ON CALCULATION OF KIRCHHOFF COEFFICIENTS FOR HELMHOLTZ RESONATOR

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Abstract

Helmholtz resonator is a shell Ω_{shell} separating a compact domain- the cavity - $\Omega_{int} \subset R_3$ from the non-compact domain $\Omega_{out} = R_3 \backslash [\Omega_{int} \cup \Omega_{shell}]$. It is assumed that a small opening in the shell connects Ω_{int} with Ω_{out}, causing transformation of real eigen-frequencies of the Neumann Laplacian on Ω_{int} into complex scattering frequencies of the corresponding stationary acoustic problem $-\Delta u = \lambda u$, in $\Omega = \Omega_{int} \cup \Omega_{out}$, with Neumann boundary condition on the C_2-smooth boundary $\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0$. The Kirchhoff model [1] gives a convenient Ansatz

$$\Psi_{out}(x,\nu,\lambda) = \Psi^N_{out}(x,\nu,\lambda) + A_{out}\, G^N_{out}(x,a,,\lambda),\ x \in \Omega_{out},$$

$$\Psi_{int}(x,\nu,\lambda) = A_{int}\, G^N_{int}(x,a,\lambda),\ x \in \Omega_{int}, \tag{1}$$

for calculation of components of the scattered wave of the acoustic problem in Ω_{int}, Ω_{out} in terms of the scattered wave $\Psi^N_{out}(x,\nu,\lambda)$ and the Green functions $G^N_{int,out}(x,a,\lambda)$ of the Neumann Laplacians in $\Omega_{int,out}$. In this paper we suggest an explicit formula for the Kirchhoff coefficients $A_{int,out}$, based on construction of a fitted solvable model for the Helmholtz resonator with a narrow short channel connecting $\Omega_{int}, \Omega_{out}$. The scattering matrix of the model serves an approximation of the scattering matrix of the Helmholtz resonator on a certain *essential* spectral interval. Calculation of the scattering frequencies of the model is reduced to solution of an algebraic equation.

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Outline

1 Preliminaries

Helmholtz resonator is a compact shell in R_3, with a piece-wise smooth boundary. The shell separates the outer domain Ω_{out} from the inner domain - the cavity- Ω_{int}. We do not suppose that the shell is uniformly thin, but assume that the outer domain and the cavity are connected by the cylindrical channel Ω^δ, length H, radius $\delta << H$, with imaginable "upper" and "lower" lids Γ_H, Γ, separating the channel from Ω_{out}, Ω_{int} respectively. In this note we assume that the lids are flat and orthogonal to the axis of the channel. On the domain $\Omega = \Omega_{out} \cup \Omega_{int} \cup \Omega^\delta$ we consider the Neumann Laplacian $L^N =: L$. The Meixner conditions are imposed on the edges of the lids in form $\mathcal{D}(L^N) \subset W_2^1(\Omega)$. We consider full stationary scattering problem for the Neumann Laplacian L in Ω. We are interested in scattering data on an "essential spectral interval" situated on the positive semi-axis, in the range of relatively small wave-numbers $kH < \pi/2$ or, equivalently, of large wavelengths, compared with the length H of the connecting channel. We obtain essential part of analytic results of the paper assuming that the channel is short and thin $kH << \pi/2$, $\delta/H << 1$. Thin short channel can be considered as a "point-wise" opening at the point a, coincident with the centers x_Γ, $x_H \approx a$ of the lower and upper lids Γ, Γ_H, $|x_\Gamma - x_H| = H << \pi/2k$,$x_\Gamma \approx x_H \approx a$, see Fig. 1 below. A convenient Ansatz for the Green function of the above spectral problem, with the "point-wise" opening at $a_\Gamma = a_H =: a$, was suggested by Kirchhoff, [1], in form of a linear combination of the unperturbed Green functions. Hereafter we consider the corresponding scattering Ansatz

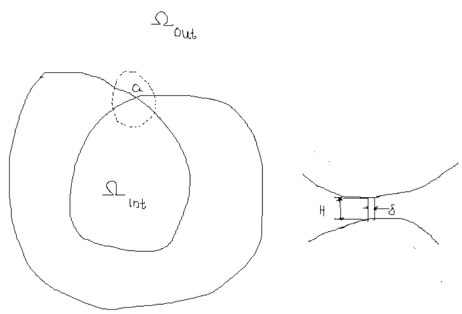

Figure 1: The opening is point-wise for the waves length k^{-1}, if $\pi/2 >> kH,\ H >> \delta$.

(1), with undefined coefficients A_{int}, A_{out}. This Ansatz obviously satisfies the Helmholtz equation

$$-\Delta\Psi = \lambda\Psi$$

in $\Omega_{int,out}$, and the Neumann homogeneous boundary condition on $\partial\Omega$, but has singularity at a. The Kirchhoff coefficients were never rigorously calculated. Note that the problem of calculation of scattered waves and scattering frequencies - resonances - was posed by Rayleigh in the beginning of previous century, see [2], but estimates [3] and approximate formulae for them in form of asymptotic series were found much later, see for instance [4, 5, 6, 7, 8].

Kirchhoff coefficients can be found easily when replacing the resonator by the corresponding solvable model. A solvable model of the Helmholtz resonator was constructed in [9] as a self-adjoint extension, see [10], of the Neumann Laplacian, restricted onto smooth functions vanishing near a. The domain of the extended operator contains singular deficiency elements, and is characterized by some asymptotic boundary conditions at the opening a.

With the boundary conditions properly selected, the model gives explicit formulae for the scattering matrix and the scattered waves and permits to reduce the calculation of the scattering frequencies to the solution of an algebraic equation. The general question on fitting of all parameters of the model [9] remained unsolved until now, because these parameters do not have any naive physical interpretation. It was conjectured in [4] that the model based on the operator extensions can emulate the resonance scattering by Helmholtz resonator with a small opening, if the parameters of the model are "properly selected", but no general procedure of the choice was suggested at that moment.

In this paper we suggest a procedure of fitting of the slightly modified solvable model [9], based on analysis of the Neumann-to-Dirichlet map (ND-maps) of the Laplacian in $\Omega_{int,out}$, and the transport properties of the short thin channel, connecting the outer domain with the cavity. The scattering matrix of the fitted model serves an approximation, on a certain spectral interval, of the scattering matrix of the Helmholtz resonator with a short thin channel. In particular we suggest an explicit formula for the Kirchhoff coefficients $A_{int,out}$. The zeros of the model scattering matrix- the resonances - can be found approximately as solutions of some algebraic equation. We conjecture that the resonances of the model are situated near to the resonances of the original scattering problem, but the proof of his statement supposedly based on operator - valued Rouche theorem, see [11] requires more precise asymptotic estimates techniques.

2 Solvable model of the Helmholtz Resonator.

In this section we assume that the shell separating the inner domain from the outer domain is thinning at the opening, see Fig. 1 . The upper and the lower lids Γ_H, Γ of the channel Ω^δ are the parts of the common boundary of the shell, of the outer domain Ω_{out} and the inner domain Ω_{int} respectively: $\Gamma \in \partial\Omega_{int}$, $\Gamma_H \in \partial\Omega_{out}$. Restrict the inner and the outer Neumann Laplacean$L_{int,out} \to (L_{int,out})_0$ onto smooth functions vanishing near the centers of the upper and lower lids $x_{\Gamma_H} = x_H \in \partial\Omega_{out}$, $x_\Gamma \in \partial\Omega_{int}$. The deficiency indices of the restricted operators L^{int}, L^{out} are $(1,1)$, and the deficiency elements at any complex point of the spectral parameter $\bar{\lambda}$ are the Green functions $G_{int}(x, x_\Gamma, \lambda)$, $G_{out}(x, x_H, \lambda)$, see [9]. The asymptotic formulae for the Green functions $G_{int,out}$ with the poles at x_H, x_Γ include details of the shape of the boundary and the spectral characteristics of the inner and outer operators. Indeed, following [9] one can calculate, based on Hilbert identity for some regular M, the normal limits of the inner and the outer Neumann Green functions :

$$G_{int}(x, x_\Gamma, \lambda) = \frac{1}{2\pi|x - x_\Gamma|} - \alpha_{int} \ln \frac{1}{|x - x_\Gamma|} +$$

$$C_\Gamma(x_\Gamma, M) + (\lambda - M) \sum_{s=1}^{\infty} \frac{\psi_s(x)\, \psi_s(x_\Gamma)}{(\lambda_s - \lambda)(\lambda_s - M)},$$

$$G_{out}(x, x_H, \lambda) = \frac{1}{2\pi|x - x_H|} + \alpha_{out} \ln \frac{1}{|x - x_H|} + C_H(x_H, M) +$$

$$\frac{(\lambda - M)}{8\pi^3} \int_0^\infty |k|^2 d|k| \int_{\Sigma_1} d\omega \frac{\bar{\psi}_\omega(x, |k|)\, \psi_\omega(x_H, |k|)}{(|k|^2 - \lambda)(|k|^2 - M)}. \qquad (2)$$

Here ψ_s are real normalized eigenfunctions, λ_s are the eigenvalues of the Neumann Laplacian in Ω_{int} and ψ_ω are the eigenfunctions of the continuous spectrum of the Neumann Laplacian in Ω_{out} - the scattered waves. The constants α_{int}, α_{out} have a certain geometrical meaning, see [9]. Indeed, consider the equation of the (curved) lid $\Gamma \subset \partial\Omega_\int$ near the center a_Γ represented in terms of coordinates connected to the outer normal n (with respect to Ω_{int}) and the tangent plane $T(a_\Gamma) = \{t_1, t_2\}$ at the point $a_\Gamma = (0, 0)$, where $\nabla\Phi(a_\Gamma) := \nabla\Phi(0, 0) = 0$. We write this equation as $n = \Phi(\vec{t})$, and represent the corresponding second quadratic form as $\Phi(t_1, t_2) = \frac{t_2^2}{R_1} + \frac{t_2^2}{R_2}$, $|t| << 1$. Assume that the mean curvature $C(a_\Gamma) := \frac{1}{R_1} + \frac{1}{R_1}$ at a_Γ is not equal to zero. Then α_{int} does not depend on the spectral parameter and is calculated as

$$\alpha_{int} = \frac{1}{8\pi} \left[\frac{1}{R_1^{int}} + \frac{1}{R_2^{int}} \right].$$

Similarly the corresponding term in the asymptotic of the outer Green function can be calculated:

$$\alpha_{out} = \frac{1}{8\pi} \left[\frac{1}{R_1^{out}} + \frac{1}{R_2^{out}} \right].$$

The spectrum of L_{int} is discrete, and the spectrum of L_{out} is absolutely continuous. The eigenfunctions of the discrete spectrum are real, and the eigenfunctions of the absolutely continuous spectrum are usually selected such that $\bar{\psi}_{\omega}(x,|k|) = \bar{\psi}_{out}(x,\omega,|k|) = \psi_{out}(x,-\omega,|k|)$.

