as

$$\mathcal{L}_a \circ \mathcal{L}_b = \mathcal{L}_{{}^a b} \circ \mathcal{L}_{a^b} \quad \forall a, b \in X.$$

It follows then from Definition 5.17 that the group $\mathcal{G}(\Gamma_1)$ is an abelian subgroup of $Sym(X_1)$. We recall the following fact about finite abelian groups.

Fact 5.19. *Basis Theorem* [Sc, Ch. 2] *Let G be a finitely generated abelian group. If s is the least integer such that G is generated by s elements, then G is the direct sum (in abelian notation) of s cyclic subgroups.*

Remark 5.20. Note that, in general, the presentation of a finite abelian group G as a direct product of cyclic subgroups is not an invariant of the group, it becomes invariant if one considers only direct products of cyclic subgroups of prime power order. (For example one has $G = \mathbb{Z}_6 = \mathbb{Z}_2 \times \mathbb{Z}_3$, here $s(G) = 1$).

However, the least number $s = s(G)$ of a generating set of a finite group is well determined, and we shall use it to introduce the notion of *basis of labels* for Γ_1.

It follows by Fact 5.19, that the group $\mathcal{G}_1 = \mathcal{G}(\Gamma_1)$ is the direct product of cyclic subgroups:

$$(5.14) \qquad \mathcal{G}_1 = (\mathcal{L}_{a_1|X_1}) \times (\mathcal{L}_{a_2|X_1}) \times \cdots \times (\mathcal{L}_{a_s|X_1}),$$

where $s = s(\Gamma_1)$ is the least integer such that $\mathcal{G}_1$ is generated by s elements. Note that s is also an invariant of Γ_1.

Definition 5.21. A set of labels $\mathbb{B} = \{a_1, \cdots, a_s\} \subseteq \mathcal{E}(\Gamma_1)$ is *a basis of of labels for Γ_1* if (5.14) holds and s is the least number of a generating set of $\mathcal{G}_1$.

For computations it is convenient to use the arrows in Γ_1 labeled by $a_i, 1 \leq i \leq s$ as "axes" of an s-dimensional discrete "coordinate system". Let $k_i = k(a_i)$ as in Theorem 5.22 part (4). We can chose an arbitrary vertex x of Γ_1 as an "origin". It has coordinates $x = (0 \cdots 0)$. Then each vertex $y \in X_1$ has s "coordinates" which we define as follows.

$$\mathcal{L}_{a_1}(00 \cdots 0) = (10 \cdots 0),\ \mathcal{L}_{a_2}(00 \cdots 0) = (01 \cdots 0),\ \cdots,\ \mathcal{L}_{a_s}(00 \cdots 0) = (00 \cdots 1).$$

Clearly, every vertex y of Γ_1 can be obtained by applying a finite sequence of left actions:

$$(5.15) \qquad y = (\mathcal{L}_{a_1})^{m_1} \circ (\mathcal{L}_{a_2})^{m_2} \circ \cdots \circ (\mathcal{L}_{a_s})^{m_s}((0 \cdots 0)) = (m_1 m_2 \cdots m_s),$$

where $0 \leq m_i \leq k_i$.

We will now give a detailed description of the shape of a graph of first type.

Theorem 5.22. *Let (X, r) be a finite nondegenerate square-free symmetric set and suppose the connected component Γ_1 of $\Gamma(X, r)$ is a graph of first type. Let $X_1 = \mathcal{V}(\Gamma_1)$, and $\mathcal{G}_1 = \mathcal{G}(\Gamma_1)$, as in (5.13). Then the following conditions hold.*

(1) *If the nontrivial edge $x \xrightarrow{a} y, x \neq y$ occurs in Γ_1 then for every vertex z in$\mathcal{V}(\Gamma_1)$ there is an edge $z \xrightarrow{a} t, z \neq t \in \mathcal{V}(\Gamma_1)$.*

(2) *X_1 is r-invariant, and (X_1, r_1) with $r_1 = r_{|X_1 \times X_1}$ is a trivial solution.*

