Vibration modes of $3n$-gaskets and other fractals

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Abstract. We study eigenvalues and eigenfunctions (vibration modes) on the class of self-similar symmetric finitely ramified fractals which includes $3n$-gaskets. We consider such examples as the Sierpinski gasket, a non-p.c.f. analog of the Sierpinski gasket, the level-3 Sierpinski gasket, a fractal 3-tree, the hexagasket, and one dimensional fractals. We develop a theoretical matrix analysis, including analysis of singularities, which allows us to compute eigenvalues, eigenfunctions and their multiplicities exactly. We support our theoretical analysis by symbolic and numerical computations.

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1. Introduction

In this paper we study eigenvalues and eigenfunctions (vibration modes) on the class of self-similar fully symmetric finitely ramified fractals. Such studies originated in [40, 41], where it was observed that on the Sierpiński there are highly localized eigenfunctions corresponding to eigenvalues of very high multiplicity. Later the spectrum of the Laplacian on the Sierpiński gasket was studied in detail in [13], and an example of the modified Koch curve was studied in [34, 33]. The main purpose of our paper is to develop a theoretical matrix analysis, including analysis of singularities, which allows the computation of eigenvalues, eigenfunctions and their multiplicities for a large class of more complicated fractals.

Our analysis, in particular, allows the computation of the spectral zeta function on fractals (see [8, 49]) and the limiting distribution of eigenvalues (i.e. integrated density of states). The latter is a pure point measure, except in the examples which are based on the one dimensional interval. This support has a representation

$$\mathrm{supp}(\kappa) = \mathcal{J}_R \bigcup \mathcal{D},$$

where $\mathcal{J}_R$ is the Julia set of a rational function, which we compute, and $\mathcal{D}$ is a possibly empty set of isolated points (if $\mathcal{D}$ is infinite, it accumulates to $\mathcal{J}_R$). Also, our analysis allows the computation of eigenvalues and eigenfunctions by a highly accurate hierarchical iterative procedure, which does not involve large matrix calculations‡ and is illustrated in Figures 1, 2 and 3.

‡ see http://www.math.uconn.edu/~teplyaev/fractals/

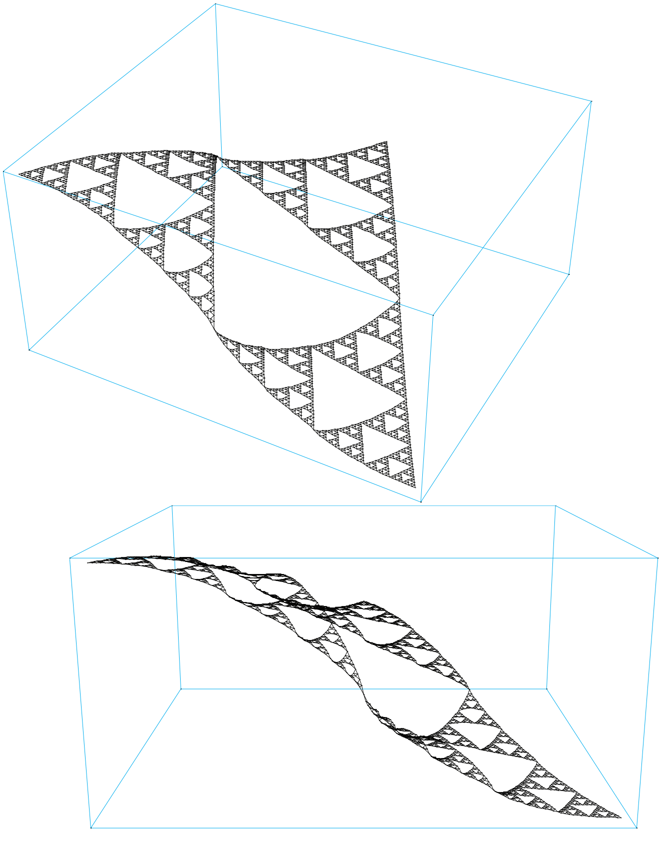

Figure 1. A basic Neumann eigenfunction on the Sierpiński gasket, three dimensional views.

There is a large body of physics and mathematics literature devoted to analysis on fractals. A small sample of it, containing many references, is [2, 5, 14, 44] and [23, 24, 25, 26, 43, 45, 46, 47, 48, 50]. In particular, tools for the numerical analysis of the Sierpiński gasket were developed in [7, 16], and fractal antenae were considered in [12, 21, 37, 39].

Our study is closely related to the analysis of self-similar graphs [27, 28, 29, 35, 36, 42, and references therein], quantum graphs [30, 31, and references therein], self-similar groups [4, 17, 18, 19, 38, 51, and references therein], and the relation between electrical circuits and Markov chains [6, 10, 11, and references therein].

This paper is organized as follows. In Section 2 we give the definition of the finitely ramified fractals with full symmetry, on which the graphs which we consider are based. In Section 3 we introduce spectral self-similarity, Schur complement and a Drichlet-to-Neumann map, and show how the resolvent of the Laplacian can be computed by an iterative procedure. In Section 4 we analyze the singularities of our map and obtain general formulas for eigenvalues and their multiplicites. We also obtain formulas for corresponding eigenprojectors. In the subsequent sections we use our general method to analyze the following examples: the Sierpiński gasket (Section 5), a non-p.c.f. analog of the Sierpiński gasket (Section 6), the level-3 Sierpiński gasket (Section 7), a fractal 3-tree (Section 8), the hexagasket (Section 9), the unit interval as a self-similar set

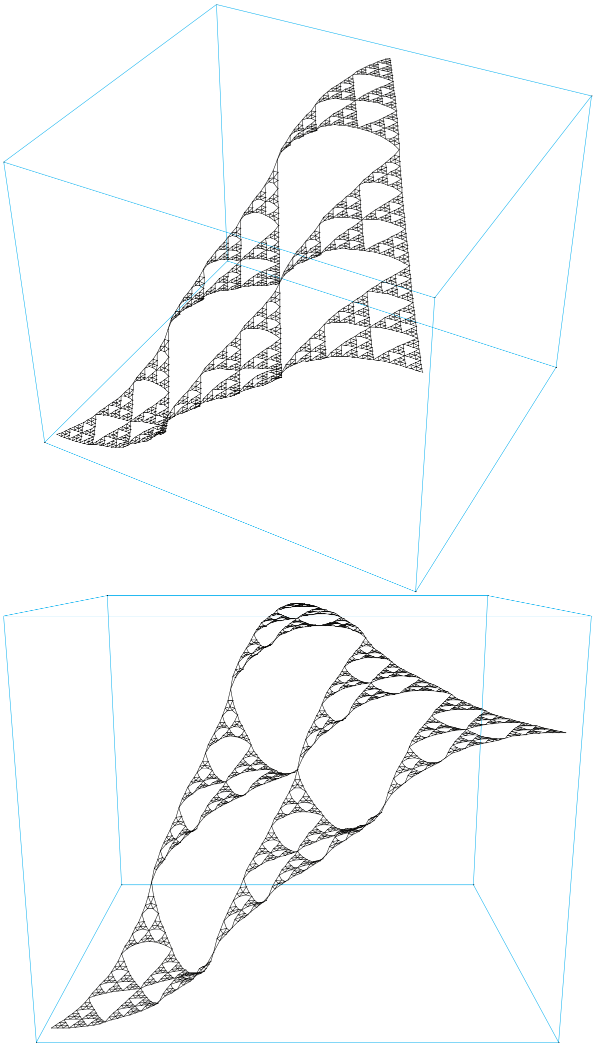

Figure 2. A basic Neumann eigenfunction on the level-3 Sierpiński gasket, three dimensional views.

(Section 10), and the diamond fractal (Section 11).

2. Finitely ramified fractals with full symmetry.

A compact connected metric space F is called a finitely ramified self-similar set if there are injective contraction maps

$$\psi_1, ..., \psi_m : F \to F$$

such that

$$F = \bigcup_{i=1}^{m} \psi_i(F)$$

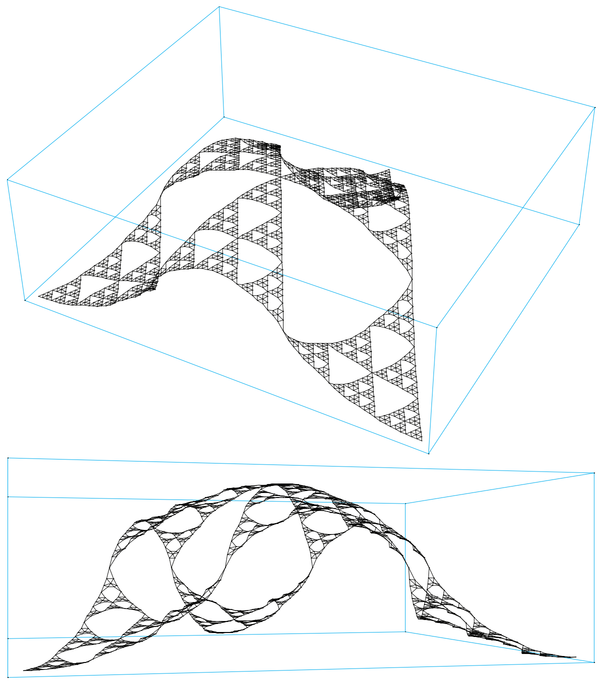

Figure 3. A Neumann eigenfunction on the level-3 Sierpiński gasket, three dimensional views.

and for any n and for any two distinct words $w, w' \in W_n = \{1, ..., m\}^n$ we have

$$F_w \cap F_{w'} = V_w \cap V_{w'},$$

where $F_w = \psi_w(F)$ and $V_w = \psi_w(V_0)$. It is assumed that V_0 is a finite set of at least two points, which often is called the boundary of F. Here for a finite word $w = w_1 ... w_n \in W_n$ we denote

$$\psi_w = \psi_{w_1} \circ ... \circ \psi_{w_n}.$$

We define

$$V_n = \bigcup_{i=1}^{m} \psi_i(V_{n-1}) = \bigcup_{w \in W_n} V_w$$

and call this set the vertices of level or depth n.

There is a natural infinite self-similar sequence of "fractal" finite graphs G_n with vertex set V_n defined as follows. For each $n \geqslant 0$ and $w \in W_n$ we define G_w as a complete graph with vertices V_w. Then, by definition,

$$G_n = \bigcup_{w \in W_n} G_w.$$

Note that G_n has no loops, but is allowed to have multiple edges, depending on the structure of the fractal F, as in Section 6. The degree of a vertex x in graph G_n is denoted by $\deg_n(x)$. The degrees of vertices are uniformly bounded in all our examples except the non-p.c.f. analog of the Sierpiński gasket in Section 6.

The main object of our study are eigenvalues and eigenfunctions on the probabilistic graph Laplacians Δ_n on G_n, which are defined by

$$\Delta_n f(x) = f(x) - \frac{1}{\deg_n(x)} \sum_{(x,y)\in E(G_n)} f(y)$$

where $E(G_n)$ denotes the set of edges of the graph G_n. For convenience we denote the matrix of Δ_n by M_n in the standard basis of functions on V_n.

Our main geometric assumption is that for any permutation $\sigma : V_0 \to V_0$ there is an isometry $g_\sigma : F \to F$ that maps any $x \in V_0$ into $\sigma(x)$ and preserves the self-similar structure of F. This means that there is a map $\widetilde{g}_\sigma : W_1 \to W_1$ such that

$$\psi_i \circ g_\sigma = g_\sigma \circ \psi_{\widetilde{g}_\sigma(i)}$$

for all $i \in W_1$. The group of isometries g_σ is denoted by $\mathcal{G}$.

It is well know that the eigenvalues and eigenfunctions of Δ_n describe vibration modes of so called cable systems modeled on the graph G_n. They are also can be considered as discrete approximations to eigenvalues and eigenfunctions of a continuous self-similar Laplacian Δ_μ on F. This continuous self-adjoint Laplacian is the generator of a self-similar diffusion process on F which can be defined in the standard way in terms of a self-similar resistance (Dirichlet) form on F, that is for any f in a suitably defined domain $Dom\Delta_\mu$ of the Neumann Laplacian we have

$$\mathcal{E}(f,f) = \int_F f \Delta_\mu f d\mu$$

where μ is the standard suitably normalized self-similar (Hausdorff, Bernoulli) measure on F.

A $\mathcal{G}$-invariant resistance form $\mathcal{E}$ on F is self-similar with energy renormalization factor ρ if for any $f \in Dom(\mathcal{E})$ we have

$$\mathcal{E}(f,f) = \rho \sum_{i=1}^{m} \mathcal{E}(f_i, f_i).$$

Here we use the notation $f_w = f \circ \psi_w$ for any $w \in W_n$. Such resistance forms in the case of p.c.f. fractals were studied in detail in [23]. The finitely ramified case can be studied in a similar way because of the general results in [24]. In particular, existence and uniqueness, up to a scalar multiplier, of the local regular self-similar $\mathcal{G}$-invariant resistance form $\mathcal{E}$ is shown in [50]. Moreover, one can show that

$$\mathcal{E} = \lim_{n\to\infty} \rho^{-n} \mathcal{E}_n$$

where the usual graph energy is

$$\mathcal{E}_n(f,f) = \sum_{(x,y)\in E(G_n)} \big(f(x) - f(y)\big)^2$$

and that

$$(\rho m)^{-n} \Delta_n f(x) \xrightarrow[n\to\infty]{} \Delta_\mu f(x)$$

for any function f for which $\Delta_\mu f \in C(F)$ and any $x \in V_* = \cup_{n\geqslant 0} V_0$. In addition, one has a relation

$$\rho m = \frac{d}{dz} R(0) > 1$$

where $R(z)$ is the rational function that appears in the spectral decimation process, and is one of the most important objects in our study.

