NONDEGENERACY, FROM THE PROSPECT OF WAVE-WAVE REGULAR INTERACTIONS OF A GASDYNAMIC TYPE : SOME REMARKS

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Abstract

∎ An *analogue* of the *genuinely nonlinear* character of an one-dimensional simple waves solution is identified and *essentially* used in the construction of some *multi-dimensional extensions* (simple waves solutions, regular interactions of simple waves solutions). ∎ A *class* of exact multidimensional gasdynamic solutions is constructed whose interactive elements are regular. ∎ Some *specific* aspects of Burnat's multidimensional "algebraic" description [which uses a *dual* connection between the hodograph and physical characteristic details] are identified and classified with an admissibility criterion – selecting a "genuinely nonlinear" type where other ("hybrid") types are formally possible. ∎ A *parallel* is constructed between Burnat's "algebraic" approach and Martin's "differential" approach [centered on a Monge–Ampère type representation] regarding their contribution to describing some nondegenerate one-dimensional gasdynamic regular interaction solutions. ∎ The present approach parallels, from a *local* prospect, some details of the two-dimensional *global* approach of [16].

Introduction. Burnat's "algebraic" approach

For the multidimensional first order hyperbolic system of a gasdynamic type [whose coefficients only depend on u]

$$\sum_{j=1}^{n}\sum_{k=0}^{m} a_{ijk}(u)\frac{\partial u_j}{\partial x_k} = 0, \quad 1 \le i \le n \tag{1}$$

the "algebraic" approach (Burnat [1]) starts with identifying *dual* pairs of directions $\vec{\beta}, \vec{\kappa}$ [we write $\vec{\kappa} \rightharpoondown \vec{\beta}$] connecting [via their duality relation] the hodograph [= in the hodograph space H of the entities u] and physical [= in the physical space E of the independent variables] characteristic details. The duality relation at $u^* \in H$ has the form:

$$\sum_{j=1}^{n}\sum_{k=0}^{m} a_{ijk}(u^*)\beta_k\kappa_j = 0, \quad 1 \le i \le n. \tag{2}$$

Here $\vec{\beta}$ is an *exceptional* direction [= orthogonal in the physical space E to a characteristic character]. A direction $\vec{\kappa}$ dual to an exceptional direction $\vec{\beta}$ is said to be a *hodograph characteristic* direction.

EXAMPLE 1. For the *one-dimensional* strictly hyperbolic version of system (1) a *finite* number n of dual pairs $\vec{\kappa}_i \rightharpoondown \vec{\beta}_i$ consisting in $\vec{\kappa}_i = \vec{R}_i$ and $\vec{\beta}_i = \Theta_i(u)[-\lambda_i(u), 1]$, where $\vec{R}_i$ is a right eigenvector of the $n \times n$ matrix a and λ_i is an eigenvalue of a, are available ($i = 1, ..., n$). Each dual pair associates in this case, at each $u^* \in \mathcal{R}$ [for a suitable $\mathcal{R} \subset H$], to a vector $\vec{\kappa}$ a *single* dual vector $\vec{\beta}$.

EXAMPLE 2 (Peradzyński [13]). For the *two-dimensional* version (3) of (1), corresponding to an isentropic description (in usual notations: c is the sound velocity, v_x, v_y are fluid velocities)

$$\begin{cases} \dfrac{\partial c}{\partial t} + v_x\dfrac{\partial c}{\partial x} + v_y\dfrac{\partial c}{\partial y} + \dfrac{\gamma-1}{2}c\left(\dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y}\right) = 0 \\ \dfrac{\partial v_x}{\partial t} + v_x\dfrac{\partial v_x}{\partial x} + v_y\dfrac{\partial v_x}{\partial y} + \dfrac{2}{\gamma-1}c\dfrac{\partial c}{\partial x} = 0 \\ \dfrac{\partial v_y}{\partial t} + v_x\dfrac{\partial v_y}{\partial x} + v_y\dfrac{\partial v_y}{\partial y} + \dfrac{2}{\gamma-1}c\dfrac{\partial c}{\partial y} = 0\,, \end{cases} \tag{3}$$

an *infinite* number of dual pairs are available at each $u^* \in H$. Each dual pair associates, at the mentioned u^*, to a vector $\vec{\kappa}$ a *single* dual vector $\vec{\beta}$.

EXAMPLE 3 (Peradzyński [14]). For the *three-dimensional* version of (3) an *infinite* number of dual pairs are available at each $u^* \in H$. Each dual pair associates, at the mentioned u^*, to a vector $\vec{\kappa}$ a *finite* [constant, $\neq 1$] number of k independent exceptional dual vectors $\vec{\beta}_j$, $1 \le j \le k$; and therefore has the structure $\vec{\kappa} \rightharpoondown (\vec{\beta}_1, ..., \vec{\beta}_k)$.

DEFINITION 4 (Burnat [1]). A curve $\mathcal{C} \subset H$ is said to be *characteristic* if it is tangent at each point of it to a characteristic direction $\vec{\kappa}$. A hypersurface $\mathcal{S} \subset H$ is said to be *characteristic* if it possesses at least a characteristic system of coordinates.

Genuine nonlinearity. Simple waves solution. Regular interaction of simple waves solutions

REMARK 5. As it is well-known, in case of an one-dimensional strictly hyperbolic version of (1) any hodograph characteristic curve $\mathcal{C} \subset \mathcal{R} \subset H$, of index i, is said to be *genuinely nonlinear (gnl)* if the dual constructive pair $\vec{\kappa}_i \rightharpoondown \vec{\beta}_i$ is restricted by $\vec{\kappa}_i(u) \diamond \vec{\beta}_i(u) \equiv \vec{R}_i(u) \cdot \mathrm{grad}_u \lambda_i(u) \neq 0$ in $\mathcal{R}$; see Example 1. This condition transcribes the requirement $\frac{\mathrm{d}\vec{\beta}}{\mathrm{d}\alpha} \neq 0$ along $\mathcal{C}$.