(3) *Each $a \in X$ acts on X_1 as an automorphism, and $\mathcal{G}_1 = \mathcal{G}(\Gamma_1)$ is a normal subgroup of $Aut(X_1, r_1)$.*

(4) *For every $a \in \mathcal{E}(\Gamma_1)$, there exists an integer $k = k(a, \Gamma_1)$, which is an invariant of a and Γ_1 such that for every vertex $x \in X_1$ the cycle $\mathcal{L}_a^x$ is of length k. Moreover*

$$\mathcal{L}_{a|X_1} = (x_1 \cdots x_k)(y_1 \cdots y_k) \cdots (z_1 \cdots z_k),$$

where each vertex of Γ_1 occurs exactly once and the product contains all disjoint cycles with edges labeled a.

(5) *All vertices $x_1, \cdots, x_N \in X_1$ have equivalent stars, $Star(x_1) \approx Star(x_i), 1 \leq i \leq N$.*

(6) *Let $\mathbb{B} = \{a_1, \cdots, a_s\}$ be a basis of labels for Γ_1, and let $k_i = k(a_i, \Gamma_1), 1 \leq i \leq s$, as in 4. Then Γ_1 is a "multi-dimensional cube" of dimension s, and the order of X_1 (i.e. the size of the cube) is exactly the product $N = k_1 k_2 \cdots k_s$.*

Proof. We start with part (1). By hypothesis Γ_1 is a connected component of Γ, and all its edges commute. Then it follows from Lemma 5.8 that for any "neighbour" z of x, $(z = y$ is also possible) with

$$
\begin{array}{ccc}
x & \xrightarrow{\ a\ } & y \\
{\scriptstyle b}\downarrow & & \\
z & &
\end{array}
$$

Γ_1 contains the subgraph:

(5.16)
$$
\begin{array}{ccc}
x & \xrightarrow{\ a\ } & y \\
{\scriptstyle b}\downarrow & & \downarrow{\scriptstyle b} \\
z & \xrightarrow{\ a\ } & t.
\end{array}
$$

Furthermore, by Lemma 5.8 part (3), $x \neq y$ implies $z \neq t$, which verifies (1).

(2). X_1 is the set of vertices of a connected component of $\Gamma(X, r)$, so by Lemma 5.6 it is r-invariant. Let $r_1 = r_{|X_1 \times X_1}$, we will show that (X, r_1) is a trivial solution, that is $r(x, y) = (y, x)$, for all $x, y \in X_1$. In view of **lri** this is equivalent to ${}^y x = x$, for all $x, y \in X_1$. Assume the contrary, there exist $x, y \in X_1$, such that ${}^y x \neq x$. Then Γ_1 contains the nontrivial edge $x \xrightarrow{y} \xi$. This by (1) implies that every vertex $z \in X_1$ has an edge $z \xrightarrow{y} t, t \neq z$. In particular $y \xrightarrow{y} \zeta, \zeta \neq y$, which is impossible, since (X, r) is a square-free solution, thus ${}^y x = y, \forall x, y \in X$. This implies (2).

The lemma below proves part (3) and will be needed also for the proof of other statements in the sequel. It remains to prove the last three items of the theorem. Suppose $x \in X_1, a, b \in \mathcal{E}(\Gamma_1)$, then by Lemma 5.11 the cycles $\mathcal{L}_a^x$ and $\mathcal{L}_a^{b_x}$ are of the

same length, say k. This and part (1) of the theorem imply that for every vertex $z \in X_1$ the cycle $\mathcal{L}_a^z$ has length k which proves (4). Clearly, (1) and (4) $\Longrightarrow$ (5) $\Longrightarrow$ (6). $\qquad\square$

Lemma 5.23. *Under the hypothesis of Theorem 5.22,*

(1) *The following equalities hold:*
$$^{a^y}x = {}^a x = {}^{{}^y a}x \quad \forall x, y \in X_1, \text{ and } \forall a \in X.$$

(2) *Each $a \in X$ acts on X_1 as an automorphism, and $\mathcal{G}(\Gamma_1)$ is a normal subgroup of $Aut(X_1, r_1)$.*

Proof. Suppose $x, y \in X_1, a \in X$. We use part (2) of the theorem and **l1** on (X, r) to obtain
$$^a x =^{(2)} {}^a({}^y x) =^{\mathbf{l1}} {}^{a^y}({}^{a^y} x) =^{(2)} {}^{a^y}x.$$

This imply $^{a^y}x = {}^a x$, for every $x, y \in X_1$, and $a \in X$, which verifies the left hand side of the equality of part (1). From there, using **lri** and the non-degeneracy of r one easily deduces the righthand side of the required equality.