The standard and almost trivial example of the self-similar energy and Laplacian in a finitely ramified situation is the case of $F = [0,1]$. In this case we can take $m = 2$ with $\psi_1(x) = \frac{1}{2}x$ and $\psi_2(x) = \frac{1}{2}x + \frac{1}{2}$, the self-similar measure μ is the usual Lebesgue measure, $\Delta_\mu f = -f''$ and

$$\mathcal{E}(f,f) = \int_0^1 (f'(x))^2 dx = \int_0^1 -ff'' dx = \int_F f\Delta_\mu f \, d\mu$$

for any $f \in Dom(\Delta_\mu) = \{f : f' \in L^2[0,1], f'(0) = f'(1) = 0\}$. Then we of course have $\rho = 2$ and

$$4^{-n}\Delta_n f(x) = \frac{2f(x) - f(x - \frac{1}{2^n}) - f(x + \frac{1}{2^n})}{4^n} \xrightarrow[n\to\infty]{} -f''(x)$$

for any $f \in C^2[0,1]$. The cases $F = [0,1]$ with $m = 3$ and $m = 4$ are discussed in Section 10.

Although in general the fractal F is an abstract metric space, in our examples $F \subset \mathbb{R}^2$ and the metric on F is the restriction of the usual Euclidean metric in $\mathbb{R}^2$. Moreover, the isometries g_σ are restrictions of isometries of $\mathbb{R}^2$ that maps F into itself and preserves the self-similar structure of F. We do not require that contractions ψ_i are similitudes (see Section 6). One can easily construct more involved and higher dimensional examples for which our methods apply.

3. Spectral self-similarity, Schur complement and Drichlet-to-Neumann map

If we have a matrix M given in a block form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{1}$$

then its Schur complement is

$$A - BD^{-1}C. \tag{2}$$

In our work one of the most important objects is the Schur complement of the matrix $M - z$ which is defined by

$$S(z) = A - z - B(D - z)^{-1}C. \tag{3}$$

Note that we use a convention that $M - z$ denotes $M - zI$ where I is the identity matrix of the same size as M. Similarly, $A - z$ and $D - z$ denote the matrices A and D minus z times the identity matrix of the appropriate size.

Our interest in $S(z)$ can be explained as follows. As the initial step in our calculations, we would like to relate the eigenvalues and eigenvectors of the larger Laplacian matrix $M = M_1$ and the eigenvalues and eigenvectors of a smaller Laplacian matrix M_0. In our setup, the blocks A and D in (1) correspond to outer (boundary) and interior vertices respectively.

Suppose v is an eigenvector of M which is partitioned into its boundary part v_0 and interior part v_1'. Then eigenvalue equation

$$Mv = zv$$

can be written as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v_0 \\ v_1' \end{bmatrix} = z \begin{bmatrix} v_0 \\ v_1' \end{bmatrix} \tag{4}$$

or as two equations

$$\begin{array}{rcl} Av_0 + Bv_1' & = & zv_0 \\ Cv_0 + Dv_1' & = & zv_1' \end{array} \tag{5}$$

From the second equation we obtain $v_1' = -(D-z)^{-1}Cv_0$, provided $z \notin \sigma(D)$, which implies

$$S(z)v_0 = 0. \tag{6}$$

If v_0 is also an eigenvector of M_0 with an eigenvalue z_0, then we would like to relate (6) with

$$(M_0 - z_0)v_0 = 0. \tag{7}$$

According to [47, 36], we can write $z_0 = R(z)$ if we solve what is our main equation

$$S(z) = \phi(z)\Big(M_0 - R(z)\Big), \tag{8}$$

where $\phi(z)$ and $R(z)$ are scalar (meaning not matrix-valued) rational functions.

Proposition 3.1. *For a given fully symmetric self-similar structure on a finitely ramified fractal F there is a unique rational function $\phi(z)$ and $R(z)$ that solve equation (8).*

Proof. Clearly $S(z)$ is a matrix valued rational function. By our main symmetry assumption in the previous section, for any z the matrix $S(z)$ is a linear combination of the identity matrix and M_0, which implies the proposition. □

Remark 3.2. *From the calculations above one can see that $S(\lambda)$ is the so called Drichlet-to-Neumann map for the Laplacian Δ_1.*

In our examples M_0 is a matrix that has 1 on the diagonal and $-\frac{1}{N_0-1}$ off the diagonal. Therefore we have that

$$\phi(z) = -(N_0 - 1)S_{1,2}(z)$$

and

$$R(z) = 1 - \frac{S_{1,1}(z)}{\phi(z)}.$$

Here N_0 is the number of boundary vertices, which is the number of points in V_0.

From the calculations above we have the following theorem.

Theorem 3.1. *Suppose that z is not an eigenvalue of D, and not a zero of ϕ. Then z is an eigenvalue of M with an eigenvector v if and only if $R(z)$ is an eigenvalue of M_0 with an eigenvector v_0, and $v = \begin{bmatrix} v_0 \\ v' \end{bmatrix}$ where*

$$v' = -(D - z)^{-1} C v_0.$$

This implies, in particular, that there is an one-to-one map from the eigenspace of M_0 corresponding to $R(z)$ onto the eigenspace of M corresponding to z

$$v_0 \mapsto v = T(z) v_0$$

where

$$T(z) = I_0 - (D - z)^{-1} C.$$

Naturally, the map $v_0 \mapsto v$ is called the eigenfunction extension map, and $T(z)$ is called the eigenfunction extension matrix.

The theorem above suggest the following definition of the so called exceptional set

$$E(M_0, M) = \sigma(D) \cup \{z : \phi(z) = 0\}.$$

Once we have computed the functions $R(z)$ and $\phi(z)$ using the smaller matrices M_0 and $M = M_1$, we can compute the spectrum of much larger matrices M_n by induction using the following results.

We use notation

$$M_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}$$

for the block decomposition of M_n corresponding to the representation

$$V_n = V_{n-1} \bigcup V_n'$$

where $V_n' = V_n \backslash V_{n-1}$.

Theorem 3.2. *For all $n > 0$ we have a relation*

$$P_{n-1} (M_n - z)^{-1} P_{n-1}^* = \frac{1}{\phi(z)} (M_{n-1} - R(z))^{-1},$$

where P_{n-1} is defined as the restriction operator from V_n to V_{n-1}. We often identify P_{n-1} with the orthogonal projection from $\ell^2(V_n)$ onto the subspace of functions with support in V_{n-1}.

Suppose that $z_n \notin E(M_0, M)$. Then z_n is an eigenvalue of M_n with an eigenvector v_n if and only if

$$z_{n-1} = R(z_n)$$

is an eigenvalue of M_{n-1} with an eigenvector v_{n-1}, and $v_n = \begin{bmatrix} v_{n-1} \\ v_n' \end{bmatrix}$ where

$$v_n' = -(D_n - z_n)^{-1} C_n v_{n-1}.$$

In such a situation v_n' is called the continuation of the eigenfunction v_{n-1} from V_{n-1} to $V_n \backslash V_{n-1}$.

One can obtain information about the extension of eigenfunctions and eigenprojectors from V_{n-1} to V_n by the following theorem.

Theorem 3.3. *Let P_{n,z_n} be the eigenprojector of M_n corresponding to an eigenvalue $z_n \notin E(M_0, M)$, and $P_{n-1,z_{n-1}}$ be the eigenprojector of M_{n-1} corresponding to eigenvalue $z_{n-1} = R(z_n)$. Then*

$$P_{n,z_n} = \frac{1}{\phi(z_n)\frac{d}{dz}R(z_n)} T_n(z_n)\, P_{n-1,z_{n-1}}\, (P_{n-1} - B_n(D_n - z_n)^{-1}P'_n) \tag{9}$$

where

$$T_n(z) = (P_{n-1} - (D_n - z)^{-1}C_n)$$

and P'_n is defined as the restriction operator from V_n to $V_n \backslash V_{n-1}$. We often identify P'_n with the orthogonal projection from $\ell^2(V_n)$ onto the subspace of functions that vanish on V_{n-1}. In this case $P'_n = I_n - P_{n-1}$.

Proof. First we will prove the key formula for the proof of these theorems. This formula is not related to spectral similarity and is a known fact. Essentially, it shows how to find the inverse of a matrix given in a two-by-two block form. To simplify notation we assume that $n = 1$ and $M_1 = M$.

Suppose that matrices $D - x$ and $A - x - B(D - x)^{-1}C$ are invertible. Then $M - x$ is invertible and

$$\begin{aligned}(M - x)^{-1} &= (D - x)^{-1} + \\ &+ (P_0 - (D - x)^{-1}C)(A - x - B(D - x)^{-1}C)^{-1}(P_0 - B(D - x)^{-1})\end{aligned} \tag{10}$$

It is enough to prove this formula for $x = 0$, i.e. to prove

$$M^{-1} = D^{-1} + (P_0 - D^{-1}C)(A - BD^{-1}C)^{-1}(P_0 - BD^{-1}) \tag{11}$$

provided that D and $A - BD^{-1}C$ are invertible.

We have

$$MD^{-1} = (P'_1 + P_0)MD^{-1}P'_1 = P'_1 + P_0MD^{-1}P'_1$$

and

$$P_0M(P_0 - D^{-1}C) = MP_0 - P'_1MP_0 - P_0MD^{-1}C = P_0(A - BD^{-1}C).$$

Thus

$$\begin{aligned}M(D^{-1}P'_1 + (P_0 - D^{-1}C)(A - BD^{-1}C)^{-1}(P_0 - BD^{-1}P'_1)) &= \\ = P'_1 + P_0MD^{-1}P'_1 + P_0(P_0 - BD^{-1}P'_1) &= P'_1 + P_0 = I.\end{aligned}$$

That is what (11) says.

To obtain the proof Theorem 3.2, note that (10) implies

$$\begin{aligned}(M - x)^{-1} &= (D - x)^{-1}P'_1 + \\ &+ (P_0 - (D - x)^{-1}C)(\phi(x)M_0 - \phi_1(x))^{-1}(P_0 - B(D - x)^{-1}P'_1),\end{aligned} \tag{12}$$

where $\phi_1(z) = \phi(z)R(z)$. The statements of Theorem 3.3 follow if we use the standard spectral representation

$$M = \sum_{z \in \sigma(M)} zP_z.$$

and pass to the limit as $x \to z$ in this formula. □

Remark 3.3. *For $n = 1$ these theorems are also true for the adjacency matrix graph Laplacian. For $n > 1$ it is important that we consider probabilistic graph Laplacian, or a multiple of it. For instance, [9, 46] and related works usually consider the Laplacian, Δ_n, multiplied by 4.*

4. Analysis of the exceptional values.

It is not enough to restrict ourself to values of z outside of the exceptional set $E(M_0, M)$. In fact, this set is very interesting because it often contains eigenvalues of high multiplicity, which in turn often correspond to localized eigenfunctions.

We first formulate a proposition that gives the multiplicities of such eigenvalues, and is used extensively to analyze examples in the rest of the paper. Then we prove a theorem which implies the proposition.

We write $\mathrm{mult}_n(z)$ for the multiplicity of z as an eigenvalue of M_n. By definition, $\mathrm{mult}_n(z) = 0$ if z is not an eigenvalue. Notation $\dim_n$ is used for the dimension of $\ell^2(V_n)$ which is the same as the number of points in V_n.

Proposition 4.1. *(i) If $z \notin E(M_0, M)$, then*

$$\mathrm{mult}_n(z) = \mathrm{mult}_{n-1}(R(z)), \tag{13}$$

and every corresponding eigenfunction at depth n is an extension of an eigenfunction at depth $n-1$.

(ii) If $z \notin \sigma(D)$, $\phi(z) = 0$ and $R(z)$ has a removable singularity at z, then

$$\mathrm{mult}_n(z) = \dim_{n-1}, \tag{14}$$

and every corresponding eigenfunction at depth n is localized.

(iii) If $z \in \sigma(D)$, both $\phi(z)$ and $\phi_1(z)$ have poles at z, $R(z)$ has a removable singularity at z, and $\frac{d}{dz}R(z) \neq 0$, then

$$\mathrm{mult}_n(z) = m^{n-1}\mathrm{mult}_D(z) - \dim_{n-1} + \mathrm{mult}_{n-1}(R(z)), \tag{15}$$

and every corresponding eigenfunction at depth n vanishes on V_{n-1}.

(iv) If $z \in \sigma(D)$, but $\phi(z)$ and $\phi_1(z)$ do not have poles at z, and $\phi(z) \neq 0$, then

$$\mathrm{mult}_n(z) = m^{n-1}\mathrm{mult}_D(z) + \mathrm{mult}_{n-1}(R(z)). \tag{16}$$

In this case $m^{n-1}\mathrm{mult}_D(z)$ linearly independent eigenfunctions are localized, and $\mathrm{mult}_{n-1}(R(z))$ more linearly independent eigenfunctions are extensions of corresponding eigenfunction at depth $n-1$.

(v) If $z \in \sigma(D)$, but $\phi(z)$ and $\phi_1(z)$ do not have poles at z, and $\phi(z) = 0$, then

$$\mathrm{mult}_n(z) = m^{n-1}\mathrm{mult}_D(z) + \mathrm{mult}_{n-1}(R(z)) + \dim_{n-1} \tag{17}$$

provided $R(z)$ has a removable singularity at z. In this case there are $m^{n-1}\mathrm{mult}_D(z) + \dim_{n-1}$ localized and $\mathrm{mult}_{n-1}(R(z))$ non-localized corresponding eigenfunctions at depth n.

(vi) If $z \in \sigma(D)$, both $\phi(z)$ and $\phi_1(z)$ have poles at z, $R(z)$ has a removable singularity at z, and $\frac{d}{dz}R(z) = 0$, then

$$\mathrm{mult}_n(z) = \mathrm{mult}_{n-1}(R(z)), \tag{18}$$

provided there are no corresponding eigenfunctions at depth n that vanish on V_{n-1}. In general we have

$$\mathrm{mult}_n(z) = m^{n-1}\mathrm{mult}_D(z) - \dim_{n-1} + 2\mathrm{mult}_{n-1}(R(z)) \tag{19}$$

(vii) If $z \notin \sigma(D)$, $\phi(z) = 0$ and $R(z)$ has a pole z, then $\mathrm{mult}_n(z) = 0$ and z is not an eigenvalue.