DEFINITION 6. We naturally extend the *gnl* character of a hodograph curve to the cases corresponding to Examples 2 and 3, by requiring along $\mathcal{C}$

$$\left|\frac{\mathrm{d}\vec{\beta}}{\mathrm{d}\alpha}\right| \neq 0 \tag{4}$$

and, respectively,

$$\sum_{\mu=1}^{k}\left|\frac{\mathrm{d}\vec{\beta}_\mu}{\mathrm{d}\alpha}\right| \neq 0. \tag{5}$$

DEFINITION 7. A solution of (1) whose hodograph is laid along a *gnl* characteristic curve is said to be a *simple waves solution (sws)*. A solution of (1) whose hodograph is laid on a characteristic hypersurface is said to be a *regular interaction of sws* if its hodograph possesses a *gnl* system of coordinates.

A class of solutions of the system (1). Wave-wave "algebraic" regular interactions. Riemann−Burnat invariants

Let $R_1, \ldots, R_p$ be characteristic coordinates on a given p-dimensional characteristic region $\mathcal{R}$ of a hodograph hypersurface $\mathcal{S}$. Solutions of the system

$$\frac{\partial u_l}{\partial x_s} = \sum_{k=1}^{p} \eta_k \kappa_{kl}(u)\beta_{ks}(u),\ u \in \mathcal{R};\ 1 \le l \le n,\ 0 \le s \le m \tag{6}$$

appear to concurrently satisfy the system (1). This indicates an "algebraic" importance of the concept of dual pair [see (2)]. We formally call these solutions *wave-wave regular interactions*.

- If a set of *Riemann−Burnat invariants* $R(x)$ exists, structuring the dependence on x of the solution u in the class above by a *regular* interaction representation

$$u_l = u_l[R_1(x_0, ..., x_m), ..., R_p(x_0, ..., x_m)],\ 1 \le l \le n, \tag{7}$$

it is easy to see that $R_i(x)$ must fulfil an (overdetermined and Pfaff) system

$$\frac{\partial R_k}{\partial x_s} = \eta_k \beta_{ks}[u(R)],\ \ 1 \le k \le p,\ 0 \le s \le m. \tag{8}$$

- Sufficient restrictions for solving (8) are proposed in [5], [13], [14]. Also see [3], [4].

A class of exact solutions of the system (3). Nondegeneracy. Linear degeneracy

A class of (local) exact solutions of (3) of the form

$$c = c(\xi, \eta),\ \ v_x = v_x(\xi, \eta),\ \ v_y = v_y(\xi, \eta), \tag{9}$$

where

$$\xi = \frac{x - x_0}{t - t_0}, \quad \eta = \frac{y - y_0}{t - t_0} \tag{10}$$

is exhaustively presented in [3], §6. The interactive elements of this class appear to correspond to some *regular* interactions. Two highly nontrivial and very suggestive regular interaction elements in the mentioned class are considered in [3], §7 in every detail.

The hodograph associated to the first of these elements is shown to possess three *gnl* characteristic systems of coordinates; such an element puts together three distinct representations of regular interactions of multidimensional *sws*. We also present in [3], §7 the physical details of each of these three representations.

The hodograph associated to the second of these mentioned elements also possesses three characteristic systems of coordinates; still, in this case two of these three systems of coordinates are proven to be [partially] *linearly degenerate (ldg)*. Again, three *distinct* and *multidimensionally coherent* [regular interaction] representations are concurrently present; only one of them would correspond to a *sws* interaction yet.

REMARK 8. A regular interaction of *sws* reflects the *nondegenerate* nature of the *gnl* hodographs of the interacting *sws*. The "algebraic" characterization of a regular interaction of *sws* will be regarded to correspond to a case of ["algebraic"] nondegeneracy. The second example above also includes a case of "algebraic" degeneracy ([3]). We therefore notice that representation (7) is not made of *sws* generally. We need a *criterion* to select the nondegenerate regular interaction solutions (see [3]).

Martin's one-dimensional "differential" approach

Next, we consider the gasdynamic system

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial(\rho v_x)}{\partial x} = 0 \\ \dfrac{\partial(\rho v_x)}{\partial t} + \dfrac{\partial}{\partial x}\left(\rho v_x^2 + p\right) = 0 \\ \dfrac{\partial(\rho S)}{\partial t} + \dfrac{\partial(\rho v_x S)}{\partial x} = 0\ ,\ \ S = S(p, \rho) \end{cases} \tag{11}$$

(in usual notations: ρ, v_x, p, S are respectively the mass density, fluid velocity, pressure and entropy density).

- We use, to begin with, the first two equations $(11)_{1,2}$ to introduce (Martin [11]), the functions ψ, $\tilde{\xi}$ and ξ cf.

$$\rho = \frac{\partial \psi}{\partial x},\ \rho v_x = -\frac{\partial \psi}{\partial t};\ \rho v_x = \frac{\partial \tilde{\xi}}{\partial x},\ \rho v_x^2 + p = -\frac{\partial \tilde{\xi}}{\partial t};\ \xi = \tilde{\xi} + pt. \tag{12}$$

We get

$$\mathrm{d}x = \frac{1}{\rho}\mathrm{d}\psi + v_x \mathrm{d}t,\ \ \mathrm{d}\tilde{\xi} = v_x \mathrm{d}\psi - \rho \mathrm{d}t,\ \ \mathrm{d}\xi = v_x \mathrm{d}\psi + t\mathrm{d}p. \tag{13}$$

• We seek for solutions of (11) which fulfil the [natural; see Appendix 1] requirement

$$\frac{\partial p}{\partial t}\frac{\partial \psi}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial \psi}{\partial t} \neq 0. \tag{14}$$

• Two types will be then considered here below for these mentioned solutions – distinguished by their details concerning the fulfilment of $(11)_3$: a *continuous* strictly adiabatic flow and, respectively, an isentropic flow.

