THE MODULUS OF CONTINUITY OF WEGNER ESTIMATES FOR RANDOM SCHRÖDINGER OPERATORS ON METRIC GRAPHS

MICHAEL J. GRUBER AND IVAN VESELIĆ

ABSTRACT. We consider an alloy type potential on an infinite metric graph. We assume a covering condition on the single site potentials. For random Schrödingers operator associated with the alloy type potential restricted to finite volume subgraphs we prove a Wegner estimate which reproduces the modulus of continuity of the single site distribution measure. The Wegner constant is independent of the energy.

1. INTRODUCTION

We study the distribution of eigenvalues of alloy-type random Schrödinger operators on finite metric graphs. More precisely, we prove a Wegner estimate on the average number of eigenvalues in a given energy interval. Our (upper) bound is proportional to the volume of the finite metric graph and reproduces the modulus of continuity of the single site random coupling constants. The volume is the sum of the lengths of all edges. In particular, for equilateral graphs it is the number of edges times the single edge length.

In the case that there exists a selfaveraging integrated density of states our result implies estimates on its modulus of continuity.

Wegner estimates have been studied for Anderson-type operators on $l^2(\mathbb{Z}^d)$ and alloy-type operators on $L^2(\mathbb{R}^d)$ starting with the paper [16]. The strategy presented here follows the line of argument of [7]. There the same type of Wegner bound as in this note was proven for discrete operators on $l^2(\mathbb{Z}^d)$. For alloy-type operators on $L^2(\mathbb{R}^d)$ a similar upper bound was derived. There however, a logarithmic correction term due to the (possible) singularity of the spectral shift function appears.

In the recent [2] a Wegner estimate for random Schrödinger operators on $L^2(\mathbb{R}^d)$ without the logarithmic correction in the energy interval length is given. The approach chosen there does not use the spectral shift function and thus avoids the logarithmic term. For Anderson models on $l^2(\mathbb{Z}^d)$ with Hölder continuous distribution of the potential in [12] an alternative proof of the results in [7] was presented.

The bounds proven in this paper for quantum graph operators have a Wegner constant which is independent of the position of the energy interval. The same situation is encountered for discrete operators, whereas for alloy-type operators on $L^2(\mathbb{R}^d)$ the Wegner constant grows with the energy. Note that this uniformity in our case is only possible since we assume

a covering condition on the single site potentials. For site potentials of small support the uniformity does not hold, cf. [6].

For more background on the general theory of random Schrödinger operators see for instance [1, 13] and for more information on Wegner estimates and the integrated density of states [9, 15], e.g.

The next section contains the results of the paper and the last section the proofs.

2. Model and results

Definition 1. Let V and E be countable sets and C^Γ a map

$$C^\Gamma \colon E \to V \times V \times [0, \infty), \quad e \mapsto (\iota(e), \tau(e), l_e).$$

We call the triple $G = (V, E, C^\Gamma)$ a metric graph, elements of $V = V(G)$ vertices, elements of $E = E(G)$ edges, $\iota(e)$ the initial vertex of e, $\tau(e)$ the terminal vertex of e and l_e the length of e. Both $\iota(e)$ and $\tau(e)$ are called endvertices of e, or incident to e. The two endvertices of an edge are allowed to coincide. The number of edges incident to the vertex v is called the *degree of v*. We assume that all vertices have finite degree.

Each edge e will be identified with the open interval $(0, l_e)$, where the point 0 corresponds to the vertex $\iota(e)$ and l_e to $\tau(e)$. The identification of edges by intervals allows us to define in a canonical way the length of a path between two points in G. Taking the infimum over the lengths of paths connecting two given points in G, one obtains a distance function $d \colon G \times G \to [0, \infty)$. Since we assumed that each vertex of G has bounded degree, the map d is indeed a metric, cf. for instance Section 2.2 in [14]. Thus we have turned G into a metric space (G, d).

For a finite subset $\Lambda \subset E$ we define the subgraph G_Λ by deleting all edges $e \in E \setminus \Lambda$ and the arising isolated vertices. We denote the set of vertices of G_Λ by V_Λ, the set of vertices $v \in V_\Lambda$ with $\deg_{G_\Lambda} v < \deg_G v$ by V_Λ^∂, and its complement $V_\Lambda \setminus V_\Lambda^\partial$ by V_Λ^i. Elements of V_Λ^∂ are called boundary vertices of G_Λ and elements of V_Λ^i interior vertices of G_Λ.