If the lids are situated on flat pieces of the boundary of the shell, then $R_{1,2}^{int,out} = \infty$, so that the logarithmic singular terms are trivial: $\alpha_{int,out} = 0$. Hereafter we assume that this is the case. Then the asymptotic of G_{int}, G_{out} at $x_\Gamma \approx x_H \approx a$ is defined by the higher order addendum of the iterated resolvent equation. For instance after three iterations we obtain:

$$G(x,a,\lambda) - G(x,a,M) - (\lambda - M)G(x,*,\lambda)\, G(*,a,M) =$$

$$(\lambda - M)^2 \left[G(x,*,M)\, G(*,*,M)\, G(*,a,M)\right] +$$
$$(\lambda - M)^3 \left[G(x,*,M)G(*,*,\lambda)\, G(*,*,M)\, G(*,a,M)\right]$$

For the Laplace equation in R_3 the higher order addenda are continuous with respect to x and their values at the point $a \in \Omega_{int} \cup \partial\Omega_{int}$ can play a role of the local spectral characteristic. For instance, if $a = a_\Gamma \in \Gamma$:

$$G_{int}(x,a_\Gamma,\lambda) =$$

$$G_{int}(x,a_\Gamma,M) + C_\Gamma(x,a_\Gamma,M) + (\lambda - M)^3 \sum_{s=1}^{\infty} \frac{\psi_s(x)\,\psi_s(a_\Gamma)}{(\lambda_s - \lambda)(\lambda_s - M)^3} =$$

$$G_{int}(x,a_\Gamma,M) + \tilde{\mathcal{M}}_{int}(a_\Gamma,\lambda). \tag{3}$$

Here $C_\Gamma(x,a_\Gamma,M)$ $x \in \Gamma$ is a generalized kernel of a bounded operator acting on Γ and $\tilde{\mathcal{M}}_{int}(a_\Gamma,\lambda)$ is represented by a convergent spectral series

$$(\lambda - M)^3 \sum_{s=1}^{\infty} \frac{\psi_s(x)\,\psi_s(a_\Gamma)}{(\lambda_s - \lambda)(\lambda_s - M)^3}.$$

Similar spectral characteristics of the outer problem is represented in form of the spectral integral over scattered waves of L_{out} at the point a_H on the upper section:

$$G_{out}(x,a_H,\lambda) = G_{out}(x,a_H,M) +$$

$$C_H(x,a_H,M) + (\lambda - M)^3 \frac{1}{8\pi^3} \int_0^\infty |k|^2 d|k| \int_{\Sigma_1} d\omega \frac{\bar{\psi}_\omega(x,|k|)\,\psi_\omega(a_H,k)}{(|k|^2 - \lambda)(|k|^2 - M)^3} \approx$$

$$G_{out}(x,a_H,M) + \mathcal{M}_{out}(a_H,\lambda), \text{ if } x \to a_H\,,\ \Im\lambda \neq 0. \tag{4}$$

Both "kernels" $C_{\Gamma}(x, a_{\Gamma}, M), C_{H}(x, a_H, M)$ admit appropriate spectral representation, for instance

$$C_{\Gamma}(x, a_{\Gamma}, M) = C_{\Gamma}\ (a_{\Gamma}) + M \sum_{s} \frac{\psi_{s}(x)\, \psi_{s}(a_{\Gamma})}{\lambda_{s}(\lambda_{s} - M)},$$

with some operator function $C_{\Gamma}\ (a_{\Gamma}, M)$ which smoothly depends M, and the series summarized with Abel procedure. After appropriate re-normalization of the constants we can represent the asymptotic of the inner and the outer Green functions at the points a_{Γ}, a_H situated on flat pieces of the boundary, $\partial\Omega_{int,out}$, without the logarithmic terms:

$$G_{int}(x, a_{\Gamma}, M) = \frac{1}{2\pi |x - a_{\Gamma}|} + \mathcal{M}_{int}(a_{\Gamma}, \lambda) + \dots,$$

$$G_{out}(x, a_H, M) = \frac{1}{2\pi |x - a_H|} + \mathcal{M}_{out}(a_H, \lambda) + \dots, \tag{5}$$

where the dots stay for the terms vanishing at a_{Γ}, a_H. The structure of the limit of $\mathcal{M}_{out}(\lambda_0 + i0)$ on the real axis, $\lambda_0 = k_0^2 > 0$ is defined by the Plejel formula. For instance, the integral term in the right side of (4) in the formula for the outer Green -function is represented as an integral over the spectral measure:

$$\lim_{\lambda \to k_0^2 + i0} \int_0^\infty \frac{k^2 dk}{8\pi^3} \int_{\Sigma_1} d\omega \frac{\bar{\psi}_{\omega}(x, k)\, \psi_{\omega}(a_H, k)}{(k^2 - \lambda)(k^2 - M)^3} = \lim_{\lambda \to k_0^2 + i0} \int_0^\infty \frac{\frac{d\mathcal{E}}{d\mu}(x, a_H, \mu) d\mu}{(\mu - \lambda)(\mu - M)^3} =$$

$$\frac{i\pi\, k_0^2}{8\pi^3} \int_{\Sigma_1} d\omega \frac{\bar{\psi}_{\omega}(x, k_0)\, \psi_{\omega}(a_H, k_0)}{2k_0 (k_0^2 - M)^3} + \frac{V.P.}{8\pi^3} \int_0^\infty k^2 dk \int_{\Sigma_1} d\omega \frac{\bar{\psi}_{\omega}(x, k)\, \psi_{\omega}(a_H, k)}{(k^2 - k_0^2)(k^2 - M)^3} =:$$

$$\frac{i\pi}{(k_0^2 - M)^3} \frac{\partial \mathcal{E}}{\partial \lambda}(x, a_H) + \frac{I}{(k_0^2 - M)^3} \mathcal{M}_{VP}(x, a_H, \lambda), \tag{6}$$

where the first addendum in the last formula is proportional to the kernel of the derivative of the spectral measure of the Neumann Laplacian in the outer domain and the last term is a V.P. integral of a singular integrand with the pole at $k = k_0$. For the non-perturbed Laplacian this term is reduced to an integral operator in $L_2(\Gamma_H)$ with the kernel

$$i\pi \frac{\sin k_0 |x - y|}{|x - y|} \frac{1}{4\pi^2 (k_0^2 - M)^3} \approx \left(1 - \frac{k_0^2 |x - y|^2}{3!} + \dots\right) \frac{i\, k_0}{4\pi (k_0^2 - M)^3}.$$

One can derive a similar asymptotic for the perturbed Neumann Laplacian on an arbitrary outer domain, everywhere on some essential spectral interval Δ, except, probably. a finite set of spectral points where the derivative of the kernel of the spectral measure degenerates, $\frac{\partial \mathcal{E}}{\partial \lambda}(x_H, x_H, \lambda) = 0$. Finiteness of the set follows from the uniqueness theorem for smooth analytic functions with a positive imaginary part, see [13].

Assumption *Hereafter we assume that, for given small flat upper lid centered at x_H, there are no degenerate points on the selected essential interval Δ. It means that for $\lambda = k^2 \in \Delta$, $x, y \in \Gamma$:*

$$\Im G_{out}(x, y, \lambda + i0) \approx \pi \, \frac{\partial \mathcal{E}}{\partial \lambda}(x_H, x_H) \neq 0, \tag{7}$$

Combining the asymptotic formula (4) for the Green function at the pole x_H in the center of the upper section Γ_H with the asymptotic of the imaginary part of $G_{out}(x, x_H, \lambda + i0)$ for $x \to x_H$ we obtain the asymptotic of the outer Green - function when $x \to x_H$:

$$G_{out}(x, x_H, M) =$$

$$\frac{1}{2\pi |x - x_H|} + C(x_H, M) + i\pi \, \frac{\partial \mathcal{E}}{\partial \lambda}(x_H, x_H) + \mathcal{M}_{VP}(x_H, x_H, \lambda) + \dots, \tag{8}$$

where the dots stay for terms vanishing at x_H, $C(x_H, M)$, $\mathcal{M}_{VP}(a_H, a_H, \lambda)$ are real constants and $i\pi \, \frac{\partial \mathcal{E}}{\partial \lambda}(x_H, x_H)$ is a nonzero purely imaginary addendum. The obtained asymptotic of the non-polar term

$$C(x_H, M) + i\pi \, \frac{\partial \mathcal{E}}{\partial \lambda}(x_H, x_H) + \mathcal{M}_{VP}(x_H, x_H, \lambda) =: \mathcal{M}_{out}(x_H) \tag{9}$$

will be used in course of discussion of properties of the fitted solvable model of the Helmholtz resonator, see section 5.

