A convergent adaptive method for elliptic eigenvalue problems

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Abstract

We prove the convergence of an adaptive linear finite element method for computing eigenvalues and eigenfunctions of second order symmetric elliptic partial differential operators. The weak form is assumed to yield a bilinear form which is bounded and coercive in H^1. Each step of the adaptive procedure refines elements in which a standard a posteriori error estimator is large and also refines elements in which the computed eigenfunction has high oscillation. The error analysis extends the theory of convergence of adaptive methods for linear elliptic source problems to elliptic eigenvalue problems, and in particular deals with various complications which arise essentially from the nonlinearity of the eigenvalue problem. Because of this nonlinearity, the convergence result holds under the assumption that the initial finite element mesh is sufficiently fine.

Keywords: Second-order Elliptic Problems, Eigenvalues, Adaptive finite element methods, convergence

Mathematics Subject Classification: 65N12, 65N25, 65N30, 65N50

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1 Introduction

In the last decades, mesh adaptivity has been widely used to improve the accuracy of numerical solutions to many scientific problems. The basic idea is to refine the mesh only where the error is high, with the aim of achieving an accurate solution using an optimal number of degrees of freedom. There is a large numerical analysis literature on adaptivity, in particular on reliable and efficient a posteriori error estimates (e.g. [1]). Recently the question of convergence of adaptive methods has received intensive interest and a number of convergence results for the adaptive solution of boundary value problems have appeared (e.g. [10, 16, 8, 7]). As far as

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we are aware, there is no convergence result for eigenvalue problems and the present paper is a first contribution to this topic.

We prove here the convergence of an adaptive linear finite element algorithm for computing eigenvalues and eigenvectors of scalar symmetric elliptic partial differential operators in bounded polygonal or polyhedral domains, subject to Dirichlet boundary data. Such problems arise in many applications, e.g. resonance problems, nuclear reactor criticality and the modelling of photonic band gap materials, to name but three.

Our refinement procedure is based on two locally defined quantities, firstly a standard a posteriori error estimator and secondly a measure of the variability (or "oscillation") of the computed eigenfunction. (Measures of "data oscillation" appear in the theory of adpativity for boundary value problems (e.g. [16]). In the eigenvalue problem the computed eigenvalue and eigenfunction on the present mesh plays the role of "data" for the next iteration of the adaptive procedure.) Our algorithm performs local refinement on all elements on which the minimum of these two local quantities is sufficiently large. We prove that the adaptive method converges provided the initial mesh is sufficiently fine. The latter condition, while absent for adaptive methods for linear symmetric elliptic boundary value problems, commonly appears for nonlinear problems and can be thought of as a manifestation of the nonlinearity of the eigenvalue problem.

The outline of the paper is as follows. In Section 2 we briefly describe the model elliptic eigenvalue problem and the numerical method and in Section 3 we describe a priori estimates, most of which are classical. Section 4 describes the a posteriori estimates and the adaptive algorithm. Section 5 proves that proceeding from one mesh to another ensures error reduction (up to oscillation of the computed eigenfunction) while the convergence result is presented in Section 6. Numerical experiments illustrating the theory are presented in Section 7.

2 Eigenvalue problem and numerical method

Throughout, Ω will denote a bounded domain in $\mathbb{R}^d$ ($d = 2$ or 3). In fact Ω will be assumed to be a polygon ($d = 2$) or polyhedron ($d = 3$). We will be concerned with the problem of finding an eigenvalue $\lambda \in \mathbb{R}$ and eigenfunction $0 \neq u \in H_0^1(\Omega)$ satisfying

$$a(u, v) := \lambda\, b(u, v)\ , \quad \text{for all} \quad v \in H_0^1(\Omega)\ , \tag{2.1}$$

where

$$a(u, v) = \int_\Omega \nabla u(x)^T \mathcal{A}(x) \nabla v(x) dx \quad \text{and} \quad b(u, v) = \int_\Omega \mathcal{B}(x) u(x) v(x) dx\ . \tag{2.2}$$

Here, the matrix-valued function $\mathcal{A}$ is required to be uniformly positive definite, i.e.

$$\underline{a}\ \leq\ \xi^T \mathcal{A}(x)\xi\ \leq\ \overline{a} \quad \text{for all} \quad \xi \in \mathbb{R}^d \quad \text{with} \quad |\xi| = 1 \quad \text{and all} \quad x \in \Omega. \tag{2.3}$$

The methods which we describe below can be extended to piecewise smooth coefficients $\mathcal{A}$, but to reduce technical detail in the proofs we will assume in fact that $\mathcal{A}$ is piecewise constant on Ω and that the jumps in $\mathcal{A}$ are aligned with the meshes $\mathcal{T}_n$ (introduced below), for all n. The scalar function $\mathcal{B}$ is required to be bounded above and below by positive constants for all $x \in \Omega$, i.e.

$$\underline{b}\ \leq\ \mathcal{B}(x)\ \leq\ \overline{b} \quad \text{for all} \quad x \in \Omega. \tag{2.4}$$

Throughout the paper, for any polygonal (polyhedral) subdomain of $D \subset \Omega$, and any $s \in [0,1]$, $\| \cdot \|_{s,D}$ and $| \cdot |_{s,D}$ will denote the standard norm and seminorm in the Sobolev space $H^s(D)$. Also $(\cdot,\cdot)_{0,D}$ denotes the $L_2(D)$ inner product. We also define the energy norm induced by the bilinear form a:

$$|||u\,|||_\Omega^2 := a(u,u) \quad \text{for all } u \in H_0^1(\Omega)\ ,$$

which, by (2.3), is equivalent to the $H^1(\Omega)$ seminorm. (The equivalence constant depends on the contrast $\overline{a}/\underline{a}$, but we are not concerned with this dependence in the present paper.) We also introduce the L_2 weighted norm:

$$\|u\|_{0,\mathcal{B},\Omega}^2 \;=\; b(u,u) \;=\; \int_\Omega \mathcal{B}(x)|u(x)|^2\,dx\ ,$$

and note the norm equivalence

$$\sqrt{\underline{b}}\|v\|_{0,\Omega} \le \|v\|_{0,\mathcal{B},\Omega} \le \sqrt{\overline{b}}\|v\|_{0,\Omega}\ . \tag{2.5}$$

Rewriting the eigenvalue problem (2.1) in standard normalised form, we seek $(\lambda, u) \in \mathbb{R} \times H_0^1(\Omega)$ such that

$$\left.\begin{array}{rcl} a(u,v) &=& \lambda\, b(u,v), \quad \text{for all} \quad v \in H_0^1(\Omega) \\ \|u\|_{0,\mathcal{B},\Omega} &=& 1 \end{array}\right\} \tag{2.6}$$

By the continuity of a and b and the coercivity of a on $H_0^1(\Omega)$ it is a standard result that (2.6) has a countable sequence of non-decreasing positive eigenvalues λ_j, $j = 1,2,\ldots$ with corresponding eigenfunctions $u_j \in H_0^1(\Omega)$ [3, 12, 21].

In this paper we will need some additional regularity for the eigenfunctions u_j, which will be achieved by making the following regularity assumption for the elliptic problem induced by a:

Assumption 2.1 *We assume that there exists a constant $C_{ell} > 0$ and $s \in [0,1]$ with the following property. For $f \in L_2(\Omega)$, if $v \in H_0^1(\Omega)$ solves the problem $a(v,w) = (f,w)_{0,\Omega}$ for all $w \in H_0^1(\Omega)$, then $\|v\|_{1+s,\Omega} \le C_{ell}\|f\|_{0,\Omega}$.*

Assumption 2.1 is satisfied with $s = 1$ when $\mathcal{A}$ is constant (or smooth) and Ω is convex. In a range of other practical cases $s \in (0,1)$, for example Ω non-convex (see [4]), or $\mathcal{A}$ having a discoutinuity across an interior interface (see [2]). Under Assumption 2.1 it follows that the eigenfunctions u_j of the problem (2.6) satisfy $\|u_j\|_{1+s,\Omega} \le C_{ell}\lambda_j\sqrt{\overline{b}}$.

To approximate problem (2.6) we use the continuous linear finite element method. Accordingly, let $\mathcal{T}_n$, $n = 1,2,\ldots$ denote a family of conforming triangular ($d = 2$) or tetrahedral ($d = 3$) meshes on Ω. Each mesh consists of elements denoted $\tau \in \mathcal{T}_n$. We assume that for each n, $\mathcal{T}_{n+1}$ is a refinement of $\mathcal{T}_n$. For a typical element τ of any mesh, its diameter is denoted H_τ and the diameter of its largest inscribed ball is denoted ρ_τ. For each n, let H_n denote the piecewise constant mesh function on Ω, whose value on each element $\tau \in \mathcal{T}_n$ is H_τ and let $H_n^{\max} = \max_{\tau\in\mathcal{T}_n} H_\tau$. Throughout we will assume that the family of meshes $\mathcal{T}_n$ is shape regular, i.e. there exists a constant C_{reg} such that

$$H_\tau \le C_{\text{reg}}\rho_\tau, \quad \text{for all} \quad \tau \in \mathcal{T}_n \quad \text{and all} \quad n = 1,2,\ldots\ . \tag{2.7}$$

In the later sections of the paper the $\mathcal{T}_n$ will be produced by an adaptive process which ensures shape regularity.