Anisentropic details of Martin's approach

A continuous [smooth] strictly adiabatic [anisentropic] flow results behind a shock discontinuity of non-constant continuous [smooth] velocity which penetrates into a region of uniform flow ([9], [11]). For such a flow, entropy $S(p,\rho)$ in $(11)_3$ is a function of ψ alone, $F(\psi)$, determined by the shock conditions. Prescription of F provides an algebraic relation between p, ρ, ψ throughout the adiabatic flow region. We select [Martin] p and ψ as new independent variables [see (14)] in place of x and t and compute from (13)

$$\frac{\partial x}{\partial \psi} = v_x\frac{\partial t}{\partial \psi} + \frac{1}{\rho},\ \frac{\partial x}{\partial p} = v_x\frac{\partial t}{\partial p},\ v_x = \frac{\partial \xi}{\partial \psi},\ t = \frac{\partial \xi}{\partial p}. \tag{15}$$

On eliminating x from $(15)_{1,2}$ and taking $(15)_{3,4}$ into account it results that ξ must fulfil the hyperbolic Monge–Ampère equation

$$\frac{\partial^2 \xi}{\partial p^2}\frac{\partial^2 \xi}{\partial \psi^2} - \left(\frac{\partial^2 \xi}{\partial p \partial \psi}\right)^2 = -\zeta^2(p,\psi) \equiv \frac{\partial}{\partial p}\left(\frac{1}{\rho}\right) \equiv -\frac{1}{\rho^2 c^2}. \tag{16}$$

where $\rho = \rho(p,\psi)$ and $c(p,\psi) = \sqrt{\left(\frac{\partial \rho}{\partial p}\right)_S^{-1}}$ is an ad hoc sound speed. Finally, we compute from (15)

$$x = \int \left(\frac{\partial \xi}{\partial \psi}\frac{\partial^2 \xi}{\partial p \partial \psi} + \frac{1}{\rho}\right) \mathrm{d}\psi + \left(\frac{\partial \xi}{\partial \psi}\frac{\partial^2 \xi}{\partial p^2}\right) \mathrm{d}p. \tag{17}$$

REMARK 9. For *any* smooth solution $\xi(p,\psi)$ of (16) we get from (15), (17)

$$v_x = v_x(p,\psi),\ x = x(p,\psi),\ t = t(p,\psi). \tag{18}$$

On reversing $(18)_{2,3}$ into $p = p(x,t)$, $\psi = \psi(x,t)$, via (14), and carrying this into $(18)_1$ we get a form $p(x,t)$, $v_x(x,t)$, $\psi(x,t)$ of the corresponding anisentropic solution of (11).

REMARK 10 [see Appendix 1]. (*a*) On prescribing F we will not find the streamlines $\mathcal{C}_0$ as characteristics of (11). (*b*) The two families of characteristics $\overline{\mathcal{C}}_\mp$ of (16) in the plane p, ψ appear to correspond to the two families of sound characteristics $\mathcal{C}_\pm$ in the plane x, t. ∎ The aspects (*a*), (*b*) are typical for the isentropic context. We notice that they *persist* in the Martin *anisentropic* context.

Martin linearization. Pseudo simple waves solution. Regular interaction of pseudo simple waves solutions. Riemann–Martin invariants. Pseudo nondegeneracy

Next, we restrict the general Remark 9 to a *particular construction* – useful for identifying solutions of (16). To

$$\zeta = \frac{\psi^{\nu-1}}{p^{\nu+1}} \quad (\nu \neq 0) \tag{19}$$

in (15) we associate [Martin] two intermediate integrals

$$\mathcal{F}_\pm \equiv p\frac{\partial \xi}{\partial p} + \psi\frac{\partial \xi}{\partial \psi} - \xi \pm \frac{1}{\nu}\left(\frac{\psi}{p}\right)^\nu \tag{20}$$

for which $\mathcal{F}_\pm$=constant$_\pm$=$R_\pm$ along a characteristic $\overline{\mathcal{C}}_\pm$ [see Appendix 2 for an exhaustive list of circumstances similar to (19), (20)].

• We have to distiguish then between the cases (*a*) when $R_\pm$ depend on the charateristic $\overline{\mathcal{C}}_\pm$, and (*b*) when R_+ or R_- are overall constants.

• In the case (*a*) we may use (Martin [11]) $R_\pm$ as new independent variables. It can be shown in this case (Martin [11]) that the entities p^{-1}, v_x, ψ^{-1}, t fulfil various Euler–Poisson–Darboux *linear* equations

$$\frac{\partial^2 w}{\partial R_+ \partial R_-} - \frac{\nu}{R_+ - R_-}\left(\frac{\partial w}{\partial R_+} - \frac{\partial w}{\partial R_-}\right) = 0, \quad \text{constant } \nu$$

to which well-known representations are associated [see Appendix 3]; we present these representations by

$$p = p(R_+,R_-),\ \psi = \psi(R_+,R_-),\ v_x = v_x(R_+,R_-), \\ t = t(R_+,R_-),\ x = x(R_+,R_-). \tag{21}$$

where $x(R_+,R_-)$ results by quadratures via (15), (17) [see Appendix 2]. Reversing $(21)_2$ into $R_\pm = R_\pm(x,t)$ will induce a form of solution (21), parallel to (7) [as $R_\pm$ have a characteristic nature]. We call $R_\pm(x,t)$ *Riemann–Martin invariants*.

• In the case (*b*) we notice that a solution $\xi(p,\psi)$ of the *linear* equation $\mathcal{F}_+ \equiv R_+$ or $\mathcal{F}_- \equiv R_-$ [see (20)] will automatically fulfil (16). We have to follow, in this case, Remark 9 to describe a solution of (11); we call such a solution *pseudo simple waves solution*.