For any $\Lambda \subset E$ the Hilbert spaces $L_2(G_\Lambda)$ have a natural direct sum representation $L_2(G_\Lambda) = \oplus_{e \in \Lambda} L_2(0, l_e)$. In particular for $\Lambda = E$ we have $L_2(G) = \oplus_{e \in E} L_2(0, l_e)$, and for $\tilde\Lambda \subset \Lambda \subset E$ we have $L_2(G_{\tilde\Lambda}) = \oplus_{e \in \tilde\Lambda} L_2(0, l_e) \subset L_2(G_\Lambda) = \oplus_{e \in \Lambda} L_2(0, l_e)$.

For a function $\phi \colon G \to \mathbb{C}$ and an edge $e \in E$ we denote by $\phi_e := \phi|_e$ its restriction to e (which is identified with $(0, l_e)$). We denote by $C(G)$ the space of continuous, complex-valued functions on the metric space (G, d). Similarly, $C(G_\Lambda)$ denotes the space of continuous, complex-valued functions on the metric sub-space (G_Λ, d). For each $v \in V$, any edge e incident to v, and function $f \in W^{2,2}(e) \subset C^1(e) \cong C^1(0, l_e)$ we define the boundary value $f(v)$ by continuity and the derivatives by

$$(1) \qquad \partial_e f(v) := \partial_e f(0) := \lim_{\epsilon \searrow 0} \frac{f(\epsilon) - f(0)}{\epsilon} \qquad\qquad \text{if } v = \iota(e)$$

and

$$(2) \qquad \partial_e f(v) := \partial_e f(l_e) := \lim_{\epsilon \searrow 0} \frac{f(l_e - \epsilon) - f(l_e)}{\epsilon} \qquad\qquad \text{if } v = \tau(e).$$

Note that, since $f|_e \in W^{2,2}(e)$ the function f is not only continuously differentiable on the open segment $(0, l_e)$, but also its derivative has well defined limits at both boundaries 0 and l_e.

Our sign convention ensures that $\partial_e f(v)$ is the inward normal derivative, and is independent of the orientation of the edge induced by ι, τ.

For any $\Lambda \subset E$ it will be convenient to use the following Sobolev space

$$W^{2,2}(\Lambda) := \oplus_{e \in \Lambda} W^{2,2}(e) \subset C^1(\Lambda) := \oplus_{e \in \Lambda} C^1(e)$$

$$\text{with the norm } \|\phi\|_{W^{2,2}(\Lambda)}^2 := \sum_{e \in \Lambda} \|\phi_e\|_{W^{2,2}(0, l_e)}^2.$$

Note that this space is defined on the edge set only and does not see the graph structure of G.

For $f \in W^{2,2}(\Lambda)$ and each vertex v we gather the boundary values $f_e(v)$ over all edges e adjacent to v in a vector $f(v) := \{f_e(v) : e \in E, v \text{ incident to } e\}$. Similarly, we gather the boundary values of $\partial_e f_e(v)$ over all edges e adjacent to v in a vector $\partial f(v)$.

Given the boundary values of functions, we can now dicuss the concept of boundary condition. Here we use material from [10, 5] to which we refer for further details and proofs. A single-vertex boundary condition at $v \in V$ is a choice of subspace S_v of $\mathbb{C}^{\deg v} \times \mathbb{C}^{\deg v}$ with dimension $\deg v$ such that

$$\eta((s, s'), (t, t')) := \langle s', t \rangle - \langle s, t' \rangle$$

vanishes for all $(s, s'), (t, t') \in S_v$. An $f \in W^{2,2}(\Lambda)$ is said to satisfy the single-vertex boundary condition S_v at v if $(f(v), \partial f(v))$ belongs to S_v. A field of single-vertex boundary conditions $S := \{S_v : v \in V_\Lambda\}$ will be called boundary condition. Given such a field, we obtain a selfadjoint realization Δ^Λ of the Laplacian Δ on $L^2(E^\Lambda)$ by choosing the domain

$$\mathcal{D}(\Delta^\Lambda) := \{f \in W^{2,2}(\Lambda) : \forall v : (f(v), \partial f(v)) \in S_v\}.$$

Particularly relevant boundary conditions are Dirichlet boundary conditions with subspace S^D consisting of all those (s, s') with $s = 0$, Neumann conditions with subspace S^N consisting of all those (s, s') with $s' = 0$, and Kirchhoff (also known as free or standard) boundary conditions S^K consisting of all (s, s') with s having all components equal and s' having the sum over its components equal to 0.

