EQUALITY OF LIFSHITZ AND VAN HOVE EXPONENTS ON AMENABLE CAYLEY GRAPHS

TONĆI ANTUNOVIĆ AND IVAN VESELIĆ

Abstract. We study the low energy asymptotics of periodic and random Laplace operators on Cayley graphs of amenable, finitely generated groups. For the periodic operator the asymptotics is characterised by the van Hove exponent or zeroth Novikov-Shubin invariant. The random model we consider is given in terms of an adjacency Laplacian on site or edge percolation subgraphs of the Cayley graph. The asymptotic behaviour of the spectral distribution is exponential, characterised by the Lifshitz exponent. We show that for the adjacency (Laplacian) the two invariants/exponents coincide. For combinatorial Laplacians one has a different universal behaviour of the low energy asymptotics of the spectral distribution function, which can be actually established on quasi-transitive graphs without an amenability assumption.

1. Introduction

Operators on Euclidean space which are invariant under a group action have a well defined *integrated density of states* (IDS), also known as the *spectral distribution function.* Prominent examples are Laplace and Schrödinger operators. Their IDS exhibits a *van Hove singularity* at the bottom of the spectrum. This means that it vanishes polynomially as the energy parameter approaches the lowest spectral edge, the exponent being equal to the space dimension divided by two. The factor two is due to the fact that the considered operators are elliptic of second order.

The IDS can be defined also for operators having a more general type of equivariance property, namely for ergodic operators. Two prominent classes of such operators are random and almost periodic ones. Among the pioneering works which have studied the IDS of such models are [27], respectively [29].

Several well-studied types of random operators on $L^2(\mathbb{R}^d)$ and $\ell^2(\mathbb{Z}^d)$ exhibit a Lifshitz tail at the bottom of the spectrum, meaning that the IDS vanishes exponentially fast. In particular, the spectral density is very sparse in this region and spectral values are due to extremely rare configurations of the randomness. Hence such spectral edges are called *fluctuation boundaries.* In Euclidean space the Lifshitz exponent is quite universal. In particular, for Laplacians with a variety of random i.i.d. non-negative perturbations it equals $d/2$, cf. for instance the survey [19] and the references therein.

Historically, physicists have introduced the IDS as a limit of spectral distribution functions of finite volume operators. For this approximation to hold true, the underlying space or group

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needs to have some amenability property. However, for the purposes of the present paper the approximation property is not relevant and we may rather consider the IDS as given by a *Shubin-Pastur trace formula* (2).

In the present paper we want to analyse whether the Lifshitz exponent equals the van Hove exponent for operators on more general geometries as well. Of course, for this to hold there needs to be a proper relation between the considered periodic and random operator, in the sense that the random operator results from its periodic counterpart by addition of stochastically independent, positive perturbations. The periodic objects we study are Laplace operators on Cayley graphs. We consider two different types of random perturbations thereof: the adjacency and the combinatorial Laplacians on random subgraphs generated by a subcritical percolation process. While the first type of operators indeed shows a coincidence of van Hove and Lifshitz exponents, the second ones exhibit a different type of universal behaviour, the reason being, that the random perturbation is not positive in this case.

Our motivation to study this question is threefold: to extend the results of [20] on lattice bond percolation models, to study the relation between van Hove and Lifshitz exponents, as done at internal spectral edges of random Schrödinger operators e.g. in [21], and to clarify some of the links between geometric L^2-invariants and the IDS, see e.g. [23, 12]. Note in particular that the van Hove exponent equals the Novikov-Shubin invariant of order zero. Our strategy of proof is coined after the one in [20]. The description of the asymptotic behaviour of the IDS at spectral boundaries of random operators plays a key role in the proof of Anderson localisation, see e.g. [15]. For more background on the IDS of percolation Hamiltonians on Cayley graphs see the discussion in [3].

In the next section we state our theorems. Thereafter, in Section 3 we present abstract upper and lower bounds on the IDS. Section 4 is devoted to eigenvalue inequalities. Section 5 contains the proofs of the theorems in the case of adjacency Laplacians on groups with polynomial growth and combinatorial Laplacians on general quasi-transitive graphs. In the last section we prove the statements concerning Lamplighter groups.

2. Definitions and results

We describe the type of graphs, the percolation process and the operators we will be considering.

Let Γ be a discrete, finitely generated group, S a finite, symmetric set of generators not containing the unit element ι of Γ and $G = (V, E)$ the associated Cayley graph. It is k-regular with $k = |S|$. The ball around ι of radius n is denoted by $B(n)$ and its volume by $V(n)$. From [5, 17, 30] it is known that either there are $d \in \mathbb{N}, a, b > 0$ such that $a\, n^d \le V(n) \le b\, n^d$, in which case Γ is called to be of *polynomial growth of order* d; or for every $d \in \mathbb{N}$ and every $b \in \mathbb{R}$ there exist only finitely many integers n such that $V(n) \le b\, n^d$, in which case Γ is called to be of *superpolynomial growth.* The growth type depends only on the group and not on the choice of the set of generators used to define the Cayley graph. Cayley graphs are a particular case of *quasi-transitive* graphs, i.e. graphs that decompose under the action of the automorphism group into finitely many orbits. Most of our results are valid only for Cayley graphs. An exception is Theorem 12 which applies to general quasi-transitive graphs with finite vertex degree.

Next we introduce site percolation on infinite, connected, quasi-transitive graphs. For $p \in [0, 1]$, let $\omega_x, x \in V$ be an i.i.d. sequence of Bernoulli random variables each taking the value 1 with probability p and the value 0 with probaility $1 - p$. The set of possible configurations

$\omega = (\omega_x)_{x\in V}$ is denoted by Ω and the corresponding product probability measure with $\mathbb{P}$. We call $V(\omega) := \{x \in V \mid \omega_x = 1\}$ the set of *open sites*. The induced subgraph of G with vertex set $V(\omega)$ is denoted by G_ω and called the *percolation subgraph* in the configuration ω. The connected components of G_ω are called *clusters*. For a fixed vertex $o \in V$ we denote by $C_o(\omega)$ the connected component which contains it. The bond percolation process is defined analogously. In this case the percolation subgraph G_ω is the graph whose edge set $E(\omega)$ is the set of all $e \in E$ with $\omega_e = 1$ and whose vertex set $V(\omega)$ consist of all vertices in V which are incident to an element of $E(\omega)$. For both site and bond percolation there exists a *critical parameter* $0 < p_c \le 1$ such that for $p < p_c$ there is no infinite cluster almost surely and for $p > p_c$ there is an infinite cluster almost surely. The first case is called the *subcritical phase* and the second *supercritical phase*. The theorems of this paper concern only the subcritical percolation phase. We will denote the expectation with respect to $\mathbb{P}$ by $\mathbb{E}\{\dots\}$.

In the following we assume throughout that G is an infinite quasi-transitive graph with bounded vertex degree and that there exist a group of automorphisms acting freely and cofinitely on G. In particular it may be a Cayley graph. Let $G' = (V', E')$ be an arbitrary subgraph of G, possibly G itself. Note that even if G is regular, G' need not be. We denote the degree of the vertex $x \in V'$ in G' by $\deg_{G'}(x)$. If two vertices $x, y \in V'$ are adjacent in the subgraph G' we write $y \sim_{G'} x$.

For G and G' as above we define the following operators on $\ell^2(G') := \ell^2(V')$.