It follows from part (1) and Lemma 2.7 that
$$\mathcal{L}_{a|\Gamma_1} \in Aut(X_1, r_1), \forall a \in X$$

thus $\mathcal{G}(\Gamma_1)$ is a subgroup of $Aut(X_1, r_1)$. Now Lemma 2.3, and the equality (2.1) imply that $\mathcal{G}(\Gamma_1)$ is a normal subgroup of $Aut(X_1, r_1)$. The lemma has been proved, which also completes the proof of the theorem. $\qquad\square$

Theorem 5.24. *Let (X, r) be a finite nondegenerate square-free symmetric set of order ≥ 2, $G = G(X, r), \mathcal{G} = \mathcal{G}(X, r)$, $\Gamma = \Gamma(X, r)$, $Aut(X, r)$ in the usual notation. Let $\Gamma_1, \Gamma_2, \cdots, \Gamma_s$ be all connected components, and $X_i = \mathcal{V}(\Gamma_i), 1 \leq i \leq s$, respectively, be their sets of vertices. The following conditions are equivalent.*

(1) *(X, r) is multipermutation solution with $mpl(X, r) = 2$.*
(2) *$\mathcal{G}(X, r)$ is an abelian group of order ≥ 2.*
(3) *$\{id_X\} \neq \mathcal{G}(X, r)$ is a subgroup of the automorphism group $Aut(X, r)$.*
(4) *The set of nontrivial components is nonempty. Suppose these are $\Gamma_1, \Gamma_2, \cdots, \Gamma_p$, with $1 \leq p \leq s$. Every nontrivial connected component Γ_i is a graph of first type. Furthermore, in this case, for each pair $i, j, 1 \leq i, j \leq s$ and each $a, b \in X_j$ one has $\mathcal{L}_{a|X_i} = \mathcal{L}_{b|X_i}$.*
(5) *(X, r) can be split into disjoint r-invariant subsets $X_i, 1 \leq i \leq s$, where each $(X_i, r_{|X_i})$ is a trivial solution and $X = X_1 \natural X_2 \natural \cdots \natural X_s$, in the sense that we can put parentheses and "apply" $\natural$ in any order. In particular, for any pair i, j r induces $X_i \natural X_j$ and*

$$(5.17) \qquad X = X_i \natural (\bigcup_{1 \leq j \leq s, j \neq i} X_j)$$

Proof. We shall show that
$$(1) \Longleftrightarrow (2) \Longleftrightarrow (3) \text{ and } (2) \Longleftrightarrow (4) \Longrightarrow (5) \Longrightarrow (2).$$

Under the hypothesis of the theorem we start with the following easy lemma.

Lemma 5.25. *The following conditions are equivalent.*

(i) $mpl(X, r) = 2$

(ii) $mpl(Ret(X, r)) = 1$,

(iii) $Ret(X, r) = ([X], r_{[X]})$ is the trivial solution.

(iv) $^{[x]}[a] = [^x a] = [a]$, for all $a, x \in X$

(v) There is an equality:

$$\text{(5.18)} \qquad\qquad \mathcal{L}_{^x a} = \mathcal{L}_a \quad \forall x, a \in X$$

(vi) There are equalities

$$\text{(5.19)} \qquad\qquad \mathcal{L}_{^x a} = \mathcal{L}_a = \mathcal{L}_{a^x}, \quad \forall x, a \in X.$$