We define the linear operator

$$-\Delta^\Lambda \colon \mathcal{D}(\Delta^\Lambda) \to L^2(\Lambda)$$

by the rule

$$(-\Delta^\Lambda f)(x) := -\frac{\partial^2 f_e(x)}{\partial x^2}$$

if $x \in G^\Lambda$ is contained in the edge e. This way the function $-\Delta^\Lambda f$ is defined on the set $E^\Lambda \subset G^\Lambda$, whose complement $V^\Lambda = G^\Lambda \setminus E^\Lambda$ in the metric space G^Λ has Hausdorff measure zero.

In our application for the Wegner estimate, we will start with boundary conditions defined on the graph G and then restrict to the induced subgraph G_Λ for a finite subset $\Lambda \subset E(G)$ as described above. On V_Λ^i the boundary conditions will be induced by those on $V(G)$, but on V_Λ^∂ there is no canonical choice. We choose to put Dirichlet conditions on V_Λ^∂ in order to define the restriction $-\Delta^\Lambda$ of $-\Delta^{E(G)}$ unambiguously.

An *alloy-type potential* is a stochastic process $\mathcal{V} \colon \Omega \times G \to \mathbb{R}$ of the form $\mathcal{V}_\omega = \sum_{e \in E} \omega_e u_e$, with the conditions outlined in the following.

The *coupling constants* $\omega_e, e \in E$, are a sequence of bounded random variables which are independent and identically distributed with distribution μ. We call μ the *single site*

distribution since in our model each edge e is the site of a single perturbation controlled by ω_e. Note that u_e is not necessarily supported on e, see below. The expectation of the product measure $\mathbb{P} := \bigotimes_{e \in E} \mu$ is denoted by $\mathbb{E}$.

The family of single site potentials $u_e, e \in E$, is assumed to fulfill a covering condition and a summability condition:

Definition 2. The family of single site potentials $u_e, e \in E$, is said to fulfill a *covering condition* with lower bound $\kappa > 0$ if, for each finite set of edges Λ, there is a finite set of edges Λ^u such that
$$\sum_{e \in \Lambda^u} u_e \geq \kappa$$
holds on the graph G_Λ.

For the following definition, recall that for a metric graph $\tilde{G}$ with finite set of edges $\tilde{E}$ and length function $e \mapsto l_e$ the volume is given by $\operatorname{vol} \tilde{G} = \sum_{e \in \tilde{E}} l_e$. In contrast to this, $|\Lambda|$ denotes the number of edges in $\Lambda \subset E$.

Definition 3. Denote by Λ_e the minimal set of edges containing the support of $u_e|_\Lambda$ and by V_e^∂ the boundary vertices of the induced subgraph G_{Λ_e}. (Here for simplicity we suppress the dependence of V_e^∂ on Λ.) Then, the family of single site potentials $u_e, e \in E$ is called *summable* if there are constants $C_j, j = 1, 2, 3$, such that

(3)
$$\sum_{e \in \Lambda^u} \sum_{v \in V_e^\partial} \deg v \leq C_1 |\Lambda| \quad \text{(finite degree property)},$$
$$\sum_{e \in \Lambda^u} \sqrt{\|u_e\|_\infty} \operatorname{vol} G_{\Lambda_e} \leq C_2 |\Lambda| \quad (L^2\text{-boundedness}),$$
$$\sum_{e \in \Lambda^u} |\Lambda_e| \leq C_3 |\Lambda| \quad \text{(volume growth)}$$

for each finite set of edges Λ.

In particular, this holds if u_e is supported on e, uniformly bounded above and away from 0 (so that $\Lambda^u = \Lambda$, $\Lambda_e = e$) and there are uniform bounds on vertex degrees and edge lengths. But our definition is much more general. For instance, decreasing edge lengths can compensate for potential growth and vice versa.

In the following we consider for a finite subset $\Lambda \subset E$ and an alloy-type potential whose family of single site potentials fulfills the covering condition and is summable the restriction $\mathcal{V}_\omega^\Lambda = \mathcal{V}_\omega \chi_\Lambda$. On $\mathcal{D}(\Delta^\Lambda)$ we define a *random Schrödinger operator of alloy-type* by $H_\omega^\Lambda = -\Delta^\Lambda + \mathcal{V}_\omega^\Lambda$. Since the potential is bounded, H_ω^Λ is selfadjoint with boundary conditions described by the field $\{S_v : v \in V_\Lambda\}$ and lower semi-bounded. In the following we will be dealing exclusively with Schrödinger operators H_ω^Λ on *finite* edge sets Λ.