Definition 1. (a) The identity operator on $\ell^2(V')$ is denoted by Id.
(b) The *degree operator* acts on $\varphi \in \ell^2(V')$ according to

$$[D(G')\varphi](x) := \deg_{G'}(x)\varphi(x).$$

(c) The *adjacency operator* is defined as

$$[A(G')\varphi](x) := \sum_{y\in V', y\sim_{G'} x} \varphi(y).$$

(d) The *combinatorial Laplacian* is defined as

$$H^N(G') := D(G') - A(G').$$

If G is a k-regular graph we define additionaly:

Definition 2. (e) The *adjacency Laplacian* on G' is defined as

$$H^A(G') := k \operatorname{Id} - A(G').$$

(f) The *boundary potential* is the multiplication operator

$$W^{b.c.}(G') = k \operatorname{Id} - D(G')$$

(g) The *Dirichlet Laplacian* is defined as

$$H^D(G') := H^A(G') + W^{b.c.}(G') = 2k \operatorname{Id} - D(G') - A(G').$$

Note that $H^N(G') = H^A(G') - W^{b.c.}(G')$. Of course, it is possible to define the operators (e) – (g) also for non regular graphs, but then there is no canonical choice for the value k which would give them a geometric meaning.

It follows that the quadratic forms of the combinatorial, Dirichlet, and adjacency Laplacian are given by

$$\begin{aligned}
&\langle H^N(G')\phi, \phi\rangle = \sum_{(x,y)\in E'} |\phi(x)-\phi(y)|^2 \\
(1)\qquad &\langle H^D(G')\phi, \phi\rangle = 2\sum_{x\in V'} (k-\deg_{G'}(x))\,|\phi(x)|^2 + \sum_{(x,y)\in E'} |\phi(x)-\phi(y)|^2 \\
&\langle H^A(G')\phi, \phi\rangle = \sum_{x\in V'} (k-\deg_{G'}(x))\,|\phi(x)|^2 + \sum_{(x,y)\in E'} |\phi(x)-\phi(y)|^2
\end{aligned}$$

and satisfy $H^N(G') \le H^A(G') \le H^D(G')$ in the sense of quadratic forms.

Remark 3 (Terminology). If $G' = G$ and G is regular then the operators H^A, H^N, H^D coincide and we denote them simply by H. If G is the Cayley graph of an amenable group the spectral bottom of H equals zero. Usually in the graph theory literature the adjacency matrix and the combinatorial Laplacian are the objects of study. For the first operator one is (among others) interested in the properties related to the upper edge of the spectrum, whereas for the second operator one considers the low-lying spectrum. In order to be able to treat both operators in parallel it is convenient to consider consider H^A rather than A. Of course, spectral properties of H^A directly translate to those of A.

Motivated by the Dirichlet-Neumann bracketing for Laplacians in the continuum, in [28] the terminology of Neumann H^N and Dirichlet H^D Laplacians was introduced. This is the reason why we use the superscript N for the combinatorial Laplacian. While in the continuum the boundary conditions are necessary to define a selfadjoint operator, in the discrete setting they correspond to a boundary potential $W^{b.c.}$, which is either added or subtracted to/from the Laplacian without boundary term, i.e. the adjacency Laplacian H^A. Note however, that the term Neumann Laplacian is sometimes, e.g. in [8], used for a different operator. Likewise, the operator H^A is often called Dirichlet Laplacian, e.g. in [7], while in [20] it is called Pseudo-Dirichlet Laplacian.

Given a (site or bond) percolation subgraph $G_\omega \subset G$ we use the following abbreviations for operators on $\ell^2(V(\omega))$: $\deg_\omega(x) = \deg_{G_\omega}(x), A_\omega = A(G_\omega), H^A_\omega = H^A(G_\omega), H^N_\omega = H^N(G_\omega), H^D_\omega = H^D(G_\omega), W^{b.c.}_\omega = W^{b.c.}(G_\omega)$. Any one of the operators $H^\#_\omega, \# \in \{A, N, D\}$ will be called a percolation Laplacian. If G is a Cayley graph we consider all three types H^A, H^N, H^D, while in the case of a quasi-transitive graph we will derive results only for the combinatorial Laplacian H^N.

Next we define the IDS. Let G be a quasi-transitive graph equipped with a subgroup Γ of its automorphism group which acts freely and cofinitely on G. Denote by $\mathcal{F}$ an arbitrary, but fixed Γ-fundamental domain, i.e. subset of G, which contains exactly one element of each Γ-orbit. The IDS of the random operator $(H^\#_\omega)_\omega$ may be defined by the following trace formula:

$$N^\#(E) := \frac{1}{|\mathcal{F}|}\mathbb{E}\big\{\operatorname{Tr}[\chi_{\mathcal{F}}\,\chi_{]-\infty,E]}(H^\#_\omega)]\big\}. \tag{2}$$

If Γ acts transitively on G the expression (2) simplifies to $\mathbb{E}\{\langle\delta_x, \chi_{]-\infty,E]}(H^\#_\omega)\delta_x\rangle\}$, where x denotes an arbitrary vertex in G and δ_x its characteristic function. If moreover $p = 1$, i.e. we consider the IDS of the Laplacian H on G itself, the formula simplifies further to $N_{\mathrm{per}}(E) = \langle\delta_x, \chi_{]-\infty,E]}(H)\delta_x\rangle$. We denote by N_{per} the IDS of the periodic operator H, while $N^\#$ is reserved for the IDS of the random operator $H^\#$.

Remark 4. Several properties of the random family $(H^{\#}_{\omega})_{\omega}$ of operators play a role in the definition of the IDS. These hold for any of the boundary types $\# \in \{A, N, D\}$. Firstly, $(H^{\#}_{\omega})_{\omega}$ is a measurable family of operators in the sense of [22] (which extends the notion introduced in [18]). Secondly, each operator is bounded, selfadjoint and non-negative.

If the group Γ is amenable, it is possible to approximate the IDS by its analogs associated to operators restricted to finite graphs along a Følner (van Hove) sequence. This has been shown for periodic operators in [14, 24] and for site percolation Hamiltonians in [31]. For bond percolation Hamiltonians the same proof applies. For bond percolation on the lattice these results were proven in [20].

A finitely generated, discrete group is amenable if and only if it contains an increasing Følner sequence, i.e. an increasing sequence of finite subsets $I_n \subset \Gamma$ such that

$$\lim_{j\to\infty} \frac{|I_j \bigtriangleup F \cdot I_j|}{|I_j|} = 0, \quad \text{for any finite } F \subset \Gamma.$$

Any increasing Følner sequence induces a monotone exhaustion $\Lambda_n, n \in \mathbb{N}$ consisting of finite subsets Λ_n of the vertex set of G, such that if we denote by $H^{\#,n}_{\omega}$ the restriction of $H^{\#}_{\omega}$ to $\ell^2(\Lambda_n \cap V(\omega))$ the convergence

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \operatorname{Tr}[\chi_{]-\infty,E]}(H^{\#,n}_{\omega})] = N^{\#}(E) \tag{3}$$

holds for almost all ω and all continuity points E of $N^{\#}$. Note that since $H^{\#,n}_{\omega}$ is a finite dimensional operator its spectrum consists entirely of eigenvalues and hence $\operatorname{Tr}[\chi_{]-\infty,E]}(H^{\#,n}_{\omega})]$ equals the number of eigenvalues of $H^{\#,n}_{\omega}$ not exceeding E. Using the inequalities for the quadratic forms (1) and Weyl's monotonicity principle it follows that $\operatorname{Tr}[\chi_{]-\infty,E]}(H^{N,n}_{\omega})] \geq \operatorname{Tr}[\chi_{]-\infty,E]}(H^{A,n}_{\omega})] \geq \operatorname{Tr}[\chi_{]-\infty,E]}(H^{D,n}_{\omega})]$. Passing to the limit $n \to \infty$ one obtains $N^N \geq N^A \geq N^D$.

There are other important properties of $(H^{\#}_{\omega})_{\omega}$ which are appropriate to mention here although they are not necessary for the formulation of our definitions or theorems. The spectrum of $H^{\#}_{\omega}$ is almost surely ω-independent, cf. [22, 31]. We denote it by $\Sigma^{\#}$ in the sequel. The same holds for the measure-theoretic components of the sectrum. The topological support of the measure whose distribution function is $N^{\#}$ coincides with $\Sigma^{\#}$. Using the same arguments as in [20] one can show that $\Sigma^{\#} \supset \sigma(H)$. The IDS of percolation Hamiltoninas has a rich set of discontinuities [6], and a characterisation of this set is given in [32]. It is also possible to extend the percolation Hamiltoninans to the removed vertices $V \setminus V(\omega)$ by a constant. This is just a matter of convention and does not alter the results essentially. For a broader discussion of the above facts see [3].