We let V_n denote the usual finite dimensional subspace of $H_0^1(\Omega)$, consisting of all continuous piecewise linear functions with respect to the mesh $\mathcal{T}_n$. Then the discrete formulation of problem (2.6) is to seek the eigenpairs $(\lambda_n, u_n) \in \mathbb{R} \times V_n$ such that

$$\left.\begin{array}{rcll} a(u_n, v_n) & = & \lambda_n\, b(u_n, v_n), & \text{for all} \quad v_n \in V_n \\ \|u_n\|_{0,\mathcal{B},\Omega} & = & 1\ . & \end{array}\right\} \tag{2.8}$$

The problem (2.8) has $N = \dim V_n$ positive eigenvalues (counted according to multiplicity) which we denote in non-decreasing order as $\lambda_{n,1} \leq \lambda_{n,2} \leq \ldots \leq \lambda_{n,N}$. As we shall see in §3, for each j, $\lambda_{n,j} \to \lambda_j$ as $H_n^{\max} \to 0$ and (by the minimax principle - see e.g. [21, §6.1]) the convergence of the $\lambda_{n,j}$ is monotone decreasing i.e.

$$\lambda_{n,j} \ \geq \ \lambda_{m,j} \ \geq \ \lambda_j\ , \quad \text{for all} \quad j = 1, \ldots, N, \quad \text{and all} \quad m \geq n\ . \tag{2.9}$$

Thus it is clear that there exists a *separation constant* $\rho > 0$ (depending on the spectrum of (2.6)) with the following property: If $\lambda_\ell = \lambda_{\ell+1} = \ldots = \lambda_{\ell+R-1}$ is any eigenvalue of (2.6) of multiplicity $R \geq 1$, then

$$\frac{\lambda_\ell}{|\lambda_{n,j} - \lambda_\ell|} \ \leq \ \rho, \qquad j \neq \ell, \ell+1,\ \ldots, \ell+R-1\ , \tag{2.10}$$

provided $H_n^{\max}$ is sufficiently small.

The a priori error analysis for our eigenvalue problem is classical (see, e.g. [3], [12] and [21]). In the next section we briefly recall some of the main known results and also prove a non-classical result (Theorem 3.2) which is essential to the proof of convergence of our adaptive scheme.

3 A priori analysis

First, manipulating the definitions of the eigenproblems (2.6) and (2.8), we obtain the important basic identity:

$$\begin{aligned} a(u_j - u_{n,j}, u_j - u_{n,j}) &= a(u_j, u_j) + a(u_{n,j}, u_{n,j}) - 2a(u_j, u_{n,j}) \\ &= \lambda_j + \lambda_{n,j} - 2\lambda_j\ b(u_j, u_{n,j}) \\ &= \lambda_{n,j} - \lambda_j + \lambda_j\ (2 - 2b(u_j, u_{n,j})) \\ &= \lambda_{n,j} - \lambda_j + \lambda_j\ b(u_j - u_{n,j}, u_j - u_{n,j})\ . \end{aligned} \tag{3.1}$$

Using this and (2.9), we obtain

$$|||u_j - u_{n,j}|||_\Omega^2 = |\lambda_j - \lambda_{n,j}| \ + \ \lambda_j\ \|u_j - u_{n,j}\|_{0,\mathcal{B},\Omega}^2\ . \tag{3.2}$$

The following theorem investigates the convergence of discrete eigenpairs. Although parts of it are very well-known, we do not know a suitable reference for all the results given below, so a brief proof is given for completeness. In the proof we make use of the orthogonal projection

Q_n of $H_0^1(\Omega)$ onto V_n with respect to the inner product induced by $a(\cdot,\cdot)$, which is defined by requiring, for each $u \in H_0^1(\Omega)$, $Q_n u \in V_n$ and

$$a(u - Q_n u, v_n) \;=\; 0 \quad \text{for all} \quad v_n \in V_n. \tag{3.3}$$

Note that it follows immediately from (2.1) that

$$a(Q_n u, v_n) \;=\; \lambda\, b(u, v_n) \quad \text{for all} \quad v_n \in V_n, \tag{3.4}$$

Theorem 3.1 *Let s be as given in Assumption 2.1. Then for all $1 \le j \le N$,*

(i)

$$|\lambda_j - \lambda_{n,j}| \;\le\; |||u_j - u_{n,j}|||_\Omega^2; \tag{3.5}$$

(ii) There is a constant $C_{adj} > 0$ such that

$$\|u_j - u_{n,j}\|_{0,\mathcal{B},\Omega} \;\le\; C_{adj}(H_n^{\max})^s\; |||u_j - Q_n u_j\;|||_\Omega \;\le\; C_{adj}(H_n^{max})^s\; |||u_j - u_{n,j}\;|||_\Omega; \tag{3.6}$$

(iii) For sufficiently small H_n^{max} there is a constant C_{spec} such that

$$|||u_j - u_{n,j}|||_\Omega \;\le C_{spec}(H_n^{max})^s\;. \tag{3.7}$$

The constants C_{adj} and C_{spec} both depend on the spectral information λ_l, u_l, $l = 1, \ldots, j$.

Proof. The estimate (3.5) follows directly from (3.2).

The proof of (3.6) is obtained by a reworking of the results in [21]. First the arguments in [21, Theorem 6.2] (extended to our case) show that

$$\|u_j - u_{n,j}\|_{0,\mathcal{B},\Omega} \le 2(1+\rho)\|u_j - Q_n u_j\|_{0,\mathcal{B},\Omega}, \tag{3.8}$$

with ρ defined in (2.10). In fact the proof in [21] is for simple eigenvalues, but it extends to the multiple eigenvalue case without difficulty (details are given in [11]).

Now, applying a standard argument from approximation theory, for any $v \in H^{1+s}(\Omega)$, there exists a function $v_n \in V_n$ such that

$$|v - v_n|_{1,\Omega} \le C_{app}(H_n^{max})^s |v|_{1+s,\Omega}, \tag{3.9}$$

where the constant C_{app} may also depend on the constant C_{reg}. Since by (3.4), $a(Q_n u_j, v_n) = b(\lambda_j u_j, v_n)$, it follows that $Q_n u_j$ is the finite element solution of the source problem $a(u^*, v) = b(\lambda_j u_j, v)$ with the exact solution $u^* = u_j$. The usual Aubin-Nitsche duality argument can be applied to obtain the L_2 convergence for $u_j - Q_n u_j$. Let us denote $e_{n,j} := u_j - Q_n u_j$ and let us define φ to be the solution of the linear problem

$$a(\varphi, w) \;=\; b(e_{n,j}, w) \quad \text{for all} \quad w \in H_0^1(\Omega). \tag{3.10}$$

We have

$$\|e_{n,j}\|_{0,\mathcal{B},\Omega}^2 = a(\varphi, e_{n,j}) = a(\varphi - v_n, e_{n,j}) \quad \text{for all} \quad v_n \in V_n,$$

where in the last step we used the orthogonality of $e_{n,j}$ to the space V_n. Then applying Cauchy-Schwarz we obtain

$$\|e_{n,j}\|^2_{0,\mathcal{B},\Omega} \leq \overline{a}\ |\varphi - v_n|_{1,\Omega}\ |e_{n,j}|_{1,\Omega}, \quad \text{for all} \quad v_n \in V_n. \tag{3.11}$$

Using (3.9) (together with the Assumption 2.1) in (3.11) we get

$$\begin{aligned}
\|e_{n,j}\|^2_{0,\mathcal{B},\Omega} &\leq \overline{a}\ C_{app}\ (H_n^{max})^s |\varphi|_{1+s,\Omega} |e_{n,j}|_{1,\Omega} \\
&\leq \overline{a}\ C_{app}\ C_{ell} (H_n^{max})^s \|\mathcal{B} e_{n,j}\|_{0,\Omega} |e_{n,j}|_{1,\Omega} \\
&\leq \overline{a}\sqrt{\overline{b}}\ C_{app}\ C_{ell} (H_n^{max})^s \|e_{n,j}\|_{0,\mathcal{B},\Omega} |e_{n,j}|_{1,\Omega}.
\end{aligned} \tag{3.12}$$

The last step of the argument consists of dividing both sides of (3.12) by $\|e_{n,j}\|_{0,\mathcal{B},\Omega}$ and applying the coercivity of the bilinear form $a(\cdot,\cdot)$

$$\|e_{n,j}\|_{0,\mathcal{B},\Omega} \leq \overline{a}\ \sqrt{\overline{b}}\ \underline{a}^{-1/2}\ C_{app} C_{ell} (H_n^{max})^s\ |||e_{n,j}\ |||_\Omega \tag{3.13}$$

Combining (3.8) and (3.13) we obtain

$$\|u_j - u_{n,j}\|_{0,\mathcal{B},\Omega} \leq C (H_n^{max})^s\ |||u_j - Q_n u_j\ |||_\Omega, \tag{3.14}$$

for a constant C in the required form.

The proof of (iii) for the simple eigenvalue case can be found in [21]. The extension to the multiple eigenvalue case (which is mentioned in [21]) is given in [11].

■

The next theorem is a generalisation to eigenvalue problems of the standard monotone convergence property for linear symmetric elliptic PDEs, namely that if one enriches the finite dimensional space, then the error is bound to decrease. This result fails to hold for eigenvalue problems (even for symmetric elliptic partial differential operators), because of the nonlinearity of such problems. The best that we can do is to show that if the finite dimensional space is enriched, then the error will not increase very much. This is the subject of Theorem 3.2.