(viii) If $z \in \sigma(D)$, but $\phi(z)$ and $\phi_1(z)$ do not have poles at z, $\phi(z) = 0$, and $R(z)$ has a pole z, then

$$\mathrm{mult}_n(z) = m^{n-1}\mathrm{mult}_D(z) \tag{20}$$

and every corresponding eigenfunction at depth n vanishes on V_{n-1}.

In the next theorem we establish the relation between eigenprojectors of spectrally similar operators. Namely, we show how one can find the eigenprojector $P_{n,z}$ of M_n corresponding to an eigenvalue z, if the eigenprojector $P_{n-1,R(z)}$ of M_{n-1} corresponding to eigenvalue $R(z)$ is known.

We state this theorem for $n = 1$ and $M = M_1$, and the analogous relation holds for any $n \geqslant 1$. As before, we define $\phi_1(z) = \phi(z)R(z)$.

Theorem 4.1. *(i) In the case of Proposition 4.1(i),*

$$P_z = \frac{1}{\phi(z)\frac{d}{dz}R(z)}(P_0 - (D-z)^{-1}C)P_{0,R(z)}(P_0 - B(D-z)^{-1}). \tag{21}$$

(ii) In the case of Proposition 4.1(ii),

$$P_z = (P_0 - (D-z)^{-1}C)(\psi_0(z)M_0 - \psi_1(z))^{-1}(P_0 - B(D-z)^{-1}) \tag{22}$$

where $\psi_0(x) = \phi(x)/(z-x)$ and $\psi_1(x) = \phi_1(x)/(z-x)$. This implies, in particular, that there is an one-to-one map $v_0 \mapsto v = v_0 - (D-z)^{-1}Cv_0$ from $\ell^2(V_0)$ onto the eigenspace of M corresponding to z.

(iii) In the case of Proposition 4.1(iii), the poles of $\phi(z)$ and ϕ_1 are simple and so $R(z)$ has a removable singularity at z, $P_zP_{D,z} = P_z$ and $P_0P_z = 0$, which means that the corresponding eigenfunctions of M vanish on V_0.
Moreover,

$$rank P_{D,z} - rank P_z = rank(\psi_0(z)M_0 - \psi_1(z)I_0) = corank P_{0,R(z)}$$

where $\psi_0(x) = \phi(x)(z-x)$ and $\psi_1(x) = \phi_1(x)(z-x)$.
In addition, the following relations hold

$$P_z = P_{D,z} + \frac{1}{\psi_0(z)}P_{D,z}C(M_0 - R(z))^{-1}(I_0 - P_{0,R(z)})BP_{D,z} \tag{23}$$

and $P_{D,z}CP_{0,R(z)} = 0$. Note that $I_0 - P_{0,R(z)}$ is the projector from $\ell^2(V_0)$ onto the space, where $(D-z)^{-1}$ is a well defined bounded operator.

(iv) *In the case of Proposition 4.1(iv),*

$$P_z = P_{D,z} + \frac{1}{\phi(z)\frac{d}{dz}R(z)}(P_0 - (D-z)^{-1}C)P_{0,R(z)}(P_0 - B(D-z)^{-1}) \quad (24)$$

and the projector $P_{D,z}$ is orthogonal to the second term in the right hand side of this formula. In particular, $P_zP_{D,z} = P_{D,z}$.

(v) *In the case of Proposition 4.1(v), P_z is the sum of the right hand sides in (22) and (24).*

(vi) *In the case of Proposition 4.1(vi), provided there are no corresponding eigenfunction at depth n that vanish on V_{n-1}, we have*

$$P_z = \frac{2}{\psi(z)\frac{d^2}{dz^2}R(z)}(P_0 - (D-z)^{-1}C)P_{0,R(z)}(P_0 - B(D-z)^{-1}). \quad (25)$$

In general, this formula is combined with 23.

(vii) *In the case of Proposition 4.1(vii) we formally have $P_z = 0$.*

(viii) *In the case of Proposition 4.1(viii) we have $P_z = P_{D,z}$.*

Proof. Item (i) is the same as Theorem 3.3; it is inserted here also for the sake of completeness.

To prove item (ii), we pass to the limit as $x \to z$ in the formula 12, which can be re-written as

$$(M-x)^{-1} = (D-x)^{-1} + \\ + \frac{1}{z-x}(P_0 - (D-x)^{-1}C)(\psi_0(x)M_0 - \psi_1(x))^{-1}(P_0 - B(D-x)^{-1}). \quad (26)$$

Then the statements to be proved follow if we pass to the limit as $x \to z$ in this formula.

To prove item (iii), we again pass to the limit as $x \to z$ in formula (12). We see that $P_0P_z \neq 0$ if and only if

$$\lim_{x\to z}(x-z)^2(\psi_0(x)M_0 - \psi_1(x)I_0)^{-1} \neq 0,$$

that is only possible if $\frac{d}{dz}R(z) = 0$. Therefore $P_0P_z = 0$ in our case. Relation (23) follows from (12).

Note that

$$\psi_0(z)M_0 - \psi_1(z)I_0 = -P_0MP_{D,z}MP_0$$

if $z \in \sigma(D)$. Hence $rank(\psi_0(z)M_0 - \psi_1(z)I_0) = rank(P_{D,z} - P_z)$. In addition, we have that $\psi_0(z)M_0 - \psi_1(z)I_0$ is nonpositive.

Also we see that $P_0(M-z)^{-1}P_0$ is a bounded operator on $\ell^2(V_0)$ and so we have $P_0(M-z)^{-1}P_0 = \lim_{x\to z}(z-x)(\psi_0(x)M_0 - \psi_1(x)I_0)^{-1}$. Hence $P_0(M-z)^{-1}P_0 = 0$ if and only if $R(z)$ has a pole at z or $R(z) \in \rho(M_0)$. If $R(z)$ has a removable singularity at z then

$$\psi_0(z)\frac{d}{dz}R(z)P_0(M-z)^{-1}P_0 = P^0_{R(z)}.$$

To prove item (iv), note that the relation $P_zP_{D,z} = P_{D,z}$ easily follows from the fact that ϕ and ϕ_1 do not have poles. Then, if we restrict everything to the orthogonal complement of the image of $P_{D,z}$, we can apply item (i) of this theorem.

Item (v) follows from items (ii) and (iv). The proof of item (vi) is a combination of the proofs of items (i) and (iii). Items (vii) and (viii) easily follow from (12). □

5. Sierpiński gasket.

Spectral analysis on the Sierpiński gasket originates from physics papers [40, 41] and is well known [7, 13, 46, 47]. In this section we show how one can study it using our methods. Note that recently Sierpiński lattices appeared as the Schreier graphs of so called Hanoi towers groups [19, 38, 51].

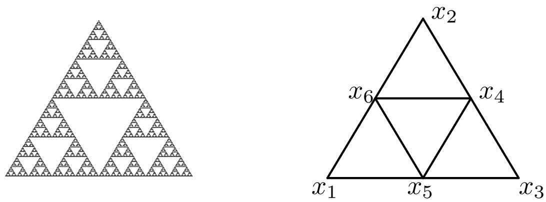

Figure 4. The Sierpiński gasket and its V_1 network.

Figure 4 shows the depth one approximation to the Sierpiński gasket. The depth 1 Laplacian matrix $M = M_1$, which is obtained from the above figure, is

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & 0 & -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 0 & -\frac{1}{4} & -\frac{1}{4} & 1 \end{pmatrix}$$

The eigenfunction extension map is

$$(D-z)^{-1}C = \begin{pmatrix} \frac{1}{-5+2(7-4z)z} & \frac{2(-1+z)}{5+2z(-7+4z)} & \frac{2(-1+z)}{5+2z(-7+4z)} \\ \frac{2(-1+z)}{5+2z(-7+4z)} & \frac{1}{-5+2(7-4z)z} & \frac{2(-1+z)}{5+2z(-7+4z)} \\ \frac{2(-1+z)}{5+2z(-7+4z)} & \frac{2(-1+z)}{5+2z(-7+4z)} & \frac{1}{-5+2(7-4z)z} \end{pmatrix}$$

From these we have that

$$\phi(z) = \frac{3-2z}{5-14z+4z^2}$$

and

$$R(z) = (5-4z)z.$$

The eigenvalues of M written with multiplicities are

$$\sigma(M) = \left\{\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{4}, \frac{3}{4}, 0\right\}$$

and the corresponding eigenvectors are {-1, -1, 0, 0, 0, 1}, {-1, 0, -1, 0, 1, 0}, {0, -1, -1, 1, 0, 0}, {2, 0, -2, -1, 0, 1}, {2, -2, 0, -1, 1, 0}, {1, 1, 1, 1, 1, 1}. The eigenvalues of D written with multiplicities are

$$\sigma(D) = \left\{\frac{5}{4}, \frac{5}{4}, \frac{1}{2}\right\}$$

and the corresponding eigenvectors are {-1, 0, 1}, {-1, 1, 0}, {1, 1, 1}. The equation $\varphi = 0$ has as its solution $\{\frac{3}{2}\}$ so the exceptional set is

$$E(M_0, M) = \left\{\frac{5}{4}, \frac{1}{2}, \frac{3}{2}\right\}.$$

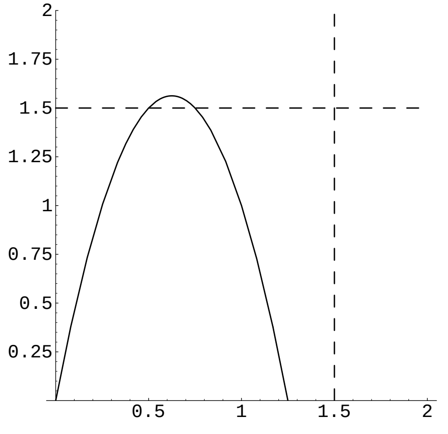

Figure 5. The graph of $R(z)$ for the Sierpiński gasket.

We can find the multiplicities of these exceptional values by using Proposition 4.1.

For the value $\frac{5}{4}$, which is a pole of $\phi(z)$ and in $\sigma(D)$, we use Proposition 4.1(iii) to find the multiplicities:

$$\begin{aligned} \text{mult}_1(\tfrac{5}{4}) &= 2 - 3 + 1 = 0, \\ \text{mult}_2(\tfrac{5}{4}) &= 6 - 6 + 1 = 1, \\ \text{mult}_3(\tfrac{5}{4}) &= 18 - 15 + 1 = 4, \end{aligned}$$

For the value $\frac{1}{2}$, which is also a pole of $\phi(z)$ and in $\sigma(D)$, we again use Proposition 4.1(iii) to find the multiplicities:

$$\begin{aligned} \text{mult}_1(\tfrac{1}{2}) &= 1 - 3 + 2 = 0, \\ \text{mult}_2(\tfrac{1}{2}) &= 3 - 6 + 3 = 0, \\ \text{mult}_3(\tfrac{1}{2}) &= 9 - 15 + 6 = 0, \end{aligned}$$

For the value $\frac{3}{2}$, since $\frac{3}{2} \notin \sigma(D)$ and $\phi(z) = 0$, we use Proposition 4.1(ii) to find the multiplicities. Here the multiplicity of $\frac{3}{2}$ in the n^{th} depth is equal to the dimension at depth $n - 1$.

$$\begin{aligned} \text{mult}_1(\tfrac{3}{2}) &= 3, \\ \text{mult}_2(\tfrac{3}{2}) &= 6, \\ \text{mult}_3(\tfrac{3}{2}) &= 15, \end{aligned}$$

Table 1 shows the ancestor-offspring structure of the eigenvalues of the Sierpiński gasket. The symbol * indicates branches

$$\xi_1(z) = \frac{5 - \sqrt{25 - 16z}}{8}$$

and

$$\xi_2(z) = \frac{5 + \sqrt{25 - 16z}}{8}$$

of the inverse function $R^{-1}(z)$ computed at the ancestor value z. By Proposition 4.1(i) the ancestor and the offspring have the same multiplicity. The empty columns represent exceptional values. If they are eigenvalues of the appropriate M_n, then the multiplicity is shown in the right hand part of the same row.

<table>
<tr><td>$z \in \sigma(M_0)$
$\text{mult}_0(z)$</td><td colspan="4">$\mathbf{0}$
1</td><td colspan="7">$\mathbf{\frac{3}{2}}$
2</td></tr>
<tr><td>$z \in \sigma(M_1)$
$\text{mult}_1(z)$</td><td colspan="3">$\mathbf{0}$
1</td><td rowspan="3">$\mathbf{\frac{5}{4}}$</td><td colspan="4">$\mathbf{\frac{3}{4}}$
2</td><td rowspan="3">$\mathbf{\frac{1}{2}}$</td><td colspan="3">$\mathbf{\frac{3}{2}}$
3</td></tr>
<tr><td>$z \in \sigma(M_2)$
$\text{mult}_2(z)$</td><td colspan="2">$\mathbf{0}$
1</td><td rowspan="2">$\mathbf{\frac{5}{4}}$</td><td colspan="2">$\boldsymbol{\xi_1(\frac{3}{4})}$
2</td><td colspan="2">$\boldsymbol{\xi_2(\frac{3}{4})}$
2</td><td colspan="2">$\mathbf{\frac{3}{4}}$
3</td><td rowspan="2">$\mathbf{\frac{1}{2}}$</td><td colspan="2">$\mathbf{\frac{3}{2}}$
6</td><td colspan="2">$\mathbf{\frac{5}{4}}$
1</td></tr>
<tr><td>$z \in \sigma(M_3)$
$\text{mult}_3(z)$</td><td>$\mathbf{0}$
1</td><td>$\mathbf{\frac{5}{4}}$</td><td>$*$
2</td><td>$*$
2</td><td>$*$
2</td><td>$*$
2</td><td>$*$
3</td><td>$*$
3</td><td>$\mathbf{\frac{3}{4}}$
6</td><td>$\mathbf{\frac{1}{2}}$</td><td>$*$
1</td><td>$*$
1</td><td>$\mathbf{\frac{3}{2}}$
15</td><td>$\mathbf{\frac{5}{4}}$
4</td></tr>
</table>

Table 1. Ancestor-offspring structure of the eigenvalues on the Sierpiński gasket

By induction one can obtain the following proposition, which is known in the case of the Sierpiński gasket (see [13, 46, 47]).