• We notice that solution (21) might be regarded as *pseudo nondegenerate* [a formal regular interaction of *pseudo sws*].

• The cases (*a*) and (*b*) appear to correspond to a *Martin linearization* – here associated to (19), (20). In Martin [11] it is proven that for each circumstance similar to (19), (20) [on an exhaustive list included in Appendix 2] a Martin linearization is [similarly] available via Euler–Poisson–Darboux representations and quadratures.

• The image of a characteristic $\mathcal{C} \subset E$ on the hodograph of a solution of (11) will be said to be a *M*- characteristic. The hodograph of a formal regular interaction of pseudo *sws* will be then made by glueing, along suitable *M*-characteristics, a hodograph (21) with some suitable hodographs of pseudo *sws* [see Figure 1].

Pseudo simple waves solution: an example

EXAMPLE 11. We satisfy $\mathcal{F}_+ \equiv R_+ = 0$ [see (20)] by

$$\xi = \frac{1}{\nu}\left(\frac{\psi}{p}\right)^{\nu}, \quad \text{integral } \nu \tag{22}$$

and calculate from (15), (17)

$$\begin{aligned} x &= -\frac{\nu+1}{2\nu+1}\frac{\psi^{2\nu-1}}{p^{2\nu+1}}, \quad t = -\frac{\psi^{\nu}}{p^{\nu+1}}, \\ v_x &= \frac{\psi^{\nu-1}}{p^{\nu}}, \quad \rho = (2\nu+1)\frac{p^{2\nu+1}}{\psi^{2\nu-2}}, \\ \mathcal{F}_-&\left(p,\psi,\xi,\frac{\partial\xi}{\partial p},\frac{\partial\xi}{\partial\psi}\right) \equiv -\frac{2}{\nu}\left(\frac{\psi}{p}\right)^{\nu}, \end{aligned} \tag{23}$$

which leads to

$$\begin{aligned} p &= -\left(\frac{\nu+1}{2\nu+1}\right)^{\nu}\frac{t^{2\nu-1}}{(-x)^{\nu}}, \quad v_x = \frac{2\nu+1}{\nu+1}\frac{x}{t}, \\ \psi &= -\left(\frac{\nu+1}{2\nu+1}\right)^{\nu+1}\frac{t^{2\nu+1}}{(-x)^{\nu+1}}, \\ \mathcal{F}_-&\left(p,\psi,\xi,\frac{\partial\xi}{\partial p},\frac{\partial\xi}{\partial\psi}\right) \equiv -\frac{2}{\nu}\frac{\nu+1}{2\nu+1}\left[\frac{t^2}{(-x)}\right]^{\nu}. \end{aligned} \tag{24}$$

This is a [local] pseudo *sws* of (11) corresponding to a certain region $\mathcal{D} \subset E$ (for example, a region of $t > 0$, $x < 0$). For this solution the assumption (14) holds, cf.

$$\frac{\partial p}{\partial t}\frac{\partial\psi}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial\psi}{\partial t} \equiv -\left(\frac{\nu+1}{2\nu+1}\right)^{2\nu+1}\frac{t^{4\nu-1}}{(-x)^{2(\nu+1)}} \neq 0 \text{ in } \mathcal{D}. \tag{25}$$

Now, we proceed with the details concerning the present solution. We calculate from (19), (23)

$$c = \frac{1}{\zeta\rho} = \frac{1}{2\nu+1}\frac{\psi^{\nu-1}}{p^{\nu}} = \frac{1}{2\nu+1}v_x \tag{26}$$

and notice that the *explicit* equations of the [physical] field lines $\mathcal{C}_-, \mathcal{C}_+, \mathcal{C}_0$ [of these, only $\mathcal{C}_\pm$ have a characteristic character (see Remark 10)] through a point $(x^*, t^*) \in \mathcal{D}$ result, cf. (24), (26), by integrating respectively the differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v_x(x,t) - c(x,t) = k_-\frac{x}{t} \quad \text{along } \mathcal{C}_- \tag{27$_-$}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v_x(x,t) + c(x,t) = k_+\frac{x}{t} \quad \text{along } \mathcal{C}_+ \tag{27$_+$}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v_x(x,t) = k_0\frac{x}{t} \quad \text{along } \mathcal{C}_0 \tag{27$_0$}$$

with

$$\begin{cases} k_- = \dfrac{2\nu}{\nu+1} = -2\dfrac{\gamma-1}{\gamma+1}; \ -\dfrac{1}{2} < k_- < 0 \quad \text{for } 1 < \gamma < \dfrac{5}{3} \\ k_+ = 2; \\ k_0 = \dfrac{2\nu+1}{\nu+1} = \dfrac{2}{\gamma+1}; \ 0 < k_0 < 1 \quad \text{for } 1 < \gamma < \dfrac{5}{3}. \end{cases} \tag{28}$$

We get from (27), (28)

$$|x| = K_-|t|^{k_-}, \ K_- = \log\frac{|x^*|}{|t^*|^{k_-}} \quad \text{along } \mathcal{C}_- \ni (x^*,t^*) \tag{29$_-$}$$

$$|x| = K_+|t|^{k_+}, \ K_+ = \log\frac{|x^*|}{|t^*|^{k_+}} \quad \text{along } \mathcal{C}_+ \ni (x^*,t^*) \tag{29$_+$}$$

$$|x| = K_0|t|^{k_0}, \ K_0 = \log\frac{|x^*|}{|t^*|^{k_0}} \quad \text{along } \mathcal{C}_0 \ni (x^*,t^*). \tag{29$_0$}$$