Theorem 3.2 *For any $1 \leq j \leq N$, there exists a constant $q > 1$ such that, for $m \geq n$, the corresponding computed eigenpair $(\lambda_{m,j}, u_{m,j})$ satisfies:*

$$|||u_j - u_{m,j}\ |||_\Omega \leq q\ |||u_j - u_{n,j}\ |||_\Omega\ . \tag{3.15}$$

Proof. From Theorem 3.1 (ii), we obtain

$$\|u_j - u_{m,j}\|_{0,\mathcal{B},\Omega} \leq C_{adj} (H_m^{\max})^s\ |||u_j - Q_m u_j\ |||_\Omega \tag{3.16}$$

Since $\mathcal{T}_m$ is a refinement of $\mathcal{T}_n$, it follows that $V_n \subset V_m$ and so the best approximation property of Q_m ensures that

$$|||u_j - Q_m u_j\ |||_\Omega\ \leq |||u_j - Q_n u_j\ |||_\Omega\ .$$

Hence from (3.16) and using the fact that $H_m^{max} \le H_n^{max}$, we have

$$\|u_j - u_{m,j}\|_{0,\mathcal{B},\Omega} \;\le\; C_{adj}(H_n^{max})^s \;|||u_j - Q_n u_j\;|||_\Omega . \tag{3.17}$$

Combining with (3.2) and then using (2.9) we obtain

$$\begin{aligned} |||u_j - u_{m,j}\;|||_\Omega^2 \;&\le\; |\lambda_j - \lambda_{m,j}| \;+\; \lambda_j C_{adj}^2 (H_n^{max})^{2s}\;|||u_j - Q_n u_j\;|||_\Omega^2 \\ &\le\; |\lambda_j - \lambda_{n,j}| \;+\; \lambda_j C_{adj}^2 (H_n^{max})^{2s}\;|||u_j - Q_n u_j\;|||_\Omega^2 . \end{aligned} \tag{3.18}$$

Hence, from (3.5) we obtain

$$|||u_j - u_{m,j}\;|||_\Omega^2 \quad\le\quad |||u_j - u_{n,j}\;|||_\Omega^2 \;+\; \lambda_j C_{adj}^2 (H_n^{max})^{2s}\;|||u_j - Q_n u_j\;|||_\Omega^2 . \tag{3.19}$$

But since Q_n yields the best approximation in the energy norm, we have

$$|||u_j - u_{m,j}\;|||_\Omega^2 \;\le\; (1 + \lambda_j C_{adj}^2 (H_0^{max})^{2s})\;|||u_j - u_{n,j}\;|||_\Omega^2 \;, \tag{3.20}$$

which is in the required form. ■

Remark 3.3 *From now on we will be concerned with a true eigenpair* (λ_j, u_j) *and its approximation on the mesh* $\mathcal{T}_n$ $(\lambda_{j,n}, u_{j,n})$ *in the sense described in Theorem 3.1. So we can drop the subscript* j *and we simply write* (λ, u) *for the eigenpair of* (2.6) *and* (λ_n, u_n) *for the corresponding eigenpair of* (2.8)*.*

4 A posteriori analysis

This section contains our a posteriori error estimator and the definition of the mesh adaptivity algorithm for which convergence will be proved in the following sections.

Recalling the mesh sequence $\mathcal{T}_n$ defined above, we let $\mathcal{S}_n$ denote the set of all the edges (or the set of faces in 3D) of the elements of the mesh $\mathcal{T}_n$. For each $S \in \mathcal{S}_n$, we assume that we have already chosen a preorientated unit normal vector $\vec{n}_S$ and we denote by $\tau_1(S)$ and $\tau_2(S)$ the elements sharing S (i.e. $\tau_1(S) \cap \tau_2(S) = S$). In addition we write $\Omega(S) = \tau_1(S) \cup \tau_2(S)$. Elements, faces and edges are to be considered closed. Furthermore we denote the diameter of S by H_S.

Notation 4.1 *We write* $A \lesssim B$ *when* A/B *is bounded by a constant which may depend on the functions* $\mathcal{A}$ *and* $\mathcal{B}$ *in* (2.2)*,on* $\underline{a}$*,* $\overline{a}$*,* $\underline{b}$ *and* $\overline{b}$*, on* C_{ell} *in Assumption 2.1,* C_{reg} *in* (2.7) *and on* C_{app} *in (3.9). The notation* $A \cong B$ *means* $A \lesssim B$ *and* $A \gtrsim B$*.*

All the constants depending on the spectrum, namely ρ *in (2.10),* q *in (3.15),* C_{adj} *in (3.6) and* C_{spec} *in (3.7), are handled explicitly. Similarly all mesh size dependencies are explicit. Note that all eigenvalues of (2.6) satisfy* $\lambda_n \gtrsim 1$*, since* $\lambda_n \ge \lambda_1 = a(u_1, u_1) \gtrsim |u_1|_{1,\Omega}^2 \gtrsim \|u_1\|_{0,\Omega}^2 \gtrsim \|u_1\|_{0,\mathcal{B},\Omega}^2 = 1$*.*

The error estimator which we shall use is obtained by adapting the standard estimates for source problems to the eigenvalue problem. Analogous estimates for eigenvalue problems can be found in [5] (for the Laplace problem) and [22] (for linear elasticity). Related results are in [13].

For a function g, which is piecewise continuous on the mesh $\mathcal{T}_n$, we introduce its jump across an edge (face) $S \in \mathcal{S}_n$ by:

$$[g]_S(x) := \left(\lim_{\substack{\tilde{x}\in\tau_1(S)\\ \tilde{x}\to x}} g(\tilde{x}) - \lim_{\substack{\tilde{x}\in\tau_2(S)\\ \tilde{x}\to x}} g(\tilde{x}) \right), \quad \text{for } x \in \operatorname{int}(S).$$

Then for any function v with piecewise continuous gradient on $\mathcal{T}_n$ we define, for $S \in \mathcal{S}_n$

$$J_S(v)(x) := [\vec{n}_S \cdot \mathcal{A}\nabla v]_S(x), \quad \text{for} \quad x \in \operatorname{int}(S).$$

The error estimator η_n on the mesh $\mathcal{T}_n$ is defined as

$$\eta_n^2 := \sum_{S\in S_n} \eta_{S,n}^2 \ , \tag{4.1}$$

where each term $\eta_{S,n}$, which is the local contribution to the residual, is defined by

$$\eta_{S,n}^2 := \|H_n\lambda_n u_n\|_{0,\mathcal{B},\Omega(S)}^2 + \|H_S^{1/2} J_S(u_n)\|_{0,S}^2 \ . \tag{4.2}$$

The following lemma is proved, in a standard way, by adapting the usual arguments for source problems.

Lemma 4.2 (Reliability)

$$|||u - u_n\,|||_\Omega \ \lesssim \ \eta_n + G_n, \tag{4.3}$$

and

$$G_n := \frac{1}{2}(\lambda + \lambda_n)\frac{\|u - u_n\|_{0,\mathcal{B},\Omega}^2}{|||u - u_n\,|||_\Omega}. \tag{4.4}$$

Proof. To ease readability we set $e_n = u - u_n$ in the proof. Note first that, since (λ, u) and (λ_n, u_n) respectively solve the eigenvalue problems (2.1) and (2.8), we have, for all $w_n \in V_n$,

$$\begin{aligned}
|||e_n|||_\Omega^2 &= a(e_n, e_n) \\
&= a(e_n, e_n - w_n) \ + \ a(e_n, w_n) \\
&= a(e_n, e_n - w_n) \ + \ a(u, w_n) \ - \ a(u_n, w_n) \\
&= a(e_n, e_n - w_n) \ + \ b(\lambda u - \lambda_n u_n, w_n) \\
&= a(e_n, e_n - w_n) \ - \ b(\lambda u - \lambda_n u_n, e_n - w_n) \ + \ b(\lambda u - \lambda_n u_n, e_n) \ .
\end{aligned} \tag{4.5}$$

We shall estimate each of the three terms on the right-hand side of (4.5).

First, we deal with the first two of these terms. Using again (2.1), and then elementwise integration by parts (recall $\mathcal{A}$ is assumed constant on each element), we have for all $v \in H_0^1(\Omega)$,

$$
\begin{aligned}
a(e_n, v) - b(\lambda u - \lambda_n u_n, v) &= -a(u_n, v) \ + \ \lambda_n b(u_n, v) \\
&= -\sum_{\tau \in \mathcal{T}_n} \int_\tau (\mathcal{A}\nabla u_n).\nabla v \ + \ \lambda_n b(u_n, v) \\
&= -\sum_{S \in \mathcal{S}_n} \int_S [\vec{n}_S \cdot \mathcal{A}\nabla u_n]_S \ v \ + \ \lambda_n b(u_n, v) \\
&= -\sum_{S \in \mathcal{S}_n} \int_S J_S(u_n) v \ + \ \lambda_n b(u_n, v) \ . \qquad (4.6)
\end{aligned}
$$

Hence for all $w_n \in V_n$,

$$
a(e_n, e_n - w_n) \ - \ b(\lambda u - \lambda_n u_n, e_n - w_n) = -\sum_{S \in \mathcal{S}_n} \int_S J_S(u_n)(e_n - w_n) \ + \ \lambda_n b(u_n, e_n - w_n). \quad (4.7)
$$

Now we choose $w_n = I_n e_n$ where I_n is the Scott-Zhang quasi-interpolation operator. In [19] it is shown that, for all $v \in H^1(\Omega)$,

$$
\|v - I_n v\|_{0,\tau} \lesssim H_n |v|_{1,\omega_\tau}, \qquad (4.8)
$$

$$
\|v - I_n v\|_{0,S} \lesssim H_S^{\frac{1}{2}} |v|_{1,\omega_S}, \qquad (4.9)
$$

where ω_τ is the union of all elements sharing at least a point with τ, and ω_S is the union of all elements sharing at least a point with S. Now substituting $w_n = I_n e_n$ in (4.7) and using Cauchy-Schwarz, together with the inequalities (4.8) and (4.9), we obtain:

$$
a(e_n, e_n - w_n) \ - \ b(\lambda u - \lambda_n u_n, e_n - w_n) \lesssim \ \eta_n |||e_n|||_\Omega. \qquad (4.10)
$$

Finally, to deal with the third term in (4.5), we simply observe that due to the normalisation in each of the eigenvalue problems (2.1) and (2.8) we have

$$
b(\lambda u - \lambda_n u_n, e_n) = (\lambda + \lambda_n)(1 - b(u, u_n)) = \frac{1}{2}(\lambda + \lambda_n)\|e_n\|^2_{0,\mathcal{B},\Omega}. \qquad (4.11)
$$

Now, combine (4.10) and (4.11) with (4.5) and divide by $|||e_n|||_\Omega$ to obtain the result. ■

Remark 4.3 *We shall see below that G_n defined above constitutes a "higher order term".*

For mesh refinement based on the local contributions to η_n, we use the same marking strategy as in [10] [16]. The idea is to refine a subset of the elements of $\mathcal{T}_n$ whose side residuals sum up to a fixed proportion of the total residual η_n.