In [9] a solvable model of the Helmholtz resonator was suggested in form of a self-adjoint extension of the orthogonal sum $(L_{in})_0 \oplus (L_{out})_0$ of the restricted Neumann Laplacians in $\Omega_{int,out}$. The domain of the extension is obtained via imposing a special boundary condition onto the asymptotic boundary values A, B at $x \to x_\Gamma$, x_H. In [9] we assumed that $kH \approx 0$, hence $x_\Gamma \approx x_H$. Hence the asymptotic boundary values $A_{in,out}$, $B_{in,out}$ are defined as the coefficients in front of the leading terms at x_Γ, x_H:

$$u_{int} = \frac{A_{int}}{2\pi |x - x_\Gamma|} + B_{int} + \dots, \quad u_{out} = \frac{A_{out}}{2\pi |x - x_H|} + B_{out} + \dots.$$

of elements from the domain of the corresponding adjoint operators $(L^{int})_0^+ \oplus (L^{out})_0^+$. The boundary forms of the adjoint operators are calculated in terms of the boundary values A, B. For instance, if $u_{out} \sim (A^u_{out}, B^u_{out})$ and $v_{out} \sim (A^v_{out}, B^v_{out})$, then the integration by parts on the complement of a small ball $B_\varepsilon : |x - x_\Gamma| < \varepsilon$, with subsequent passing to the limit $\varepsilon \to 0$, gives an expression for the outer boundary form:

$$\mathcal{J}_{out}(u, v) := \lim_{\varepsilon \to 0} \int_{\Omega_{out} \setminus B_\varepsilon} [-\Delta \bar{u}\, v + \bar{u}\, \Delta v]\ dx^3 = \bar{B}^u_{out}\, A^v_{out} - \bar{A}^u_{out}\, B^v_{out}. \quad (10)$$

Similar formula is true for the inner boundary form. The sum of the inner and the outer boundary forms vanishes if the asymptotic boundary values are submitted to some self-adjoint boundary condition, for instance :

$$\begin{pmatrix} \beta_{00} & \beta_{01} \\ \beta_{10} & \beta_{11} \end{pmatrix} \begin{pmatrix} B_{out} \\ A_{int} \end{pmatrix} = \begin{pmatrix} A_{out} \\ -B_{int} \end{pmatrix}. \quad (11)$$

with an Hermitian 2×2 matrix β. The Neumann Laplacian L_β in $\Omega_{int} \cup \Omega_{out}$ defined by this boundary condition is self-adjoint.

More difficult part of the problem is *fitting* of the parameters of the model, in particular: physically reasonable choice of the matrix β defining the connection between the inner and the outer space. It is not clear a-priori, if the elements of the matrix have any physical meaning. Eventually we will fit the boundary parameters based on comparison of an *explicit* expression for the model scattering amplitude and an approximate expression for the amplitude of the original problem, with a thin short channel, represented in a similar form.

First of all we attempt to fit the above solvable model, as it was represented in [9], before any modification. The scattered wave are characterized by the asymptotic at infinity in the direction ω, $x \to \omega\infty$, $|\omega| = 1$, and involve so-called *scattering amplitude* $a(\omega, \nu, \sqrt{\lambda}) = a(k, l)$for $k = \omega|k|$, $l = \nu|k|$, $|k| = \sqrt{\lambda}$:

$$\psi(x, k) = e^{-i|k|<x,\nu>} - 2\pi^2 \frac{e^{i|k||x|}}{|x|} a(k, l) + o(\frac{1}{|x|}). \quad (12)$$

We choose the Ansatz for the scattered waves of the model operator in Kirchhoff form:

$$\psi^\beta(x) = \psi_{out}(x) + A_H G_{out}(x, x_H, \lambda) \quad (13)$$

in Ω_H, and

$$\psi^{\beta}(\omega, x) = A_\Gamma G_{int}(x, x_\Gamma, \lambda), \tag{14}$$

where $G_{out}(x, x_H, \lambda)$, $G_{int}(x, x_\Gamma, \lambda)$ are, respectively the limit values of the Green functions of L_{out}, L_{int} from the upper half-plane: $\lambda := \lambda + i0$. The asymptotic boundary values of the Ansatz are calculated, due to (5), as $B_{out} = \psi_{out}(x_H) + A_H \mathcal{M}_{out}(x_H,\ \lambda)$; $A_{int} = A_\Gamma$, $B_{int} = A_\Gamma \mathcal{M}_{int}(a_\Gamma, \lambda)$. Then, inserting the asymptotic boundary values into the boundary condition (11) we obtain, after an elementary calculation, the following expression for A_H:

$$A_H = -\frac{|\beta_{01}|^2 - \beta_{00}(\beta_{11} + \mathcal{M}_{int})}{(1 - \beta_{00}\mathcal{M}_{out})(\beta_{11} + \mathcal{M}_{int}) + |\beta_{01}|^2 \mathcal{M}_{out}} \psi_{out}(x_H, \nu) =: -\mathcal{A}\psi_{out}(x_H, \nu) \tag{15}$$

Inserting this expression into (13), and taking the limit $x \to \omega\infty$ with use of the asymptotic $G_{out}(x, y, \lambda) \approx (4\pi)^{-1}\, e^{i\sqrt{\lambda}|x|}|x|^{-1}\Psi_{out}(y, \omega)$, we obtain the formula for the additional term of the amplitude, caused by the opening:

$$\psi_{out}(x_H, \omega)\frac{\mathcal{A}}{8\pi^3}\psi_{out}(x_H, \nu). \tag{16}$$

The obtained expression (15) for the Kirchhoff coefficient and the corresponding expression for the additional term of the amplitude contain four parameters β_{st}. Fitting of the constructed model is reduced to the appropriate choice of the parameters. It is natural to say, that *the model is fitted* on an essential spectral interval Δ, if the model scattering matrix serves on Δ an approximation of the scattering matrix of the original perturbed operator, see an extended discussion of fitting below, in section 5.

Unfortunately the above naive model, suggested in [9] can't be fitted, as follows from comparison of (15,16) with an approximate expressions for the original Helmholtz resonator obtained in section 4. The reason for it is the non-zero length of the channel $H > 0$. Luckily, for thin short channel another *modified* model can be constructed, based on the same outer operator $L^N_{out} =: L_{out}$ and some finite matrix $\mathbf{A} : E \to E$ acting in an abstract space E attached to the point-wise opening $a = x_H$ with a lead length H. The modified model is constructed as a "zero-range model with an inner structure", see for instance [14] and also a recent paper [15]. This model *can be fitted on a certain essential spectral interval* Δ, but certainly not on the whole spectrum. Fitting of a similar model in case of quantum networks is based on spectral properties of the quantum dot, see [17]. In the case of Helmholtz resonator we will also fit the model based on properly re-normalized spectral data of the cavity.

We will use here the standard notations for the zero-range models with inner structure, see for instance [17]. The role of coordinates is played by symplectic coordinates $\xi_\pm \in N_i$ of elements from the defect $N = N_i + N_{-i}$ - the sum of the deficiency subspaces. The boundary form in the inner space is represented as

$$\mathcal{J}_{int}(u, v) = \langle \xi_+^u, \xi_-^v \rangle_E - \langle \xi_-^u, \xi_+^v \rangle_E. \tag{17}$$

The symplectic coordinates $\xi_\pm \in N_i$ of the solution $u = u_0 + \frac{\mathbf{A}}{\mathbf{A}-iI}\xi_+ - \frac{I}{\mathbf{A}-iI}\,\xi_-$ of the adjoint homogeneous equation $(\mathbf{A}_0^+ - \lambda I)u = 0$ are connected, see [16] via the Krein matrix-function $\mathcal{M}$ as

$$\xi_- = -P\frac{I + \lambda\mathbf{A}}{\mathbf{A} - \lambda I}P\ \xi_+ =: -\mathcal{M}\,\xi_+. \tag{18}$$

Here P is an orthogonal projection onto N_i. Generally the Krein function and its inverse $-\mathcal{M}^{-1}$ are matrix R-functions which admit the standard Herglotz representation for a rational matrix R-function:

$$\mathcal{M}(\lambda) = \mathcal{M}_0 + \mathcal{M}_1\lambda + \sum_{l=1}^{N}\frac{1 + \lambda A_l}{A_l - \lambda I}q_l, \tag{19}$$

where $\mathcal{M}_0$ is an hermitian matrix in N_i, $\mathcal{M}_1$- a positive hermitian matrix, A_l- the eigenvalues of $\mathbf{A}$, and $q_l = PQ_lP = q_l^+$ - the spectral projections Q_l of $\mathbf{A}$, framed by the projections P onto the selected deficiency subspace N_i.

To construct the fittable model of the Helmholtz resonator we attach the inner structure $\{E, A\}$ to the point-wise opening a via a one-dimensional channel length H, see the Fig. 2. If the unperturbed inner Hamiltonian $\mathbf{A}$ is just a second order differential operator, $\mathbf{A}u = -u'' + qu$ in $L_2(0, 1)$ on the interval $(0, 1)$ attached to x_Γ with its right end $x = 1$. We assume that the link (x_Γ, x_H), playing the role of the channel in the model, is direct and one-dimensional, with the operator $-d^2$ on it. Then the transmission of the Neumann/Dirichlet data u'_Γ, $u_\Gamma \longrightarrow u'_H$, u_H from the lower end x_Γ of the link to the upper end $\Gamma_H = a$ is defined by the 1-d Neumann-to-Dirichlet map of the link:

$$\mathcal{ND}_\Gamma = \frac{u_\Gamma}{u'_\Gamma} \longrightarrow \frac{u_H}{u'_H} = \mathcal{ND}_H = \frac{\lambda^{-1/2}\tan\sqrt{\lambda}H + \mathcal{ND}_\Gamma}{1 - \sqrt{\lambda}\tan\sqrt{\lambda}H\mathcal{ND}_\Gamma} \tag{20}$$

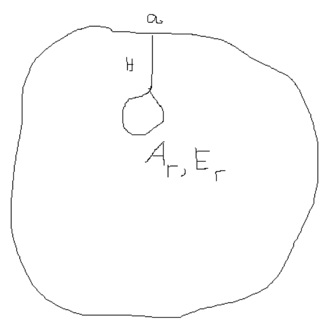

Figure 2: Inner structure for the cavity of the resonator, attached to the opening a via a one-dimensional channel.