Proof. It follows from Definition 3.5 and Lemma 3.8 that (i) $\Longleftrightarrow$ (ii). Lemma 3.10 gives the equivalence (ii) $\Longleftrightarrow$ (iii). For (iii) $\Longleftrightarrow$ (vi), the condition that the retract $([X], r_{[X]})$ is the trivial solution, implies that each pair $x, a \in X$ satisfies

$$^{[x]}[a] = [^x a] = [a] = [a]^{[x]} = [a^x]$$

These equalities, "translated" in terms of the left actions give (5.19). (iii) $\Longleftrightarrow$ (iv) follows from Definition 3.2 and Lemma 3.10 again. By Definition 3.1 the equality (5.18) is just expression of $[^x a] = [a]$ in terms of the left action, thus (iv) $\Longleftrightarrow$ (v). □

We continue with the proof of the theorem. (1) $\Longrightarrow$ (2). Suppose $mplX = 2$. Recall that $\mathcal{G}(X, r)$ is the subgroup of $Sym(X)$ generated by the permutations $\mathcal{L}_a, a \in X$. By Lemma 5.25 the equalities (5.19) are in force, so we yield:

$$\mathcal{L}_a \circ \mathcal{L}_b =^{\mathbf{l1}} \mathcal{L}_{^a b} \circ \mathcal{L}_{a^b} =^{(5.19)} \mathcal{L}_b \circ \mathcal{L}_a, \quad \forall a, b \in X$$

Thus $\mathcal{G}(X, r)$ is an abelian group. The following implications are clear

$$(2) \Longrightarrow (5.18) \Longrightarrow^{\text{by Lemma 5.25}} (1)$$

The implications (5.18) $\Longleftrightarrow$ (3) follow from Lemma 2.7, and together with Lemma 5.25 yield (1) $\Longleftrightarrow$ (3).

(2) $\Longleftrightarrow$ (4). Suppose (2) holds. Then $\mathcal{G}(X, r) \neq id_X$, so there is a nontrivial connected component of Γ. In addition by (5.18) each such a component Γ_i is a graph of first type.

Lemma 5.26. (4) $\Longrightarrow$ (5.18).

Proof. Suppose (4) holds and assume the contrary, there exist a pair $a, x \in X$, such that $\mathcal{L}_{^x a} \neq \mathcal{L}_a$.

Hence there exist a t, for which

$$\text{(5.20)} \qquad\qquad ^{^x a}t \neq {}^a t.$$

Clearly, then the connected component Γ_t which contains t is nontrivial. Denote by X_0 the set of vertices of Γ_t, the set of labels we denote by $\mathcal{E}_0 = \mathcal{E}(\Gamma_t)$. By our assumption Γ_t is a graph of first type, and therefore all its edges commute, so at

least one of a and x is not in $\mathcal{E}_0$. Suppose a is not in $\mathcal{E}_0$. Then ${}^{x^a}t \neq {}^at = t$, implies that

$$(5.21) \qquad\qquad {}^xa \in \mathcal{E}_0.$$

Two cases are possible. **a)** $x \in \mathcal{E}_0$. Then, since Γ_t is a graph of first type, $\mathcal{L}_{{}^xa}$ and $\mathcal{L}_x$ commute on Γ_t. Now the equalities

$$(5.22) \qquad\qquad \mathcal{L}_{({}^xa|\Gamma_1)} = {}^{\text{Lemma 5.18}}\ \mathcal{L}_{(({}^xa)^x|\Gamma_1)} = {}^{\textbf{lri}}\ \mathcal{L}_{a|\Gamma_1}$$

give a contradiction with (5.20). **b)** x is not in $\mathcal{E}_0$, so ${}^xt = t$, and ${}^x({}^at) = {}^at$. By **l1** one has ${}^x({}^at) = {}^{({}^xa)}(({}^{x^a})t)$, so (5.20) imply ${}^{(x^a)}t \neq t$. Thus $x^a \in \mathcal{E}_0$, and by (5.21) Lemma 5.18 one has

$$(5.23) \qquad\qquad \mathcal{L}_{({}^xa|\Gamma_1)} = {}^{\text{by Lemma 5.18}}\ \mathcal{L}_{(({}^xa)^{x^a}|\Gamma_1)} = \mathcal{L}_{a|\Gamma_1}.$$