Notation $R_{-n}A$ is used for the preimage of a set A under the n-th composition power of the function R.

Proposition 5.1. (i) $\sigma(M_0) = \{0, \frac{3}{2}\}$.

(ii) *For any* $n \geqslant 0$

$$\sigma(M_n) \subset \bigcup_{m=0}^{n} R_{-m}\{0, \tfrac{3}{2}\}$$

and for any $n \geqslant 1$ *we have*

$$\sigma(M_n) = \{\tfrac{3}{2}\} \bigcup \left(\bigcup_{m=0}^{n-1} R_{-m}\{0, \tfrac{3}{4}\} \right).$$

In particular, for $n \geqslant 2$

$$\sigma(M_n) = \{0, \tfrac{3}{2}\} \bigcup \left(\bigcup_{m=0}^{n-1} R_{-m}\{\tfrac{3}{4}\} \right) \bigcup \left(\bigcup_{m=0}^{n-2} R_{-m}\{\tfrac{5}{4}\} \right).$$

(iii) *For any* $n \geqslant 0$, $\dim_n = \frac{3^{n+1}+3}{2}$.

(iv) *For any* $n \geqslant 0$, $\text{mult}_n(0) = 1$.

(v) *For any* $n \geqslant 0$, $\text{mult}_n(\frac{3}{2}) = \frac{3^n+3}{2}$.

(vi) *If* $z \in R_{-k}\{\frac{3}{4}\}$ *then* $\mathrm{mult}_n(z) = \frac{3^{n-k-1}+3}{2}$ *for* $n \geqslant 1$, $0 \leqslant k \leqslant n-1$.

(vii) *If* $z \in R_{-k}\{\frac{5}{4}\}$ *then* $\mathrm{mult}_n(z) = \frac{3^{n-k-1}-1}{2}$ *for* $n \geqslant 2$, $0 \leqslant k \leqslant n-2$.

Corollary 5.2. *The normalized limiting distribution of eigenvalues (the integrated density of states) is a pure point measure* κ *with the the set of atoms*

$$\{\tfrac{3}{2}\} \bigcup \left(\bigcup_{m=0}^{\infty} R_{-m}\{\tfrac{3}{4}\} \right) \bigcup \left(\bigcup_{m=0}^{\infty} R_{-m}\{\tfrac{5}{4}\} \right).$$

Moreover,

$$\kappa\left(\{\tfrac{3}{2}\}\right) = \frac{1}{3},$$

and

$$\kappa(\{z\}) = 3^{-m-1}$$

if $z \in R_{-m}\{\frac{3}{4}, \frac{5}{4}\}$.

For the Sierpiński gasket we also demonstrate how one can compute the eigenprojectors for the two most interesting eigenvalues, $z = \frac{3}{2}$ and $z = \frac{5}{4}$. For the former case we use Theorem 4.1(ii). We compute $\psi_0(\frac{3}{2}) = 1$ and $\psi_1(\frac{3}{2}) = -\frac{3}{2}$ and so

$$P_{n+1,\frac{3}{2}} = \Big(P_n - \big(D_n - \tfrac{3}{2}\big)^{-1} C_n\Big)\Big(M_n + \tfrac{3}{2}\Big)^{-1}\Big(P_n - B_n\big(D_n - \tfrac{3}{2}\big)^{-1}\Big). \tag{27}$$

For the case $z = \frac{5}{4}$ we use Theorem 4.1(iii) with $R(\frac{5}{4}) = 0$ and $\psi_0(\frac{5}{4}) = \frac{1}{12}$ and so

$$P_{n+1,\frac{5}{4}} = P_{D_n,\frac{5}{4}} + 12\, P_{D_n,\frac{5}{4}} C_n M_n^{-1} B_n P_{D_n,\frac{5}{4}}. \tag{28}$$

Note that one can show that the term $I_n - P_{n,0}$ is the projector to the orthogonal complement to constants and so can be omitted in this case. Note also that $P_{D_n,\frac{5}{4}}$ has a simple block structure with blocks

$$\frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

and that D_n has a block structure with blocks

$$\frac{1}{4}\begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}.$$

The matrices of C_n and B_n also have similarly simple block structure with block equivalent, depending on the labeling of vertices, to

$$\frac{1}{4}\begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}.$$

except for boundary vertices. The computation of the eigenprojectors using Theorem 4.1(i) plays an important role in [47].

6. A non-p.c.f. analog of the Sierpiński gasket.

Several non-p.c.f. analogs of the Sierpiński gasket were introduced in [50]. Here we analyze the simplest one of them. This fractal can be constructed as a self-affine fractal in $\mathbb{R}^2$ using 6 affine contractions, as shown in [50]. It is finitely ramified but not p.c.f. in the sense of Kigami. Figure 6 shows the V_1 network for this fractal.

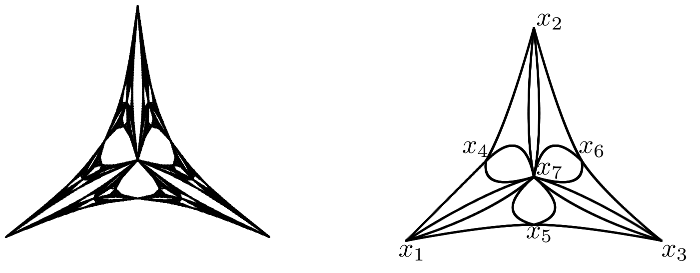

Figure 6. The non-p.c.f. analog of the Sierpiński gasket and its V_1 network.

The matrix of the depth-1 Laplacian $M_1 = M$ is

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{4} & -\frac{1}{4} & 1 & 0 & 0 & -\frac{1}{2} \\ -\frac{1}{4} & 0 & -\frac{1}{4} & 0 & 1 & 0 & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & 1 \end{pmatrix}$$

and the eigenfunction extension map is

$$(D-z)^{-1}C = \begin{pmatrix} -\frac{1}{6-18z+12z^2} & \frac{-5+6z}{12(1-3z+2z^2)} & \frac{-5+6z}{12(1-3z+2z^2)} \\ \frac{-5+6z}{12(1-3z+2z^2)} & -\frac{1}{6-18z+12z^2} & \frac{-5+6z}{12(1-3z+2z^2)} \\ \frac{-5+6z}{12(1-3z+2z^2)} & \frac{-5+6z}{12(1-3z+2z^2)} & -\frac{1}{6-18z+12z^2} \\ \frac{1}{-3+6z} & \frac{1}{-3+6z} & \frac{1}{-3+6z} \end{pmatrix}.$$

Moreover, we compute that

$$\phi(z) = \frac{15-14z}{24-72z+48z^2}$$

and

$$R(z) = -\frac{24z(z-1)(2z-3)}{14z-15}.$$

The eigenvalues of D, written with multiplicites, are

$$\sigma(D) = \left\{\frac{3}{2}, 1, 1, \frac{1}{2}\right\}$$

with corresponding eigenvectors {-1, -1, -1, 1}, {-1, 0, 1, 0}, {-1, 1, 0, 0}, {1, 1, 1, 1}. One can also compute

$$\sigma(M) = \left\{\frac{3}{2}, \frac{3}{2}, \frac{5}{4}, \frac{5}{4}, \frac{3}{4}, \frac{3}{4}, 0\right\}$$

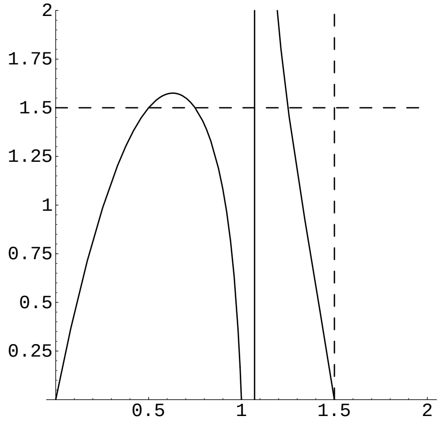

Figure 7. The graph of $R(z)$ for the non-p.c.f. analog of the Sierpiński gasket.

with the corresponding eigenvectors {-1, -1, -1, 0, 0, 0, 1}, {-1, -1, -1, 1, 1, 1, 0}, {-1, 0, 1, -1, 0, 1, 0}, {-1, 1, 0, -1, 1, 0, 0}, {1, 0, -1, -1, 0, 1, 0}, {1, -1, 0, -1, 1, 0, 0}, {1, 1, 1, 1, 1, 1, 1}.

It is easy to see that $\phi(z) = 0$ has one solution $\{\frac{15}{14}\}$. Thus, the exceptional set is

$$E(M_0, M) = \left\{ \frac{3}{2}, 1, \frac{1}{2}, \frac{15}{14} \right\}.$$

$z \in \sigma(M_0)$	**0**					$\frac{3}{2}$												
$\text{mult}_0(z)$	*1*					*2*												
$z \in \sigma(M_1)$	**0**			**1**	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{4}$			$\frac{5}{4}$			$\frac{3}{2}$					
$\text{mult}_1(z)$	*1*						*2*			*2*			*2*					
$z \in \sigma(M_2)$	**0**	**1**	$\frac{3}{2}$				$*$	$*$	$*$	$*$	$*$	$*$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{5}{4}$	$\frac{1}{2}$	**1**	$\frac{3}{2}$
$\text{mult}_2(z)$	*1*						*2*	*2*	*2*	*2*	*2*	*2*		*2*	*2*	*1*	*6*	*7*

Table 2. Ancestor-offspring structure of the eigenvalues on the non-p.c.f. analog of the Sierpiński gasket.

To begin the analysis of the exceptional values, note that $\frac{15}{14}$ is a pole of $R(z)$ and therefore is not an eigenvalue by Proposition 4.1(vii). We are interested in the values of $R(z)$ in the other exceptional points, which are

$$R(1) = R(\tfrac{3}{2}) = 0 \quad \text{and} \quad R(\tfrac{1}{2}) = \tfrac{3}{2}.$$

It is easy to see that 1 and $\frac{1}{2}$ are poles of $\phi(z)$ and so we can use Proposition 4.1(iii) to compute the multiplicities. We obtain

$$\begin{aligned}\mathrm{mult}_1(1) &= 2-3+1=0,\\ \mathrm{mult}_1(\tfrac{1}{2}) &= 1-3+2=0,\\ \mathrm{mult}_2(1) &= 12-7+1=6,\end{aligned}$$

and

$$\mathrm{mult}_2(\tfrac{1}{2}) = 6-7+2=1.$$

Since $\frac{3}{2}$ is not a pole of $\phi(z)$, we can use Proposition 4.1(iv) to compute the multiplicities

$$\mathrm{mult}_1(\tfrac{3}{2}) = 1+1=2$$

and

$$\mathrm{mult}_2(\tfrac{3}{2}) = 6+1=7.$$

The ancestor-offspring structure of the eigenvalues on the non-p.c.f. analog of the Sierpiński gasketis shown in Table 2. The symbol * indicates branches of the inverse function $R^{-1}(z)$ computed at the ancestor value. The multiplicity of the ancestor is the same as that of the offspring by Proposition 4.1(i). The empty columns correspond to the exceptional values. If they are eigenvalues of the appropriate M_n, then the multiplicity is shown in the right hand part of the same row.

Theorem 6.1. (i) *For any* $n \geqslant 0$ *we have that* $\sigma(\Delta_n) \subset \bigcup_{m=0}^{n} R_{-m}(\{0,\frac{3}{2}\})$ *and* $\sigma(\Delta_1) = \{0,\frac{3}{4},\frac{5}{4},\frac{3}{2}\}$.
(ii) *For* $n \geqslant 2$ *we have that*

$$\sigma(\Delta_n) = \{0,\tfrac{3}{2}\} \bigcup \left(\bigcup_{m=0}^{n-1} R_{-m}\{\tfrac{3}{4},\tfrac{5}{4}\}\right) \bigcup \left(\bigcup_{m=0}^{n-2} R_{-m}\{\tfrac{1}{2},1\}\right).$$

(iii) *For any* $n \geqslant 0$ *we have* $\dim_n = \dfrac{11+4\cdot 6^n}{5}$.
(iv) *For any* $n \geqslant 0$ *we have* $\mathrm{mult}_n(0)=1$.
(v) *For any* $n \geqslant 1$ *we have* $\mathrm{mult}_n(\frac{3}{2}) = 6^{n-1}+1$.
(vi) *For any* $n \geqslant 1$ *and* $z \in R_{1-n}\{\frac{3}{4},\frac{5}{4}\}$ *we have that* $\mathrm{mult}_n(z)=2$.
(vii) *For any* $0 \leqslant m \leqslant n-2$ *and* $z \in R_{-m}\{\frac{3}{4},\frac{5}{4}\}$ *we have that*

$$\mathrm{mult}_n(z) = \mathrm{mult}_{n-m-1}(\tfrac{3}{2}) = 6^{n-m-2}+1.$$

(viii) *For any* $0 \leqslant m \leqslant n-2$ *and* $z \in R_{-m}\{\frac{1}{2}\}$ *we have* $\mathrm{mult}_n(\frac{1}{2}) = \dfrac{11\cdot 6^{n-m-2}-6}{5}$.
(ix) *For any* $0 \leqslant m \leqslant n-2$ *and* $z \in R_{-m}\{1\}$ *we have* $\mathrm{mult}_n(1) = \dfrac{6^{n-m}-6}{5}$.