The next statement characterises the asymptotic behaviour of the IDS of the periodic Laplacian H at the spectral bottom and can be inferred either form [33] or [23]. For groups of polynomial growth it exhibits a van Hove singularity, while in the case of superpolynomial growth one encounters a different type asymptotics which may be interpreted as corresponding to a van Hove exponent equal to infinity.

Theorem 5. *Let Γ be a finitely generated, amenable group, H the Laplace operator on a Cayley graph of Γ and N_{per} the associated IDS. If Γ has polynomial growth of order d then*

$$\lim_{E\searrow 0} \frac{\ln N_{\mathrm{per}}(E)}{\ln E} = \frac{d}{2}. \tag{4}$$

and if Γ *has superpolynomial growth then*

$$\lim_{E\searrow 0}\frac{\ln N_{\mathrm{per}}(E)}{\ln E}=\infty. \tag{5}$$

Next we state our result about the low energy asymptotics of $(H^A_\omega)_\omega$ and $(H^D_\omega)_\omega$ and compare it with the asymptotic behaviour of the Laplacian H on the full Cayley graph. Here and in the sequel we restrict ourselves to the subcritical phase of (site or bond) percolation, i.e. we consider a percolation parameter $p < p_c$. The asymptotic behaviour of the IDS of the adjacency and the Dirichlet percolation Laplacian on a Cayley graph at low energies is as follows:

Theorem 6. *Let* G *be a Cayley graph of an amenable, finitely generated group* Γ*. Let* $(H^A_\omega)_\omega$ *and* $(H^D_\omega)_\omega$ *be the adjacency, respectively the Dirichlet percolation Laplacian for subcritical site or bond percolation on* G*.*

Assume that G *has polynomial growth and* $V(n)\sim n^d$*. Then there are positive constants* $\alpha^+_D(p)$ *and* $\alpha^-_D(p)$ *such that for all positive* E *small enough*

$$e^{-\alpha^-_D(p)E^{-d/2}}\le N^D(E)\le N^A(E)\le e^{-\alpha^+_D(p)E^{-d/2}}. \tag{6}$$

Assume that G *has superpolynomial growth. Then*

$$\lim_{E\searrow 0}\frac{\ln|\ln\, N^D(E)|}{|\ln E|}=\lim_{E\searrow 0}\frac{\ln|\ln\, N^A(E)|}{|\ln E|}=\infty. \tag{7}$$

Remark 7. The inequality $N^D(E)\le N^A(E)$ in (6) is deduced from the convergence of the finite volume eigenvalue counting functions to the IDS which is explained in Remark 4. This is slightly inconsistent with our approach that we want to deduce the asymptotic behaviour of the IDS from the trace formula (2) alone. Note however that our proof of Theorem 6 shows that even *without the use of the finite volume approximation* we have

$$e^{-\alpha^-_D(p)E^{-d/2}}\le N^D(E)\le e^{-\tilde\alpha^+_D(p)E^{-d/2}}\text{ and }e^{-\tilde\alpha^-_D(p)E^{-d/2}}\le N^A(E)\le e^{-\alpha^+_D(p)E^{-d/2}}$$

with some positive constants $\tilde\alpha^+_D,\tilde\alpha^-_D$. Thus even without the knowledge that the IDS has finite volume approximations the correct asymptotic behaviour of the IDS may be deduced. An analogous remark applies to equation (7).

Remark 8. Theorem 6 is a generalisation of the results in [20] on subcritical bond percolation on the lattice and consistent with the Lifshitz asymptotics for various other types of random Schrödinger operators in Euclidean space, cf. e.g. [19]. In particular, Lifshitz tails have been proven for the Anderson model, i.e. the discrete random Schrödinger operator on $\ell^2(\mathbb{Z}^d)$ with an i.i.d. potential. The first proofs of this result were given in [25, 28]. It is not clear whether the proof of [28] can be adapted to general amenable Calyey graphs. One obstacle for this extension is the fact one has to bound the IDS in terms of the eigenvalues of the random operator restricted to a finite graph. In Euclidean space this can be established using the fact that cubes are a very neat Følner sequence which are at the same time fundamental domains of sublattices. For general amenable groups such sequences do not exist necessarily. The other reason is that the eigenvalue estimates for the Anderson models restricted on finite graphs are established using the Temple inequality, which in turn to be applied efficiently needs lower bounds on the distance between the two lowest eigenvalues. This lower bound on the spectral gap is immediate in the Euclidean case, while for more general transitive graphs it may be inferred from a strengthened version of the Cheeger inequality. Taking these considerations into account one may hope that the proof of Lifshitz tails for the Anderson model on $\mathbb{Z}^d$ can be adapted for Cayley graphs of polynomial growth. They are, apart from being amenable, residually finite and thus admit an approximation by finite transitive graphs.

Theorem 6 implies in particular that the IDS is very sparse near the bottom of the spectrum $E = 0$ and consequently zero is a fluctuation boundary. Relation (6) implies that in the case of polynomial growth the Lifshitz exponent coincides with the van Hove exponent of the Laplacian on the full Cayley graph. In particular, we have

$$\lim_{E\searrow 0} \frac{\ln|\ln N^D(E)|}{|\ln N_{\text{per}}(E)|} = \lim_{E\searrow 0} \frac{\ln|\ln N^A(E)|}{|\ln N_{\text{per}}(E)|} = 1$$

In the case of superpolynomial growth we have that both exponents are infinite. One may ask whether the limits defining them diverge at the same rate and whether, for instance, the relation

$$\lim_{E\searrow 0} \frac{\ln\ln|\ln N^D(E)|}{\ln|\ln N_{\text{per}}(E)|} = \lim_{E\searrow 0} \frac{\ln\ln|\ln N^A(E)|}{\ln|\ln N_{\text{per}}(E)|} = 1$$

holds. We are not able prove this in general, but at least for the case of the Lamplighter groups $\mathbb{Z}_m \wr \mathbb{Z}$. These groups are amenable, but of exponential growth.

Theorem 9. *Let G be a Cayley graph of the Lamplighter group $\mathbb{Z}_m \wr \mathbb{Z}$. There are positive constants a_1^+ and a_2^+ such that*

$$N_{\text{per}}(E) \le a_1^+ e^{-a_2^+ E^{-1/2}}, \text{ for all } E \text{ small enough.}$$

Moreover for every $r > 1/2$ there are positive constants $a_{r,1}^-$ and $a_{r,2}^-$ such that

$$N_{\text{per}}(E) \ge a_{r,1}^- e^{-a_{r,2}^- E^{-r}}, \text{ for all } E \text{ small enough.}$$

Thus we have an exponential behaviour of the IDS at the bottom of the spectrum, in particular:

$$\lim_{E\searrow 0} \frac{\ln|\ln N_{\text{per}}(E)|}{|\ln E|} = \frac{1}{2}. \tag{8}$$

Theorem 10. *Let G be an arbitrary Cayley graph of the Lamplighter group $\mathbb{Z}_m \wr \mathbb{Z}$. For every $p < p_c$ there are positive constants b_1, b_2, c_1, c_2, such that the IDS of the adjacency and Dirichlet (site or bond) percolation Laplacian satisfies the following inequality*

$$e^{-c_1 e^{c_2 E^{-1/2}}} \le N^A(E) \le N^D(E) \le e^{-b_1 e^{b_2 E^{-1/2}}}, \text{ for all } E > 0 \text{ small enough.} \tag{9}$$

Remark 11. Our proofs show that the lower bounds $e^{-\alpha_D^-(p)E^{-d/2}} \le N^D(E) \le N^A(E)$ in Theorem 6 and $e^{-c_1 e^{c_2 E^{-1/2}}} \le N^A(E) \le N^D(E)$ in Theorem 10 are valid for all values of the percolation parameter $p \in]0, 1]$

Let us now turn to combinatorial Laplacians $(H_\omega^N)_\omega$, i.e. Laplacians with the third type of boundary term which we did not discuss yet. In the case of Neumann boundary conditions the energy zero is not a fluctuation boundary. The IDS has a discontinuity at zero, thus one may say that the zeroth L^2-Betti number of the random operator $(H_\omega^N)_\omega$ does not vanish. For the combinatorial Laplacian we are able to treat general quasi-transitive graphs. In particular, G does not need to be neither amenable nor a Cayley graph. The following again generalises a result of [20] on $\mathbb{Z}^d$-bond percolation.