Definition 4.4 (Marking Strategy 1) *Given a parameter $0 < \theta < 1$, the procedure is: mark the sides in a minimal subset $\hat{\mathcal{S}}_n$ of $\mathcal{S}_n$ such that*

$$
\left(\sum_{S \in \hat{\mathcal{S}}_n} \eta^2_{S,n} \right)^{1/2} \geq \theta \eta_n \ . \qquad (4.12)
$$

To satisfy the condition (4.12), we need first of all to compute all the "local residuals" $\eta_{S,n}$ and sort them according their values. Then the edges (faces) S are inserted into $\hat{\mathcal{S}}_n$ in decreasing order of $\eta_{S,n}$, starting from the edge (face) with the biggest local residual, until the condition (4.12) is satisfied. Note that a minimal subset $\hat{\mathcal{S}}_n$ may not be unique. Then we construct another set $\hat{\mathcal{T}}_n$, containing all the elements of $\mathcal{T}_n$ which share at least one edge (face) $S \in \hat{\mathcal{S}}_n$.

In order to prove the convergence of the adaptive method, we require an additional marking strategy, which will be defined in Definition 4.6 below. The latter marking strategy is driven by oscillations. The same argument has been already used in many papers about convergence for source problems (see [10], [16], [15], [8] and [7]), but to our knowledge has not yet been used for analysing convergent algorithms for eigenvalue problems.

The concept of "oscillation" is just a measure of how well a function may be approximated by piecewise constants on a particular mesh. For any function $v \in L_2(\Omega)$, and any mesh $\mathcal{T}_n$, we introduce its orthogonal projection $P_n v$ onto piecewise constants defined by:

$$(P_n v)|_\tau = \frac{1}{|\tau|} \int_\tau v_n, \quad \text{for all} \quad \tau \in \mathcal{T}_n. \tag{4.13}$$

Then we make the definition:

Definition 4.5 (Oscillations) *On a mesh $\mathcal{T}_n$, we define*

$$\operatorname{osc}(v, \mathcal{T}_n) := \|H_n(v - P_n v)\|_{0,\mathcal{B},\Omega}. \tag{4.14}$$

Note that

$$\operatorname{osc}(v, \mathcal{T}_n) = \left(\sum_{\tau \in \mathcal{T}_n} H_\tau^2 \|v - P_n v\|_{0,\mathcal{B},\tau}^2 \right)^{1/2}.$$

and that (by standard approximation theory and the ellipticity of $a(\cdot,\cdot)$),

$$\operatorname{osc}(v, \mathcal{T}_n) \ \lesssim \ (H_n^{max})^2 |||v|||_\Omega \ , \quad \text{for all} \quad v \in H_0^1(\Omega) \ . \tag{4.15}$$

The second marking strategy (introduced below) aims to reduce the oscillations corresponding to a particular approximate eigenfunction u_n.

Definition 4.6 (Marking Strategy 2) *Given a parameter $0 < \tilde{\theta} < 1$: mark the sides in a minimal subset $\tilde{\mathcal{T}}_n$ of $\mathcal{T}_n$ such that*

$$\operatorname{osc}(u_n, \tilde{\mathcal{T}}_n) \ \geq \ \tilde{\theta} \operatorname{osc}(u_n, \mathcal{T}_n) \ . \tag{4.16}$$

Note that a minimal subset $\tilde{\mathcal{T}}_n$ may not be unique. To satisfy the condition (4.16), we need first of all to compute all the local terms $H_\tau^2 \|(u_n - P_n u_n)\|_{0,\mathcal{B},\tau}^2$ forming $\operatorname{osc}(u_n, \mathcal{T}_n)$ and sort them according their values. Then the elements τ are inserted into $\tilde{\mathcal{T}}_n$ in decreasing order of the size of those local terms, until the condition (4.16) is satisfied.

Our adaptive algorithm can then be stated:

Algorithm 1 Converging algorithm

Require: $0 < \theta < 1$
Require: $0 < \tilde{\theta} < 1$
loop
Solve the Problem (2.8) for (λ_n, u_n)
Mark the elements using the first marking strategy (Definition 4.4)
Mark any additional unmarked elements using the second marking strategy (Definition 4.6)
Refine the mesh $\mathcal{T}_n$ and construct $\mathcal{T}_{n+1}$
end loop

In 2D at the $n-th$ iteration in Algorithm 1 each element in the set $\hat{\mathcal{T}}_n \cup \tilde{\mathcal{T}}_n$ is refined using the "bisection5" algorithm (see, e.g. [16]), as illustrated in Figure 1c. An advantage of this technique is the creation of a new node in the middle of each marked side in $\hat{\mathcal{S}}_n$ and also a new node in the interior of each marked element.

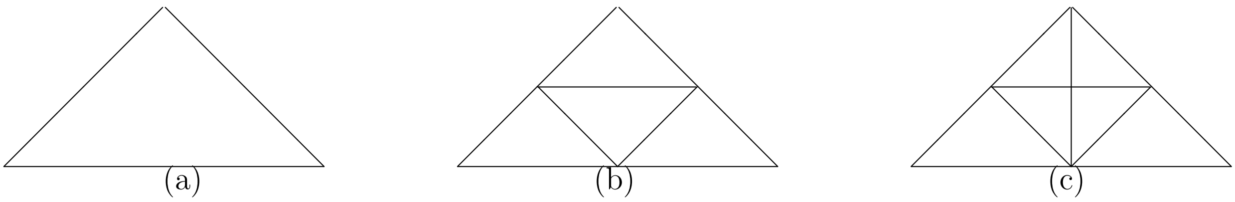

Figure 1: The refinement procedure applied to an element of the mesh. In (a) the element before the refinement, in (b) after the three sides as been refined and in (c) after the bisection of one of the three new segments.

In the 3D-case we use a suitable refinement that creates a new node on each marked face in $\hat{\mathcal{S}}_n$ and a node in the interior of each marked element. These requirements are analogous to the requirements satisfied by bisection5 in 2D-case.

In [16] and [15] it has been shown for linear source problems that the reduction of the error, as the mesh is refined, is triggered by the decay of data oscillations on the sequence of constructed meshes. This is if we are solving $a(u,v) = (f,v)_{0,\Omega}$ for a given function f ("the data"), then reduction of error is achieved when the oscillation of f on the adaptively refined meshes is sufficiently small.

For the eigenvalue problem (2.1) the quantity λu plays the role of data and in principle we have to ensure that oscillations of this quantity (or more precisely of its finite element approximation $\lambda_n u_n$), are sufficiently small. However $\lambda_n u_n$ may change if the mesh changes and so the proof of error reduction for eigenvalue problems is not as simple as it is for linear source problems. This is the essence of the theoretical problems solved in this paper.

5 Error Reduction

In this section we give the proof of error reduction for Algorithm 1. The proof has been inspired by the corresponding theory for source problems in [16]. However the nonlinearity of the eigenvalue problem introduces new complications and there are several lemmas before the main theorem (Theorem 5.5).

For the rest of the section let (λ_n, u_n) be an approximate eigenpair on a mesh $\mathcal{T}_n$, let $\mathcal{T}_{n+1}$ be the mesh obtained by one iteration of Algorithm 1 and let (λ_{n+1}, u_{n+1}) be the corresponding eigenpair in the sense made precise in Remark 3.3.

The first lemma is similar to [16, Lemma 4.2] for the 2D case. The extension of this lemma to the 3D case is treated in Remark 5.2.

Lemma 5.1 *Consider the 2D case. Let $\hat{\mathcal{S}}_n$ be as defined in Definition 4.4 and let P_n be as defined in (4.13). For any $S \in \hat{\mathcal{S}}_n$, there exists a function $\Phi_S \in V_{n+1}$ such that* $\operatorname{supp}(\Phi_S) = \Omega(S)$ *and also*

$$\lambda_n \int_{\Omega_S} \mathcal{B}(P_n u_n)\Phi_S - \int_S J_S(u_n)\Phi_S \;=\; \lambda_n \|H_n P_n u_n\|^2_{0,\mathcal{B},\Omega(S)} + \|H_S^{1/2} J_S(u_n)\|^2_{0,S}, \tag{5.1}$$

and

$$|||\Phi_S|||^2_{\Omega(S)} \;\lesssim\; \|H_n P_n u_n\|^2_{0,\mathcal{B},\Omega(S)} + \|H_S^{1/2} J_S(u_n)\|^2_{0,S}. \tag{5.2}$$

Proof.

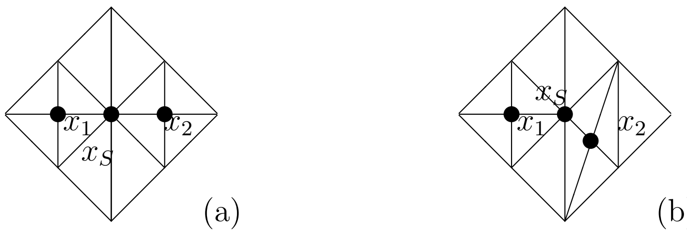

Figure 2: Two cases of refined couples of elements .

Figure 2 illustrates two possible configurations of the domain Ω_S: in Figure 2a we have that both the green-refinements are applied to the shared edge, while Figure 2b shows the case where the green-refinements are applied to different edges. The point x_S is the node created by the red-refinement in the middle of the shared edge S while the points x_1 and x_2 are the nodes created in the interior of the refined elements $\tau_1(S)$ and $\tau_2(S)$ respectively.

The two situations in Figure 2 do not exhaust all the possible configurations for couples of adjacent refined elements. There could be other possible configurations different from Figure 2b, in which the green-refinements are applied to different edges. However, the way in which the green-refinements split the elements is irrelevant for the proof, since the only important thing is the existence of an new node on the shared edge and two nodes in the interior of the elements. So, we choose Figure 2 just to illustrate the possibility in practise that the green-refinements could be applied or could not be applied to the same edge.