Remark 4. For random Schrödinger operators on the whole, infinite graph G one needs to impose more restrictive conditions if one wants to ensure that the Schrödinger operator is lower semi-bounded. For instance, Dirichlet, Neumann, standard/free/Kirchhoff conditions and others with "$L_+ = 0$" lead to positive graph Laplacians (see [11]).

Finally, we introduce the *modulus of continuity* of the distribution of the single site distribution, for $\varepsilon > 0$, as

(4)
$$s(\mu, \varepsilon) = \sup\{\mu([\lambda - \varepsilon, \lambda + \varepsilon]) \mid \lambda \in \mathbb{R}\}.$$

With these definitions, we can formulate

Theorem 5. *Let $\mathcal{V}_\omega$ be an alloy-type potential. Then there exists a constant C_W such that for all $\lambda \in \mathbb{R}$, all finite sets of edges Λ and all $\varepsilon \leq 1/2$*

(5) $$\mathbb{E}\{\mathrm{Tr}[\chi_{[\lambda-\varepsilon,\lambda+\varepsilon]}(H_\omega^\Lambda)]\} \leq C_W \; s(\mu,\varepsilon)\,|\Lambda|\,.$$

Remark 6 (integrated density of states). For the discussion in this remark we assume that all edge lengths $l_e \equiv l$ are equal, that all single site potentials u_e have the same shape, that the boundary condition S_v at the vertex $v \in V$ depends only on the degree $\deg(v)$, and that there is a uniform bound $d_+ := \sup_{v \in V} \deg(v) < \infty$ on the vertex degree. In that case H_ω^Λ is selfadjoint and lower semi-bounded even for $\Lambda = E$, and the domain of the operator is independent of $\omega \in \Omega$. For an energy $\lambda \in \mathbb{R}$ and an exhaustion $\Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \subset \ldots$ of the edge set E consider the sequence of random variables

$$N_\omega^n(\lambda) := \frac{1}{\mathrm{vol}\, G_{\Lambda_n}} \mathrm{Tr}[\chi_{(-\infty,\lambda]}(H_\omega^{\Lambda_n})].$$

Under certain additional conditions one can show that the sequence of distribution functions $N_\omega^n(\lambda), n \in \mathbb{N}$ converges and that the resulting limiting distribution function N is independent of $\omega \in \Omega$ almost surely. If this is the case Theorem 5 implies the following continuity property of N:

$$\forall \varepsilon \in [0,1/2], \forall \lambda \in \mathbb{R} \quad : \quad N(\lambda + \epsilon) - N(\lambda) \leq C_W \; s(\mu,\varepsilon)$$

Thus the integrated density of states inherits the continuity modulus of the single site measure μ. Let us stress that the Wegner constant C_W is energy independent.

For a particular example of a random Schrödinger operator on a metric graph with $\mathbb{Z}^d$-structure the construction of the integrated density of states has been carried out in [6] following the strategy of [8].

Wegner estimates play an important role in the proof of spectral localization for random Schrödinger operators via the so called multiscale analysis. For an alloy type model on a $\mathbb{Z}^d$-metric graph localization has been proven in [3] using (weaker) Wegner estimates and multiscale analysis.

3. Proofs

Consider a pair of selfadjoint, lower semi-bounded operators H_1, H_2. If the spectrum of both H_1 and H_2 is purely discrete, the spectral shift function (SSF) $\xi(\cdot) = \xi(\cdot, H_2, H_1)$ is defined as the difference of the eigenvalue counting functions, i.e.

$$\xi(\lambda) := \mathrm{Tr}[\chi_{(-\infty,\lambda]}(H_2) - \chi_{(-\infty,\lambda]}(H_2)].$$

If we merely assume that the difference $H_2 - H_1$ is trace class, then there is still a unique function ξ such that *Krein's trace identity*

(6) $$\mathrm{Tr}\,[\rho(H_2) - \rho(H_1)] = \int \rho'(\lambda)\,\xi(\lambda, H_2, H_1)\,d\lambda$$

holds for all $\rho \in C^\infty$ with compactly supported derivative. In the case of operators with discrete spectrum both definitions of the function ξ coincide. We can weaken the assumption on the operator pair further. Assume that there exists a monotone, smooth function $g \colon \mathbb{R} \to [0,\infty)$ which is bounded on the spectra of H_1 and H_2 and such that $g(H_2) - g(H_1)$ is trace class. In that case the definition

(7) $$\xi(\lambda, H_2, H_1) := \mathrm{sign}(g')\,\xi\big(g(\lambda), g(H_2), g(H_1)\big).$$

makes sense and is independent of the choice of the function g.