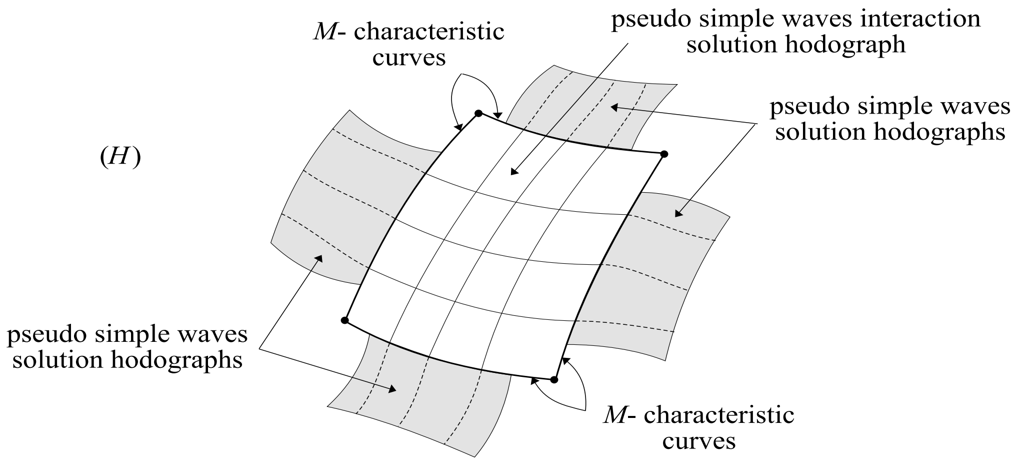

FIGURE 1

We remark from $(24)_3$, $(28)_2$ and $(29)_+$ that for the Riemann−Martin invariant $R_-(x,t)$ we get along a characteristic $\overline{\mathcal{C}}_-$, and hence along a characteristic $\mathcal{C}_+$ [see Appendix 1],

$$R_-(x,t) = -\frac{2}{\nu}\frac{\nu+1}{2\nu+1}\varepsilon^\nu\left(\frac{|t|^{k_+}}{|x|}\right)^\nu = -\frac{2}{\nu}\frac{\nu+1}{2\nu+1}\varepsilon^\nu K_+^\nu = \text{constant}_-.$$

In Figure 1 the M- characteristics corresponding to values of R dependent on $\overline{\mathcal{C}}$ in the four contributing pseudo *sws* [of two distinct Martin indices] are depicted by broken lines.

REMARK 12. We notice ([4]) that [in contrast with a *sws*] a *pseudo sws* has a two-dimensional hodograph [see (25)] and for it none of the characteristic fields $\mathcal{C}_\pm$ [see Remark 10] in the physical plane x, t is made of straight-lines generally [see (28), (29)].

REMARK 13. The images of characteristics $\mathcal{C}_-$ are labelled by a *common* $R_+ (\equiv 0)$ [as associated to correspondent characteristics $\overline{\mathcal{C}}_+$]. Still this will not affect the linearization which is already active [cf. the case (*b*) above]. ∎ In case (*a*) mentioned above, the linearization results from the interaction of the *two* available coordinates $R_\pm$.

"Algebraic" approach and "differential" approach: a parallel. Martin's approach is strictly "nonalgebraic"

We show, via a comparison with Burnat's "algebraic" approach, that the "differential" representation (21) is *strictly "nonalgebraic"*.

- We begin by presenting the Burnat context connected with the system

$$\begin{cases} \dfrac{\partial p}{\partial t} + v_x\dfrac{\partial p}{\partial x} + \rho(p,S)c^2(p,S)\dfrac{\partial v_x}{\partial x} = 0 \\ \dfrac{\partial v_x}{\partial t} + v_x\dfrac{\partial v_x}{\partial x} + \dfrac{1}{\rho(p,S)}\dfrac{\partial p}{\partial x} = 0 \\ \dfrac{\partial S}{\partial t} + v_x\dfrac{\partial S}{\partial x} = 0. \end{cases} \tag{30}$$

This system is smoothly equivalent with (11) [Appendix 1]; also see (1.4) in [3].

- The Burnat context is constructed from the eigenelements of (30):

$$\lambda_\pm(u) = v_x \pm c(p,S),\ \lambda_0(u) = v_x;$$
$$\vec{\kappa}_\pm \equiv \overset{\pm}{R} = \Lambda_\pm\left(\pm\tfrac{1}{\zeta},1,0\right)^t,\ \vec{\kappa}_0 \equiv \overset{0}{R} = (0,0,1)^t \tag{31}$$

where

$$u \overset{\text{def}}{=} (p,v_x,S)^t;\ \ \zeta = \frac{1}{\rho(p,S)c(p,S)}. \tag{32}$$

- Next, we take into account the characteristic details associated to (16) [Appendix 1] to compute the M- characteristic vectors $\vec{\mu}$ at the points of the solution hodograph:

$$\vec{\mu}_\pm = \left(\frac{\partial p}{\partial R_\pm},\frac{\partial v_x}{\partial R_\pm},\frac{\partial S}{\partial R_\pm}\right)^t = \frac{\partial v_x}{\partial R_\pm}\left(\mp\frac{1}{\zeta},1,0\right)^t + \frac{\partial S}{\partial R_\pm}(0,0,1)^t = \eta_\mp\vec{\kappa}_\mp + \eta_0\vec{\kappa}_0 \tag{33}$$

where

$$S(R_+,R_-) \equiv F[\psi(R_+,R_-)],\ \eta_\mp = \frac{1}{\Lambda_\mp}\frac{\partial v_x}{\partial R_\pm},\ \eta_0 = \frac{\partial S}{\partial R_\pm}$$

and we notice that $\vec{\kappa}_0$ is no more a hodograph characteristic direction, cf. Remark 10(*a*).