For the right hand side of (5.23) we used the following equalities which come from the cyclic condition **cl1**, see Definition 1.8, and **lri**:

$$({}^xa)^{x^a} = {}^{\text{by } \textbf{cl1}} = ({}^{x^a}a)^{x^a} = {}^{\text{by } \textbf{lri}}\ a.$$

This way (5.23) gives a contradiction with (5.20). It follows then that $a \in \mathcal{E}_0$. Next (5.20) implies that x is not in $\mathcal{E}_0$, so it commutes with every element of X_0 and in particular with ${}^at$. Hence one has

$$ {}^at = {}^x({}^at) = {}^{({}^xa)}(({}^{x^a})t),$$

and therefore $x^a \in \mathcal{E}_0$. It follows then that

$$\mathcal{L}_{a|\Gamma_1} = \mathcal{L}_{(({}^{x^a}a)|\Gamma_1)} = {}^{\text{by } \textbf{cl1}}\ \mathcal{L}_{({}^xa|\Gamma_1)}.$$

(Remind that due to **cl1** one has ${}^{x^a}a = {}^xa$, which was used in the right hand side of the above equality). It follows than that (5.20) is impossible. We have shown that for each $a, x \in X$ one has $\mathcal{L}_{{}^xa} = \mathcal{L}_a$. This proves the lemma $\qquad\square$

It follows from Lemmas 5.25 and 5.26 that $(4) \implies (2)$

Next we show $(4) \implies (5)$. So, assume now (4) and consider the sets $X_i, 1 \leq i \leq s$. By Theorem 5.22 (X_i, r_i) is a trivial solution of order ≥ 2 for all $i, 1 \leq i \leq p$ and (in case that $p < s$) it is one element solution for $i, p+1 \leq i \leq s$. by Lemma 5.26 the equalities (5.18) hold, and it is easy to see that (5.18) implies (5).

Conversely, suppose (5) holds. Let $a \in X$. Then there exist unique $i, 1 \leq i \leq s$ such that $a \in X_i$. It follows then from (5.17) that $\mathcal{L}_a$ acts on $X \backslash X_i$ as an automorphism. Note that since X_i is the trivial solution, or one element solution thus in both cases one has $\mathcal{L}_{a|X_i} = id_{X_i}$, which clearly implies $\mathcal{L}_a = \mathcal{L}_{a|X \backslash X_i}$. It follows then that $\mathcal{L}_a \in Aut(X, r)$. We have shown $(5) \implies (2)$. The theorem has been proved. $\qquad\square$

Proposition 5.27. *Let (X, r) be a nondegenerate square-free symmetric set. Let $\Gamma_1, \Gamma_2, \cdots, \Gamma_s$ be all connected components, and $X_i = \mathcal{V}(\Gamma_i), 1 \leq i \leq s$, respectively, be their sets of vertices. Suppose $mpl(X, r) = 3$. Then for each $i, 1 \leq i \leq s,$, (X_i, r_i) with $r_i = r_{|X_i \times X_i}$ is a multipermutation solutions of level $mpl(X_i) \leq 2$. Firthermore, for each pair $i \neq j, 1 \leq i, j \leq s$, one has $X_i \natural X_j$.*

Proof. Proposition 5.6 implies that for each $i, 1 \leq i \leq s$ and each $x \in X_i$ one has $X_i \subseteq \mathcal{O}(x, 2)$, and therefore $mpl(X_i, r_i) \leq 2$.

Chose now an arbitrary pair $i \neq j, 1 \leq i, j \leq s$. We claim that $X_i \natural X_j$. It will be enough to show

(5.24) $$\qquad \qquad {}^{\alpha^y}x = {}^\alpha x \quad {}^{x^\beta}\alpha = {}^x\alpha \quad \forall x, y \in X_i, \alpha, \beta \in X_j.$$

Let $x, y \in X_i$, $\alpha \in X_j$. Then $\alpha^y \sim \alpha^x$, so

$$ {}^{\alpha^y}x = {}^{\alpha^x}x = {}^{\textbf{cl1}} {}^\alpha x,$$

which proves the left hand side equality of (5.24). Similar argument verifies the remaining equality in (5.24). $\hfill\square$

We will illustrate our theory on Examples 2.5 and 2.8. To begin with, it follows straightforwardly from the definition of (X, r) in these examples that for $Ret(X, r) = ([X], r_{[X]})$ one has $[X] = \{[x_1], [b], [c]\}$, and $r_{[X]})$ the trivial solution. Therefore $mpl(X) = 2$. Moreover, $Z = X \natural \{a\}$, where $\mathcal{L}_a = \mathcal{L}_b \circ \mathcal{L}_c \in Aut(X, r)$. We leave the reader to verify $mpl(Z, r_Z) = 3$.