Proof. For this fractal we have $\sigma(\Delta_0) = \{0,\frac{3}{2}\}$ with $\mathrm{mult}_0(\frac{3}{2})=2$ and, for the purposes of Proposition 4.1, $m=6$.

Item (i) is obtained above in this section.

Item (ii) follows from the subsequent items.

Item (iii) is straightforward by induction.

Item (iv) follows from Proposition 4.1(i) because 0 is a fixed point of $R(z)$.

Item (v) easily follows from Proposition 4.1(iv).
Items (vi) and (vii) follows from the items above.
Items (viii) and (ix) follows from Proposition 4.1(iii) because

$$\mathrm{mult}_n(\tfrac{1}{2}) = 6^{n-1}\cdot 1 - \frac{11+4\cdot 6^{n-1}}{5} + 6^{n-2} + 1 = \frac{11\cdot 6^{n-2}-6}{5},$$

$$\mathrm{mult}_n(1) = 6^{n-1}\cdot 2 - \frac{11+4\cdot 6^{n-1}}{5} + 1 = \frac{6^n-6}{5}.$$

□

Corollary 6.1. *The normalized limiting distribution of eigenvalues (the integrated density of states) is a pure point measure κ with the the set of atoms*

$$\bigcup_{m=0}^{\infty} R_{-m}\{\tfrac{3}{2},1\},$$

where $\kappa\left(\{\tfrac{3}{2}\}\right) = \dfrac{5}{24}$ *and*

$$\begin{aligned} \kappa(\{z\}) &= \frac{5}{4}6^{-m-2} && \text{if} \quad z\in R_{-m}\{\tfrac{3}{4},\tfrac{5}{4}\};\\ \kappa(\{z\}) &= \frac{11}{4}6^{-m-2} && \text{if} \quad z\in R_{-m}\{\tfrac{1}{2}\};\\ \kappa(\{z\}) &= \frac{1}{4}6^{-m} && \text{if} \quad z\in R_{-m}\{1\}. \end{aligned}$$

7. Level-3 Sierpiński gasket.

The level-3 Sierpiński gasket is shown in Figure 8. It had been used as an example in several works [3, 20, 46, and references therein]. In particular, the spectrum is computed in the recent paper [9] independently of our work.

The matrix for the depth-1 Laplacian $M_1 = M$ is

$$M = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0\\ -\frac{1}{4} & 0 & 0 & 1 & -\frac{1}{4} & 0 & 0 & 0 & -\frac{1}{4} & -\frac{1}{4}\\ -\frac{1}{4} & 0 & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & 0 & 0 & -\frac{1}{4}\\ 0 & -\frac{1}{4} & 0 & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & 0 & -\frac{1}{4}\\ 0 & -\frac{1}{4} & 0 & 0 & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & -\frac{1}{4}\\ 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{4}\\ 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & -\frac{1}{4} & 1 & -\frac{1}{4}\\ 0 & 0 & 0 & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & 1 \end{pmatrix}.$$

and the eigenfunction extension map $(D-z)^{-1}C$ is

$$\begin{pmatrix}
\frac{-24+109z-132z^2+48z^3}{3(1-6z+4z^2)(15-32z+16z^2)} & \frac{-9+7z}{3(1-6z+4z^2)(15-32z+16z^2)} & \frac{-4+3z}{3(-5+34z-4z^2+16z^3)} \\
\frac{-24+109z-132z^2+48z^3}{3(1-6z+4z^2)(15-32z+16z^2)} & \frac{-4+3z}{3(-5+34z-4z^2+16z^3)} & \frac{-9+7z}{3(1-6z+4z^2)(15-32z+16z^2)} \\
\frac{-4+3z}{3(-5+34z-4z^2+16z^3)} & \frac{-24+109z-132z^2+48z^3}{3(1-6z+4z^2)(15-32z+16z^2)} & \frac{-9+7z}{3(1-6z+4z^2)(15-32z+16z^2)} \\
\frac{-9+7z}{3(1-6z+4z^2)(15-32z+16z^2)} & \frac{-24+109z-132z^2+48z^3}{3(1-6z+4z^2)(15-32z+16z^2)} & \frac{-4+3z}{3(-5+34z-4z^2+16z^3)} \\
\frac{-9+7z}{3(1-6z+4z^2)(15-32z+16z^2)} & \frac{-4+3z}{3(-5+34z-4z^2+16z^3)} & \frac{-24+109z-132z^2+48z^3}{3(1-6z+4z^2)(15-32z+16z^2)} \\
\frac{-4+3z}{3(-5+34z-4z^2+16z^3)} & \frac{-9+7z}{3(1-6z+4z^2)(15-32z+16z^2)} & \frac{-24+109z-132z^2+48z^3}{3(1-6z+4z^2)(15-32z+16z^2)} \\
-\frac{1}{3-18z+12z^2} & -\frac{1}{3-18z+12z^2} & -\frac{1}{3-18z+12z^2}
\end{pmatrix}.$$

Moreover, we compute that

$$\phi(z) = \frac{(2z-3)(6z-7)}{3(4z-5)(4z-3)(1-6z+4z^2)}$$

and

$$R(z) = \frac{6z(z-1)(4z-5)(4z-3)}{6z-7}.$$

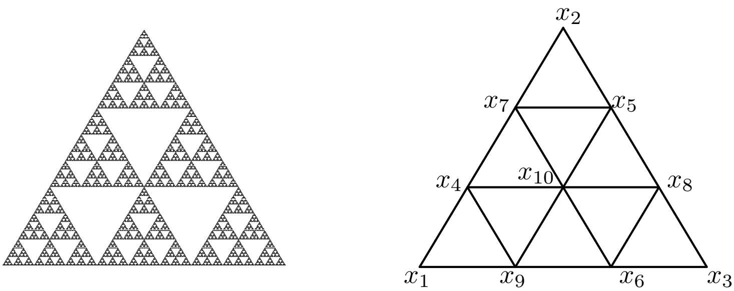

Figure 8. The level-3 Sierpiński gasketand its V_1 network.

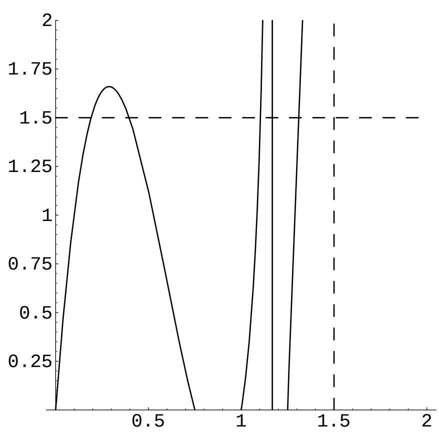

Figure 9. The graph of $R(z)$ for the level-3 Sierpiński gasket.

The eigenvalues of D, written with multiplicities are

$$\sigma(D) = \left\{\frac{3}{2}, \frac{1}{4}(3+\sqrt{5}), \frac{5}{4}, \frac{5}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4}(3-\sqrt{5})\right\}$$

One can also compute

$$\sigma(M)=\left\{\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{1}{4}(3+\sqrt{2}),\frac{1}{4}(3+\sqrt{2}),1,\frac{1}{4}(3-\sqrt{2}),\frac{1}{4}(3-\sqrt{2}),0\right\}$$

We find that $\phi(z)=0$ has two solutions $\{\frac{7}{6}\}$, $\{\frac{3}{2}\}$. Thus, the exceptional set is

$$E(M_0,M)=\left\{\frac{3}{2},\frac{1}{4}(3+\sqrt{5}),\frac{5}{4},\frac{3}{4},\frac{1}{4}(3-\sqrt{5}),\frac{7}{6}\right\}.$$

<table>
<tr><td>$z\in\sigma(M_0)$</td><td colspan="10">$\mathbf{0}$</td><td colspan="10">$\mathbf{\frac{3}{2}}$</td><td colspan="7"></td></tr>
<tr><td>$\text{mult}_0(z)$</td><td colspan="10">*1*</td><td colspan="10">*2*</td><td colspan="7"></td></tr>
<tr><td>$z\in\sigma(M_1)$</td><td colspan="4">$\mathbf{0}$</td><td colspan="4">$\mathbf{1}$</td><td>$\mathbf{\frac{3}{4}}$</td><td>$\mathbf{\frac{5}{4}}$</td><td colspan="8">$\mathbf{\frac{3\pm\sqrt{2}}{4}}$</td><td colspan="2">$\mathbf{\frac{3\pm\sqrt{5}}{4}}$</td><td colspan="4">$\mathbf{\frac{3}{2}}$</td><td colspan="3"></td></tr>
<tr><td>$\text{mult}_1(z)$</td><td colspan="4">*1*</td><td colspan="4">*1*</td><td></td><td></td><td colspan="4">*2*</td><td colspan="4">*2*</td><td></td><td></td><td colspan="4">*4*</td><td colspan="3"></td></tr>
<tr><td>$z\in\sigma(M_2)$</td><td>$\mathbf{0}$</td><td>$\mathbf{1}$</td><td>$\mathbf{\frac{3}{4}}$</td><td>$\mathbf{\frac{5}{4}}$</td><td>*</td><td>*</td><td>*</td><td>*</td><td></td><td></td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td></td><td></td><td colspan="2">$\mathbf{\frac{3\pm\sqrt{2}}{4}}$</td><td colspan="2">$\mathbf{\frac{3\pm\sqrt{5}}{4}}$</td><td>$\mathbf{\frac{3}{2}}$</td><td>$\mathbf{\frac{3}{4}}$</td><td>$\mathbf{\frac{4}{5}}$</td></tr>
<tr><td>$\text{mult}_2(z)$</td><td>*1*</td><td>*1*</td><td></td><td></td><td>*1*</td><td>*1*</td><td>*1*</td><td>*1*</td><td></td><td></td><td>*2*</td><td>*2*</td><td>*2*</td><td>*2*</td><td>*2*</td><td>*2*</td><td>*2*</td><td>*2*</td><td></td><td></td><td>*4*</td><td>*4*</td><td></td><td></td><td>*16*</td><td>*3*</td><td>*3*</td></tr>
</table>

Table 3. Ancestor-offspring structure of the eigenvalues on the level-3 Sierpiński gasket.

To begin the analysis of the exceptional values, note that find the poles of R(z) and see if it is an exceptional value It is easy to see that $\frac{3}{4}$, $\frac{5}{4}$, $\frac{1}{4}(3-\sqrt{5})$ and $\frac{1}{4}(3+\sqrt{5})$ are poles of $\phi(z)$ and so we can use Proposition 4.1(iii) to compute the multiplicities. We obtain

$$\begin{aligned}
\text{mult}_1(\tfrac{3}{4}) &= 2-3+1=0,\\
\text{mult}_2(\tfrac{3}{4}) &= 12-10+1=3,\\
\text{mult}_1(\tfrac{5}{4}) &= 2-3+1=0,\\
\text{mult}_2(\tfrac{5}{4}) &= 12-10+1=3,\\
\text{mult}_1(\tfrac{3\pm\sqrt{5}}{4}) &= 1-3+2=0,\\
\text{mult}_2(\tfrac{3\pm\sqrt{5}}{4}) &= 6-10+4=0.
\end{aligned}$$

Note that $R(\frac{3}{4})=R(\frac{5}{4})=0$ and $R(\frac{3\pm\sqrt{5}}{4})=\frac{3}{2}$. Also, $\frac{3}{2}$ is not a pole of $\phi(z)$ but $\phi(\frac{3}{2})=0$ and therefore we use Proposition 4.1(v) to compute the multiplicities. We obtain

$$\begin{aligned}
\text{mult}_1(\tfrac{3}{2}) &= 1+0+3=4,\\
\text{mult}_2(\tfrac{3}{2}) &= 6+0+10=16.
\end{aligned}$$

The ancestor-offspring structure of the eigenvalues on the level-3 Sierpiński gasket is shown in Table 3. The multiplicity of the ancestor is the same as that of the offspring by Proposition 4.1(i). The empty columns correspond to the exceptional values. If they are eigenvalues of the appropriate M_n, then the multiplicity is shown in the right hand part of the same row.

Theorem 7.1. (i) *For any* $n \geqslant 0$ *we have that* $\sigma(\Delta_n) \subset \bigcup_{m=0}^{n} R_{-m}(\{0, \frac{3}{2}\})$ *and* $\sigma(\Delta_1) = \{\frac{3}{2}, \frac{1}{4}(3 \pm \sqrt{2}), \frac{5}{4}, \frac{3}{4}\}$.
(ii) *For* $n \geqslant 0$ *we have that*

$$\sigma(\Delta_n) = (R_{-n}(0)) \bigcup \left(R_{-(n-1)} \left(\frac{3 \pm \sqrt{5}}{4} \right) \right) \bigcup \left\{ \frac{3}{2} \right\}.$$

(iii) *For* $n \geqslant 0$ *we have* $\dim_n = 3 + \frac{7}{5}(6^n - 1)$.
(iv) *For* $n \geqslant 0$ *we have that* $\mathrm{mult}_n(0) = \mathrm{mult}_n(1) = 1$.
(v) *For* $n \geqslant 2$ *and for* $z \in R_{-k}(1)$, $0 \geqslant k \leqslant 2$ *we have that* $\mathrm{mult}_n(z) = 1$.
(vi) *For* $n \geqslant 0$ *we have that* $\mathrm{mult}_n(\frac{3}{2}) = \dfrac{2 \cdot 6^n + 8}{5}$.
(vii) *For* $n \geqslant 2$ *and* $0 \leqslant k \leqslant n - 2$ *we have for* $z \in R_{-k}\{\frac{3}{4}, \frac{5}{4}\}$ *that*

$$\mathrm{mult}_n(z) = \tfrac{3}{5}(6^{n-k-1} - 1).$$

Note as a special case $k = 0$ *which gives the multiplicities of* $\frac{3}{4}$ *and* $\frac{5}{4}$.
(viii) *For* $n \geqslant 1$ *with* $0 \leqslant k \leqslant n - 1$ *we have that for* $z \in R_{-k}(\frac{3 \pm \sqrt{2}}{4})$

$$\mathrm{mult}_n(z) = \mathrm{mult}_{n-k-1}(\tfrac{3}{2}) = \frac{2 \cdot 6^{n-k-1} + 8}{5}.$$

(ix) *For any* $n \geqslant 1$ *with* $0 \leqslant k \leqslant n - 1$ *we have that for* $z \in R_{-k}(\frac{3 \pm \sqrt{5}}{4})$

$$\mathrm{mult}_n(z) = 0.$$

Proof. For this fractal we have $\sigma(\Delta_0) = \{0, \frac{3}{2}\}$ with $\mathrm{mult}_0(\frac{3}{2}) = 2$ and, for the purposes of Proposition 4.1, $m = 6$.