Here we define the direction of differentiation by the vector $(x_\Gamma,\, x_H)$. Then the ND coefficient $\mathcal{ND}_\Gamma$ is connected with the Weyl function $\mathcal{M}$ of the inner Hamiltonian by the formula

$$\mathcal{ND}_\Gamma = -1/\mathcal{M}. \tag{21}$$

Comparing the boundary forms $J_{\Gamma,\Gamma_H}(u,v) = \bar{u}v' - \bar{u}'v\Big|_{\Gamma,\Gamma_H}$ with the boundary form of the inner structure $J_{int} = \bar{\xi}^u_+ \xi^v_- - \bar{\xi}^u_- \xi^v_+$ we extend the above formula (21) to the case of an abstract inner structure $\mathbf{A}$ via the identification: $u_\Gamma = \xi^\Gamma_+$, $u'_\Gamma = \xi^\Gamma_-$ and the Weyl function $\mathcal{M}$ substituted by the corresponding Krein function $\mathcal{M}_\Gamma$:

$$\frac{u_\Gamma}{u'_\Gamma} = \mathcal{ND}_\Gamma = \frac{\xi^\Gamma_+}{\xi^\Gamma_-} = -\frac{1}{\mathcal{M}_\Gamma},$$

$$\frac{u_H}{u'_H} = \mathcal{ND}_H = \frac{\xi^H_+}{\xi^H_-} = -\frac{1}{\mathcal{M}_H}.$$

Lemma 2.1 *The symplectic coordinates $\xi^H_\pm$ of the solution of the adjoint homogeneous equation of the inner structure with the one-dimensional channel* *attached are connected via* $\mathcal{M}_H$ *as* $\xi_-^H = -\mathcal{M}_H \xi_+^H$ *with*

$$\mathcal{M}_H = -\frac{1-(\lambda)^{-1/2}\tan\sqrt{\lambda}H\ \mathcal{M}_\Gamma}{\mathcal{M}_\Gamma + \sqrt{\lambda}\tan\sqrt{\lambda}H} =$$

$$\frac{\tan\sqrt{\lambda}H}{\sqrt{\lambda}} - \frac{1}{\cos^2\sqrt{\lambda}H[\mathcal{M}_\Gamma + \sqrt{\lambda}\tan\sqrt{\lambda}H]}. \tag{22}$$

For any rational R-function $\mathcal{M}_\Gamma$*, the Krein function* $\mathcal{M}_H$ *of the inner structure with the 1-d link included, is also R-function: it is analytic with respect* λ *in the upper half-plane and lower half-planes* $\Im\lambda > 0$, $\Im\lambda < 0$*, and it's first derivative exists on the real axis everywhere except a discrete set of simple poles and is positive.*

Proof is obtained based on the transfer matrix of the 1-d channel

$$u_H = u_\Gamma \cos\sqrt{\lambda}H + u'_\Gamma \lambda^{-1/2}\sin\sqrt{\lambda}H,$$

$$u'_H = -\sqrt{\lambda}u_\Gamma \sin\sqrt{\lambda}H + u'_\Gamma \cos\sqrt{\lambda}H$$

and the above identification. The positive derivative $\mathcal{M}'_H$ is obtained via direct calculation.

End of the proof

Remark 1 For thin short channel $\lambda H^2 << 1$ we substitute $\tan\sqrt{\lambda}H$ by $\sqrt{\lambda}H$ and obtain an approximate expression for $\mathcal{M}_H$ on Δ:

$$\mathcal{M}_H \approx \frac{H\mathcal{M}_\Gamma - 1}{\mathcal{M}_\Gamma + \lambda H} = H - \frac{1-\lambda H^2}{\mathcal{M}_\Gamma + \lambda H} \approx H - \frac{1}{\mathcal{M}_\Gamma + \lambda H} \tag{23}$$

The boundary form of the inner structure at the upper contact point $a = a_H$, with the one-dimensional channel attached, is

$$\bar{\xi}_+^H(u)\xi_-^H(v) - \bar{\xi}_-^H(u)\xi_+^H(v) = \mathcal{J}^H(u,v).$$

The sum of the boundary forms of the adjoint outer Neumann Laplacian and the inner structure at the upper contact point is, due to (10)

$$\mathcal{J}^H(u,v) + \mathcal{J}_{out}(u,v) = \bar{\xi}_+^H(u)\xi_-^H(v) - \bar{\xi}_-^H(u)\xi_+^H(v) + \bar{B}^u_{out}\,A^v_{out} - \bar{A}^u_{out}\,B^v_{out}.$$

We will construct a *fittable* solvable model of the Helmholtz resonator as a common self-adjoint extension of the orthogonal sum of the restricted outer

Neumann Laplacian and the restricted operator $\mathbf{A}$ of the inner structure with a one-dimensional channel attached, via choosing a Lagrangian plane $\mathcal{L}$ in the space of the boundary data $\xi^H_\pm$, A_{out}, B_{out} such that the sum $\mathcal{J}^H(u,v) + \mathcal{J}_{out}(u,v)$ of the boundary forms vanishes on $\mathcal{L}$. In particular we can connect the boundary data with an hermitian matrix β

$$\begin{pmatrix} \beta_{00} & \beta_{01} \\ \beta_{10} & \beta_{11} \end{pmatrix} \begin{pmatrix} B_{out} \\ \xi^H_+ \end{pmatrix} = \begin{pmatrix} A_{out} \\ \xi^H_- \end{pmatrix}. \tag{24}$$

The boundary condition (24) defines a selfadjoint operator L_β - a solvable model of the Helmholtz resonator,- which will be fitted later, see section 5. Inserting the boundary data of the solutions of the adjoint homogeneous equation $\left[\mathbf{A}_0^+ \oplus L_0^+ - \lambda I\right] u = 0$ into (24) we obtain an equation for the Kirchhoff coefficient A_{out}:

$$\begin{pmatrix} \beta_{00} & \beta_{01} \\ \beta_{10} & \beta_{11} \end{pmatrix} \begin{pmatrix} \psi_{out}(a) + A_{out}\mathcal{M}_{out} \\ \xi^H_+ \end{pmatrix} = \begin{pmatrix} A_{out} \\ -\mathcal{M}_H \xi^H_+ \end{pmatrix} \tag{25}$$

Theorem 2.1 *The outer Kirchhoff constant of the constructed model with* $\beta_{00} = 0$ *is found as*

$$A_{out} = -\frac{\psi_{out}(a)}{\mathcal{M}_{out} + |\beta_{01}|^{-2}\mathcal{M}_H + \beta_{11}|\beta_{01}|^{-2}}.$$

Proof Elimination of ξ^H_+ from the second equation gives

$$\xi^H_+ = -\frac{1}{\beta_{11} + \mathcal{M}_H}\ \beta_{10}\left[\psi_{out}(a) + A_{out}\mathcal{M}_{out}\right].$$

Substitution of the result into the first equation gives

$$A_{out} = -\frac{(\beta_{11} + \mathcal{M}_H)\beta_{00} - |\beta_{01}|^2}{\mathcal{M}_{out}[(\beta_{11} + \mathcal{M}_H)\beta_{00} - |\beta_{01}|^2] - (\beta_{11} + \mathcal{M}_H)}\ \psi_{out}(a) = -\mathcal{A}\psi_{out}$$

and is transformed to the announced form if $\beta_{00} = 0$.

The end of the proof

Remark 2 Similarly to (16) we conclude that the additional term of the amplitude of the solvable model with the inner structure is

$$\frac{\psi_{out}(\omega, a)\psi_{out}(\nu, a)}{\mathcal{M}_{out} + |\beta_{01}|^{-2}\mathcal{M}_H + \beta_{11}|\beta_{01}|^{-2}}. \tag{26}$$

The constructed model is parametrized by the matrix $\mathbf{A}$ and by the matrix elements β_{ik}. One may guess that the eigenvalues of the matrix $\mathbf{A}$ should simulate properly renormalized eigenvalues of the cavity. The boundary parameters β_{ik} do not have any naive physical meaning, so they can't be defined trivially, and even the connection between the eigenvalues of the inner structure $\mathbf{A}$ with the eigenvalues of the cavity is not simple, see sections 4 and 5.

The aim of this paper is: to fit the constructed model L_β on a certain essential spectral interval Δ, that is - to select the boundary parameters β_{st} and the eigenvalues of the operator $\mathbf{A}$ such that the scattering matrix of the model serves an approximation of the "full" scattering matrix of the spectral problem for Helmholtz resonator on the selected essential spectral interval Δ. Then the Kirchhoff coefficients of the model are calculated explicitly.

We will fit the model for Helmholtz resonator with a short narrow channel Ω^δ, based on spectral data of the inner and outer Neumann problems and the geometry of the channel.

3 Scattering matrix via Neumann-to-Dirichlet map.

We proceed in this section by considering an extended inner domain $\Omega_{int} \cup \Omega^\delta =: \Omega^*_{int}$ obtained via inclusion of the channel into it. Consider two boundary problems for the Neumann Laplacean L_{int}, L^*_{int} in Ω_{int}, Ω^*_{int} respectively, assuming that the normal is directed outward - to Ω_{out}:

$$-\Delta u = \lambda u,\ \frac{\partial u}{\partial n}\Big|_\Gamma = \rho_\Gamma,\ x \in \Omega_{int},\ -\Delta u = \lambda u,\ \frac{\partial u}{\partial n}\Big|_{\Gamma_H} = \rho_H,\ x \in \Omega^*_{int}.$$

Solutions of these problem are given by integral transforms with kernels defined by the corresponding Neumann Green functions G_{int}, G^*_{int} of the inner and the extended inner problems on the lids Γ, Γ_H,respectively, for instance:

$$[\mathbf{Q}^*_{int}(\lambda)\ \ \rho_H\,](x) = \int_{\Gamma_H} G^*_{int}(x,\, y,\, \lambda)\rho_H(y)d\Gamma,\ \ x \in \Omega^*_{int}. \tag{27}$$

Traces of $\mathbf{Q}_{int}\rho_\Gamma$ and $\mathbf{Q}^*_{int}\rho_{\Gamma_H}$ on Γ and Γ_H respectively define the restrictions of the standard Neumann-to-Dirichlet maps in $\Omega_{int}, \Omega^*_{int}$ onto Γ, Γ_H:

$$\frac{\partial \mathbf{Q}_{int}\rho_\Gamma}{\partial n}\Big|_\Gamma = \rho_\Gamma,\ \ \frac{\partial \mathbf{Q}_{int}\rho_\Gamma}{\partial n}\Big|_{\partial\Omega_{int}\setminus\Gamma} = 0.$$

This implies:

$$\mathbf{Q}_{int}\rho_\Gamma = \mathcal{ND}_\Gamma\rho_\Gamma\Big|_\Gamma = \mathcal{ND}\Big|_\Gamma\rho_\Gamma.$$

Similarly, for $\mathbf{Q}^*_{int}$ we have:

$$\frac{\partial \mathbf{Q}^*_{int}\rho_H}{\partial n}\Big|_{\Gamma_H} = \rho_H,\ \frac{\partial \mathbf{Q}^*_{int}\rho_H}{\partial n}\Big|_{\Omega^*_{int}\setminus\Gamma_H} = 0.$$

Hence the trace of $\mathbf{Q}^*_{int}\rho_H$ on Γ_H coincides with $\mathcal{Q}^*_{int}\rho_H$ that is, with ND-map $\mathcal{ND}^*_{int}\rho_H$. It is important to notice, that the inverse map $[\mathcal{ND}_\Gamma]^{-1}$ exist, if λ does not coincide with the eigenvalue of the corresponding mixed boundary problem

$$-\Delta u = \lambda u,\ \frac{\partial u}{\partial n}|_{\Omega_{int}\setminus\Gamma} = 0,\ u|_\Gamma = 0$$

and coincides with the associated *relative* Dirichlet-to-Neumann map obtained as restriction onto Γ of the boundary current of the solution of the *relative* Dirichlet boundary problem:

$$-\Delta u = \lambda u,\ \frac{\partial u}{\partial n}|_{\Omega_{int}\setminus\Gamma} = 0,\ u|_\Gamma = u_\Gamma,\ \frac{\partial u}{\partial n}|_\Gamma =: \mathcal{DN}^\Gamma u_\Gamma :$$

$$\mathcal{DN}^\Gamma\mathcal{ND}_\Gamma\rho_\Gamma = \rho_\Gamma. \tag{28}$$

Similar statement holds for Γ_H.