Clearly the graph $\Gamma(Z, r_Z)$ is obtained by adding to $\Gamma(X, r)$ a new vertex and two new edges labeled by a.

Next we shall describe the group $Aut(Z, r_Z)$. Each automorphism $\varphi \in Aut(Z, r_Z)$ is a product $\varphi = \varphi'\varphi''$ where $\varphi' \in Sym(a, b, c), \varphi'' \in Sym(x_1, x_2, x_3, x_4)$. We claim that φ is uniquely determined by the data $(\varphi', \varphi''(x_1))$. Indeed, knowing the image $\varphi''(x_1)$, one applies the equalities

$$\varphi \circ \mathcal{L}_a = \mathcal{L}_{\varphi(a)} \circ \varphi, \quad \varphi \circ \mathcal{L}_b = \mathcal{L}_{\varphi(b)} \circ \varphi, \quad \varphi \circ \mathcal{L}_c = \mathcal{L}_{\varphi(c)} \circ \varphi.$$

to find $\varphi(x_i), 2 \leq i \leq 4$. We can then tabulate the automorphisms with row corresponding to $\varphi' \in S_3$ and column corresponding to the value of $\varphi''(x_1)$, which in all except the last row (where we use a different notation) this value supplies the index used. For simplicity we write $1, 2, \cdots$ instead of $x_1, x_2, \cdots$. Then the full list of automorphisms is:

$$\begin{aligned}
&\tau_1 = (bc)(23), \quad \tau_2 = (bc)(1243), \tau_3 = (bc)(1342), \tau_4 = (bc)(14) \\
&\pi_1 = (abc)(234), \pi_2 = (abc)(124), \pi_3 = (abc)(132), \pi_4 = (abc)(143) \\
&\eta_1 = (acb)(243), \eta_2 = (acb)(123), \eta_3 = (acb)(134), \eta_4 = (acb)(142) \\
&\rho_1 = (ab)(24), \quad \rho_2 = (bc)(1234), \rho_3 = (ab)(13), \quad \rho_4 = (ab)(1432) \\
&\sigma_1 = (ac)(34), \quad \sigma_2 = (ac)(12), \quad \sigma_3 = (ac)(1324), \sigma_4 = (ac)(1423) \\
&id_Z = (), \qquad \mathcal{L}_b = (12)(34), \quad \mathcal{L}_c = (13)(24) \quad \mathcal{L}_a = (14)(23).
\end{aligned}$$

For example, π_2 corresponds to the data $(\pi_{2|\{abc\}} = (abc), \pi_2(x_1) = x_2)$. We will do the first step of the "search" for the missing information to see how this was obtained.

$$\pi_2 \circ \mathcal{L}_b(x_1) = \mathcal{L}_{\pi_2(b)} \circ \pi_2(x_1), \quad \pi_2(x_2) = \mathcal{L}_c(x_2) = x_4$$

so π_2 acts as $x_1 \mapsto x_2 \mapsto x_4$. Similar computation with $\pi_2 \circ \mathcal{L}_a(x_4)$ shows that $\pi_2(x_4) = x_1$. Hence $\pi_2 = (abc)(x_1 x_2 x_4)$. Note that the η row consists of the inverses of the π row. In the same way one obtains the rest of the table. This list was illustrated in Figure 2.

We claim that $Aut(Z, r_z)$ is isomorphic to the symmetric group S_4. Indeed, we know that S_4 is generated by $\{(1234), (12)\}$. Direct computation shows that the assignment

$$(1234) \longrightarrow (ab)(1234) = \rho_2, \quad (12) \longrightarrow (ac)(12) = \sigma_2$$

extends to an isomorphism of groups $S_4 \longrightarrow Aut(X, r)$.