We then define

$$\Phi_S := \alpha_S\varphi_S + \beta_1\varphi_1 + \beta_2\varphi_2, \tag{5.3}$$

where φ_S and φ_i are the nodal basis functions associated with the points x_S and x_i on $\mathcal{T}_{n+1}$, and α_S, β_i are defined by

$$\alpha_S = \begin{cases} -\dfrac{\|H_S^{1/2}J_S(u_n)\|_{0,S}^2}{\int_S J_S(u_n)\varphi_S} & \text{if } J_S(u_n) \neq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{5.4}$$

and

$$\beta_i = \begin{cases} \dfrac{\|H_nP_nu_n\|_{0,\mathcal{B},\tau_i(S)}^2 - \alpha_S\int_{\tau_i(S)}\mathcal{B}(P_nu_n)\varphi_S}{\int_{\tau_i(S)}\mathcal{B}(P_nu_n)\varphi_i} & \text{if } P_nu_n|_{\tau_i(S)} \neq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{5.5}$$

for $i = 1, 2$.

Using the fact that $\text{supp}(\varphi_i) = \tau_i(S)$, for $i = 1, 2$ we can easily see that the above formulae imply

$$\alpha_S\int_S J_S(u_n)\varphi_S = -\|H_S^{1/2}J_S(u_n)\|_{0,S}^2, \tag{5.6}$$

$$\int_{\Omega(S)}\mathcal{B}(P_nu_n)(\alpha_S\varphi_S + \beta_1\varphi_1 + \beta_2\varphi_2) = \|H_nP_nu_n\|_{0,\mathcal{B},\Omega(S)}^2, \tag{5.7}$$

(and that these formulae remain true even if $J_S(u_n)$ or $P_nu_n|_{\tau_i(S)}$ vanish). Hence

$$\lambda_n\int_{\Omega(S)}\mathcal{B}(P_nu_n)\Phi_S - \int_S J_S(u_n)\Phi_S = \lambda_n\int_{\Omega(S)}\mathcal{B}(P_nu_n)(\alpha_S\varphi_S+\beta_1\varphi_1+\beta_2\varphi_2) - \int_S J_S(u_n)\alpha_S\varphi_S$$

and (5.1) follows immediately on using (5.6) and (5.7).

To prove (5.2), use (5.3), and the fact that $|\varphi_S|_{1,\Omega(S)} \lesssim 1$ and $|\varphi_i|_{1,\Omega(S)} \lesssim 1$ to obtain

$$|||\Phi_S|||_{\Omega(S)}^2 \lesssim |\alpha_S|^2 + |\beta_1|^2 + |\beta_2|^2 . \tag{5.8}$$

Now, since $J_S(u_n)$ is constant on S and $\int_S \varphi_S \sim H_S$, we have

$$|\alpha_S| \lesssim \frac{|J_S(u_n)|\|H_S^{1/2}\|_{0,S}^2}{H_S} \lesssim |J_S(u_n)|H_S \sim \|H_S^{1/2}J_S(u_n)\|_{0,S} . \tag{5.9}$$

Also since P_nu_n is constant on each $\tau_i(S)$ and since $\int_{\tau_i(S)}\mathcal{B}\phi_i \sim H_{\tau_i(S)}^2$, we have

$$\begin{aligned} |\beta_i| &\lesssim \frac{|P_nu_n|_{\tau_i(S)}|\ \|H_n\|_{0,\mathcal{B},\tau_i(S)}^2 + |\alpha_S|H_{\tau_i(S)}^2}{H_{\tau_i(S)}^2} \\ &\lesssim |P_nu_n|_{\tau_i(S)}|\ H_{\tau_i(S)}^2 + |\alpha_S| \sim \|H_nP_nu_n\|_{0,\mathcal{B},\tau_i(S)} + |\alpha_S| \end{aligned}$$

This implies

$$|\beta_i|^2 \lesssim \|H_n P_n u_n\|^2_{0,\mathcal{B},\tau_i(S)} + |\alpha_S|^2 \lesssim \|H_n P_n u_n\|^2_{0,\mathcal{B},\tau_i(S)} + \|H_S^{1/2} J_S(u_n)\|^2_{0,S} , \qquad (5.10)$$

and the proof is completed by combining (5.8) with (5.9) and (5.10). ∎

Remark 5.2 *To extend the results in Lemma 5.1 to the 3D-case we need to use a refinement procedure for tetrahedra that creates a new node on each marked face in $\hat{\mathcal{S}}_n$ and a node in the interior of each marked element. The proof in the 3D-case is similar to the proof in the 2D-case: for each couple of refined elements we define*

$$\Phi_S := \alpha_S \varphi_S + \beta_1 \varphi_1 + \beta_2 \varphi_2,$$

where φ_S is the nodal basis function associated to the new node on the shared face and φ_i are the nodal basis functions associated to the new nodes in the interior of the elements. The coefficients α_S, β_1 and β_2 can be chosen in the same way as in Lemma 5.1 and the rest of the proof goes on similarly.

In the next lemma we bound the local error estimator above by the local difference of two discrete solutions coming from consecutive meshes, plus higher order terms. This kind of result is called "discrete local efficiency" by many authors.

Recall that $\mathcal{T}_{n+1}$ is the refinement of $\mathcal{T}_n$ obtained by applying Algorithm 1.

Lemma 5.3 *For any $S \in \hat{\mathcal{S}}_n$, we have*

$$\begin{aligned}\eta^2_{S,n} \lesssim \; & |||u_{n+1} - u_n|||^2_{\Omega(S)} + \|H_n(\lambda_{n+1}u_{n+1} - \lambda_n P_n u_n)\|^2_{0,\mathcal{B},\Omega(S)} \\ & + \|H_n \lambda_n (u_n - P_n u_n)\|^2_{0,\mathcal{B},\Omega(S)} .\end{aligned} \qquad (5.11)$$

Proof. Since the function Φ_S defined in Lemma 5.1 is in V_{n+1} and $\mathrm{supp}(\Phi_S) = \Omega(S)$, we have

$$a(u_{n+1} - u_n, \Phi_S) = a(u_{n+1}, \Phi_S) - a(u_n, \Phi_S) = \lambda_{n+1} \int_{\Omega_S} \mathcal{B} u_{n+1} \Phi_S - a(u_n, \Phi_S). \qquad (5.12)$$

Now applying integration by parts to the last term on the right-hand side of (5.12), we obtain

$$a(u_{n+1} - u_n, \Phi_S) = \lambda_{n+1} \int_{\Omega_S} \mathcal{B} u_{n+1} \Phi_S - \int_S J_S(u_n) \Phi_S. \qquad (5.13)$$

Combining (5.13) with (5.1), we obtain

$$\begin{aligned} a(u_{n+1} - u_n, \Phi_S) - \int_{\Omega_S} \mathcal{B}(\lambda_{n+1}u_{n+1} - \lambda_n P_n u_n)\Phi_S & \\ = \lambda_n \int_{\Omega_S} \mathcal{B}(P_n u_n)\Phi_S - \int_S J_S(u_n)\Phi_S & \\ = \lambda_n \|H_n P_n u_n\|^2_{0,\mathcal{B},\Omega(S)} + \|H_S^{1/2} J_S(u_n)\|^2_{0,S}. & \qquad (5.14) \end{aligned}$$

Rearranging this, and then applying the triangle and Cauchy-Schwarz inequalities, we obtain

$$\begin{aligned}
&\|H_n\lambda_n P_n u_n\|^2_{0,\mathcal{B},\Omega(S)} + \|H_S^{1/2} J_S(u_n)\|^2_{0,S} \\
&\quad\leq\ |a(u_{n+1}-u_n,\Phi_S)| + \left|\int_{\Omega(S)} \mathcal{B}(\lambda_{n+1}u_{n+1} - \lambda_n P_n u_n)\Phi_S\right| \\
&\quad\leq\ |||u_{n+1}-u_n|||_{\Omega(S)}|||\Phi_S|||_{\Omega(S)} + \|\lambda_{n+1}u_{n+1} - \lambda_n P_n u_n\|_{0,\mathcal{B},\Omega(S)}\|\Phi_S\|_{0,\mathcal{B},\Omega(S)} \\
&\quad\lesssim\ \Big(|||u_{n+1}-u_n|||_{\Omega(S)} + \|H_n(\lambda_{n+1}u_{n+1} - \lambda_n P_n u_n)\|_{0,\mathcal{B},\Omega(S)}\Big)|||\Phi_S|||_{\Omega(S)}\ . \qquad (5.15)
\end{aligned}$$

In the final step of (5.15) we made use of the Poincaré inequality $\|\Phi_S\|_{0,\mathcal{B},\Omega_S} \lesssim H_S|||\Phi_S|||_{\Omega(S)}$ and also the shape-regularity of the meshes. In view of (5.2), the fact that $\lambda_n \geq \lambda_1 \gtrsim 1$ (see Notation 4.1) yields

$$\begin{aligned}
&\|H_n\lambda_n P_n u_n\|^2_{0,\mathcal{B},\Omega(S)} + \|H_S^{1/2} J_S(u_n)\|^2_{0,S} \\
&\qquad\lesssim\ \Big(|||u_{n+1}-u_n|||_{\Omega(S)} + \|H_n(\lambda_{n+1}u_{n+1} - \lambda_n P_n u_n)\|_{0,\mathcal{B},\Omega(S)}\Big)^2 \\
&\qquad\lesssim\ |||u_{n+1}-u_n|||^2_{\Omega(S)} + \|H_n(\lambda_{n+1}u_{n+1} - \lambda_n P_n u_n)\|^2_{0,\mathcal{B},\Omega(S)}. \qquad (5.16)
\end{aligned}$$

From the definition of $\eta_{S,n}$ in (4.2), and the triangle inequality, we have

$$\eta^2_{S,n}\ \lesssim\ \|H_n\lambda_n P_n u_n\|^2_{0,\mathcal{B},\Omega(S)} + \|H_S^{1/2} J_S(u_n)\|^2_{0,S} + \|H_n\lambda_n(u_n - P_n u_n)\|^2_{0,\mathcal{B},\Omega(S)}. \qquad (5.17)$$

The required inequality (5.11) now follows from (5.16) and (5.17). ■

In the main result of this section, Theorem 5.5 below, we will be interested in achieving an error reduction result of the form $|||u-u_{n+1}|||_\Omega \leq \alpha|||u-u_n|||_\Omega$ for some $\alpha < 1$. In the case of source problems (e.g. [16]) this is approached by writing

$$\begin{aligned}
|||u-u_n\ |||^2_\Omega &= |||u-u_{n+1}+u_{n+1}-u_n\ |||^2_\Omega \\
&= |||u-u_{n+1}\ |||^2_\Omega + |||u_{n+1}-u_n\ |||^2_\Omega + 2a(u-u_{n+1}, u_{n+1}-u_n).
\end{aligned} \qquad (5.18)$$

and making use of the fact that the last term on the right-hand side vanishes due to Galerkin orthogonality. However this approach is not available to us in the eigenvalue problem. Therefore a more technical approach is needed to bound the two terms on the right-hand side of (5.18) from below. The main technical result is in the following lemma. Recall the convention in Notation 4.1.