One can consider a similar boundary problem in Ω_{out}, with the normal on Γ_H directed to Ω_{out}:

$$-\Delta u = \lambda u,\ \frac{\partial u}{\partial n}|_{\Gamma_H} = -\rho_H,\ \frac{\partial u}{\partial n}|_{\partial\Omega_{out}\setminus\Gamma_H} = 0.$$

Solution of this problem is given by the integral transform:

$$[\mathbf{Q}\rho_H](x) = \int_{\Gamma_H} G^N_{out}(x,s)\rho_H(s)d\Gamma_H,$$

because, with the normal defined above,

$$\frac{\partial \mathbf{Q}\rho_H}{\partial n}|_H = \rho_H.$$

Hence the corresponding *standard* Neumann-to-Dirichlet map $\mathcal{ND}_{out}$, associated with the normal on Γ_H directed outside Ω_{out} is defined by the trace of $\mathbf{Q}\rho_H$ onto Γ_H:

$$\mathcal{Q}_{out}\rho_H = \mathrm{T}race_{\Gamma_H}\mathbf{Q}_{out}\rho_H.$$

Again, it is convenient to notice that the inverse $[\mathcal{ND}^*_{\Gamma_H}]^{-1}$ of the ND-map associated with Γ_H is obtained as *relative* Dirichlet-to-Neumann map for the boundary problem on Ω_{out} with relative Dirichlet boundary data, for instance

$$-\Delta u = \lambda u,\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega_{out}\setminus\Gamma_H} = 0,\ u\Big|_{\Gamma_H} = u_{\Gamma_H},\ \frac{\partial u}{\partial n}\Big|_{\Gamma_H} =: \mathcal{DN}_H\ u_{\Gamma_H}.$$

Then, with matching outer normals on Γ we have

$$\mathcal{DN}^H_{out}\mathcal{ND}_{out}\Big|_{\Gamma_H}\ \rho_{\Gamma_H} = \rho_{\Gamma_H}. \tag{29}$$

It is proved in [18] that the singularities if $\mathcal{DN}_{int}(\lambda)$ as an unbounded operator in $W_2^{3/2}(\Gamma)$, and the poles of $\mathcal{DN}_{int}(\lambda)$ at the eigenvalues of the inner Dirichlet problems can be separated, see the theorems 3.1, 3.2 below. These statements are valid both in case of the classical DN-maps and in case of the relative DN-map due to above statements (28,29). Hence in the following theorems 3.1,3.2, quoted from [18] we mean both standard and relative DN-maps, associated with Dirichlet of relative Dirichlet boundary problems:

Theorem 3.1 *Consider the Dirichlet Laplacian L^D_{int} or relative Dirichlet Laplacian in $L_2(\Omega_{int})$ on a compact domain $\Omega_{int}\subset R_3$ with a smooth boundary $\partial\Omega = \Gamma$ or $\partial\Omega\supset\Gamma$ respectively. The DN-map (relative DN-map) of L^D_{int} has the following representation on the complement of the corresponding spectrum σ^D_{int} in complex plane λ, with $M>0$:*

$$\mathcal{DN}^\Gamma(\lambda) =$$

$$= \mathcal{DN}^\Gamma(M) - (\lambda - M)\mathcal{P}^+(M)\mathcal{P}(M) - (\lambda - M)^2\mathcal{P}^+(M)R_\lambda\mathcal{P}(M), \tag{30}$$

where R_λ is the resolvent of L^D_{int}, and $\mathcal{P}(-M)$ is the corresponding Poisson kernel. Similar formula is true for the ND map, after two iterations of the resolvent equation we obtain:

$$\mathcal{ND}_\Gamma(\lambda) =$$

$$= \mathcal{ND}_\Gamma(M)+(\lambda-M)\mathbf{Q}^+_{int}(M)\mathbf{Q}_{int}(M)+(\lambda-M)^2\mathbf{Q}^+_{int}(M)R_\lambda\mathbf{Q}_{int}(M). \tag{31}$$

Here $\mathcal{ND}_\Gamma(\lambda)$ is obtained as the trace of $\mathbf{Q}(\lambda)\rho$ on Γ. The operators

$$\mathcal{DN}^\Gamma(M),\ \ \mathcal{P}^+(M)\mathcal{P}(M)$$

are respectively bounded from $W_2^{3/2}(\Gamma)$ *onto* $W_2^{1/2}(\Gamma)$ *and bounded in* $W_2^{3/2}(\Gamma)$*, and the operator-function*

$$\left[\mathcal{P}^+(M)R_\lambda\mathcal{P}(M)\right](x_\Gamma, y_\Gamma) = \sum_{\lambda_s\in\Sigma_L} \frac{\frac{\partial\varphi_s}{\partial n}(x_\Gamma)\frac{\partial\varphi_s}{\partial n}(y_\Gamma)}{(\lambda_s - M)^2(\lambda_s - \lambda)} \tag{32}$$

is compact in $W_2^{3/2}(\Gamma)$*.*

On the continuous spectrum, $\lambda \geq 0$, the Dirichlet-to-Neumann map is defined as the boundary current of the outgoing solution of the corresponding boundary problem, which is obtained as $\lim_{\varepsilon\to 0} u_{\lambda+i\varepsilon}$. The statement similar to the above theorem remains true for the DN-map of the Laplacian on Ω^*_{int}, and, after appropriate re-formulation, for the corresponding ND-map. We calculate the boundary currents for both inner and outer domain via differentiation of the outgoing solution of the corresponding boundary problem in the outward direction on Γ_H. A statement similar to the above theorem 3.1 is also true for the DN-map of the exterior domain. Again we choose the outward normal on Γ_H. Then we obtain:

$$\mathcal{DN}_{out}(\lambda) = \tag{33}$$

$$\mathcal{DN}_{out}(M) - (\lambda - M)\mathcal{P}^+(M)\mathcal{P}(M) - (\lambda - M)^2\mathcal{P}^+(M)R_\lambda\mathcal{P}(M),$$

with only difference that first terms of the decomposition contain the DN-map and Poisson kernel for the exterior domain and the generalized kernel in the last term is represented via the integral over the absolutely continuous spectrum $\sigma_L^a = [0,\infty)$, and the integrand combined of normal derivatives of the scattered waves $\psi(x, |k|, \nu)$, $k = |k|\nu$, $|\nu| = 1, \Im\lambda \neq 0$:

$$\mathcal{P}^+(M)R_\lambda\mathcal{P}(M)(x_\Gamma,\, y_\Gamma) =$$

$$\frac{1}{(2\pi)^3}\int_{|k|^2\,\in\,\Sigma_L^a} \frac{\frac{\partial\bar\psi}{\partial n}(x_\Gamma,\, |k|, \nu)\frac{\partial\psi_s}{\partial n}(y_\Gamma\, |k|, \nu)}{(|k|^2 - M)^2(|k|^2 - \lambda)} d^3k.$$

The absolutely-continuous spectra $\sigma_{out}^{D,N}$ of both Dirichlet and Neumann Laplacean $L_{out}^{D,N}$ fill the positive semi-axis $0 \leq \lambda < \infty$ with infinite multiplicity and the scattered waves $\psi(x,k)$- are parametrized by the energy $\lambda > 0$, $|k| = \sqrt{\lambda}$, and the direction $\nu, |\nu| = 1$, or just by the *momentum* $k = |k|\nu \in R^3$.

The normal limit values of $\mathcal{ND}_{out}$ can be calculated, due to absolute continuity of the spectrum of L_{out} based on the Plejel formula:

$$\lim_{\lambda \to \lambda + i0} \mathcal{ND}_{out} = i\pi \frac{d\mathcal{E}}{d\lambda} + VP\mathcal{Q}_{out},$$

where $VP\mathcal{Q}_{out}$ is calculated as VP-value of the sum of the spectral integrals. For instance, with $M > 0$,

$$VP \;\; \mathcal{Q}_{out}(M) \;\; R_\lambda \;\; \mathcal{Q}_{out}(M) =$$

$$\frac{1}{(2\pi)^3} VP \int_0^\infty d|k| \int_{\Sigma_1} d\Sigma_\nu \frac{\psi(x_\Gamma, \, |k|, \nu)\psi(y_\Gamma \, |k|, \nu)}{(|k|^2 - M)^2(|k|^2 - \lambda)} |k|^2,$$

where the VP-limit of integrals is taken over the complements of a sequence of small nesting intervals centered at $\lambda = \bar{\lambda}$. Following [18] we can connect the resolvent of the self-adjoint operator $\mathcal{L}$ on the composite domain $\Omega^*_{int} \cup \Omega_{out}$ with the resolvents of the orthogonal sum of the self-adjoint operators $L^*_{int} \oplus L_{out}$ defined in $L_2(\Omega^*_{int}) \oplus L_2(\Omega_{out})$ by the homogeneous Neumann boundary conditions.

Theorem 3.2 *The resolvent kernel $G_\lambda(x, y)$ of the operator $\mathcal{L}$ for regular λ and x, y in Ω_{out} is represented by the Krein formula*

$$G(x, y, \lambda) = G_{out}(x, y, \lambda) -$$

$$G_{out}(x, *, \lambda) \left[\mathcal{Q}^*_{int}(\lambda) + \mathcal{Q}_{out}(\lambda)\right]^{-1} G_{out}(*, y, \lambda), \tag{34}$$

where the asterisk stays for the argument on Γ_H.