The computation of $Aut(X, r)$ as a set is done analogously, and the relations show directly that there is a group isomorphism $Aut(X, r) \approx D_4$. Clearly, $Aut(X, r)$ is a subgroup of $Aut(Z, r_Z)$ as expected.

References

[AS] ARTIN, M. AND SCHELTER, W., *Graded algebras of global dimension 3*, Adv. Math. **66** (1987), 171–216.

[Dri] DRINFELD, V., *On some unsolved problems in quantum group theory*, Lecture Notes in Mathematics **1510** (1992), 1–8.

[ESS] ETINGOF, P., SCHEDLER, T. AND SOLOVIEV, A., *Set-theoretical solutions to the quantum Yang-Baxter equation*, Duke Math. J. **100** (1999), 169–209.

[GI94] GATEVA-IVANOVA, T., *Noetherian properties of skew polynomial rings with binomial relations*, Trans. Amer. Math. Soc. **343** (1994), 203–219.

[GI96-1] GATEVA-IVANOVA, T., *Skew polynomial rings with binomial relations*, J. Algebra **185** (1996), 710–753.

[GI96-2] GATEVA-IVANOVA, T., *Regularity of the skew polynomial rings with binomial relations*, Preprint (1996).

[GI00] GATEVA-IVANOVA, T., *Set theoretic solutions of the Yang-Baxter equation*, Mathematics and education in Mathematics, Proc. of the Twenty Ninth Spring Conference of the Union of Bulgarian Mathematicians, Lovetch (2000), 107-117.

[GI04] T. Gateva-Ivanova, *Quantum binomial algebras, Artin-Schelter regular rings, and solutions of the Yang-Baxter equations*, Serdica Math. J. **30** (2004), 431-470.

[GI] GATEVA-IVANOVA, T., *A combinatorial approach to the set-theoretic solutions of the Yang-Baxter equation*, J.Math.Phys., **45** (2004), 3828–3858.

[GIM] T. GATEVA-IVANOVA, T. AND S. MAJID, *Matched pairs approach to set theoretic solutions of the Yang-Baxter equation*, arXiv:math.QA/0507394, revised 12/06.

[GB] GATEVA-IVANOVA, T. AND VAN DEN BERGH, M., *Semigroups of I-type*, J. Algebra **206** (1998), 97–112.

[GJO] GATEVA-IVANOVA, T., JESPERS, E. AND OKNINSKI, J., *Quadratic algebras of skew polynomial type and underlying semigroups*, arXiv:math.RA/0210217 v1; J.Algebra, **270** (2003), 635-659.

[JO] JESPERS, E. AND OKNINSKI, J., *Binomial Semigroups*, J. Algebra **202** (1998), 250–275.

[LYZ] LU, J., YAN, M. AND ZHU, Y., *On the set-theoretical Yang-Baxter equation*, Duke Math. J. **104** (2000), 1–18.

[M90] MAJID, S., Matched pairs of Lie groups associated to solutions of the Yang-Baxter equations. *Pac. J. Math.*, **141** (1990) 311–332.

[M02] MAJID, S., *A Quantum Groups Primer*, L.M.S. Lect. Notes **292** (2002) 179pp.

[M] MANIN, YU., Quantum Groups and Non Commutative Geometry Montreal University Report No. CRM- 1561, 1988

[R] RUMP, W., *A decomposition theorem for square-free unitary solutions of the quantum Yang-Baxter equation*, Advances in Mathematics, **193** (2005), 40–55.

[Sc] SCHENKMAN, E., *Group Theory*, Robert E. Krieger Publishing Company, Huntington, New York (1975).

[T] TAKEUCHI, M., *Survey on matched pairs of groups. An elementary approach to the ESS-LYZ theory*, Banach Center Publ. **61** (2003), 305–331.

[Wei] WEINSTEIN, P. AND XU, P., *Classical solutions of the quantum Yang-Baxter equation*, Comm. Math. Phys. **148** (1992), 309–343.

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