Item (i) is obtained above in this section.

Item (ii) follows from the subsequent items.

Item (iii) is straightforward by induction.

Item (iv) follows from Proposition 4.1(i) because 0 is a fixed point of $R(z)$ and because $R(1) = 0$.

Item (v) easily follows from Proposition 4.1(i) and Item (iv).

Item (vi) follows from the previous items and Proposition 4.1(v).

Item (vii) follows from Proposition 4.1(iii).

Item (viii) follows from Proposition 4.1(i).

Item (ix) follows from Proposition 4.1(iii), and as a consequence none of these values appear in the spectrum. □

Corollary 7.1. *The normalized limiting distribution of eigenvalues (the integrated density of states) is a pure point measure κ with the the set of atoms*

$$\left\{\frac{3}{2}\right\} \bigcup \left(\bigcup_{m=0}^{\infty} R_{-m}\{\tfrac{3}{4}, \tfrac{5}{4}, \tfrac{3\pm\sqrt{2}}{4}\} \right).$$

Moreover, $\kappa\left(\{\tfrac{3}{2}\}\right) = \dfrac{2}{7}$ *and*

$$\kappa(\{z\}) = \tfrac{3}{7}6^{-m-1} \quad \textit{if} \quad z \in R_{-m}\{\tfrac{3}{4}, \tfrac{5}{4}\};$$
$$\kappa(\{z\}) = \tfrac{2}{7}6^{-m-1} \quad \textit{if} \quad z \in R_{-m}\left\{\tfrac{3\pm\sqrt{2}}{4}\right\}.$$

8. A fractal 3-tree.

The fractal tree is a fractal that is approximated by triangles as shown in Figure 10, but in the limit is a topological tree. It appeared as the limit set of the Gupta-Sidki group, see [4, 38, and references therein].

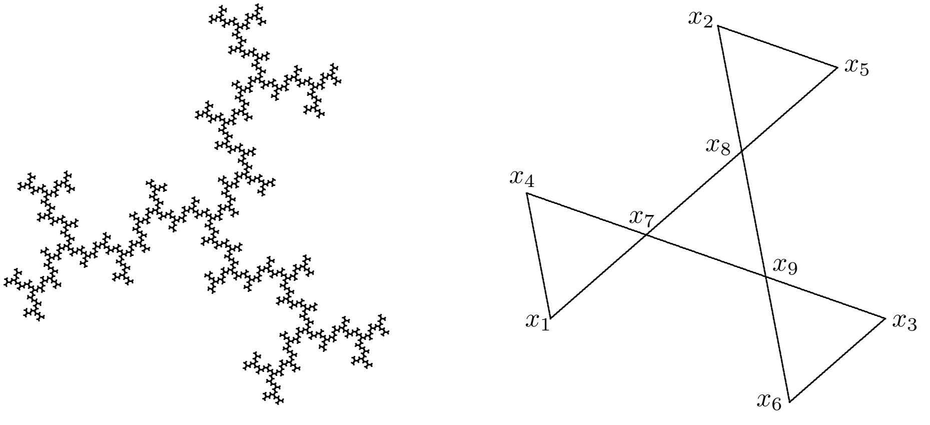

Figure 10. The fractal 3-tree and its V_1 network.

The depth-1 Laplacian matrix $M_1 = M$ is

$$M = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ -\frac{1}{4} & 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 & 0 & 0 & -\frac{1}{4} \\ -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & 1 \end{pmatrix}$$

and the eigenfunction extension map $(D-z)^{-1}C$ is

$$\begin{pmatrix} \frac{5+2z(-7+4z)}{9-8z(6+z(-9+4z))} & \frac{2(-1+z)}{(-3+4z)(3+4z(-3+2z))} & \frac{2(-1+z)}{(-3+4z)(3+4z(-3+2z))} \\ \frac{2(-1+z)}{(-3+4z)(3+4z(-3+2z))} & \frac{5+2z(-7+4z)}{9-8z(6+z(-9+4z))} & \frac{2(-1+z)}{(-3+4z)(3+4z(-3+2z))} \\ \frac{2(-1+z)}{(-3+4z)(3+4z(-3+2z))} & \frac{2(-1+z)}{(-3+4z)(3+4z(-3+2z))} & \frac{5+2z(-7+4z)}{9-8z(6+z(-9+4z))} \\ \frac{-7+8(3-2z)z}{(-3+4z)(3+4z(-3+2z))} & \frac{1}{9-8z(6+z(-9+4z))} & \frac{1}{9-8z(6+z(-9+4z))} \\ \frac{1}{9-8z(6+z(-9+4z))} & \frac{-7+8(3-2z)z}{(-3+4z)(3+4z(-3+2z))} & \frac{1}{9-8z(6+z(-9+4z))} \\ \frac{1}{9-8z(6+z(-9+4z))} & \frac{1}{9-8z(6+z(-9+4z))} & \frac{-7+8(3-2z)z}{(-3+4z)(3+4z(-3+2z))} \end{pmatrix}.$$

From here, we compute that

$$\phi(z) = \frac{3-2z}{9-48z+72z^2-32z^3}$$

and

$$R(z) = 4z(z-1)(4z-3).$$

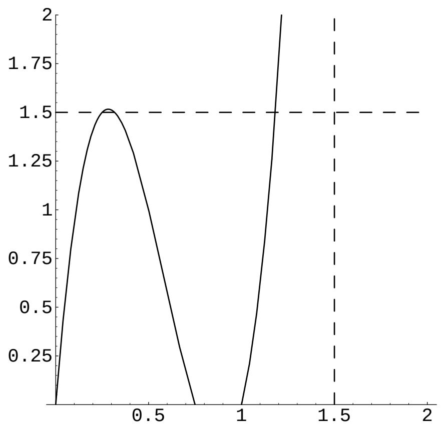

Figure 11. The graph of $R(z)$ for the fractal tree.

The eigenvalues of D written with multiplicities are

$$\sigma(D) = \left\{\frac{3}{2}, \frac{3}{2}, \frac{1}{4}\left(3+\sqrt{3}\right), \frac{3}{4}, \frac{3}{4}, \frac{1}{4}\left(3-\sqrt{3}\right)\right\}$$

and the corresponding eigenvectors are $\{1,0,-1,-1,0,1\}$, $\{1,-1,0,-1,1,0\}$, $\{\frac{1-\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}, 1, 1, 1\}$, $\{-\frac{1}{2}, 0, \frac{1}{2}, -1, 0, 1\}$, $\{-\frac{1}{2}, \frac{1}{2}, 0, -1, 1, 0\}$, and $\{\frac{1+\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}, 1, 1, 1\}$.

Computing the eigenvalues of M with multiplicities gives

$$\sigma(M) = \left\{\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1, \frac{1}{4}, \frac{1}{4}, 0\right\}$$

and the corresponding eigenvectors are $\{0,\ 0,\ -1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1\}$, $\{0,\ -1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0\}$, $\{-1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0\}$, $\{1,\ 0,\ -1,\ -1,\ 0,\ 1,\ 0,\ 0,\ 0\}$, $\{1,\ -1,\ 0,\ -1,\ 1,\ 0,\ 0,$

0, 0}, {1, 1, 1, −1, −1, −1, 1, 1, 1}, {−1, 0, 1, $-\frac{1}{2}$, 0, $\frac{1}{2}$, −1, 0, 1}, {−1, 1, 0, $-\frac{1}{2}$, $\frac{1}{2}$, 0, −1, 1, 0}, {1, 1, 1, 1, 1, 1, 1, 1, 1}.

The only solution of $\phi(z) = 0$ is $\frac{3}{2}$. As such, the exceptional set is

$$E(M_0, M) = \left\{ \frac{3}{2}, \frac{3}{4}, \frac{1}{4}(3+\sqrt{3}), \frac{1}{4}(3-\sqrt{3}) \right\}.$$

For analysis of exceptional values, one can find $R(z)$ at each exceptional point by

$$R^{-1}(0) = \left\{ 0, \frac{3}{4}, 1 \right\} \quad \text{and} \quad R^{-1}(\tfrac{3}{2}) = \left\{ \frac{1}{4}, \frac{1}{4}(3-\sqrt{3}), \frac{1}{4}(3+\sqrt{3}) \right\}.$$

Using Proposition 4.1, one can determine the multiplicities of the exceptional values. For the value $\frac{3}{2}$, which is a zero of $\phi(z)$, we use Proposition 4.1(v) to find the multiplicities.

$$\text{mult}_1(\tfrac{3}{2}) = 4^0(2) + 0 + 3 = 5,$$
$$\text{mult}_2(\tfrac{3}{2}) = 4^1(2) + 0 + 9 = 17.$$

For the value $\frac{3}{4}$, which is a pole of $\phi(z)$, we use Proposition 4.1(iii) to find the multiplicities.

$$\text{mult}_1(\tfrac{3}{4}) = 4^0(2) - 3 + 1 = 0,$$
$$\text{mult}_2(\tfrac{3}{4}) = 4^1(2) - 9 + 1 = 0.$$

For the values $\frac{1}{4}(3+\sqrt{3})$ and $\frac{1}{4}(3-\sqrt{3})$, which are poles of $\phi(z)$, we use Proposition 4.1(iii) to find the multiplicities.

$$\text{mult}_1(\tfrac{1}{4}(3 \pm \sqrt{3})) = 4^0(1) - 3 + 2 = 0,$$
$$\text{mult}_2(\tfrac{1}{4}(3 \pm \sqrt{3})) = 4^1(1) - 9 + 5 = 0.$$

$z \in \sigma(M_0)$	**0**							**$\frac{3}{2}$**						
$\text{mult}_0(z)$	*1*							*2*						
$z \in \sigma(M_1)$	**0**			**$\frac{3}{4}$**	**1**			**$\frac{1}{4}$**			**$\frac{3\pm\sqrt{3}}{4}$**	**$\frac{3}{2}$**		
$\text{mult}_1(z)$	*1*				*1*			*2*				*5*		
$z \in \sigma(M_2)$	**0**	**$\frac{3}{4}$**	**1**		$*$	$*$	$*$	$*$	$*$	$*$		**$\frac{1}{4}$**	**$\frac{3\pm\sqrt{3}}{4}$**	**$\frac{3}{2}$**
$\text{mult}_2(z)$	*1*		*1*		*1*	*1*	*1*	*2*	*2*	*2*		*5*		*17*

Table 4. Ancestor-offspring structure of the eigenvalues of the fractal tree.

The ancestor-offspring structure of the eigenvalues of the Fractal Tree is shown in Table 4. As before, the symbol * indicates branches of the inverse function $R^{-1}(z)$ computed at the ancestor value. The multiplicity of the ancestor equals that of the offspring by Proposition 4.1(i). The exceptional values are represented by the empty columns. If they are eigenvalues of the appropriate M_n, then the multiplicity is shown in the right hand part of the same row.

Theorem 8.1. (i) *For any* $n \geqslant 0$ *we have that* $\sigma(\Delta_n) \subset \bigcup_{m=0}^{n} R_{-m}(\{0, \frac{3}{2}\}) \bigcup \{\frac{3}{2}\}$ *and* $\sigma(\Delta_1) = \{\frac{3}{2}, \frac{1}{4}(3 \pm \sqrt{3}), \frac{3}{4}\}$.
(ii) *For* $n \geqslant 2$ *we have that*

$$\sigma(\Delta_n) = \left\{0, \frac{3}{2}\right\} \bigcup \left(\bigcup_{k=0}^{n-1} R_{-k} \left\{\frac{1}{4}, 1\right\} \right).$$

And for $n = 1$ *we have* $\sigma(\Delta_1) = \{0, \frac{1}{4}, 1, \frac{3}{2}\}$.
(iii) *For* $n \geqslant 0$ *we have* $\dim_n = 3 + 2(4^n - 1)$.
(iv) *For* $n \geqslant 0$ *we have* $\mathrm{mult}_n(0) = \mathrm{mult}_n(1) = 1$.
(v) *For* $n \geqslant 2$ *with* $0 \leqslant k \leqslant n-2$ *we have that if* $z \in R_{-k}(1)$ *then*

$$\mathrm{mult}_n(z) = \mathrm{mult}_{n-k}(1) = 1.$$

(vi) *For* $n \geqslant 0$ *we have that*

$$\mathrm{mult}_n(\tfrac{3}{2}) = 4^n + 1.$$

(vii) *For* $n \geqslant 1$ *with* $0 \leqslant k \leqslant n$ *we have for* $z \in R_{-k}(\frac{1}{4})$ *that*

$$\mathrm{mult}_n(z) = \mathrm{mult}_{n-k}(\tfrac{1}{4}) = \mathrm{mult}_{n-k-1}(\tfrac{3}{2}) = 4^{n-k-1} + 1.$$

(viii) *For* $n \geqslant 1$ *we have* $\mathrm{mult}_n(\frac{3}{4}) = 0$.
(ix) *For* $n \geqslant 1$ *with* $0 \leqslant k \leqslant n$ *we have that if* $z \in R_{-k}(\frac{3 \pm \sqrt{3}}{4})$ *then* $\mathrm{mult}_n(z) = 0$.