The following formula connects the scattered waves $\psi(x, \nu, \lambda)$ of the perturbed problem (with opening), in the outer domain, with the scattered waves of the outer Neumann problem,

$$\psi(x, \nu, \lambda) = \psi_{out}(x, \, \nu, \lambda) -$$

$$-G_{out}(x, *, \lambda) \left[\mathcal{Q}^*_{int}(\lambda) + \mathcal{Q}_{out}(\lambda)\right]^{-1} \psi_{out}(*, \nu, \lambda). \tag{35}$$

The expression for the scattering amplitude of the original Neumann Laplacian L is given by the formula:

$$a(\omega, \nu, \lambda) = a_{out}(\omega, \nu, \lambda) +$$

$$\frac{1}{8\pi^3}\psi_{out}(*,\,\omega,\lambda)\,[\mathcal{Q}^*_{int}(\lambda)+\mathcal{Q}_{out}(\lambda)]^{-1}\,\psi_{out}(*,\,\nu,\lambda). \qquad (36)$$

Both formulae (35,36) admit analytical continuation on the spectral sheet of the variable $\lambda = k^2$, ($\Im k > 0$), and all operator-functions involved are calculated on the real axis λ as weak limits from the upper half-plane, $\Im\lambda \to 0^+$.

Remark 3 Calculation of the amplitude based on (36) requires solution of the equation

$$[\mathcal{Q}^*_{int}(\lambda)+\mathcal{Q}_{out}(\lambda)]\,u = \psi_{out}(*,\,\nu,\lambda). \qquad (37)$$

Both operators which stay in the left side of this equation exists and act, generically, as

$$W_2^{1/2}(\Gamma) \xrightarrow{\mathcal{Q}_{out}} W_2^{3/2}(\Gamma)$$

for any λ (if $R_3\backslash\Omega_{out}$ is compact), and

$$W_2^{1/2}(\Gamma) \xrightarrow{\mathcal{Q}_{int}} W_2^{3/2}(\Gamma)$$

if λ is not an eigenvalue of the relative Dirichlet problem

$$-\Delta u = \lambda u,\; u\Big|_{\Gamma} = 0,\; \frac{\partial u}{\partial n}\Big|_{\partial\Omega_{int}\backslash\Gamma} = 0.$$

Both $\mathcal{Q}_{out}$, $\mathcal{Q}_{int}$ can be extended from $W_2^{1/2}(\Gamma_H)$ onto $L_2(\Gamma_H)$. The extended operators are compact in $L_2(\Gamma_H)$. The sum of them is a compact operator too. To construct the corresponding inverse we need to regularize the problem. We will do it following approach based on closed graph theorem, suggested in [20], where more details can be found.

If λ_0 is an eigenvalue of the relative Dirichlet problem, and u_0 is the corresponding eigenfunction, then

$$\frac{\partial u_0}{\partial n} \xrightarrow{\mathcal{Q}_{int}} 0,$$

hence $\mathcal{Q}_{int}$ has zero eigenvalue, and

$$\mathcal{Q}_{int}L_2(\partial\Omega) = L_2(\partial\Omega)\ominus\left\{\frac{\partial u_0}{\partial n}\Big|_{\Gamma}\right\}.$$

Due to absence of eigenvalues of the outer problem, the operator $\mathcal{Q}_{out}$ is invertible, and the inverse of it is an operator of the differential order 1 :

$$\mathcal{Q}^{-1}_{out} = \mathcal{DN}^H_{out} : W^{3/2}(\Gamma_H) \to W_2^{1/2}(\Gamma_H).$$

or

$$\mathcal{Q}_{out}^{-1} = \mathcal{DN}_{out}^{H} : W^1(\Gamma_H) \to L_2(\Gamma_H).$$

The inverse coincides with the relative DN-map which is associated with the boundary problem

$$-\Delta u = \lambda u,\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega_{out}\setminus\Gamma_H} = 0,\ u\Big|_{\Gamma_H} = u_H,$$

with the Meixner condition imposed on inner angles:

$$\mathcal{DN}_{out}^{H} := \mathcal{DN}_{out}^{\Gamma_H} : u_H \longrightarrow \frac{\partial u}{\partial n}\Big|_{\Gamma_H}$$

The operator $\mathcal{Q}_{int}^{*}(\lambda)$ can be represented due to (31), as:

$$\mathcal{Q}_{int}^{*}(\lambda) = \mathcal{Q}_{int}^{*}(M) + (\lambda - M)\mathbf{Q}_{int}^{+}(M)\mathbf{Q}_{int}(M) + (\lambda - M)^2\mathbf{Q}_{int}^{+}(M)R_\lambda\mathbf{Q}_{int}(M) \tag{38}$$

Lemma 3.1 *The third term in the right side of (38) can be represented as a series of one-dimensional polar terms, convergent in operator norm* $W_2^1(\Gamma) \times L_2(\Gamma)$*:*

$$(\lambda - M)^2 \sum_{l=1}^{N} \frac{\varphi_l\Big|_\Gamma\rangle\ \langle\varphi_l\Big|_\Gamma, g\rangle}{(\lambda_l - M)^2(\lambda_l - \lambda)} + (\lambda - M)^2\ O\left(\sum_{N+1}^{\infty} \frac{1}{\lambda_l^{3-\alpha/2-\alpha'/2}}\right) =:$$

$$\mathcal{Q}_{int}^{N} + (\lambda - M)^2\ O\left(\sum_{N+1}^{\infty} \frac{1}{\lambda_l^{3-\alpha/2-\alpha'/2}}\right), \tag{39}$$

with $\alpha > 1/2,\ \alpha' > 3/2,\ \alpha/2 + \alpha'/2 < 3/2.$

Proof is derived from embedding results

$$|\varphi_l|_{L_2(\Gamma)} \leq \lambda_l^{\beta/2},\ \beta > 1/2,\ |\varphi_l|_{W_2(\Gamma)} \leq \lambda_l^{\beta/2},\ \text{if}\ \ 1 + 1/2 < \alpha',$$

because, due to Weyl asymptotic for eigenvalues of L_{int}^{*} the series $\sum_{l=1}^{\infty} \lambda_l^{-\gamma}$ is converging if $\gamma > 3/2$.

The end of the proof

Note that the compact operator $\mathcal{Q}_{out} : L_2(\Gamma) \to W_2^1(\Gamma)$ is invertible. It inverse exists for any λ, $\Im\lambda \geq 0$ and acts as the relative DN-map $\mathcal{DN}_{out}$, associated with the generalized $W_2^{3/2}(\Omega_{out})$ solution of the boundary problem

$$-\Delta u = \lambda u,\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega_{out}\setminus\Gamma} = 0,\ u\Big|_\Gamma = u_\Gamma, \Im\lambda > 0.$$

The generalized $W_2^{3/2}(\Omega_{out})$- solution of this problem is unique, see [12] and the corresponding relative DN- map is a closed operator

$$\mathcal{DN}_\Gamma : W_2^1(\Gamma) \to L_2(\Gamma).$$

Then the inverse operator is a closed map $\mathcal{Q}_{out}$ onto $L_2(\Gamma) \to W_2^1(\Gamma)$.

The operator R-function in the upper half-plane $\Im\lambda \geq 0$:

$$\mathcal{Q}_{out}(\lambda) + \mathcal{Q}_{int}(M) + (\lambda - M)\mathbf{Q}_{int}^+(M)\mathbf{Q}_{int}(M) =: \mathbf{Q}(\lambda)$$

is compact in $L_2(\Gamma)$ and defines a closed operator $L_2(\Gamma) \to W_2^1(\Gamma)$. The map is "onto" if $\mathbf{Q}(\lambda)$ does not have zero eigenvalue. In this case the corresponding inverse exists and is bounded, due to closed graph theorem, see for instance [16]

$$\mathbf{Q}^{-1}(\lambda) : W_2^1(\Gamma) \to L_2(\Gamma), \parallel \mathbf{Q}^{-1}(\lambda \parallel_{W_2^1(\Gamma)\times L_2(\Gamma)} < \infty.$$

The operator R-function $\mathbf{Q}(\lambda)$ is smooth in the closed upper half-plane. Then vector zeros μ_s of it are real

$$\mathbf{Q}(\mu_s)e_s = 0, e_s \in L_2(\Gamma)$$

and, according to [13] it may have only a finite number of the vector zeros on any finite interval of the real axis of the spectral parameter. Denote by Δ_μ the finite set of all vector-zeros on the essential spectral interval Δ and select, for given rational approximation (39), a real neighborhood of Δ_μ such that on the complement of it in Δ the condition

$$\sup_{\lambda\in\Delta\backslash\Delta_\mu} (\lambda - M)^2 \parallel \mathbf{Q}^{-1}(\lambda)\mathcal{K}^N \parallel_{L_2(\Gamma)} =: q < 1 \tag{40}$$

is fulfilled. Then the operator -function $\mathbf{Q} + \mathcal{K}^N(\lambda)$ is invertible on $\Delta\backslash\Delta_\mu$ and the following statement is true:

Lemma 3.2 *The equation (37) can be re-written on $\Delta\backslash\Delta_\mu$ in the finite-dimensional form:*

$$u + \left[\mathbf{Q} + \mathcal{K}^N(\lambda)\right]^{-1} \mathcal{Q}_{int}^N u = \left[\mathbf{Q} + \mathcal{K}^N(\lambda)\right]^{-1} \psi_{out}(*, \nu, \lambda). \tag{41}$$

Proof follows directly from the above arguments.

The end of the proof

Summarizing the above results we obtain the required regularization of the problem (37):

Theorem 3.3 *The problem (37) is reduced on* $\Delta\backslash\Delta_\mu$ *to the finite-dimensional equation, and has a unique smooth solution* $u \in W_2^{1/2}(\Gamma)$ *if* $\lambda \in \Delta\backslash\Delta_\mu$ *is not a zero of the corresponding determinant:*

$$det\left[I + \left[\mathbf{Q} + \mathcal{K}^N(\lambda)\right]^{-1} \mathcal{Q}_{int}^N\right] =: \mathbf{D}(\lambda) \neq 0.$$

Proof We use the smoothness of the trace $\psi_{out}\Big|_\Gamma \in W_2^{3/2}(\Gamma)$ on Γ of the generalized eigenfunction ψ_{out} (scattered waves) of L_{out}.