Lemma 7. *Let $\tilde{G}$ be a finite or infinite metric graph, $\tilde{\Lambda}$ a finite subset of its edges, $-\Delta$ a selfadjoint realization of the Laplacian on $L^2(\tilde{G})$ and W_1, W_2 two potentials acting as bounded operators on $L^2(\tilde{G})$ such that $\operatorname{supp}(W_2 - W_1) \subset \tilde{G}_{\tilde{\Lambda}}$. Set $H_j = -\Delta + W_j, j = 1, 2$, and assume that the SSF ξ_{H_1, H_2} is well defined. Denote the restriction of H_j to $L^2(\tilde{\Lambda})$ by $h_j, j = 1, 2$. Then we have*

$$|\xi_{H_1, H_2}(\lambda)| \leq \sum_{v \in V_{\tilde{\Lambda}}^{\partial}} \deg(v) + |\xi_{h_1, h_2}(\lambda)|.$$

Proof. The basic idea is to decouple the interior of $\tilde{G}_{\tilde{\Lambda}}$ from the exterior by choosing appropriate boundary conditions on $V_{\tilde{\Lambda}}^{\partial}$; this is in the spirit of the statements around equation (9) – (11) in [6]. So, let H_j^D be H_j but with Dirichlet conditions in $V_{\tilde{\Lambda}}^{\partial}$. Then $\left|\xi_{H_j, H_j^D}(\lambda)\right| \leq \sum_{v \in V_{\tilde{\Lambda}}^{\partial}} \deg(v)$ according to Corollary 11 and Lemma 13 of [4]. Now, the H_j^D decompose into a direct sum of exterior and interior parts, the former coinciding by assumption, the latter being given by h_j. This proves the assertion.   □

We will apply this lemma for a set of edges $\tilde{\Lambda} := \Lambda_e$ containing the support of u_e, with finite $\tilde{G} = G_{\Lambda_e}$ such that the SSF exists automatically.

Lemma 8. *Let $-\Delta$ be a selfadjoint realization of the Laplacian on an arbitrary finite graph $\tilde{G}$ and let W_1, W_2 be bounded potentials on $L^2(\tilde{G})$. Set $H_j = -\Delta + W_j, j = 1, 2$. Then*

$$|\xi_{H_1, H_2}(\lambda)| \leq \left(\sqrt{\|W_1\|} + \sqrt{\|W_2\|} \right) \frac{\operatorname{vol} \tilde{G}}{\pi} + 5|E(\tilde{G})|$$

where $\operatorname{vol} \tilde{G} = \sum_{e \in E(\tilde{G})} l_e$ is the one-dimensional volume of $\tilde{G}$.

Proof. This is an extension of Lemma 14 in [4], where this is proved for metric graphs with edges of length 1. Using the same proof but keeping track of the lengths l_e yields the desired estimate.   □

Proof of Theorem 5. Let ρ be a smooth, monotone switch function $\rho := \rho_{\lambda, \varepsilon} \colon \mathbb{R} \to [-1, 0]$. By a switch function we mean that for a positive $\varepsilon \leq 1/2$, ρ has the following properties: $\rho \equiv -1$ on $(-\infty, \lambda - \varepsilon]$, $\rho \equiv 0$ on $[\lambda + \varepsilon, \infty)$ and $\|\rho'\|_\infty \leq 1/\varepsilon$. Then

$$\chi_{[\lambda - \varepsilon, \lambda + \varepsilon]}(x) \leq \rho(x + 2\varepsilon) - \rho(x - 2\varepsilon)$$

We may assume without loss of generality $\sum_{e \in \Lambda^u} u_e \geq 1$, i.e. $\kappa = 1$. By the min-max principle for eigenvalues, we conclude

$$\operatorname{Tr}[\rho(H_\omega^\Lambda + \varepsilon)] \leq \operatorname{Tr}\left[\rho\left(H_\omega^\Lambda + \varepsilon \sum_{e \in \Lambda^u} u_e\right)\right].$$