Lemma 5.4

$$|||u_{n+1}-u_n|||^2_\Omega\ \gtrsim\ \theta^2|||u-u_n|||^2_\Omega - \mathrm{osc}(\lambda_n u_n, \mathcal{T}_n)^2 - L_n^2\ , \qquad (5.19)$$

where θ is defined in the marking strategy in Definition 4.4 and L_n satisfies the estimate:

$$L_n\ \lesssim\ \hat{C}(H_n^{max})^s|||u-u_n|||_\Omega\ , \qquad (5.20)$$

where $\hat{C}$ depends on θ, λ, C_{spec}, C_{adj} and q.

Proof. By Lemma 5.3 and Definition 4.4 we have

$$\theta^2\eta_n^2 \quad \leq \quad \sum_{S\in\hat{\mathcal{S}}_n} \eta_{S,n}^2$$

$$\lesssim |||u_{n+1} - u_n\ |||_\Omega^2 + \|H_n(\lambda_{n+1}u_{n+1} - \lambda_n P_n u_n)\|_{0,\mathcal{B},\Omega}^2 + \mathrm{osc}(\lambda_n u_n \mathcal{T}_n)^2\ .$$

Hence, rearranging and making use of Lemma 4.2, we have

$$\begin{aligned} |||u_{n+1} - u_n|||_\Omega^2 \quad &\gtrsim \quad \theta^2\eta_n^2 - \|H_n(\lambda_{n+1}u_{n+1} - \lambda_n P_n u_n)\|_{0,\mathcal{B},\Omega}^2 - \mathrm{osc}(\lambda_n u_n \mathcal{T}_n)^2 \\ &\gtrsim \quad \theta^2|||u - u_n|||_\Omega^2 - \mathrm{osc}(\lambda_n u_n \mathcal{T}_n)^2 \\ &\qquad -\theta^2 G_n^2 - \|H_n(\lambda_{n+1}u_{n+1} - \lambda_n P_n u_n)\|_{0,\mathcal{B},\Omega}^2\ . \end{aligned} \tag{5.21}$$

We now estimate the last two terms in (5.21) separately.

To estimate G_n, we use (4.4), combined with the Poincaré inequality (and the H^1 - ellipticity of $a(\cdot,\cdot)$) and then Theorem 3.1(ii) to obtain

$$G_n \ \lesssim\ \frac{1}{2}(\lambda + \lambda_n)\|u - u_n\|_{0,\mathcal{B},\Omega} \ \lesssim\ \frac{1}{2}(\lambda + \lambda_n) C_{adj} (H_n^{max})^s |||u - u_n|||_\Omega. \tag{5.22}$$

To estimate the last term in (5.21), we first use the triangle inequality to obtain

$$\|H_n(\lambda_{n+1}u_{n+1} - \lambda_n P_n u_n)\|_{0,\mathcal{B},\Omega} \ \leq\ \|H_n(\lambda_{n+1}u_{n+1} - \lambda_n u_n)\|_{0,\mathcal{B},\Omega} \ +\ \mathrm{osc}(\lambda_n u_n, \mathcal{T}_n). \tag{5.23}$$

For the first term on the right-hand side of (5.23), we have

$$\|H_n(\lambda_{n+1}u_{n+1} - \lambda_n u_n)\|_{0,\mathcal{B},\Omega} \leq H_n^{max}\big(\|\lambda u - \lambda_{n+1}u_{n+1}\|_{0,\mathcal{B},\Omega} + \|\lambda u - \lambda_n u_n\|_{0,\mathcal{B},\Omega}\big). \tag{5.24}$$

Then, recalling (2.6) and Theorem 3.1, we obtain

$$\begin{aligned} \|\lambda u - \lambda_{n+1}u_{n+1}\|_{0,\mathcal{B},\Omega} \quad &\leq \quad |\lambda - \lambda_{n+1}|\|u\|_{0,\mathcal{B},\Omega} \ +\ \lambda_{n+1}\|u - u_{n+1}\|_{0,\mathcal{B},\Omega} \\ &\leq \quad |||u - u_{n+1}|||_\Omega^2 \ +\ \lambda_{n+1} C_{adj}(H_n^{max})^s |||u - u_{n+1}|||_\Omega\ . \end{aligned} \tag{5.25}$$

Using Theorem 3.1 again and then Theorem 3.2, this implies

$$\begin{aligned} \|\lambda u - \lambda_{n+1}u_{n+1}\|_{0,\mathcal{B},\Omega} \quad &\lesssim \quad (C_{spec} + \lambda_{n+1}C_{adj})(H_n^{max})^s |||u - u_{n+1}|||_\Omega \\ &\leq \quad q(C_{spec} + \lambda_{n+1}C_{adj})(H_n^{max})^s |||u - u_n|||_\Omega\ . \end{aligned} \tag{5.26}$$

An identical argument shows

$$\|\lambda u - \lambda_n u_n\|_{0,\mathcal{B},\Omega} \ \lesssim\ (C_{spec} + \lambda_n C_{adj})(H_n^{max})^s |||u - u_n|||_\Omega\ . \tag{5.27}$$

Combining (5.26) and (5.27) with (5.24), we obtain

$$\|H_n(\lambda_{n+1}u_{n+1} - \lambda_n u_n)\|_{0,\mathcal{B},\Omega} \ \lesssim\ (1+q)(C_{spec} + \lambda_n C_{adj})(H_n^{max})^{s+1} |||u - u_n|||_\Omega\ . \tag{5.28}$$

Now combining (5.28) with (5.21), (5.22) and (5.23) we obtain the result.

■

The next theorem contains the main result of this section. It shows that provided we start with a "fine enough" mesh $\mathcal{T}_n$, the mesh adaptivity algorithm will reduce the error in the energy norm.

Theorem 5.5 (Error reduction) *For each $\theta \in (0,1)$, exists a sufficiently fine mesh threshold H_{max}^n and constants $\mu > 0$ and $\alpha \in (0,1)$ (all of which may depend on θ and on the eigenvalue λ), with the following property. For any $\varepsilon > 0$ the inequality*

$$\operatorname{osc}(\lambda_n u_n, \mathcal{T}_n) \le \mu\varepsilon, \tag{5.29}$$

implies either $|||u - u_n|||_\Omega \le \varepsilon$ or

$$|||u - u_{n+1}|||_\Omega \ \le\ \alpha |||u - u_n|||_\Omega \ ,$$

where the constant α may depend also on the parameter θ and on λ.

Proof.

In view of the equation (5.18) and remembering that $u_{n+1} - u_n \in V_{n+1}$ we have

$$\begin{aligned} |||u - u_n\ |||_\Omega^2 - \ |||u - u_{n+1}\ |||_\Omega^2 \ &= |||u_{n+1} - u_n\ |||_\Omega^2 + 2a(u - u_{n+1}, u_{n+1} - u_n) \\ &= |||u_{n+1} - u_n\ |||_\Omega^2 + 2b(\lambda u - \lambda_{n+1} u_{n+1}, u_{n+1} - u_n). \end{aligned}$$

Now using Cauchy-Schwarz and the Young inequality $2ab \le \frac{1}{4}a^2 + 4b^2$ on the second term on the right hand side we get

$$\begin{aligned} |||u - u_n\ |||_\Omega^2 - \ |||u - u_{n+1}\ |||_\Omega^2 \ &\ge |||u_{n+1} - u_n\ |||_\Omega^2 \ - \ 2\|\lambda u - \lambda_{n+1}u_{n+1}\|_{0,\mathcal{B},\Omega}\|u_{n+1} - u_n\|_{0,\mathcal{B},\Omega} \\ &\ge |||u_{n+1} - u_n\ |||_\Omega^2 - \frac{1}{4}\|u_{n+1} - u_n\|_{0,\mathcal{B},\Omega}^2 - 4\|\lambda u - \lambda_{n+1}u_{n+1}\|_{0,\mathcal{B},\Omega}^2 \\ &\ge \ \frac{3}{4}\ |||u_{n+1} - u_n\ |||_\Omega^2 - 4\|\lambda u - \lambda_{n+1}u_{n+1}\|_{0,\mathcal{B},\Omega}^2. \end{aligned} \tag{5.30}$$

Hence

$$|||u - u_{n+1}|||_\Omega^2 \ \le\ |||u - u_n|||_\Omega^2 - \frac{3}{4}|||u_{n+1} - u_n|||_\Omega^2 + 4\|\lambda u - \lambda_{n+1}u_{n+1}\|_{0,\mathcal{B},\Omega}^2 \ .$$

Applying Lemma 5.4 we obtain

$$\begin{aligned} |||u - u_{n+1}|||_\Omega^2 \ \lesssim\ & \left(1 - \frac{3}{4}\theta^2 + \hat{C}^2 (H_n^{max})^{2s}\right) |||u - u_n|||_\Omega^2 \\ & + 4\|\lambda u - \lambda_{n+1}u_{n+1}\|_{0,\mathcal{B},\Omega}^2 \\ & + \operatorname{osc}(\lambda_n u_n, \mathcal{T}_n)^2 \end{aligned}$$

Then making use of (5.26) we have

$$|||u - u_{n+1}|||_\Omega^2 \ \lesssim\ \beta_n\ |||u - u_n|||_\Omega^2 \ +\ \operatorname{osc}(\lambda_n u_n, \mathcal{T}_n)^2. \tag{5.31}$$

with

$$\beta_n := \left[1 - \frac{3}{4}\theta^2 + \big(q^2(C_{spec} + \lambda_n C_{adj})^2 + \hat{C}^2\big)(H_n^{max})^{2s}\right]. \tag{5.32}$$

Note that H_n^{max} can be chosen sufficiently small so that $\beta_m \le \beta < 1$ for all $m \ge n$.