The end of the proof

Remark 4 Smoothness of the scattered waves ψ_{out} is defined by the smoothness of the boundary. In particular, for the properly smooth boundary, $\partial\Omega_{out} \in C_2$, we have at least $\psi_{out}\Big|_\Gamma \in W_2^{5/2}(\Gamma)$, then the right side of (41) belongs to $W_2^{3/2}(\Gamma)$, hence $u \in W_2^{3/2}(\Gamma) \in C(\Gamma)$.

Unfortunately the suggested general regularization does not help in practical calculation of u. In next section we develop a special regularization method for (37) based on filtering of signals by the channel, barred at a certain level frequencies.

4 Transport properties of a short thin channel.

Now we evaluate the contribution of the channel to the additional term of the amplitude (36) under assumption that the channel is relatively short and thin : $kH < \pi/2$, $\delta/H << 1$, see the Figure 2. In fact each of these condition can be loosen, see the concluding comments. In fact we will also use in this section more hard conditions $kH << \pi/2$, $\delta/H << 1$ which define "short thin" channels.

The denominator $\mathcal{Q}_{int}^*(\lambda) + \mathcal{Q}_{out}(\lambda)$ in above formulae (36), where $\mathcal{Q}_{in}^*(\lambda)$ is the ND-map of the extended inner domain. In fact we should transfer the ND map of the inner domain from the lower lid Γ_G to the upper lid along the channel Ω^δ. It may be done based on transport properties of the channel. If the channel is relatively short and thin, the final formulae appear to be convenient for explicit asymptotic calculation of the additional term of the scattering amplitude. We assume, see section 1 above, that the channel has a form of a relatively short and thin circular cylinder, see Fig.3, hight

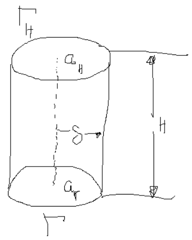

Figure 3: The short thin channel

$H,\ 0 < x < H$, radius $\delta,\ 0 < r < \delta$, with the lower lid Γ, and the upper lid Γ_H.

Denote by $\lambda_{n,s} = \nu^2_{n,s}(\delta) = \delta^{-2}\left[\nu^1_{n,s}\right]^2$ the eigenvalues of the Laplacian on the cross-section of the cylinder, with homogeneous Neumann boundary conditions at $r = \delta$, and by $P_{n,s}$ the projections onto the corresponding normalized eigenfunctions $Y_n J_n(\nu_{n,s}\, r)$, $n = 1, 2, \ldots$, with $Y_n(\varphi) = \text{Const}\ e^{\pm i n \varphi}$. The eigenvalues $\nu^2_{n,s} = \delta^{-2}\,(\nu^1)^2_{n,s}$ are defined by the zeros of the derivative of the Bessel functions: $J'_n(\nu_{n,s}\delta) = 0$. Denote by P_0 the projection onto the constant eigenfunction $Y_{0,0} = (\sqrt{2\pi}\,\delta)^{-1}$ corresponding to the eigenvalue $\nu_0^2 = 0$. Then $\sum_{(n,s)\neq(0,0)} P_{n,s} = P^{\perp}$ is the projection onto the orthogonal complement of constants on the cross-sections $\Gamma,\ \Gamma_H,\ P_0 \oplus P^{\perp} = I$ in $L_2(\Gamma),\ L_2(\Gamma_H)$. The complementary projections $P_0, P^{\perp}$ in $L_2(\Gamma), L_2(\Gamma_H)$ are represented as:

$$P_0 = \frac{\chi(x)\rangle\,\langle\chi(y)}{\pi\delta^2},\ P^{\perp} = I - \frac{\chi(x)\rangle\,\langle\chi(y)}{\pi\delta^2}. \tag{42}$$

Here $\chi(x) = \chi_\Gamma(x),\ \chi_H(x)$ is an indicator of the corresponding lid $\Gamma,\ \Gamma_H$, for instance : $\chi_H(x) = 1$, if $x \in \Gamma_H$, and 0 on the complement $\partial\Omega_{out}\backslash\Gamma_H$.

Hereafter we use spectral data for the scaled Neumann Laplacian $-\triangle_1$ on an orthogonal cross-section of the channel Ω_δ with respect to the scaled variables $\delta^{-1}\, r =: \xi,\ 0 < \xi < 1$, with the scaled eigenvalues $\lambda^1_{n,s} = (\nu^1_{n,s})^2$.

We also consider the boundary problem for the Laplacian on the channel with Neumann boundary condition at $r = \delta$ and non-homogeneous Dirichlet boundary conditions on the lids:

$$-\Delta u = \lambda u,\ u\Big|_{\Gamma} = u_{_\Gamma},\ u\Big|_{H} = u_{_H}.$$

The relative Dirichlet-to-Neumann map Λ^δ of this problem on Γ, Γ_H is defined by the normal outward derivatives with respect to the inner domain, on both sections Γ, Γ_H, and is obtained via separation of variables:

$$\Lambda^\delta =$$

$$\left(\begin{array}{cc} \Lambda^{^{HH}} & \Lambda^{^{H\Gamma}} \\ \Lambda^{^{\Gamma H}} & \Lambda^{^{\Gamma\Gamma}} \end{array} \right) = \frac{\sqrt{\lambda}}{\sin\sqrt{\lambda}\, H} \left(\begin{array}{cc} \cos\sqrt{\lambda} H & -1 \\ 1 & -\cos\sqrt{\lambda} H \end{array} \right) P_0 +$$

$$+ \sum_{n,s=1}^{\infty} \frac{\sqrt{\nu^2_{_{n,s}} - \lambda}}{\sinh\sqrt{\nu^2_{_{n,s}} - \lambda}\ \ H} \times$$

$$\times \left(\begin{array}{cc} \cosh\sqrt{\nu^2_{_{n,s}} - \lambda}\ \ H & -1 \\ 1 & -\cosh\sqrt{\nu^2_{_{n,s}} - \lambda}\ \ H \end{array} \right) P_{_{n,s}} := \Lambda^{^\delta}_{_0} + \Lambda^\delta_\perp. \qquad (43)$$

For thin channel the non-trivial (non-diagonal) component of the DN-map, responsible for the transmission of Dirichlet/Neumann data from one lid to another, is essentially defined by the constant eigenfunction of the cross-section:

$$\Lambda^\delta \approx \Lambda^{^\delta}_{_0} + \left(\begin{array}{cc} \delta^{^{-1}}\sqrt{-\triangle_{_1} - \delta^2 \lambda I}\, P^{^\perp}_{_H} & 0 \\ 0 & -\delta^{^{-1}}\sqrt{-\triangle_{_1} - \delta^2 \lambda I}\, P^{^\perp}_{_\Gamma} \end{array} \right), \qquad (44)$$

where $-\triangle_{_1}$ is the Neumann Laplacian in the orthogonal complement of constants on the lids Γ, Γ_H represented in terms of the scaled variables (on the corresponding scaled section radius 1). See below an extended discussion of the approximation suggested.

The inverse operator - Neumann-to-Dirichlet map- is calculated as

$$\mathcal{Q}^\delta := \left(\begin{array}{cc} \mathcal{Q}^\omega_{HH} & \mathcal{Q}^\omega_{H\Gamma} \\ \mathcal{Q}^\omega_{\Gamma H} & \mathcal{Q}^\omega_{\Gamma\Gamma} \end{array} \right) = \left(\begin{array}{cc} -\frac{1}{\sqrt{\lambda}\tan\sqrt{\lambda} H} & \frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda} H} \\ -\frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda} H} & \frac{1}{\sqrt{\lambda}\tan\sqrt{\lambda} H} \end{array} \right) P_0 +$$

$$\sum_{n,s}\left(\begin{array}{cc}\frac{1}{\sqrt{\lambda_{n,s}-\lambda}\tanh\sqrt{\lambda_{n,s}-\lambda}\,H} & -\frac{1}{\sqrt{\lambda_{n,s}-\lambda}\,\sinh\sqrt{\lambda_{n,s}-\lambda}\,H}\\ \frac{1}{\sqrt{\lambda_{n,s}-\lambda}\,\sinh\sqrt{\lambda_{n,s}-\lambda}\,H} & -\frac{1}{\sqrt{\lambda_{n,s}-\lambda}\,\tanh\sqrt{\lambda_{n,s}-\lambda}\,H}\end{array}\right)P_{n,s}=:$$

$$=:\mathcal{Q}_0^\delta+\mathcal{Q}_\perp^\delta.$$

It coincides with restriction of the full ND-map of the channel onto the lids. Note that the second addendum admits also the spectral representation:

$$\mathcal{Q}_\perp^\delta=\left(\begin{array}{cc}1 & e^{-\sqrt{-\Delta^\perp-\lambda I^\perp}H}\\ e^{-\sqrt{-\Delta^\perp-\lambda I^\perp}H} & 1\end{array}\right)\left(\begin{array}{cc}\frac{1}{\sqrt{-\Delta^\perp-\lambda I^\perp}H} & 0\\ 0 & -\frac{1}{\sqrt{-\Delta^\perp-\lambda I^\perp}H}\end{array}\right)$$

$$\left(\begin{array}{cc}1 & -e^{-\sqrt{-\Delta^\perp-\lambda I^\perp}H}\\ -e^{-\sqrt{-\Delta^\perp-\lambda I^\perp}H} & 1\end{array}\right)\frac{1}{1-e^{-2\sqrt{-\Delta^\perp-\lambda I^\perp}H}},$$

which shows that, for the thin channel, $\delta/H << 1$ and the values of the spectral parameter below the second threshold $\mathcal{Q}_\perp^\delta$ can be substituted, with a small error, by the diagonal matrix

$$\left(\begin{array}{cc}\frac{1}{\sqrt{-\Delta^\perp-\lambda I^\perp}\tanh\sqrt{-\Delta^\perp-\lambda I^\perp}H} & 0\\ 0 & -\frac{1}{\sqrt{-\Delta^\perp-\lambda I^\perp}\,\tanh\sqrt{-\Delta^\perp-\lambda I^\perp}H}\end{array}\right),$$

or just by

$$\left(\begin{array}{cc}\frac{1}{\sqrt{-\Delta^\perp-\lambda I^\perp}H} & 0\\ 0 & -\frac{1}{\sqrt{-\Delta^\perp-\lambda I^\perp}H}\end{array}\right),$$

because $\tanh\sqrt{-\Delta^\perp-\lambda I^\perp}H\approx 1$ for thin channel $\delta/H << 1$. Hereafter, for short thin channel

$$0<\sqrt{\lambda}H<\pi/2,\delta H^{-1}<<1, \tag{45}$$

we use the following approximation:

$$\mathcal{Q}^\delta\approx\mathcal{Q}_0^\delta+P_\perp\left(-\Delta-\lambda I\right)^{-1/2}P_\perp\left(\begin{array}{cc}1 & 0\\ 0 & -1\end{array}\right), \tag{46}$$

where the second addendum deviates from the corresponding exact term by the exponentially small error

$$\left(\begin{array}{cc}0 & \frac{1}{\sqrt{\lambda_{n,s}-\lambda}\,\sinh\sqrt{\lambda_{n,s}-\lambda}\,H}\\ -\frac{1}{\sqrt{\lambda_{n,s}-\lambda}\,\sinh\sqrt{\lambda_{n,s}-\lambda}\,H} & 0\end{array}\right)\approx e^{-\sqrt{\lambda_{n,s}^1-\delta^2\lambda}\,H/\delta}, \tag{47}$$