REMARK 14. (*i*) We notice from (33) that at the points of a solution hodograph of (11) [or (30)] the M- characteristic fields and, respectively, the Burnat characteristic fields appear to be *distinct* generally in the anisentropic context and can be shown to be *coincident* in the isentropic context (cf. $\eta_0 \equiv 0$). Let $\mathcal{S}$ be the hodograph surface corresponding to (21). We denote $\mathcal{T}_u\mathcal{S}$ the tangent space to $\mathcal{S}$ at a point $u \in \mathcal{S}$. We notice from (33) that at each point $u \in \mathcal{S}$: $\vec{\mu}_\pm \in \mathcal{T}_u\mathcal{S}$ and, concurrently, $\vec{\kappa}_\pm \notin \mathcal{T}_u\mathcal{S}$. ∎ Representation (21) corresponds to an example of hodograph surface of (11) [or (30)] which *is not* a Burnat characteristic surface [Definition 4]. Still, incidentally and essentially for the anisentropic approach, this representation is associated with an example of hodograph surface of (11) [or (30)] for which a characteristic character persists in a Martin sense. (*ii*) If a (Martin or Burnat) hodograph characteristic field is laid on a solution hodograph then it appears to be the image of a characteristic field in the domain $\mathcal{D} \subset E$ of the considered solution. This remark shows that if on a solution hodograph there exist simultaneously a Martin and, respectively, a Burnat characteristic coordinate systems then these coordinate systems must be coincident. ∎ In the isentropic case the "algebraic" nature and the "differential" nature are coincident. In particular, the pseudo *sws* / regular interaction of pseudo *sws* / pseudo nondegeneracy are replaced in an isentropic "differential" construction by a *sws* / regular interaction of *sws* / nondegeneracy.

DEFINITION 15. Given a smooth solution of (11) [or (30)], a curve (field) in H is said to be *characteristic* (with respect to this solution) if it is image of a characteristic curve (field) in E.

REMARK 16. Remark 14 suggests to parallel the *anisentropic* structures mentioned in Remark 9 or in (21) by the *isentropic* structures of Example 5.1 in [3]. ∎ In such a parallel some similarities result from Remark 10. A contrast results from Remark 13(*i*).

- In [3], §§2−5 we follow Burnat (via Example 5.1) to extend the *isentropic* context from the one-dimensional case to the multidimensional case.
- In the present paper we follow Martin [11] to extend the *one-dimensional* context from the isentropic case to the anisentropic case.

Isentropic details of Martin's approach

In an *isentropic* flow we have $S(p,\rho)\equiv$constant in $(11)_3$. We notice in this case [see Appendix 1] that restriction (14) and the Martin's differential approach are still active.

- Two intermediate integrals are available in this case − corresponding to $\zeta = \zeta(p)$ in (16) [see Appendix 2, second row of Chart 2]:

$$\mathcal{F}_\pm \equiv v_x \mp \int \zeta(p)\mathrm{d}p = \upsilon_\mp \ ; \quad v_x = \frac{\partial \xi}{\partial \psi} \, . \qquad (34)$$

- It is easy to see in this case that $\upsilon_\mp$ structure a Riemann invariance and that the "algebraic" and "differential" approaches are coincident.
- The case (*b*) above will correspond here to two families of *sws*. This issue could suggest the terminology of pseudo *sws* used above in the anisentropic approach.

Regular interaction of simple waves solutions from the prospects of Burnat's "algebraic" approach or Martin's "differential" approach	Interaction of simple waves solutions from the prospect of two-dimensional Riemann problem [16]
Local character	*Global* character
An eventual *multidimensional* character of the simple waves solutions implied	*One-dimensional* character of the simple waves solutions implied
The *gnl* character must be extended for a multidimensional approach	Usual (Lax [6]) *gnl* character of the one-dimensional simple waves solutions
Orthogonality is only *incidental* for interactions of one-dimensional simple waves solutions	The one-dimensional simple waves solutions implied are *orthogonal*
The simple waves solutions interaction is constructed to be *regular*	The simple waves solutions interaction is *not regular* generally
In a regular interaction: the *gnl* character of the contributing simple waves solutions is associated to a *Burnat nondegeneracy*	
Martin nondegeneracy of a "differential" regular interaction of pseudo simple waves solutions	
Martin's approach is strictly "nonalgebraic"	

CHART 1 Nature of the present approach: from the prospect of a parallel with the approach [16]

Nature of the present approach: details of a parallel with the approach [16]

Some details of a parallel with the approach [16] are included in Chart 1.

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Appendix 1

• ***At first, we prove that for an anisentropic description in (30) the streamlines are not characteristic.*** We consider a local system of coordinates (τ,ν) where $\frac{\partial}{\partial\tau}$ denotes differentiation in the stream direction and $\frac{\partial}{\partial\nu}$ denotes differentiation in the direction which makes an angle of +90^0 with the first. We use

$$\begin{aligned}\frac{\partial}{\partial t}&=\frac{1}{\sqrt{1+v_x^2}}\left(\frac{\partial}{\partial\tau}+v_x\frac{\partial}{\partial\nu}\right)\\ \frac{\partial}{\partial x}&=\frac{1}{\sqrt{1+v_x^2}}\left(v_x\frac{\partial}{\partial\tau}-\frac{\partial}{\partial\nu}\right)\end{aligned}\qquad(A1)$$

to find from (30) the (τ,ν) form

$$\begin{cases}\sqrt{1+v_x^2}\,\dfrac{\partial\rho}{\partial\tau}+\rho v_x\dfrac{\partial v_x}{\partial\tau}=\rho\dfrac{\partial v_x}{\partial\nu}\\ \rho\sqrt{1+v_x^2}\,\dfrac{\partial v_x}{\partial\tau}+v_x\dfrac{\partial p}{\partial\tau}=\dfrac{\partial p}{\partial\nu}\\ \dfrac{\partial S}{\partial\tau}=0,\quad S=S(p,\rho)\;.\end{cases}\qquad(A2)$$