Theorem 12. *Let G be a graph with bounded vertex degree and Γ a group of automorphisms acting freely and cofinitely on G. Consider the IDS of the Neumann percolation Hamiltonian $(H_\omega^N)_\omega$ of a subcritical site or bond percolation process. There exist positive constants $\alpha_N^+(p)$ and $\alpha_N^-(p)$ such that for all positive E small enough*

$$e^{-\alpha_N^-(p)E^{-1/2}} \leq N^N(E) - N^N(0) \leq e^{-\alpha_N^+(p)E^{-1/2}}. \tag{10}$$

The value $N^N(0)$ coincides with the average number of clusters per vertex in the random graph G_ω. After subtracting this value we can speak of (10) as a kind of 'renormalised' Lifshitz asymptotics with exponent $1/2$.

It is possible to prove Theorem (12) for a larger class of Neumann Hamiltonians, which correspond to long-range percolation in the subcritical phase. However, then one is dealing with random operators which do not have a finite hopping range and are unbounded, thus the formal setting is different.

On an abstract level Theorem 12 and its proof show that the low energy asymptotics of the combinatorial Laplacian does not depend on geometric properties of G, but only on the rate at which the linear clusters are produced by the percolation process.

3. Abstract upper and lower bounds on the IDS

To obtain upper bounds for the integrated density of states near the lower spectral edge, we have to prove that the spectrum is relatively scarce in this area. In the subcritical phase the spectrum is only pure point and consists of the eigenvalues of the operators $H^\#(G')$, where G' goes over the set of all finite subgraphs. So what one really needs are certain lower bounds for the eigenvalues of the operators $H^\#(G')$, $\# \in \{N, A, D\}$. Vice versa for lower bounds for the IDS we shall need upper bounds for these eigenvalues in some neighbourhood of the lower spectral edge. In this spirit we present Propositions 13 and 14, which are generalisations of Lemmata 2.7 and 2.9 in [20]. Denote with $\lambda^\#(G')$ the lowest nonzero eigenvalue of the operator $H^\#(G')$, $\# \in \{N, A, D\}$.

Proposition 13. *Let G be a quasi-transitive graph and $\# \in \{A, D, N\}$. Assume that there is a continuous strictly decreasing function $f\colon [1, \infty[\to \mathbb{R}^+$ such that $\lim_{s\to\infty} f(s) = 0$ and $\lambda^\#(G') \geq f(|G'|)$ for any finite subgraph G'. Then, for every $0 < p < p_c$ there is a positive constant a_p such that*

$$N^\#(E) - N^\#(0) \leq e^{-a_p f^{-1}(E)}, \tag{11}$$

for all E from the interval $]0, f(1)[$ on which the inverse function f^{-1} is well defined.

Proof. Fix $\# \in \{N, A, D\}$ and $0 < E < f(1)$. Since the subspace $\ell^2(C_x(\omega))$ is invariant for the operator $H_\omega^\#$ and the restriction on this subspace is exactly $H^\#(C_x(\omega))$ we can write

$$N^\#(E) - N^\#(0) = \frac{1}{|\mathcal{F}|} \sum_{x \in \mathcal{F}} \mathbb{E}\Big(\langle \delta_x, \chi_{]0,E]}(H^\#(C_x(\omega)))\delta_x \rangle \Big).$$

Now $\chi_{]0,E]}(H^{\#}(C_x(\omega)))$ is the zero operator if $E < \lambda^{\#}(C_x(\omega))$, in particular in the case $|C_x(\omega)| < f^{-1}(E)$. Since $\langle \delta_x, \chi_{]0,E]}(H^{\#}(C_x(\omega)))\delta_x \rangle \leq 1$ for any E and ω we can write

$$\begin{aligned} N^{\#}(E) - N^{\#}(0) &= \frac{1}{|\mathcal{F}|} \sum_{x \in \mathcal{F}} \mathbb{E}\Big(\langle \delta_x, \chi_{]0,E]}(H^{\#}(C_x(\omega)))\delta_x \rangle \, \chi_{\{|C_x(\omega)| \geq f^{-1}(E)\}}(\omega) \Big) \\ &\leq \frac{1}{|\mathcal{F}|} \sum_{x \in \mathcal{F}} \mathbb{P}(|C_x(\omega)| \geq f^{-1}(E)). \end{aligned}$$

Now the result follows from the fact that the probabilities of large subcritical clusters in quasi-transitive graphs decay exponentially, i.e. $\mathbb{P}(|C_x(\omega)| \geq n) \leq e^{-a_p n}$ for all positive integers n, all vertices x and all $p < p_c$, where a_p is a positive constant depending only on the value of the parameter p. This fact is established for quasi-transitive graphs in [2] using the methods developed in [1]. $\square$

Proposition 14. *Let G be graph with bounded vertex degree and Γ a group of isometries acting cofinitely on G and $\# \in \{A, D, N\}$. Suppose that there is a sequence of connected subgraphs $(G'_n)_n$ and a sequence $(c_n)_n$ in $\mathbb{R}^+$ such that*

(i) $\lim_{n \to \infty} |G'_n| = \infty$,
(ii) $\lim_{n \to \infty} c_n = 0$,
(iii) $\lambda^{\#}(G'_n) \leq c_n$.

For every $E > 0$ small enough define $n(E) := \min \{n; c_n \leq E\}$. Then for every $0 < p < 1$ there is a positive constant b_p such that the following inequality holds for all $E > 0$ small enough

$$N^{\#}(E) - N^{\#}(0) \geq \frac{1}{|\mathcal{F}|} \mathbb{P}(G'_{n(E)} \textit{ is a cluster in } G(\omega)) \geq e^{-b_p |G'_{n(E)}|}. \tag{12}$$

Proof. Fix $\# \in \{N, A, D\}$ and $E > 0$ small enough so that $n(E)$ is well defined. Define $\mathcal{S}_x(E) := \left\{ \tau \in \Gamma; x \in \tau G'_{n(E)} \right\}$ where $\tau G'_{n(E)}$ is the translation of the subgraph $G'_{n(E)}$ obtained by mapping each vertex of the subgraph $G'_{n(E)}$ by automorphism τ. On the set $\mathcal{S}_x(E)$ define the equivalence relation $\simeq$ in the following way $\tau_1 \simeq \tau_2 :\Leftrightarrow \tau_1 G'_{n(E)} = \tau_2 G'_{n(E)}$. Now take $\mathcal{T}_x(E)$, a subset of $\mathcal{S}_x(E)$, which contains exactly one element from each equivalence class. Note that for each vertex y in $G'_{n(E)}$ there exist a vertex $x \in \mathcal{F}$ and an automorphism