Consider now the consequences of the inequality (5.29). If $|||u - u_n|||_\Omega > \varepsilon$ then (5.31) implies

$$|||u - u_{n+1}|||_\Omega^2 \;\le\; [\beta + \mu^2]\,|||u - u_n|||_\Omega^2 \,.$$

Now choose μ small enough so that

$$\alpha := (\beta + \mu^2)^{1/2} < 1 \tag{5.33}$$

to complete the proof. ∎

6 Proof of convergence

The main result of this paper is Theorem 6.2 below which proves convergence of the adaptive method and also demonstrates the decay of oscillations of the sequence of approximate eigenfunctions. Before proving this result we need a final lemma.

Lemma 6.1 *There exists a constant $\tilde{\alpha} \in (0, 1)$ such that*

$$\mathrm{osc}(u_{n+1}, \mathcal{T}_{n+1}) \;\le\; \tilde{\alpha}\,\mathrm{osc}(u_n, \mathcal{T}_n) \;+\; (1+q)(H_n^{max})^2\,|||u - u_n\,|||_\Omega. \tag{6.1}$$

Proof. First recall that one of the key results in [16] is the proof that the oscillations of any fixed function $v\ H_0^1(\Omega)$ are reduced by applying one refinement based on Marking Strategy 2 (Definition 4.6). Thus we have (in view of Algorithm 1):

$$\mathrm{osc}(u_n, \mathcal{T}_{n+1}) \;\le\; \tilde{\alpha}\,\mathrm{osc}(u_n, \mathcal{T}_n), \tag{6.2}$$

where $0 < \tilde{\alpha} < 1$ is independent of u_h. Thus, a simple application of the triangle inequality combined with (6.2) yields

$$\begin{aligned}\mathrm{osc}(u_{n+1}, \mathcal{T}_{n+1}) &\le \mathrm{osc}(u_n, \mathcal{T}_{n+1}) \;+\; \mathrm{osc}(u_{n+1} - u_n, \mathcal{T}_{n+1})\\ &\le \tilde{\alpha}\,\mathrm{osc}(u_n, \mathcal{T}_n) \;+\; \mathrm{osc}(u_{n+1} - u_n, \mathcal{T}_{n+1})\end{aligned} \tag{6.3}$$

A further application of the triangle inequality and then (4.15) yields

$$\begin{aligned}\mathrm{osc}(u_{n+1} - u_n, \mathcal{T}_{n+1}) &\le \mathrm{osc}(u - u_{n+1}, \mathcal{T}_{n+1}) \;+\; \mathrm{osc}(u - u_n, \mathcal{T}_{n+1})\\ &\lesssim (H_{n+1}^{max})^2\,(|||u - u_{n+1}|||_\Omega + |||u - u_n|||_\Omega)\end{aligned} \tag{6.4}$$

and then combining (6.3) and (6.4) and applying Theorem 3.2 completes the proof. ∎

Theorem 6.2 *Provided the initial mesh $\mathcal{T}_0$ is chosen so that H_0^{max} is small enough, there exists a constant $p \in (0, 1)$ such that the recursive application of Algorithm 1 yields a convergent sequence of approximate eigenvalues and eigenvectors, with the property:*

$$|||u - u_n\,|||_\Omega \;\le\; C_0 q p^n, \tag{6.5}$$

and

$$\lambda_n\,\mathrm{osc}(u_n, \mathcal{T}_n) \;\le\; C_1 p^n, \tag{6.6}$$

where C_0 and C_1 are constants and q is the constant defined in Theorem 3.2.

Remark 6.3 *The initial mesh convergence threshold and the constants C_1 and C_2 may depend on θ, $\tilde{\theta}$ and λ.*

Proof. The proof of this theorem is by induction and the induction step contains an application of Theorem 5.5. In order to ensure the reduction of the error, we have to assume that the starting mesh $\mathcal{T}_0$ is fine enough and μ in Theorem 5.5 is small enough such that for the chosen value of θ, the quantity α in (5.33) satisfies $\alpha < 1$.

Then with $\tilde{\alpha}$ as in Lemma 6.1, we set

$$\max\{\alpha, \tilde{\alpha}\} \ < \ p \ < \ 1 \ .$$

We also set

$$C_1 \ = \ \mathrm{osc}(\lambda_0, u_0, \mathcal{T}_0) \quad \text{and} \quad C_0 \ = \ \max\{\mu^{-1}p^{-1}C_1, \ |||u - u_0|||_\Omega\}.$$

To perform the inductive proof, first note that by the definition of C_0 and Theorem 3.2,

$$|||u - u_0\ |||_\Omega \leq C_0 \leq C_0 q,$$

since $q > 1$. Combined with the definition of C_1 we have shown the result for $n = 0$.

Now, suppose that for some $n > 0$ the inequalities (6.5) and (6.6) hold.

Now let us consider the outcomes, depending on whether the inequality

$$|||u - u_n\ |||_\Omega \leq C_0 p^{n+1}, \tag{6.7}$$

holds or not. If (6.7) holds then we can apply Theorem 3.2 to conclude that

$$|||u - u_{n+1}\ |||_\Omega \leq q\ |||u - u_n\ |||_\Omega \leq q C_0 p^{n+1},$$

which proves (6.5) for $n + 1$.

On the other hand, if (6.7) does not hold then, by definition of C_0,

$$|||u - u_n|||_\Omega \ > \ C_0 p^{n+1} \ \geq \ \mu^{-1} C_1 p^n. \tag{6.8}$$

Also, since we have assumed (6.6) for n, we have

$$\lambda_n \ \mathrm{osc}(u_n, \mathcal{T}_n) \leq \mu\varepsilon \quad \text{with} \quad \varepsilon := \mu^{-1} C_1 p^n \ . \tag{6.9}$$

Then (6.8) and (6.9) combined with Theorem 5.5 yields

$$|||u - u_{n+1}|||_\Omega \ \leq \ \alpha |||u - u_n|||_\Omega$$

and so using the inductive hypothesis (6.5) combined with the definition of p, we have

$$|||u - u_{n+1}|||_\Omega \ \leq \ \alpha C_0 q p^n \leq q C_0 p^{n+1},$$

which again proves (6.5) for $n + 1$.

To conclude the proof, we have to show that also (6.6) holds for $n+1$. Using Lemma 6.1, (2.9) and the inductive hypothesis, we have

$$\begin{aligned}\lambda_{n+1}\ \mathrm{osc}(u_{n+1},\mathcal{T}_{n+1}) &\leq \tilde{\alpha}C_1p^n + (1+q)(H_n^{max})^2\lambda_n C_0 q p^n \\ &\leq (\tilde{\alpha}C_1 + (1+q)(H_0^{max})^2\lambda_0 C_0 q)p^n. \qquad (6.10)\end{aligned}$$

Now, (recalling that $\tilde{\alpha} < p$), in addition to the condition already imposed on H_0^{max} we can further require that

$$\tilde{\alpha}C_1 + (1+q)(H_0^{max})^2|\lambda_0|C_0 q \ \leq\ pC_1.$$

This ensures that

$$\lambda_{n+1}\ \mathrm{osc}(u_{n+1},\mathcal{T}_{n+1}) \ \leq\ C_1 p^{n+1},$$

thus concluding the proof. ■

7 Numerical Experiments

We start with a brief discussion about the implementation of our method. Algorithm 1 has been implemented in FORTRAN95. The mesh refinement has been done using the toolbox ALBERTA [20]. We used the package ARPACK [14] to compute eigenpairs via Arnoldi's method and the linear solver ME27 from the HSL [17, 18] to carry on the shift-invert solves required by Arnoldi's method.

7.1 Example: Laplace operator

In the first set of simulations we have solved the Laplace eigenvalue problem on a unit square with Dirichlet boundary conditions.

We compare different runs of Algorithm 1 using different values for θ and $\tilde{\theta}$ in Table 1. Since the problem is smooth, from Theorem 3.1 it follows that using uniform refinement the rate of convergence for eigenvalues should be $\mathcal{O}(H_n^{max})^2$, or equivalently the rate of convergence in the number of degrees of freedom (DOFs) N should be $\mathcal{O}(N^{-1})$. We measure the rate of convergence by conjecturing that $|\lambda - \lambda_n| = CN^{-\beta}$ and estimating β for each pair of computations from the formula $\beta = -\log(|\lambda-\lambda_n|/|\lambda-\lambda_{n-1}|)/\log(DOFs_n/DOFs_{n-1})$. Similarly Table 2 contains the same kind of information relative to the fourth smallest eigenvalue of the problem. As can be seen the rate of convergence is sensitive to the values of θ and $\tilde{\theta}$. Moreover, our results for the adaptive method shown a convergence rate close to $\mathcal{O}(N^{-1})$ for θ, $\tilde{\theta}$ sufficiently large.

In the theory presented in [21] it is shown how the error for eigenvalues for smooth problems is proportional to the square of the considered eigenvalue, i.e. $|\lambda-\lambda_n| \leq C\ \lambda^2\ (H_n^{max})^2$. Since the Laplace problem is very well understood, we know from the theory the exact values for the first and the fourth eigenvalues, namely: 19.7392089 and 78.9568352. Comparing errors in Tables 1 and 2, corresponding to similar numbers of degrees of freedom (DOFs), we see that the error grows roughly with the square of the eigenvalue.