We may replace $(A2)_3$ with equation

$$\frac{\partial p}{\partial\tau}-c^2(p,\rho)\frac{\partial\rho}{\partial\tau}=0,\quad c^2(p,\rho)=-\frac{\left(\frac{\partial S}{\partial\rho}\right)_p}{\left(\frac{\partial S}{\partial p}\right)_\rho}\qquad(A3)$$

and we notice that c^2 considered in $(A3)$ corresponds to c^2 previously defined in (16). Because, from $S[p,\rho(p,S)]\equiv S$ we get

$$\left(\frac{\partial\rho}{\partial p}\right)_S\overset{(16)}{=}\frac{1}{c^2}\overset{(A3)}{=}-\frac{\left(\frac{\partial S}{\partial\rho}\right)_p}{\left(\frac{\partial S}{\partial p}\right)_\rho},\quad\text{for } S=F(\psi)\;.\quad(A4)$$

A particularity of the anisentropic description above is that a relation $S(p,\rho)=F(\psi)$ is available among p,ρ,ψ with a smooth F. We use this particularity in place of $(A2)_3$ or $(A3)$ to determine $\frac{\partial\rho}{\partial\nu}$ at the points of streamline $\mathcal{C}_0$. Precisely, we compute

$$\frac{\partial S}{\partial t}=\frac{\partial S}{\partial p}\frac{\partial p}{\partial t}+\frac{\partial S}{\partial \rho}\frac{\partial \rho}{\partial t}=F'(\psi)\frac{\partial\psi}{\partial t}=-\rho v_x F'(\psi)$$

which could be transcribed cf. $(A1)$, $(A4)$, by:

$$\left[\frac{\partial p}{\partial\tau}-c^2(p,\psi)\frac{\partial\rho}{\partial\tau}\right]+v_x\left[\frac{\partial p}{\partial\nu}-c^2(p,\psi)\frac{\partial\rho}{\partial\nu}\right]$$
$$+\frac{\rho v_x\sqrt{1+v_x^2}\,F'(\psi)}{\left(\frac{\partial S}{\partial p}\right)_\rho}=0\,. \qquad (A5)$$

or, cf. $(A2)_2$, by:

$$c^2(p,\psi)v_x\frac{\partial\rho}{\partial\nu}=\left[\frac{\partial p}{\partial\tau}-c^2(p,\psi)\frac{\partial\rho}{\partial\tau}\right]$$
$$+v_x\left[\rho\sqrt{1+v_x^2}\,\frac{\partial v_x}{\partial\tau}+v_x\frac{\partial p}{\partial\tau}\right]+\frac{\rho v_x\sqrt{1+v_x^2}\,F'(\psi)}{\left(\frac{\partial S}{\partial p}\right)_\rho}\,.$$

We add this relation to $(A2)_{1,2}$ in order to *determine* $\frac{\partial p}{\partial\nu}$, $\frac{\partial v_x}{\partial\nu}$, $\frac{\partial\rho}{\partial\nu}$ at the points of a streamline. ***The proof is now complete.***

• The two families of characteristics $\mathcal{C}_\mp$ of (11) [or (30)] could be put in correspondence with the two families of characteristics $\mathcal{C}_\mp$ of (15). In fact we have

$$\begin{cases} \mathrm{d}\left(\dfrac{\partial\xi}{\partial\psi}\right)\pm\zeta\mathrm{d}p=0 \\ \mathrm{d}\left(\dfrac{\partial\xi}{\partial p}\right)\mp\zeta\mathrm{d}\psi=0 \end{cases} \quad \text{along } \overline{\mathcal{C}}_\mp \qquad (A6)$$

and therefore

$$\mp\mathrm{d}t\overset{(15)_4,(A6)_2}{=}-\zeta\mathrm{d}\psi\overset{(13)_1}{=}-\frac{1}{c}(\mathrm{d}x-v_x\mathrm{d}t)\quad\text{along } \overline{\mathcal{C}}_\mp$$

which corresponds to

$$dx=(v_x\pm c)\mathrm{d}t\quad\text{along } \mathcal{C}_\pm.$$

• Equation $(30)_3$ and the equation fulfilled by the particle function ψ are connected, cf. $S=F(\psi)$, by

$$\frac{\partial S}{\partial t}+v_x\frac{\partial S}{\partial x}=F'(\psi)\left(\frac{\partial\psi}{\partial t}+v_x\frac{\partial\psi}{\partial x}\right)\,. \qquad (A7)$$

We may use $(A7)$ and $S=F(\psi)$ to express S by ψ in (30). Cf.

$$\begin{cases} \dfrac{\partial p}{\partial t}+v_x\dfrac{\partial p}{\partial x}+\rho(p,\psi)c^2(p,\psi)\dfrac{\partial v_x}{\partial x}=0 \\ \dfrac{\partial v_x}{\partial t}+v_x\dfrac{\partial v_x}{\partial x}+\dfrac{1}{\rho(p,\psi)}\dfrac{\partial p}{\partial x}=0 \\ \dfrac{\partial\psi}{\partial t}+v_x\dfrac{\partial\psi}{\partial x}=0\,. \end{cases} \qquad (30)^\psi$$

• The natural character of requirement (14) could result from a comparation with a singular example – corresponding to $\psi=\psi(p)$ (see Stanyukovich [15]). In presence of relation $\psi=\psi(p)$ a solution of $(30)^\psi$ is found in [15] by considering the Lagrange form of $(30)^\psi$

$$\begin{cases} \dfrac{\partial p}{\partial t}+\dfrac{1}{\zeta^2(p,\psi)}\dfrac{\partial v_x}{\partial q}=0 \\ \dfrac{\partial v_x}{\partial t}+\dfrac{\partial p}{\partial q}=0 \\ \dfrac{\partial\psi}{\partial t}=0 \end{cases} \qquad (A8)$$

where q is the Lagrange mass coordinate. We get for the mentioned solution the form

$$p=kq+k',\;\; v_x=-kt+k'';\;\; \psi=\psi(p) \qquad (A9)$$

which indicates a *particular* structure. Requirement (14) and the anisentropic "differential" approach above appear to be more permissive.