$\tau \in \mathcal{T}_x(E)$ which maps y to x. Now we can write

$$\begin{aligned}
N^\#(E) - N^\#(0) &\geq \frac{1}{|\mathcal{F}|} \sum_{x \in \mathcal{F}} \mathbb{E}\Big(\langle \delta_x, \chi_{]0,E]}(H^\#(C_x(\omega)))\delta_x \rangle \, \chi_{\big\{C_x(\omega)=\tau G'_{n(E)};\tau \in \mathcal{T}_x(E)\big\}}\Big) \\
&\geq \frac{1}{|\mathcal{F}|} \sum_{x \in \mathcal{F}} \sum_{\tau \in \mathcal{T}_x(E)} \mathbb{E}\Big(\Big\langle \delta_x, \chi_{]0,\lambda^\#(G'_{n(E)})]}(H^\#(\tau G'_{n(E)}))\delta_x \Big\rangle \, \chi_{\big\{C_x(\omega)=\tau G'_{n(E)}\big\}}\Big) \\
&= \frac{1}{|\mathcal{F}|} \sum_{x \in \mathcal{F}} \sum_{\tau \in \mathcal{T}_x(E)} \Big\langle \delta_x, U_\tau^{-1} \chi_{]0,\lambda^\#(G'_{n(E)})]}(H^\#(G'_{n(E)}))U_\tau \delta_x \Big\rangle \, \mathbb{P}(C_x(\omega) = \tau G'_{n(E)}) \\
&= \frac{1}{|\mathcal{F}|} \sum_{x \in \mathcal{F}} \sum_{\tau \in \mathcal{T}_x(E)} \Big\langle \delta_{\tau^{-1}x}, \chi_{]0,\lambda^\#(G'_{n(E)})]}(H^\#(G'_{n(E)}))\delta_{\tau^{-1}x} \Big\rangle \, \mathbb{P}(C_{\tau^{-1}x}(\omega) = G'_{n(E)}) \\
&= \frac{1}{|\mathcal{F}|} \sum_{y \in G'_{n(E)}} \Big\langle \delta_y, \chi_{]0,\lambda^\#(G'_{n(E)})]}(H^\#(G'_{n(E)}))\delta_y \Big\rangle \, \mathbb{P}(C_y(\omega) = G'_{n(E)}) \sum_{x \in \mathcal{F}} \sum_{\substack{\tau \in \mathcal{T}_x(E) \\ y=\tau^{-1}x}} 1 \\
&\geq \frac{1}{|\mathcal{F}|} \mathbb{P}(G'_{n(E)} \text{ is a cluster in } G(\omega)) \sum_{y \in G'_{n(E)}} \Big\langle \delta_y, \chi_{]0,\lambda^\#(G'_{n(E)})]}(H^\#(G'_{n(E)}))\delta_y \Big\rangle \\
&\geq \frac{1}{|\mathcal{F}|} \mathbb{P}(G'_{n(E)} \text{ is a cluster in } G(\omega)).
\end{aligned}$$

Here we used the fact that for any subgraph G' and any element τ of the group Γ, $H^\#(\tau G') = U_\tau^{-1} H^\#(G') U_\tau$, where U_τ is a unitary operator on $\ell^2(G)$ defined by $U_\tau f(x) := f(\tau x)$. The operators U_τ have the property $U_\tau \delta_x = \delta_{\tau^{-1}x}$. In the last step we used the fact that $\chi_{]0,\lambda^\#(G'_{n(E)})]}(H^\#(G'_{n(E)}))$ is a non-trivial projection and thus its trace is equal to the dimension of its range which is greater or equal than one. Since we are considering independent percolation on a graph of uniformly bounded vertex degree we can find a positive constant b_p depending only on p, such that $\frac{1}{|\mathcal{F}|}\mathbb{P}(G' \text{ is a cluster in } G(\omega)) \geq e^{-b_p|G'|}$ holds for any finite subgraph G'. □

4. Bounds on eigenvalues

As we have seen in the previous section, for good upper and lower bounds for the IDS we need to estimate $\lambda^\#(G')$. Lower bounds for eigenvalues (which give upper bounds for IDS) which are sufficient for our purposes can be given in terms of the growth rate of the group. Recall that $B(n)$ denotes the ball in a Cayley graph G, of radius n around the unit element ι and $V(n)$ stands for the volume (the number of vertices) of $B(n)$. Also define $\phi(t) := \min\{n \geq 0; V(n) > t\}$.

Proposition 15. *Let $G = (V, E)$ be a Cayley graph of a finitely generated group Γ. For every finite connected subgraph G'*

$$\lambda^A(G') \geq \frac{1}{128} \frac{1}{k^2\, \phi(2|G'|)^2}. \tag{13}$$

Proof. If we prove that every non-zero φ satisfies $\dfrac{\langle \varphi, H^A(G')\varphi \rangle}{\|\varphi\|^2} \geq \dfrac{(128k^2)^{-1}}{\phi(2|G'|)^2}$, the inequality will follow by the mini-max principle, after taking the infimum over all non-zero φ. The above inequality follows from results in [10] and [33]. Namely in the course of the proof of Proposition 14.1 in [33] one proves that for any $\varphi \in \ell^2(G)$ with finite support we have

$$\frac{D_P(\varphi)}{\|\varphi\|^2} \geq \frac{1}{2\kappa^2 \mathfrak{f}(|\operatorname{supp}\varphi|)^2}, \tag{14}$$

where κ is a positive constant and $\mathfrak{f}\colon \mathbb{N} \to \mathbb{R}$ is non-decreasing and such that $\kappa|\partial_E A| \geq \dfrac{|A|}{\mathfrak{f}(|A|)}$ for all finite subsets of vertices A. Here $\partial_E A$ is the edge boundary, i.e. the set of edges which have one end-vertex in A and the other outside A and D_P is the Dirichlet sum, which in the special case when P defines the nearest neighbor simple random walk satisfies $D_P(\varphi) = \sum_{x \sim_G y} |\varphi(x) - \varphi(y)|^2$. Note that (1) implies the fact that for any finite subgraph G' of a Cayley graph G and any $\zeta \in \ell^2(G')$ we have $\left\langle H^A(G')\zeta, \zeta \right\rangle = \sum_{x \sim_G y} |\widetilde{\zeta}(x) - \widetilde{\zeta}(y)|^2$, where $\widetilde{\zeta}$ is an extension of ζ in $\ell^2(G)$ defined by setting $\widetilde{\zeta}(x)$ to be equal to 0 for every $x \notin G'$. Thus the Dirichlet sum considered in [33] satisfies

$$D_P(\widetilde{\zeta}) = \left\langle H^A(G')\zeta, \zeta \right\rangle, \tag{15}$$

in the special case where the transition matrix P corresponds to a simple nearest neighbour random walk on G. On the other hand Théorème 1 in [10] shows that for any Cayley graph of a finitely generated group

$$8k\,|\partial_V A| \geq \frac{|A|}{\phi(2|A|)}, \tag{16}$$

holds for all finite subsets of vertices A. (Here $\partial_V A$ is the inner vertex boundary of A, i.e. the set of vertices in A which have a neighbour outside A.) Since $|\partial_E A| \geq |\partial_V A|$, the conditions of Proposition 14.1 in [33] are satisfied with $\mathfrak{f}(n) = \phi(2n)$ and so (14) and (15) together imply the desired inequality. ☐

The role of the subgraphs G'_n from Proposition 14 will be played by the $B(n)$. As for the sequence c_n from the same proposition, the next proposition will give us a candidate.

Proposition 16. *Let $G = (V, E)$ be a Cayley graph of a finitely generated group with polynomial growth. The there exists a positive constant β_D^+ such that for every positive integer n we have*

$$\lambda^D(B(n)) \leq \frac{\beta_D^+}{n^2}. \tag{17}$$

Proof. From the mini-max principle we know

$$\lambda^D(B(n)) \leq \frac{\langle \varphi, H^D(B(n))\varphi \rangle}{\|\varphi\|^2}, \tag{18}$$

for every $\varphi \in \ell^2(B(n))$. For a test function φ use the radially symmetric function defined in the following way:

$$\varphi(x) := \begin{cases} n - d(\iota, x), & \text{if } d(\iota, x) \in \{\lfloor n/2 \rfloor, \dots, n\} \\ \lceil n/2 \rceil, & \text{if } d(\iota, x) < \lfloor n/2 \rfloor \\ 0, & \text{else.} \end{cases}$$

Now we have

$$\langle \varphi, H^D(B(n))\varphi \rangle = \sum_{\substack{[x,y]\in E \\ x,y\in B(n)}} |\varphi(x) - \varphi(y)|^2 + 2 \sum_{\substack{(x,y)\in E \\ x\in B(n), y\notin B(n)}} |\varphi(x) - \varphi(y)|^2 \leq kV(n)$$

$$\|\varphi\|^2 = \sum_{x\in B(n)} |\varphi(x)|^2 \geq \lceil n/2 \rceil^2 V(\lfloor n/2 \rfloor).$$

Inserting these two inequalities into (18) and using the fact that $V(n)$ grows polynomially one easily obtains (17). ☐

Now we give bounds for the eigenvalues in case of the combinatorial Laplacian on quasi-transitive graphs.