	$\theta = \tilde{\theta} = 0.2$			$\theta = \tilde{\theta} = 0.5$			$\theta = \tilde{\theta} = 0.8$		
Iteration	$\|\lambda - \lambda_n\|$	DOFs	β	$\|\lambda - \lambda_n\|$	DOFs	β	$\|\lambda - \lambda_n\|$	DOFs	β
1	0.1350	400	-	0.1350	400	-	0.1350	400	-
2	0.1327	498	0.0802	0.1177	954	0.1581	0.0529	1989	0.5839
3	0.1293	613	0.1228	0.0779	1564	0.8349	0.0176	5205	1.1407
4	0.1256	731	0.1645	0.0501	1977	1.8788	0.0073	15980	0.7877
5	0.1215	854	0.2138	0.0351	2634	1.2383	0.0024	48434	0.9836
6	0.1165	970	0.3340	0.0176	4004	0.7885	0.0009	122699	1.0673
7	0.1069	1097	0.6962	0.0121	6588	0.7217	0.0003	312591	1.0083

Table 1: Comparison of the reduction of the error and DOFs of the adaptive method for the smallest eigenvalue for the Laplace problem on the unit square.

	$\theta = \tilde{\theta} = 0.2$			$\theta = \tilde{\theta} = 0.5$			$\theta = \tilde{\theta} = 0.8$		
Iteration	$\|\lambda - \lambda_n\|$	DOFs	β	$\|\lambda - \lambda_n\|$	DOFs	β	$\|\lambda - \lambda_n\|$	DOFs	β
1	2.1439	400	-	2.1439	400	-	2.1439	400	-
2	2.0997	505	0.0895	1.8280	1016	0.1658	0.7603	2039	0.6365
3	2.0549	626	0.1004	1.0850	1636	1.1662	0.2439	6793	0.9447
4	1.9945	759	0.1548	0.7792	12254	1.0331	0.0917	18717	0.9652
5	1.9164	883	0.2638	0.4936	3067	1.4826	0.0331	54113	0.9583
6	1.7717	1017	0.5557	0.3484	4681	0.8240	0.0120	146056	1.0181
7	1.6463	1131	0.6911	0.2578	7321	0.6730	0.0046	382024	0.9970

Table 2: Comparison of the reduction of the error and DOFs of the adaptive method for the fourth smallest eigenvalue for the Laplace problem on the unit square.

7.2 Example: Elliptic operator with discontinuous coefficients

In this second example we investigate how our method copes with discontinuous coefficients. In order to do that we modified the smooth problem from the previous example. We inserted a square subdomain of side 0.5 in the center of the unit square domain. We also choose the function $\mathcal{A}$ to be piecewise constant and to assume the value 100 inside the subdomain and the value 1 outside it.

The jump in the value of $\mathcal{A}$ could produce a jump in the gradient of the eigenfunctions all along the boundary of the subdomain. So the regularity of the eigenfunctions in the sense of Assumption 2.1 is now between $3/2 \leq s+1 < 2$. From Theorem 3.1, using uniform refinement, the rate of convergence for eigenvalues should be at least $\mathcal{O}(H_n^{max})^{2s}$ or equivalently $\mathcal{O}(N^{-s})$, where N is the number of DOFs. Instead, using our method we obtain greater orders of convergence for big enough value of θ and $\tilde{\theta}$, as can be seen from Table 3. We measure the rate of convergence computing the value of β as before. In fact the rate of convergence for $\theta = \tilde{\theta} = 0.5$ or 0.8 is close to the rate of convergence for smooth problems in Table 1 and Table 2. In this case the exact eigenvalue λ is unknown, but we approximate it by computing the eigenvalue on a very fine mesh involving about half a million of DOFs.

In Figure 3 we depict the mesh coming from the fourth iteration of Algorithm 1 with $\theta =$

	$\theta = \tilde{\theta} = 0.2$			$\theta = \tilde{\theta} = 0.5$			$\theta = \tilde{\theta} = 0.8$		
Iteration	$\|\lambda - \lambda_n\|$	DOFs	β	$\|\lambda - \lambda_n\|$	DOFs	β	$\|\lambda - \lambda_n\|$	DOFs	β
1	1.1071	81	-	1.1071	81	-	1.1071	81	-
2	0.9166	108	0.6561	0.7959	216	0.3364	0.4214	362	0.6452
3	0.9036	136	0.0062	0.6075	301	0.8139	0.1955	1153	0.6628
4	0.8575	159	0.3350	0.4168	437	1.0108	0.0789	2811	1.0174
5	0.8118	186	0.3497	0.2750	643	1.0762	0.0335	6534	1.0151
6	0.8065	208	0.0582	0.1989	954	0.8212	0.0172	14059	0.8687
7	0.7580	229	0.6448	0.1236	1459	1.1186	0.0066	28341	1.3621
8	0.7447	250	0.2024	0.0935	2117	0.7504	0.0033	60148	0.9123

Table 3: Comparison of the reduction of the error and DOFs of the adaptive method for the smallest eigenvalue for the problem with discontinuous coefficients.

$\tilde{\theta} = 0.8$. This mesh is the result of multiple refinements using both marking strategies 1 and 2 each time. As can be seen the region along the interface and particularly the corners are much more refined than the rest of the mesh. This is clearly the effect of the first marking strategy, since the edge residuals have detected the discontinuity in the gradient of the eigenfunction along the interface. However also in some other parts away form the interface the mesh is relatively fine. These are the results of the use of the second marking strategy based on oscillations, which is designed to detect high gradients of the eigenfunction.

Finally in Figure 4 we depict the eigenfunction corresponding to the smallest eigenvalue of the problem with discontinuous coefficients. This eigenfunction is the one used to refine the mesh in Figure 3.

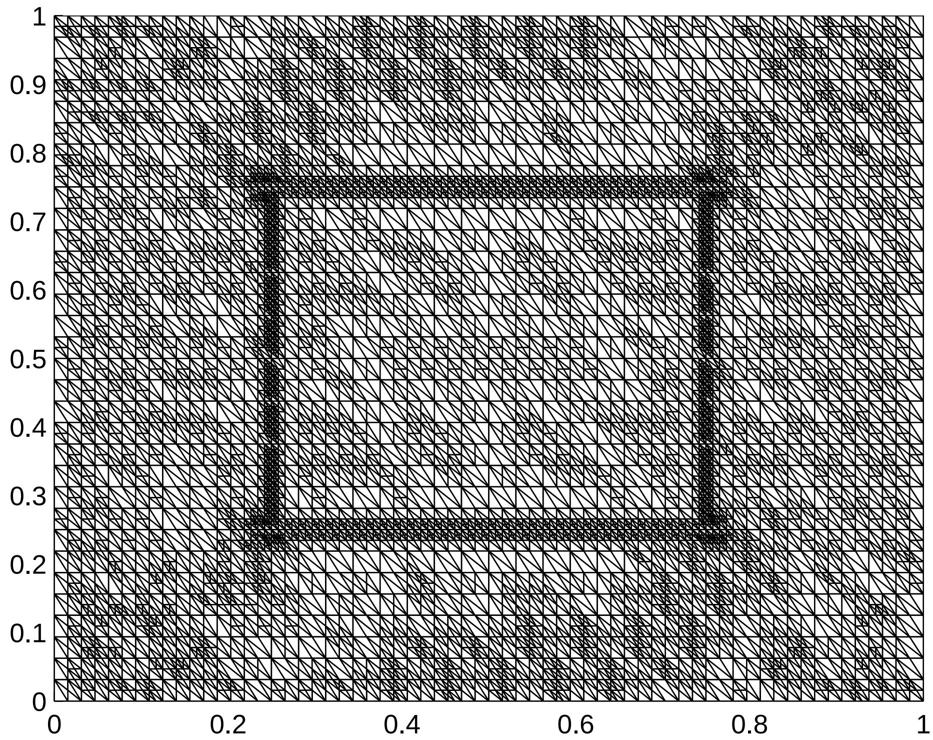

Figure 3: A refined mesh from the adaptive method corresponding to the first eigenvalue of the problem with discontinuous coefficients.

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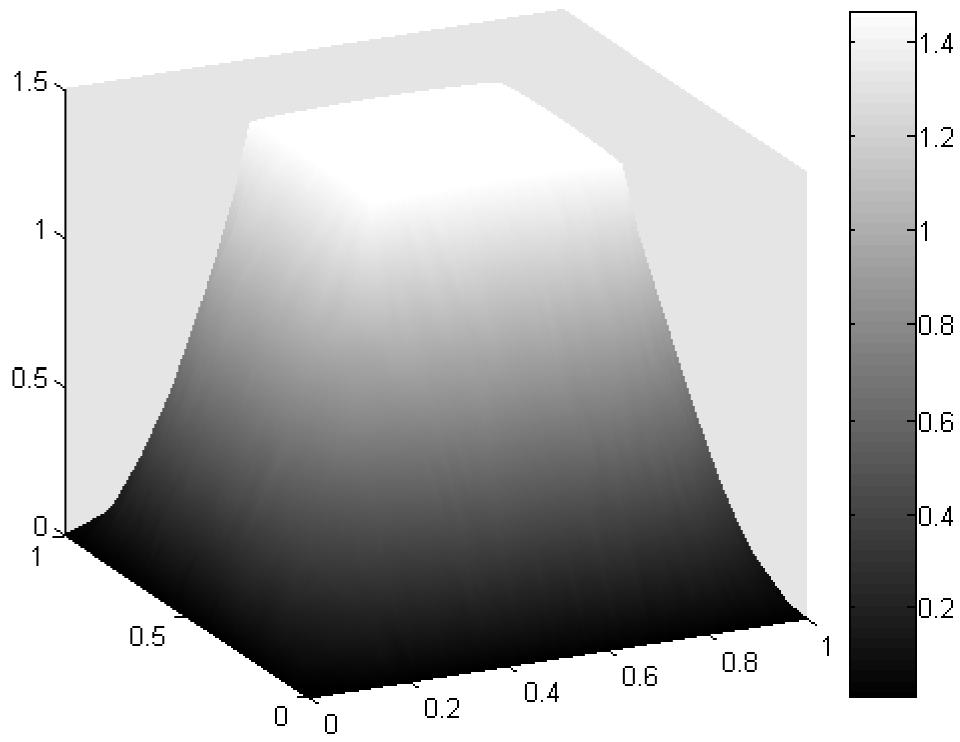

Figure 4: The eigenfunction corresponding to the first eigenvalue of the problem with discontinuous coefficients.

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