• In the isentropic case the system (30) becomes a 2×2 system – for the entities p,v_x for example – for which the streamlines are not among the characteristics. Along a streamline $\mathcal{C}_0$ the normal derivatives $\frac{\partial p}{\partial\nu}$, $\frac{\partial v_x}{\partial\nu}$ are determined. The normal derivative $\frac{\partial\rho}{\partial\nu}$ is also determined – via the dependence $\rho=\rho(p)$ – as in this case the equation $(A5)$ appears to be identically fulfilled, cf.

$$\frac{\partial p}{\partial\mu}-c^2(\rho)\frac{\partial\rho}{\partial\mu}=0,\;\;\mu=\tau,\nu;\quad F'(\psi)\equiv 0\,.$$

• In the isentropic case the equation $(30)^\psi_3$ appears to be uncoupled in $(30)^\psi$. Therefore ψ will not be regarded as a hodograph entity. Still, the requirement (14) is again useful for initiating a "differential" approach – associated with $\zeta=\zeta(p)$ in (16).

Appendix 2

• In Chart 2 we present an exhaustive list of circumstances parallel to (19), (20). In Martin [11] it is proved that all the circumstances on this list are linearizable. Precisely (Martin [11]), in each circumstance mentioned in Chart 2 [associated with two intermediate integrals] a solution of the system (11) can be found via linear equations of the first order, or Euler–Poisson–Darboux equations of the second order, and quadratures. We notice that for the circumstance (19), (20) [No.4 in Chart 2] we get for example

$$L_{\nu_+}\left[\frac{1}{p}\right]=0,\;L_{\mu_+}[v_x]=0,\;L_{\mu_-}\left[\frac{1}{\psi}\right]=0,\;L_{\nu_-}[t]=0 \qquad (A10)$$

where

$$L_\nu[w] \equiv \frac{\partial^2 w}{\partial R_+ \partial R_-} - \frac{\nu}{R_+ - R_-}\left(\frac{\partial w}{\partial R_+} - \frac{\partial w}{\partial R_-}\right) = 0, \quad \text{constant } \nu \tag{A11}$$

is an *Euler–Poisson–Darboux (EPD) equation* and for $\gamma > 1$ we compute

$$\nu_+ = \frac{3\gamma - 1}{2(\gamma - 1)} > 0, \;\; \mu_+ = -\nu_+ < 0,$$
$$\nu_- = \nu_+ - 1 = \frac{\gamma + 1}{2(\gamma - 1)} > 0, \;\; \mu_- = -\nu_- < 0\ .$$

We notice finally that $x(R_-, R_+)$ results by quadratures cf.

$$x = \int [v_x(R_-, R_+) + c(R_-, R_+)]\frac{\partial t}{\partial R_+}\mathrm{d}R_+ + [v_x(R_-, R_+) - c(R_-, R_+)]\frac{\partial t}{\partial R_-}\mathrm{d}R_-\ . \tag{A12}$$

Appendix 3

• For an *integral* ν the general solution (depending on two arbitrary functions) of the *EPD* equation (A11) has the explicit closed form

$$w = \begin{cases} \dfrac{\partial^{2\nu-2}}{\partial R_+^{\nu-1}\partial R_-^{\nu-1}}\left[\dfrac{\mathfrak{f}(R_+)+\mathfrak{g}(R_-)}{R_+-R_-}\right]; \text{ for } \nu > 0 \\ (R_+-R_-)^{2\mu+1}\dfrac{\partial^{2\mu}}{\partial R_+^{\mu}\partial R_-^{\mu}}\left[\dfrac{\mathfrak{f}(R_+)+\mathfrak{g}(R_-)}{R_+-R_-}\right]; \\ \qquad \mu = -\nu; \text{ for } \nu \le 0 \end{cases} \tag{A13}$$

where $\mathfrak{f}$ and $\mathfrak{g}$ are arbitrary functions of R_+ and R_- respectively (see Ludford [7], Ludford and Martin [8] on this subject).

• An extension of (62) to *non-integral* values of ν is available by means of certain integral representations (Mackie [9]).

No.	$\zeta = P(p)\Psi(\psi)$	$\mathcal{F}_\pm\left(p, \psi, \frac{\partial\xi}{\partial p}, \frac{\partial\xi}{\partial\psi}\right)$
1	1	$\frac{\partial\xi}{\partial p} \pm \psi = \Phi_\pm\left(\frac{\partial\xi}{\partial\psi} \mp p\right)$ (arbitrary $\Phi_\pm$)
2	$P(p) \not\equiv 0,\ \Psi(\psi) \equiv 1$	$\frac{\partial\xi}{\partial\psi} \pm \int P\mathrm{d}p$
3	$\frac{1}{p^2}$	$p\frac{\partial\xi}{\partial p} + \psi\frac{\partial\xi}{\partial\psi} - \xi \pm \frac{\psi}{p} = \Phi_\pm\left(\frac{\partial\xi}{\partial\psi} \pm \frac{1}{p}\right)$ (arbitrary $\Phi_\pm$)
4	$\frac{\psi^{\nu-1}}{p^{\nu+1}}$ $(\nu \neq 0)$	$p\frac{\partial\xi}{\partial p} + \psi\frac{\partial\xi}{\partial\psi} - \xi \pm \frac{1}{\nu}\left(\frac{\psi}{p}\right)^\nu$
5	$\frac{\psi^{-1}}{p}$	$p\frac{\partial\xi}{\partial p} + \psi\frac{\partial\xi}{\partial\psi} - \xi \pm \ln\left(\frac{\psi}{p}\right)$
6	$\mathrm{e}^p\mathrm{e}^\psi$	$\frac{\partial\xi}{\partial\psi} - \frac{\partial\xi}{\partial p} \pm \mathrm{e}^p\mathrm{e}^\psi$

CHART 2 Exhaustive list of intermediate integrals of (16) for $\zeta \not\equiv 0$ (Martin [11])