Proposition 17. *Let $G = (V, E)$ be a quasi-transitive graph with vertex degree bounded by $\widetilde{k}$. For every finite subgraph $G' = (V', E')$ we have*

$$\lambda^N(G'_n) \geq \frac{2}{\widetilde{k}\,|G'|^2}. \tag{19}$$

Proof. The Cheeger inequality (cf. Théorème 3.1.(2) in [9]) gives us

$$\lambda^N(G'_n) \geq \frac{h^2(G')}{2\widetilde{k}},$$

where $h(G')$ is the Cheeger constant of G' defined as $h(G') := \min\limits_{\substack{\widetilde{V'}\subset V \\ |\widetilde{V'}|\leq |V'|/2}} \frac{|\partial_E \widetilde{V'}|}{|\widetilde{V'}|}$. Since the Cheeger constant $h(G')$ satisfies $h(G') \geq 2/|G'|$, we get the desired bound in (19). ☐

The role of the subgraphs G'_n from Proposition 14, in the case of the Neumann Laplacian, will be played by linear subgraphs. A linear subgraph $L_n \subset G$ of length n is the subgraph induced by a path $v_1, v_2, \dots, v_{n+1}$ in the graph G, such that the distance between v_i and v_j is equal to $|j - i|$, for every $1 \leq i, j \leq n + 1$. Notice that for every connected infinite graph G and every $n \in \mathbb{N}$ there exists a linear subgraph of length n in G. To see this fix an arbitrary vertex w_0 and take any vertex w_n on the sphere of radius n with center in w_0 (this sphere is obviously non-empty). Now take a shortest path $(w_0, w_1, \dots w_{n-1}, w_n)$ between the vertices w_0 and w_n. Clearly, the vertices $\{w_0, w_1, \dots w_n\}$ are vertices of a linear subgraph L_n.

Proposition 18. *Let $G = (V, E)$ be a quasi-transitive graph with bounded vertex degree. For any integer n we have*

$$\lambda^N(L_n) \leq \frac{12}{n^2}. \tag{20}$$

Proof. We will again use the mini-max principle, i.e. $\lambda^N(L_n) \leq \dfrac{\langle \varphi, H^N(L_n)\varphi\rangle}{\|\varphi\|^2}$, for all $\varphi \in \ell^2(L_n)$, which are orthogonal to the kernel of the operator $H^N(L_n)$. Since the kernel is one dimensional and contains only constant functions, the condition that φ is orthogonal to the kernel is equivalent to $\sum_{x\in L_n} \varphi(x) = 0$. One obtains (20) by inserting the function which grows linearly along L_n having the value $-n/2$ on one end-vertex and $n/2$ on the other, see Lemma 2.6 in [20]. □

5. Proofs of the theorems for groups of polynomial growth and quasi-transitive graphs

We insert the eigenvalue bounds from the previous section into Propositions 13 and 14 to obtain the estimates on the IDS stated in Theorems 6 and 12.

Proof of Theorem 6. For the first inequality in (6) use Proposition 14 with $G'_n := B(n)$ and $c_n := \frac{\beta_D^+}{n^2}$, where β_D^+ is the constant from Proposition 16. When E approaches 0 from above, $n(E)E^{1/2}$ is bounded from above by a constant and thus the same is true for $|G'_{n(E)}|E^{d/2}$. Now using the fact that $N^D(0) = 0$ the result follows directly from Proposition 14.

For the second inequality in (6) we refer to Remark 4.

To prove the third inequality we use Proposition 13 together with the lower bounds from Proposition 15. Because of the polynomial volume growth we have $\phi(s) \leq \kappa_1 s^{1/d}$, for all $s > 1$, which implies $g(s) := \dfrac{(128k^2)^{-1}}{\phi(2s)^2} \geq \kappa_2 s^{-2/d}$, for all $s > 1$, where κ_1, κ_2 are positive constants. Now the choice $f(s) := \kappa_2 s^{-2/d}$ satisfies the conditions of Proposition 13 and, since $N^A(0) = 0$, the proof is straightforward.

Now we prove (7). By $N^D \leq N^A$ the divergence in (7) has to be proven only for the case of the adjacency Hamiltonian. In the case of superpolynomial growth we have $\lim_{n\to\infty} \dfrac{\ln n}{\ln V(n)} = 0$ from where one can easily obtain

$$\lim_{t\to\infty} \frac{\ln \phi(t)}{\ln t} = 0. \tag{21}$$

If we show that the assumptions of Proposition 13 are satisfied we can obtain for $E > 0$ small enough such that

$$\frac{\ln|\ln N^A(E)|}{\ln E} \leq \frac{\ln a_p}{\ln E} + \frac{\ln f^{-1}(E)}{\ln E}.$$

So it is enough to find an f which satisfies the conditions of Proposition 13, in particular $\lambda^A(G') \geq f(|G'|)$ for all finite subgraphs G', and such that $\lim_{E\searrow 0} \dfrac{\ln f^{-1}(E)}{\ln E} = -\infty$. The last condition is equivalent to

$$\lim_{s\to\infty} \frac{\ln s}{\ln f(s)} = -\infty. \tag{22}$$

Proposition 15 gives us the bound $\lambda^A(G') \geq g(s)$, where again $g(s) := \frac{(128k^2)^{-1}}{\phi(2s)^2}$. Now (21) easily implies

$$\lim_{s\to\infty} \frac{\ln s}{\ln g(s)} = -\infty. \tag{23}$$

The function $g(s)$ is not a good candidate for the function f in Proposition 13, since it is not continuous and strictly decreasing. Thus we define the function f as an arbitrary continuous, strictly decreasing, positive function which has the following three properties:

- $f(s) \leq g(s)$ for every positive real s,
- $f(s) = g(s)$ if s is a discontinuity point of g,
- $f(s) \geq \frac{1}{2} g(s)$ if g has no discontinuity points in $[s, 2s[$.

A function with these properties exists since g is continuous from the right. Now f satisfies the conditions of Proposition 13 as well as (22) and this proves the last part. □

Proof of the Theorem 12. By Proposition 17 we see that the assumptions of Proposition 13 are satisfied with $f(s) := \frac{2}{\widetilde{k}\, s^2}$. Moreover, by Proposition 18 we can use Proposition 14 with $G_n' = L_n$ and $c_n = \frac{12}{n^2}$. Now the bounds in (10) follow directly. □

As for the periodic case, the formulae for the limits in Theorem 5 are not new. See for instance Lemma 2.46. in [23], where this statement is derived using homological algebra. Another way to derive these bounds is the following: Consider the scaled adjacency operator $\frac{1}{k}A(G)$ and its integrated density of states $N_{\frac{1}{k}A}$. Denote the return probability after n steps of the simple random walk (X_n) which started at o by $\mathbb{P}_o(X_n = o)$. It follows that $\mathbb{P}_o(X_n = o) = \int_{\mathbb{R}} t^n dN_{\frac{1}{k}A}(t)$. Now it is possible to give sharp bounds on the behaviour of $N_{\frac{1}{k}A}$ near the upper spectral edge (i.e. $E = 1$) since the return probabilities of the simple random walk are well studied. Namely in the case of Cayley graphs of groups with polynomial growth the probabilities $\mathbb{P}_o(X_n = o)$ behave like $n^{-d/2}$ (see Corollary 14.5 and Theorems 14.12 and 14.19 in [33]). Now the desired bounds for N_{per} follow directly.