Proof. For this fractal we have $\sigma(\Delta_0) = \{0, \frac{3}{2}\}$ with $\mathrm{mult}_0(\frac{3}{2}) = 2$ and, for the purposes of Proposition 4.1, $m = 6$.

Item (i) is obtained above in this section.

Item (ii) follows from the subsequent items.

Item (iii) is straightforward by induction.

Item (iv) follows from Proposition 4.1(i) because 0 is a fixed point of $R(z)$ and because $R(1) = 0$.

Item (v) easily follows from Proposition 4.1(i) and Item (iv).

Item (vi) follows from the previous items and Proposition 4.1(v).

Item (vii) follows from Proposition 4.1(i).

Items (viii) and (ix) follow from Proposition 4.1(iii), and as a consequence none of these values appear in the spectrum. □

Corollary 8.1. *The normalized limiting distribution of eigenvalues (the integrated density of states) is a pure point measure* κ *with the the set of atoms*

$$\left\{\frac{3}{2}\right\} \bigcup \left(\bigcup_{m=0}^{\infty} R_{-m} \left\{\frac{1}{4}\right\} \right).$$

Moreover, $\kappa\left(\{\frac{3}{2}\}\right) = \frac{1}{2}$, *and* $\kappa(\{z\}) = \frac{1}{2} 4^{-m-1}$ *if* $z \in R_{-m}\{\frac{1}{4}\}$.

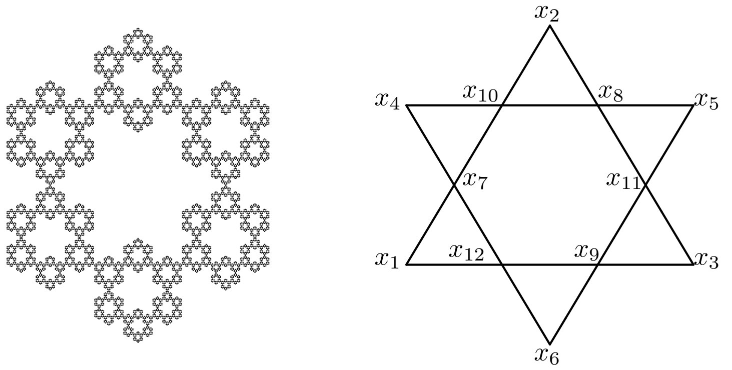

Figure 12. The hexagasket and its V_1 network.

9. Hexagasket.

The hexagasket, or the hexakun, is a fractal which in different situations [1, 6, 23, 46, 50, 52, 53, and references therein] is called a polygasket, a 6-gasket, or a (2, 2, 2)-gasket. The depth-1 approximation to it is shown in Figure 12.

The matrix of the depth-1 Laplacian $M_1 = M$ is

$$M = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & 0 & 1 & -\frac{1}{4} & 0 & 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ -\frac{1}{4} & 0 & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & 0 & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & 0 & 0 & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & 0 & 0 & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 1 \end{pmatrix}$$

and the eigenfunction extension map $(D-z)^{-1}C$ is

$$\begin{pmatrix} \frac{-4+z(23+4z(-9+4z))}{(1-6z+4z^2)(7+8z(-3+2z))} & \frac{-1+z}{(1-6z+4z^2)(7+8z(-3+2z))} & \frac{-2+(7-4z)z}{(1-6z+4z^2)(7+8z(-3+2z))} \\ \frac{-4+z(23+4z(-9+4z))}{(1-6z+4z^2)(7+8z(-3+2z))} & \frac{-2+(7-4z)z}{(1-6z+4z^2)(7+8z(-3+2z))} & \frac{-1+z}{(1-6z+4z^2)(7+8z(-3+2z))} \\ \frac{-2+(7-4z)z}{(1-6z+4z^2)(7+8z(-3+2z))} & \frac{-4+z(23+4z(-9+4z))}{(1-6z+4z^2)(7+8z(-3+2z))} & \frac{-1+z}{(1-6z+4z^2)(7+8z(-3+2z))} \\ \frac{-1+z}{(1-6z+4z^2)(7+8z(-3+2z))} & \frac{-4+z(23+4z(-9+4z))}{(1-6z+4z^2)(7+8z(-3+2z))} & \frac{-2+(7-4z)z}{(1-6z+4z^2)(7+8z(-3+2z))} \\ \frac{-1+z}{(1-6z+4z^2)(7+8z(-3+2z))} & \frac{-2+(7-4z)z}{(1-6z+4z^2)(7+8z(-3+2z))} & \frac{-4+z(23+4z(-9+4z))}{(1-6z+4z^2)(7+8z(-3+2z))} \\ \frac{-2+(7-4z)z}{(1-6z+4z^2)(7+8z(-3+2z))} & \frac{-1+z}{(1-6z+4z^2)(7+8z(-3+2z))} & \frac{-4+z(23+4z(-9+4z))}{(1-6z+4z^2)(7+8z(-3+2z))} \\ \frac{-3+4(3-2z)z}{(1-6z+4z^2)(7+8z(-3+2z))} & -\frac{1}{(1-6z+4z^2)(7+8z(-3+2z))} & \frac{-3+4(3-2z)z}{(1-6z+4z^2)(7+8z(-3+2z))} \\ \frac{-3+4(3-2z)z}{(1-6z+4z^2)(7+8z(-3+2z))} & \frac{-3+4(3-2z)z}{(1-6z+4z^2)(7+8z(-3+2z))} & -\frac{1}{(1-6z+4z^2)(7+8z(-3+2z))} \\ -\frac{1}{(1-6z+4z^2)(7+8z(-3+2z))} & \frac{-3+4(3-2z)z}{(1-6z+4z^2)(7+8z(-3+2z))} & \frac{-3+4(3-2z)z}{(1-6z+4z^2)(7+8z(-3+2z))} \end{pmatrix}$$

Moreover, we compute that

$$\phi(z) = \frac{3+4(z-2)z}{(4z^2+6z-1)\,(7+8z(2z-3))}$$

and

$$R(z) = \frac{2z(z-1)(7-24z+16z^2)}{2z-1}.$$

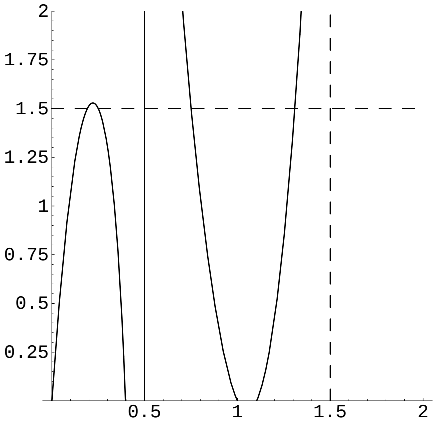

Figure 13. The graph of $R(z)$ for the hexagasket.

The eigenvalues of D, written with multiplicities, are

$$\sigma(D) = \left\{\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{4}\left(3 \pm \sqrt{5}\right), \frac{1}{4}\left(3 \pm \sqrt{2}\right), \frac{1}{4}\left(3 \pm \sqrt{2}\right)\right\}.$$

One can also compute

$$\sigma(M) = \left\{\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1, \frac{3}{4}, \frac{3}{4}, \frac{1}{4}, \frac{1}{4}, 0\right\}$$

with the corresponding eigenvectors $\{0, 1, 0, 0, 0, 0, -1, 0, 0, 0, 0, 1\}$, $\{1, 0, 0, 0, -1, 0, 0, 0, 0, 0, 1, 0\}$, $\{1, 0, 0, -1, 0, 0, 0, 0, 0, 1, 0, 0\}$, $\{1, 0, -1, -1, 0, 0, 0, 0, 1, 0, 0, 0\}$, $\{0, 1, -1, 0, 0, 0, -1, 1, 0, 0, 0, 0\}$, $\{1, -1, 0, 0, -1, 1, 0, 0, 0, 0, 0, 0\}$, $\{-1, -1, -1, 0, 0, 0, 0, 0, 0, 1, 1, 1\}$, $\{1, -1, 0, 0, \frac{1}{2}, -\frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{2}, -1, 0, 1\}$, $\{0, -1, 1, -\frac{1}{2}, \frac{1}{2}, 0, -\frac{1}{2}, \frac{1}{2}, 0, -1, 1, 0\}$, $\{-1, 1, 0, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2}, -1, 0, 1\}$, $\{0, 1, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2}, -1, -1, 1, 0\}$, $\{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}$.

It is easy to see that $\phi(z) = 0$ has two solution $\frac{1}{2}$ and $\frac{3}{2}$. Thus, the exceptional set is

$$E(M_0, M) = \left\{\frac{3}{2}, \frac{1}{4}\left(3 \pm \sqrt{5}\right), \frac{1}{4}\left(3 \pm \sqrt{2}\right), \frac{1}{2}\right\}.$$

To begin the analysis of the exceptional values, note that $\frac{1}{2}$ is the pole of $R(z)$ and therefore is not an eigenvalue by Proposition 4.1(vii).

It is easy to see that $\frac{1}{4}(3 \pm \sqrt{2})$ and $\frac{1}{4}\left(3 \pm \sqrt{5}\right)$ are the four poles of $\phi(z)$ and so we can use Proposition 4.1(iii) to compute the multiplicities. We obtain

$$\begin{aligned}
\mathrm{mult}_1(\tfrac{1}{4}(3 \pm \sqrt{2})) &= 6^0 \cdot 2 - 3 + 1 = 0, \\
\mathrm{mult}_2(\tfrac{1}{4}(3 \pm \sqrt{2})) &= 6^1 \cdot 2 - 12 + 1 = 1, \\
\mathrm{mult}_1(\tfrac{1}{4}(3 \pm \sqrt{5})) &= 6^0 \cdot 1 - 3 + 2 = 0, \\
\mathrm{mult}_2(\tfrac{1}{4}(3 \pm \sqrt{5})) &= 6^1 \cdot 1 - 12 + 6 = 0.
\end{aligned}$$

$z \in \sigma(M_0)$	$\mathbf{0}$								$\mathbf{\frac{3}{2}}$													
$\mathrm{mult}_0(z)$	*1*								*2*													
$z \in \sigma(M_1)$	$\mathbf{0}$			$\mathbf{1}$				$\mathbf{\frac{3\pm\sqrt{2}}{4}}$	$\mathbf{\frac{1}{4}}$				$\mathbf{\frac{3}{4}}$				$\mathbf{\frac{3\pm\sqrt{5}}{4}}$	$\mathbf{\frac{3}{2}}$				
$\mathrm{mult}_1(z)$	*1*			*1*					*2*				*2*					*6*				
$z \in \sigma(M_2)$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{\frac{3\pm\sqrt{2}}{4}}$	*	*	*	*		*	*	*	*	*	*	*	*		$\mathbf{\frac{1}{4}}$	$\mathbf{\frac{3}{4}}$	$\mathbf{\frac{3\pm\sqrt{5}}{4}}$	$\mathbf{\frac{3}{2}}$	$\mathbf{\frac{3\pm\sqrt{2}}{4}}$
$\mathrm{mult}_2(z)$	*1*	*1*		*1*	*1*	*1*	*1*		*2*	*2*	*2*	*2*	*2*	*2*	*2*	*2*		*6*	*6*		*30*	*1* *1*

Table 5. Ancestor-offspring structure of the eigenvalues on the hexagasket.

The exceptional value $\frac{3}{2}$ is in the spectrum $\sigma(D)$, not a pole of $\phi(z)$ and $\phi(\frac{3}{2}) = 0$. For this reason we can use Proposition 4.1(v) to compute the multiplicities.

$$\begin{aligned}\mathrm{mult}_1(\tfrac{3}{2}) &= 6^0 \cdot 3 + 0 + 3 = 6,\\ \mathrm{mult}_2(\tfrac{3}{2}) &= 6^1 \cdot 3 + 0 + 12 = 30.\end{aligned}$$

As in the other sections, the multiplicities of all eigenvalues at depths 0, 1 and 2 are shown in Table 5.

Theorem 9.1. (i) $\sigma(M_0) = \{0, \frac{3}{2}\}$.

(ii) *We have that* $\sigma(M_1) = \left\{0, \frac{1}{4}, \frac{3}{4}, 1, \frac{3}{2}\right\}$ *and for* $n \geqslant 2$ *we have*

$$\sigma(M_n) = \left\{0, \frac{3}{2}\right\} \bigcup \left(\bigcup_{m=0}^{n-1} R_{-m} \left\{1, \frac{1}{4}, \frac{3}{4}\right\} \right) \bigcup \left(\bigcup_{m=0}^{n-2} R_{-m} \left\{ \frac{3 \pm \sqrt{2}}{4} \right\} \right).$$

(iii) *For any* $n \geqslant 0$ *we have* $\dim_n = \dfrac{6 + 9 \cdot 6^n}{5}$.

(iv) *For any* $n \geqslant 0$, $\mathrm{mult}_n(0) = 1$ *and* $\mathrm{mult}_n(\frac{3}{2}) = \dfrac{6 + 4 \cdot 6^n}{5}$.

(v) *For any* $n \geqslant 1$ *and* $0 \leqslant k < n - 1$ *we have that if* $x \in R_{-k}(1)$ *then* $\mathrm{mult}_n(z) = 1$.