Let Λ^u be as above. Then $\sum_{e \in \Lambda^u} u_e(x) \geq 1$ for $x \in G_\Lambda$. Λ^u contains $L := |\Lambda^u|$ edges. We enumerate the edges in Λ^u by $e \colon \{1, \ldots, L\} \to \Lambda^u$, $n \mapsto e(n)$, and set

$$W_0 \equiv 0, \quad W_n = \sum_{m=1}^{n} u_{e(m)}, \qquad n = 1, 2, \ldots, L$$

Thus

$$\chi_{[\lambda-\varepsilon,\lambda+\varepsilon]}(H_\omega^\Lambda) \leq \rho(H_\omega^\Lambda + 2\varepsilon) - \rho(H_\omega^\Lambda - 2\varepsilon)$$

(8)
$$\leq \rho(H_\omega^\Lambda - 2\varepsilon + 4\varepsilon W_L) - \rho(H_\omega^\Lambda - 2\varepsilon)$$

$$= \sum_{n=1}^{L} \rho(H_\omega^\Lambda + 2\varepsilon + 4\varepsilon W_n) - \rho(H_\omega^\Lambda - 2\varepsilon + 4\varepsilon W_{n-1})$$

We fix $n \in \{1,\dots,L\}$, define

$$\omega^\perp := \{\omega_e^\perp\}_{e\in\Lambda^u}, \qquad \omega_e^\perp := \begin{cases} 0 & \text{if } e = e(n), \\ \omega_e & \text{if } e \neq e(n), \end{cases}$$

and set

$$\phi_n(\eta) = \mathrm{Tr}\big[\rho(H_{\omega^\perp}^\Lambda - 2\varepsilon + 4\varepsilon W_{n-1} + \eta u_{e(n)})\big], \quad \eta \in \mathbb{R}.$$

The function ϕ_n is continuously differentiable, monotone increasing and bounded. By definition of ϕ_n,

$$\mathrm{Tr}[\rho(H_\omega^\Lambda - 2\varepsilon + 4\varepsilon W_n)) - \rho(H_\omega^\Lambda - 2\varepsilon + 4\varepsilon W_{n-1})] = \phi_n(\omega_{e(n)} + 4\varepsilon) - \phi_n(\omega_{e(n)})]$$

since $\phi_n(\eta) = \mathrm{Tr}\big[\rho(H_\omega^\Lambda - 2\varepsilon + 4\varepsilon W_{n-1} + (\eta - \omega_{e(n)})u_{e(n)})\big]$, so that

$$\mathbb{E}_{\omega_{e(n)}}\{\mathrm{Tr}[\rho(H_\omega^\Lambda - 2\varepsilon + 4\varepsilon W_n)) - \rho(H_\omega^\Lambda - 2\varepsilon + 4\varepsilon W_{n-1})]\} = \int [\phi_n(\omega_{e(n)} + 4\varepsilon) - \phi_n(\omega_{e(n)})]\, d\mu(\omega_{e(n)})$$

where $\mathbb{E}_{\omega_{e(n)}}$ denotes the expectation with respect to the random variable $\omega_{e(n)}$ only. Let $\mathrm{supp}(\mu) \subset (a,b)$. Using Lemma 6 in [7] we have

$$\int [\phi_n(\omega_{e(n)} + 4\varepsilon) - \phi_n(\omega_{e(n)})]\, d\mu(\omega_{e(n)}) \leq s(\mu, 4\varepsilon)[\phi_n(b) - \phi_n(a)]$$

Denote by $\xi_{\Lambda,n}$ the SSF associated to the pair of operators $H_n(a), H_n(b)$ on $L^2(\Lambda)$ where $H_n(\eta)$ is given by $H_n(\eta) := H_\omega^\Lambda - 2\varepsilon + 4\varepsilon W_{n-1} + (\eta - \omega_{e(n)})u_{e(n)}$. Then by the Krein trace identity and the normalization of ρ

$$\phi_n(b) - \phi_n(a) = \int_a^b \rho'\, \xi_{\Lambda,n}\, d\lambda \leq \|\xi_{\Lambda,n}\|_\infty$$