The idea to relate the IDS with the return probabilities of the simple random walk will be important for studying the same problem in the case of Lamplighter groups. Here we shall refer to results in [26].

6. Estimates for Lamplighter groups

In this section we derive upper and lower bounds on the IDS for a particular class of amenable groups of superpolynomial growth, namely for Lamplighter groups.

Fix a positive integer $m \geq 2$. The Lamplighter group is defined as the wreath product $\mathbb{Z}_m \wr \mathbb{Z}$, or in other words, elements of the group are ordered pairs (φ, x), where φ is a function $\varphi\colon \mathbb{Z} \to \mathbb{Z}_m$ with finite support and $x \in \mathbb{Z}$. The multiplication is given by $(\varphi_1, x_1) * (\varphi_2, x_2) := (\varphi_1 + \varphi_2(\cdot - x_1), x_1 + x_2)$.

We shall use the following notation. For $x \in \mathbb{Z}$ let δ_x denote the function which has value 1 at x and 0 everywhere else. The zero function will be denoted by $\mathbf{0}$.

Lamplighter groups are examples of amenable groups with exponential growth. It suffices to prove these two properties for some Cayley graph of the Lamplighter group. Consider the Cayley graph of the Lamplighter group $\mathbb{Z}_m \wr \mathbb{Z}$, defined with respect to the set of generators

$\{(\mathbf{0}, \pm 1), (k\delta_0, 0); k \in \mathbb{Z}_m \setminus \{0\}\}$. To prove the amenability one only has to notice that the sequence of sets

$$\Big(\{(\varphi, x); \operatorname{supp}\varphi \subseteq \{-n, \ldots, n\}, x \in \{-n, \ldots, n\}\}\Big)_n$$

is a Følner sequence. Exponential growth follows directly from the fact that for any function φ with support in $\{1, 2, \ldots, n\}$ one is able reach the vertex (φ, n), from the zero element in at most $2n$ steps, and so ball of radius $2n$ has at least m^n elements.

Using the same ingredients as in the case of groups with polynomial growth we now prove the upper bound in (9).

Proof of the upper bound from Theorem 10. Using the fact that the growth of the Lamplighter group is exponential, we get the upper bound $\phi(s) \leq \mu_1 \ln s$, for all $s > 2$ and some $\mu_1 > 0$, where the function ϕ was defined before Proposition 16. It implies the lower bound $g(s) = \dfrac{(128k^2)^{-1}}{\phi(2s)^2} \geq \mu_2(\ln s)^{-2}$, for all $s > 2$, where μ_2 is a positive constant. Thus the function $f(s) := \mu_2(\ln s)^{-2}$ satisfies the conditions of Proposition 14 and gives the desired estimate. ☐

The lower bound in Theorem 10 requires an additional step. In the proof we shall first prove the claimed estimate in the case of a particular generator set and then we shall show how to generalize the result to arbitrary Cayley graphs.

For an arbitrary generator set K of $\mathbb{Z}_m \wr \mathbb{Z}$ denote by $(\mathbb{Z}_m \wr \mathbb{Z})_K$ the Cayley graph induced by the generator set K. Also if V' is a subset of $\mathbb{Z}_m \wr \mathbb{Z}$ denote by $G(V', K)$ the subgraph of $(\mathbb{Z}_m \wr \mathbb{Z})_K$ induced by the vertex set V'.

Define the following symmetric set of generators

$$K_0 := \{(l \cdot \delta_1, 1), l \in \mathbb{Z}_m\} \cup \{(l \cdot \delta_0, -1), l \in \mathbb{Z}_m\}. \tag{24}$$

As explained in Section 2 of [34] the Cayley graph $(\mathbb{Z}_m \wr \mathbb{Z})_{K_0}$ is the horocyclic product of two $(m+1)$-regular trees. We will briefly sketch the necessary definitions and results. For a comprehensive introduction and a graphical illustration of horocyclic products of trees we refer to [4].

Let $T = (V, E)$ be a $(m+1)$-regular rooted tree with graph metric d. Let ξ be an arbitrary but fixed end. (An end is an infinite path from the root o in which vertices do not repeat.) For each vertex x there is the unique path γ_x from o to x. With $\gamma_x \cap \xi$ denote the intersection of the paths γ_x and ξ, that is the sequence of edges which lie both in γ_x and ξ. Now the *Busemann function* of the tree T (with respect to the root o and the end ξ) is defined as $\mathfrak{h}\colon V \to \mathbb{Z}$, $\mathfrak{h}(x) := |\gamma_x| - 2|\gamma_x \cap \xi|$. For two vertices x and y which satisfy $\mathfrak{h}(y) \geq \mathfrak{h}(x)$ and $d(x, y) = \mathfrak{h}(y) - \mathfrak{h}(x)$ we shall write $x \leq y$.

Assume now that we are given two $(m+1)$-regular trees T_1 and T_2 with Busemann functions $\mathfrak{h}_1$ and $\mathfrak{h}_2$ respectively. The *horocyclic product* of the trees T_1 and T_2 is defined as the graph whose vertex set is given by $\{(x_1, x_2); x_i \in T_i, \mathfrak{h}(x_1) + \mathfrak{h}(x_2) = 0\}$, with two vertices (x_1, x_2) and (x_1', x_2') adjacent if x_i and x_i' are adjacent in T_i for $i = 1, 2$. The choice of a root and an end in the definition is irrelevant since all horocyclic product of two given trees are mutually isomorphic. As we mentioned before, the Cayley graph $(\mathbb{Z}_m \wr \mathbb{Z})_{K_0}$ is isomorphic to the horocyclic product of two $(m+1)$-regular trees.

The spectrum of the full Laplace operator on the graph $(\mathbb{Z}_m \wr \mathbb{Z})_{K_0}$ is pure point, with eigenfunctions having only finite support. This was shown for the Lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ in [16] and for more general wreath products in [11]. Here we shall follow the methods from

[4] where the same facts are proved for Diestel-Leader graphs, which include certain Cayley graphs of the Lamplighter groups $\mathbb{Z}_m \wr \mathbb{Z}$ as a particular case. Moreover, there the spectrum of the Laplace operator restricted to certain subgraphs called tetrahedrons is calculated. This is where the representation of $(\mathbb{Z}_m \wr \mathbb{Z})_{K_0}$ as a horocyclic product becomes essential.

Assume we are given a horocyclic product of two $(m+1)$-regular trees T_1 and T_2 with Busemann functions $\mathfrak{h}_1$ and $\mathfrak{h}_2$ and graph metrics d_1 and d_2 respectively. Fix a positive integer n and take two vertices $x_1 \in T_1$ and $x_2 \in T_2$ such that $\mathfrak{h}_2(x_2) = -\mathfrak{h}_1(x_1) - n$. Now the *tetrahedron* S_n with height n is defined as the subgraph of the horocyclic product of T_1 and T_2 induced by the set of vertices $\{(x_1', x_2') \in T_1 \times T_2; \mathfrak{h}_1(x_1') + \mathfrak{h}_2(x_2') = 0, x_i \leq x_i', 1 = 1, 2\}$. Note that we do not need to specify the vertices x_1 and x_2 in the definition of the tetrahedron, since all tetrahedra with height n are isomorphic.

Corollary 1 and Proposition 1 from [4] specify certain eigenvalues for the Laplacian restricted to tetrahedron with height n among which is $2m(1-\cos\frac{\pi}{n})$. Moreover there exist an eigenfunction corresponding to this eigenvalue which vanishes on the inner vertex boundary of the tetrahedron, so $2m(1-\cos\frac{\pi}{n})$ is an eigenvalue of the operators $H^\#(S_n)$ for $\# = N, A, D$. This gives us upper bounds on the lowest eigenvalue of $H^D(S_n)$ which are sharp enough to give the lower bounds for the IDS from (9).