(vi) *For any* $n \geqslant 1$ *and* $0 \leqslant k < n - 1$ *we have that if* $z \in R_{-k}\{\frac{1}{4}, \frac{3}{4}\}$ *then*

$$\mathrm{mult}_n(z) = \frac{6 + 4 \cdot 6^{n-k-1}}{5}.$$

(vii) *For any* $n \geqslant 2$ *and* $0 \leqslant k < n - 2$ *we have that if* $z \in R_{-k}(\frac{3\pm\sqrt{2}}{4})$ *then*

$$\mathrm{mult}_n(z) = \frac{6^{n-k-1} - 1}{5}.$$

(viii) *For* $n \geqslant 0$ *we have* $\mathrm{mult}_n(\frac{3\pm\sqrt{5}}{4}) = 0$.

Proof. For this fractal we have $\sigma(\Delta_0) = \{0, \frac{3}{2}\}$ with $\mathrm{mult}_0(\frac{3}{2}) = 2$ and, for the purposes of Proposition 4.1, $m = 6$.

Item (i) is obtained above in this section.

Item (ii) follows from the subsequent items.

Item (iii) is straightforward by induction.

Item (iv) follows from Proposition 4.1(i) because 0 is a fixed point of $R(z)$, and from Proposition 4.1(v).

Items (v) and (vi) follow from Proposition 4.1(i).

Items (vii) and (viii) follow from Proposition 4.1(iii). □

Corollary 9.1. *The normalized limiting distribution of eigenvalues (the integrated density of states) is a pure point measure κ with the the set of atoms*

$$\left\{\frac{3}{2}\right\} \bigcup \left(\bigcup_{m=0}^{\infty} R_{-m} \left\{ \frac{1}{4}, \frac{3}{4}, \frac{3 \pm \sqrt{2}}{4} \right\} \right).$$

Moreover, $\kappa\left(\{\frac{3}{2}\}\right) = \dfrac{4}{9}$, *and*

$$\kappa(\{z\}) = \tfrac{4}{9} 6^{-m-1} \quad \textit{if} \quad z \in R_{-m}\{\tfrac{1}{4}, \tfrac{3}{4}\};$$
$$\kappa(\{z\}) = \tfrac{1}{9} 6^{-m-1} \quad \textit{if} \quad z \in R_{-m}\{\tfrac{3 \pm \sqrt{2}}{4}\}.$$

10. One dimensional interval as a self-similar set.

In this section we show how our results allow us to recover classically known information about the spectrum of the discrete Laplacians that approximate the usual one dimensional continuous Laplacian. The unit interval [0,1] can be represented as a self-similar set in various ways. Here we consider three cases: when it subdivided into two, three or four subintervals of equal length. In our notation this means that m is 2, 3, or 4. The depth-1 networks for these cases are shown in Figure 14. The first two cases were also discussed in [49]. Note that in each case the function $R(z)$ is the same as the Chebyshev polynomial of degree m for the interval [0,2], which is the smallest interval that contains the spectrum of the matrices M_n. It is shown in [49], in particular, that the iterations of these polynomials are related in a natural way with the Riemann zeta function.

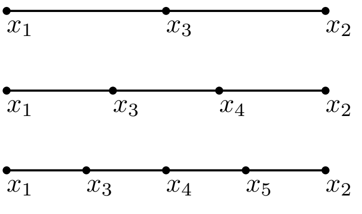

Figure 14. V_1 networks for the interval in cases $m = 2, 3, 4$ respectively.

Case $m = 2$. The matrix of the depth-1 Laplacian $M_1 = M$ is

$$M = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$$

and the eigenfunction extension map is

$$(D-z)^{-1}C = \begin{pmatrix} \frac{1}{2(z-1)} & \frac{1}{2(z-1)} \end{pmatrix}.$$

Moreover, we compute that

$$\phi(z) = \frac{1}{2(1-z)}$$

and

$$R(z) = 2z(2-z).$$

The only eigenvalue of D is $\sigma(D) = \{1\}$. One can also compute $\sigma(M) = \{2, 1, 0\}$ with the corresponding eigenvectors{{-1, -1, 1}, {-1, 1, 0}, {1, 1, 1}}. It is easy to see that $\phi(z) \neq 0$. Thus, the exceptional set is

$$E(M_0, M) = \{1\}.$$

To begin the analysis of the exceptional value, note that $R(z)$ does not have any poles. We are interested in the value of $R(z)$ at the exceptional point, which is $R(1) = 2$. It is easy to see that 1 is a pole of $\phi(z)$, $R(z)$ has a removable singularity at z, and $\frac{d}{dz}R(z) = 0$. So for all n we can use Proposition 4.1(vi) to compute its multiplicity

$$\text{mult}_n(1) = 1.$$

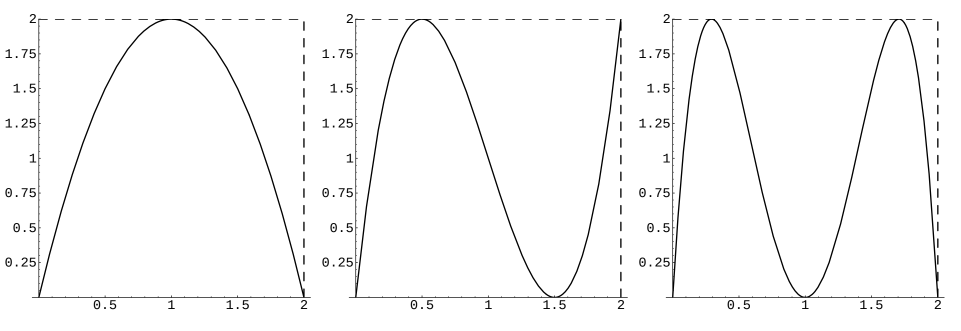

Figure 15. The graph of $R(z)$ for $F = [0, 1]$ with $m = 2$, $m = 3$ and $m = 4$ respectively.

Case $m = 3$. The matrix of the depth-1 Laplacian $M_1 = M$ is

$$M = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -\frac{1}{2} & 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$$

and the eigenfunction extension map is

$$(D-z)^{-1}C = \begin{pmatrix} \frac{2(z-1)}{3-8z+4z^2} & \frac{1}{8z-4z^2-3} \\ \frac{1}{8z-4z^2-3} & \frac{2(z-1)}{3-8z+4z^2} \end{pmatrix}.$$

Moreover, we compute that

$$\phi(z) = \frac{1}{4\left(z-\frac{3}{2}\right)\left(z-\frac{1}{2}\right)}$$

and

$$R(z) = z(3-2z)^2.$$

The eigenvalues of D, written with multiplicities, are

$$\sigma(D) = \left\{\frac{3}{2}, \frac{1}{2}\right\}$$

with corresponding eigenvectors{{-1, 1}, {1, 1}}. One can also compute

$$\sigma(M) = \left\{2, \frac{3}{2}, \frac{1}{2}, 0\right\}$$

with the corresponding eigenvectors{{1, -1, -1, 1}, {-2, -2, 1, 1}, {-2, 2, -1, 1}, {1, 1, 1, 1}}. It is easy to see that $\phi(z) \neq 0$. Thus, the exceptional set is

$$E(M_0, M) = \left\{\frac{3}{2}, \frac{1}{2}\right\}.$$

Again, note that $R(z)$ does not have any poles. We are interested in the values of $R(z)$ in the exceptional points, which are

$$R(\tfrac{3}{2}) = 0, \quad R(\tfrac{1}{2}) = 2.$$

Since $\frac{d}{dz}R(z) = 0$ in these points, we can use Proposition 4.1(vi) to obtain

$$\text{mult}_n(\tfrac{3}{2}) = \text{mult}_n(\tfrac{1}{2}) = 1$$

for all n.

Case $m = 4$**.** The matrix of the depth-1 Laplacian $M_1 = M$ is

$$M = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ -\frac{1}{2} & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

and the eigenfunction extension map is

$$(D-z)^{-1}C = \begin{pmatrix} \frac{3-8z+4z^2}{4(-1+5z-6z^2+2z^3)} & \frac{1}{4(-1+5z-6z^2+2z^3)} \\ -\frac{1}{2-8z+4z^2} & -\frac{1}{2-8z+4z^2} \\ \frac{1}{4(-1+5z-6z^2+2z^3)} & \frac{3-8z+4z^2}{4(-1+5z-6z^2+2z^3)} \end{pmatrix}$$

We compute that

$$\phi(z) = \frac{1}{4 - 20z + 24z^2 - 8z^3}$$

and

$$R(z) = 8z(z-2)(1-z)^2.$$

The eigenvalues of D, written with multiplicities, are

$$\sigma(D) = \left\{\frac{1}{2}\left(2+\sqrt{2}\right), 1, \frac{1}{2}\left(2-\sqrt{2}\right)\right\}$$

with corresponding eigenvectors

$$\left\{\left\{1,-\sqrt{2},1\right\},\{-1,0,1\},\left\{1,\sqrt{2},1\right\}\right\}.$$

One can also compute

$$\sigma(M)=\left\{2,\frac{1}{2}\left(2+\sqrt{2}\right),1,\frac{1}{2}\left(2-\sqrt{2}\right),0\right\}.$$

It is easy to see that $\phi(z)\neq 0$. Thus, the exceptional set is

$$E(M_0,M)=\left\{\tfrac{1}{2}\left(2+\sqrt{2}\right),1,\tfrac{1}{2}\left(2-\sqrt{2}\right)\right\}.$$

To begin the analysis of the exceptional values, note that $R(z)$ does not have any poles. We are interested in the values of $R(z)$ at the exceptional points, which are

$$R(\tfrac{1}{2}(2+\sqrt{2}))=2,\quad R(1)=0,\quad R(\tfrac{1}{2}(2-\sqrt{2}))=2.$$

Once again, $\frac{d}{dz}R(z)=0$ at these points, and by Proposition 4.1(vi) we have

$$\mathrm{mult}_n(\tfrac{1}{2}(2+\sqrt{2}))=\mathrm{mult}_n(1)=\mathrm{mult}_n(\tfrac{1}{2}(2-\sqrt{2}))=1$$

for all n.

11. Diamond fractal.

The diamond fractal is shown in figure 16. The diamond self-similar hierarchical lattice appeared as an example in several physics works, such as [14]. Recently the critical percolation on the diamond fractal was analyzed in [15].

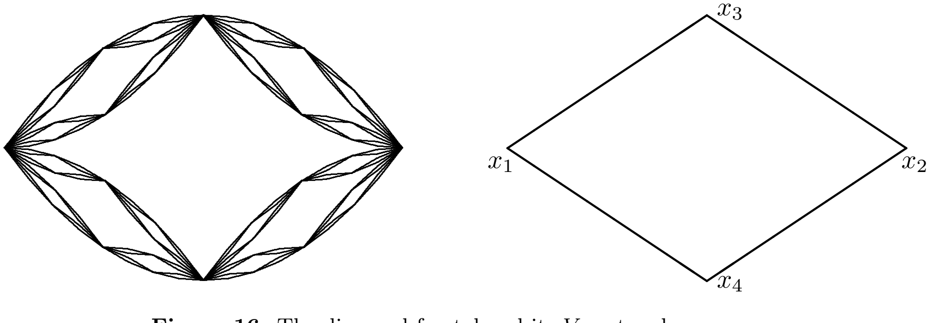

Figure 16. The diamond fractal and its V_1 network.

We can use the results obtained for the unit interval [0,1] in Section 10, case $m=2$, to develop the spectral decimation method for the diamond fractal. The matrix of the depth-1 Laplacian $M_1=M$ is

$$M=\begin{pmatrix}1 & 0 & -\frac{1}{2} & -\frac{1}{2}\\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0\\ -\frac{1}{2} & -\frac{1}{2} & 0 & 1\end{pmatrix}$$

and the eigenfunction extension map is now the square matrix with the same entries

$$(D-z)^{-1}C = \tfrac{1}{2(z-1)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

while the functions

$$\phi(z) = \frac{1}{2(1-z)}$$

and

$$R(z) = 2z(2-z)$$

are the same as for the unit interval, $\sigma(D) = \{1,1\}$ has multiplicity two, and $\sigma(M) = \{2,1,1,0\}$ with the corresponding eigenvectors {-1, -1, 1, 1}, {-1, 1, 0,0}, { 0,0,-1, 1}, {1, 1, 1,1}. The exceptional set is

$$E(M_0, M) = \{1\}\,.$$

Theorem 11.1. (i) *For any $n \geqslant 0$ we have that*

$$\sigma(\Delta_n) = \bigcup_{m=0}^{n} R_{-m}(\{0,2\}).$$

(ii) *For any $n \geqslant 0$ we have* $\dim_n = 3 + 2(4^n - 1)$.

(iii) *For any $n \geqslant 0$ we have* $\mathrm{mult}_n(0) = \mathrm{mult}_n(2) = 1$.

(iv) *For any $n \geqslant 1$ and $0 \leqslant k \leqslant n-1$ we have* $\mathrm{mult}_n(z) = \dfrac{4^{n-k}+2}{3}$ *if $z \in R_{-k}(1)$.*

Proof. Item (i) follows from (iii) and (iv). Item (ii) is obtained by induction. Item (iii) follows from Proposition 4.1(i), and the fact that $R(0) = R(2) = 0$. For the analysis of the only exceptional value $z = 1$, note that it is a pole of $\phi(z)$, $R(1) = 2$, $R(z)$ has a removable singularity at 1, and $\frac{d}{dz}R(1) = 0$. Therefore by Proposition 4.1(vi) we have

$$\mathrm{mult}_n(1) = 4^{n-1} \cdot 2 - \tfrac{2 \cdot 4^{n-1}+4}{3} + 2 = \tfrac{4^n+2}{3}$$

for all $n \geqslant 1$. This implies Item (iv). □

Corollary 11.1. *The normalized limiting distribution of eigenvalues (the integrated density of states) is a pure point measure κ with the the set of atoms*

$$\bigcup_{m=0}^{\infty} R_{-m}\{1\}$$

and $\kappa(\{z\}) = \frac{1}{2}4^{-m}$ *if $z \in R_{-m}\{1\}$.*

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