Let Λ_e, u_e, V_e^∂ and G_{Λ_e} be as in the definition of summable potentials. By $\xi_{\Lambda_{e(n)},n}$ we denote the SSF associated to the pair $H_n(a), H_n(b)$, but now considered as operators on $L^2(\Lambda_{e(n)})$. Apply Lemma 7 to obtain:

$$\|\xi_{\Lambda,n}\|_\infty \leq \sum_{v\in V_{e(n)}^\partial} \deg(v) + \|\xi_{\Lambda_{e(n)},n}\|_\infty$$

Now apply Lemma 8 successively L times to obtain

$$\mathbb{E}\{\mathrm{Tr}[\chi_{[\lambda-\varepsilon,\lambda+\varepsilon]}(H_\omega^\Lambda)]\} \leq s(\mu, 4\varepsilon) \sum_{n=1}^{L} \left(\sum_{v\in V_{e(n)}^\partial} \deg(v) + \sqrt{\|u_{e(n)}\|_\infty} \frac{\mathrm{vol}\, G_{\Lambda_{e(n)}}}{\pi} + 5|\Lambda_{e(n)}| \right)$$

$$\leq s(\mu, 4\varepsilon)(C_1 + C_2/\pi + 5C_3)|\Lambda|$$

by the summability condition on the family of single site potentials. $\qquad \square$

References

[1] R. Carmona and J. Lacroix. *Spectral Theory of Random Schrödinger Operators*. Birkhäuser, Boston, 1990.

[2] J.-M. Combes, P. D. Hislop, and F. Klopp. An optimal Wegner estimate and its application to the global continuity of the integrated density of states for random Schrödinger operators. 2006, arXiv:math-ph/0605029.

[3] P. Exner, M. Helm, and P. Stollmann. Localization on a quantum graph with a random potential on the edges. 2006, arXiv:math-ph/0612087.

[4] M. J. Gruber, D. Lenz, and I. Veselić. Uniform existence of the integrated density of states for random Schrödinger operators on metric graphs over $\mathbb{Z}^d$. 2006, arXiv:math.SP/0612743.

[5] M. Harmer. Hermitian symplectic geometry and extension theory. *J. Phys. A*, 33(50):9193–9203, 2000.

[6] M. Helm and I. Veselić. A linear Wegner estimate for alloy type Schrödinger operators on metric graphs. 2006, arXiv:math.SP/0611609.

[7] D. Hundertmark, R. Killip, S. Nakamura, P. Stollmann, and I. Veselić. Bounds on the spectral shift function and the density of states. *Comm. Math. Phys.*, 262(2):489–503, 2006.

[8] W. Kirsch and F. Martinelli. On the density of states of Schrödinger operators with a random potential. *J. Phys. A: Math. Gen.*, 15:2139–2156, 1982.

[9] W. Kirsch and B. Metzger. The integrated density of states for random Schrödinger operators. In *Spectral Theory and Mathematical Physics: A Festschrift in Honor of Barry Simon's 60th Birthday*, volume 76 of *Proceedings of Symposia in Pure Mathematics*, pages 649–698. AMS, 2007, arXiv:math-ph/0608066.

[10] V. Kostrykin and R. Schrader. Kirchhoff's rule for quantum wires. *J. Phys. A*, 32(4):595–630, 1999.

[11] V. Kostrykin and R. Schrader. Laplacians on metric graphs: eigenvalues, resolvents and semigroups. In *Quantum graphs and their applications*, volume 415 of *Contemp. Math.*, pages 201–225. Amer. Math. Soc., Providence, RI, 2006.

[12] M. Krishna. Continuity of integrated density of states – independent randomness. 2006, arXiv:math-ph/0609040.

[13] L. A. Pastur and A. L. Figotin. *Spectra of Random and Almost-Periodic Operators*. Springer Verlag, Berlin, 1992.

[14] C. Schubert. Laplace-Operatoren auf Quantengraphen. Diplomarbeit, TU Chemnitz, 2006.

[15] I. Veselić. Existence and regularity properties of the integrated density of states of random Schrödinger operators, 2006. Habilitation thesis. TU Chemnitz. To be published in the Springer LNM series.

[16] F. Wegner. Bounds on the density of states in disordered systems. *Z. Phys. B*, 44(1-2):9–15, 1981.

TU Clausthal, Institut für Mathematik, 38678 Clausthal-Zellerfeld, Germany
URL: http://www.math.tu-clausthal.de/~mjg/

TU Chemnitz, Fakultät für Mathematik, 09107 Chemnitz, Germany
URL: http://www.tu-chemnitz.de/mathematik/schroedinger/members.php