Proof of the lower bound from Theorem 10. First we shall consider the Cayley graph $(\mathbb{Z}_m \wr \mathbb{Z})_{K_0}$. Again we shall use Proposition 14 for $\# = D$. We set $G_n' = S_n$. It is easy to see that $|S_n| = (n+1)m^n$. Moreover $\lambda^D(S_n) \leq 2m(1-\cos\dfrac{\pi}{n}) \leq \dfrac{m\pi^2}{n^2}$ and thus we can set $c_n = \dfrac{m\pi^2}{n^2}$. Proposition 14 now gives the desired result.

Now take an arbitrary generator set K and consider the corresponding Cayley graph $(\mathbb{Z}_m \wr \mathbb{Z})_K$. Let V_n be a set of vertices which induces a tetrahedron with height n in the Cayley graph $(\mathbb{Z}_m \wr \mathbb{Z})_{K_0}$. The same set of vertices need not be connected in $(\mathbb{Z}_m \wr \mathbb{Z})_K$ and thus the induced subgraph in $(\mathbb{Z}_m \wr \mathbb{Z})_K$ will not be a good candidate for G_n' in Proposition 14. For this reason we consider a thickening of this set defined by $V_{n,R} := \cup_{x\in V_n} B_K(x,R)$, where $B_K(x,R)$ is the ball in $(\mathbb{Z}_m \wr \mathbb{Z})_K$ of radius R with center in x. Here R is a positive integer, large enough so that the set $V_{n,R}$ is connected in $(\mathbb{Z}_m \wr \mathbb{Z})_K$. (We can take R equal to the maximal distance in $(\mathbb{Z}_m \wr \mathbb{Z})_K$ between vertices which were neighbours in $(\mathbb{Z}_m \wr \mathbb{Z})_{K_0}$.) The set $V_{n,R}$ induces a connected subgraph $G(V_{n,R}, K)$ of $(\mathbb{Z}_m \wr \mathbb{Z})_K$. The volume of $G(V_{n,R}, K)$ is bounded above by a constant times $|V_n| = (n+1)m^n$, where for the constant we can take the volume of $B_K(x,R)$.

Next we will prove that

$$\lambda^D(G(V_{n,R}, K)) \leq \varrho\lambda^A(S_n), \tag{25}$$

for all n and some positive constant ϱ. Having in mind that $2m(1-\cos\frac{\pi}{n})$ is in the spectrum of $H^A(S_n)$ the desired estimate will follow with the choice $G_n' = G(V_{n,R}, K)$ and $c_n = \varrho\frac{m\pi^2}{n^2}$.

For each function $\varphi \in \ell^2(V_n)$ define the extension $\widetilde{\varphi}$ to $V_{n,R}$ by setting $\widetilde{\varphi}(x) = 0$ for all $x \in V_{n,R}\backslash V_n$. Theorem 3.2 in [33] implies

$$\langle H^A(G(V_{n,R}, K))\widetilde{\varphi}, \widetilde{\varphi}\rangle \leq \varrho\,\langle H^A(G(V_{n,R}, K_0))\widetilde{\varphi}, \widetilde{\varphi}\rangle\,, \tag{26}$$

for some positive constant ϱ. (To see this consider the special case of Theorem 3.2 in [33] where the supporting graph is $(\mathbb{Z}_m \wr \mathbb{Z})_{K_0}$ and the transition matrix P defines the nearest neighbour simple random walk on $(\mathbb{Z}_m \wr \mathbb{Z})_K$ and use (15)).

From (1) and the fact that $S_n = G(V_n, K_0)$ it follows that

$$\langle H^A(G(V_{n,R}, K_0))\widetilde{\varphi}, \widetilde{\varphi}\rangle = \langle H^A(G(V_n, K_0))\varphi, \varphi\rangle = \langle H^A(S_n)\varphi, \varphi\rangle \,. \tag{27}$$

Now, having in mind $\|\varphi\| = \|\widetilde{\varphi}\|$ and $H^D(G(V_{n,R}, K)) \leq H^A(G(V_{n,R}, K))$, (25) follows from (26) and (27) and the proof is finished. □

Now we are left to consider the case of the full Laplacian on the Lamplighter group, i.e. to prove Theorem 9. As we have said before we shall use the relation between the integrated density of states and return probabilities of the simple random walk. To simplify expressions we shall use the following notation. If f and g are two functions $f, g\colon \mathbb{R}^+ \to \mathbb{R}$, we shall write $f \preceq g$ if there exist an $\varepsilon > 0$ and positive constants A and B such that $f(x) \leq Ag(Bx)$ for every $x \in]0, \varepsilon[$.

Theorem 19. *Let G be a Cayley graph of a finitely generated amenable group and $(X_n)_n$ simple random walk on G, started at o. Let $\mathbb{P}_o(X_n = o)$ be the probability of the return of the simple random walk after n steps.*

(i) *Assume that there is a constant $0 < b < 1$ such that for every positive integer n we have $\mathbb{P}_o(X_{2n} = o) \preceq e^{-(2n)^b}$. Then the integrated density of the full Laplace operator N_{per} satisfies*

$$N_{\text{per}}(E) \preceq e^{-E^{-\frac{b}{1-b}}}.$$

(ii) *Assume that there is a constant $0 < b < 1$ such that $e^{-(2n)^b} \preceq \mathbb{P}_o(X_{2n} = o)$. Then, for every $r > \frac{b}{1-b}$ we have*

$$e^{-E^{-r}} \preceq N_{\text{per}}(E).$$

Proof. The proof of both parts is a minor modification of the proof of the Theorem 4.4 (parts (ii) and (iii)) in [26]. Using the notation in [26] we shall explain the adjustments which are needed to obtain Theorem 19 from the proof of [26, Thm. 4.4]. The results in [26] are formulated in terms of a certain distribution function F. First note that the value $F(\lambda)$, for any given positive λ, is nothing but $1 - \lim_{s \nearrow 1-\lambda} N_{\frac{1}{k}A}(s)$, where $N_{\frac{1}{k}A}$ is the IDS of the rescaled adjacency operator $\frac{1}{k}A$. Here k is the vertex degree in the graph. From the relation $N_{\text{per}}(\lambda) = 1 - \lim_{s \nearrow 1-\frac{1}{k}\lambda} N_{\frac{1}{k}A}(s)$ it is clear that $N_{per}(\lambda) = F(\lambda/k)$. Thus it is sufficient to prove the desired inequalities for the function F.

In the proof of the part (ii) we choose

$$n_\lambda := \left[\!\left[\Big(\frac{Cb}{\ln(\frac{1}{1-\lambda})}\Big)^{1/(1-b)}\right]\!\right].$$

This replaces the choice of

$$n_\lambda := \left[\!\left[\Big(\frac{1}{\lambda}\Big)^{1/(1-b+\varepsilon)}\right]\!\right]$$

of Oguni in [26]. This enables us to eliminate the variable ε from the calculations and to prove the wanted upper bound for $F(\lambda)$.

For the lower bounds notice that our assumptions are somewhat different than those in the part (iii) of the Theorem 4.4 in [26]. Namely we assume uniform lower bounds for the return probabilities. Following steps of the cited proof, one can prove the same inequalities for all positive λ small enough (i.e. we do not need to define the sets Λ_C). This is exactly what we wanted. $\square$

Proof of Theorem 9. Since the return probabilities of the simple random on any Cayley graph of the Lamplighter group $\mathbb{Z}_m \wr \mathbb{Z}$ satisfy the conditions from both parts of the preceding theorem with $b = 1/3$ (see Theorem 15.15 in [33]), the proof is straightforward from Theorem 19. $\square$

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Emmy-Noether-Programme of the Deutsche Forschungsgemeinschaft
& Fakultät für Mathematik, 09107 TU Chemnitz, Germany
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