For the short channel $\sqrt{\lambda}H << \pi/2$ further simplification is possible:

$$\frac{1}{\lambda H}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} P_0 + P_\perp \left(-\Delta - \lambda I\right)^{-1/2} P_\perp \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

but now the deviation of first term of the second line of (46) from the corresponding exact term contains powers of $\lambda^{-1/2}$.

Hereafter we neglect the exponentially small terms (47) in the second addendum of the first line of (46), but *retain exact expression for the first term* $\mathcal{Q}_0$:

$$\mathcal{Q}^\delta_{appr} = \begin{pmatrix} -\frac{1}{\sqrt{\lambda}\tan\sqrt{\lambda}H} & \frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}H} \\ -\frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}H} & \frac{1}{\sqrt{\lambda}\tan\sqrt{\lambda}H} \end{pmatrix} P_0 +$$

$$\sum_{n,s}\begin{pmatrix} \frac{1}{\sqrt{\lambda_{n,s}-\lambda}\tanh\sqrt{\lambda_{n,s}-\lambda}\,H} & 0 \\ 0 & -\frac{1}{\sqrt{\lambda_{n,s}-\lambda}\,\tanh\sqrt{\lambda_{n,s}-\lambda}\,H} \end{pmatrix} P_{n,s} =:$$

$$\begin{pmatrix} -\gamma_t & \gamma_s \\ -\gamma_s & \gamma_t \end{pmatrix} P_0 + \begin{pmatrix} d^\perp & 0 \\ 0 & -d^\perp \end{pmatrix} =: \begin{pmatrix} \mathcal{Q}^\delta_{HH} & \mathcal{Q}^\delta_{H\Gamma} \\ \mathcal{Q}^\delta_{\Gamma H} & \mathcal{Q}^\delta_{\Gamma\Gamma} \end{pmatrix}. \qquad (48)$$

Hereafter the following notations are used:

$$\sqrt{\lambda}\,H\tan^{-1}\sqrt{\lambda}\,H =: \gamma_t,\ \sqrt{\lambda},\ \ H\sin^{-1}\sqrt{\lambda}\,H =: \gamma_s,$$

$$\sum_{n,s}\frac{1}{\sqrt{\lambda_{n,s}-\lambda}\tanh\sqrt{\lambda_{n,s}-\lambda}\ H}P_{n,s} =$$

$$\frac{1}{\sqrt{-\Delta_\perp - \lambda P_\perp}\tanh\sqrt{-\Delta_\perp - \lambda P_\perp}\ H} =: d^\perp, \qquad (49)$$

and the exponentially small non-diagonal elements of $\mathcal{Q}^\delta_\perp$ are neglected. Note that the diagonal elements $\mathcal{Q}^\delta_{\Gamma\Gamma}, \mathcal{Q}^\delta_{\Gamma_H,\Gamma_H}$ are invertible if the conditions (45) are fulfilled. Substituting, for thin channel $\delta/H << 1$, of the exact ND-map of Ω^δ by the above approximation $\mathcal{Q}^\delta_{appr}$ we admit an exponentially small error $O(e^{-H/\delta})$. In all calculations with precision $o(\delta/H), o(\sqrt{\lambda}H)$ we may just replace $\mathcal{Q}^\delta$ by $\mathcal{Q}^\delta_{appr}$. In particular, based on the above approximation (48) for $\mathcal{Q}^\delta$, we will calculate the restriction onto Γ_H of the ND - map of the extended inner domain $\Omega^*_{int} := \Omega_{int}\cup\Omega^\delta$.

One can construct, based on [11] an approximate spectral representation for the DN-map of the basic domain Ω_{int}:

$$\mathcal{DN}_\Gamma = \sum_{l=1}^{N}\frac{\frac{\partial\varphi_l}{\partial n}\rangle\,\langle\frac{\partial\varphi_l}{\partial n}}{\lambda-\lambda_l} + \mathcal{K}^N =: \mathcal{DN}^N_\Gamma + \mathcal{K}^N$$

with respect to Γ, on an essential spectral interval Δ and its complex neighborhood G_Δ. The number N can be selected such that $\parallel \mathcal{Q}^\delta_{\Gamma\Gamma}\mathcal{K}^N \parallel_{W_2^{3/2}\Gamma_H} < 1$. Then the operators

$$I - \mathcal{Q}^\delta_{\Gamma\Gamma}\mathcal{K}^N, \ \mathcal{Q}^\delta_{\Gamma\Gamma} - \mathcal{Q}^\delta_{\Gamma\Gamma}\mathcal{K}^N \mathcal{Q}^\delta_{\Gamma\Gamma} \tag{50}$$

are invertible. Denote

$$[\mathcal{Q}^\delta_{\Gamma\Gamma} - \mathcal{Q}^\delta_{\Gamma\Gamma}\mathcal{K}^N \mathcal{Q}^\delta_{\Gamma\Gamma}]^{-1} =: V = [\mathcal{Q}^\delta_{\Gamma\Gamma}]^{-1} \frac{I}{I - \mathcal{Q}^\delta_{\Gamma\Gamma}\mathcal{K}^N}$$

Denote by $E^N = \bigvee_{l=1}^N \varphi_l$ an invariant subspace of L_{int} which corresponds to the eigenfunctions $\{\varphi_l\}_{l=1}^N$ and introduce the spectral projection $P^N = \sum_{l=1}^N \varphi_l\rangle\,\langle\varphi_l$ and the part

$$L^N_{int} = \sum_{l=1}^N \lambda_l \varphi_l\rangle\,\langle\varphi_l$$

of L_{int} in E^N. Hereafter we use also the maps

$$T := \sum_{l=1}^N \varphi_l\rangle\,\langle\frac{\partial\varphi_l}{\partial n}, \ \mathcal{Q}^\delta_{\Gamma\Gamma} *\rangle, \ T^+ := \sum_{l=1}^N \mathcal{Q}^\delta_{\Gamma\Gamma}\frac{\partial\varphi_l}{\partial n}\rangle\ \langle\varphi_l, *\rangle. \tag{51}$$

The following statement describes the transmission of the Dirichlet / Neumann data from the lower lid Γ of the channel Ω_δ to the upper lid Γ_H:

Theorem 4.1 *The approximation (48) for $\mathcal{Q}^\delta$ implies the following approximate formula for the relative ND-map $\mathcal{ND}_H$ of Ω^*_{int} on Γ_H, for thin channel $\delta/H << 1$:*

$$\mathcal{ND}_H = d^\perp + \frac{\tan\sqrt{\lambda}H}{\sqrt{\lambda}}\ P_0 -$$

$$\frac{I}{\lambda\sin^2\sqrt{\lambda}H} P_0 \left[VT^+ \frac{I}{\lambda I^N - L^N - TVT^+} TV + V\right] P_0, \tag{52}$$

Proof The DN-map of L_{int} with respect to Γ is connected to ND map of L^*_{int} on the extended domain Ω^*_{int} by the linear system:

$$\begin{pmatrix} \mathcal{Q}^\delta_{HH} & \mathcal{Q}^\delta_{H\Gamma} \\ \mathcal{Q}^\delta_{\Gamma H} & \mathcal{Q}^\delta_{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} \rho_H \\ \mathcal{DN}_\Gamma u_\Gamma \end{pmatrix} = \begin{pmatrix} \mathcal{ND}_H\ \rho_H \\ u_\Gamma \end{pmatrix}. \tag{53}$$

This system implies the representation:

$$\mathcal{N}\mathcal{D}_H = \left[\mathcal{Q}^{\delta}_{HH} - \mathcal{Q}^{\delta}_{H\Gamma}\left[\mathcal{Q}^{\delta}_{\Gamma\Gamma}\right]^{-1}\mathcal{Q}^{\delta}_{\Gamma H}\right] + \mathcal{Q}^{\delta}_{H\Gamma}\left[\mathcal{Q}^{\delta}_{\Gamma\Gamma} - \mathcal{Q}^{\delta}_{\Gamma\Gamma}\mathcal{D}\mathcal{N}_{\Gamma}\mathcal{Q}^{\delta}_{\Gamma\Gamma}\right]^{-1}\mathcal{Q}^{\delta}_{\Gamma H}, \tag{54}$$

which can be simplified based on explicit expressions for $\mathcal{Q}^{\delta}$ substituted by $\mathcal{Q}^{\delta}_{appr}$, with an exponentially small error. In particular, the first addendum in the right side of (54) is simplified to

$$\mathcal{Q}^{\delta}_{HH} - \mathcal{Q}^{\delta}_{H\Gamma}\left[\mathcal{Q}^{\delta}_{\Gamma\Gamma}\right]^{-1}\mathcal{Q}^{\delta}_{\Gamma H} = d^{\perp} + \frac{\tan\sqrt{\lambda}H}{\sqrt{\lambda}}\, P_0.$$

To calculate the second addendum of (54) we should solve the equation:

$$\left[\mathcal{Q}^{\delta}_{\Gamma\Gamma} - \mathcal{Q}^{\delta}_{\Gamma\Gamma}\mathcal{D}\mathcal{N}_{\Gamma}\mathcal{Q}^{\delta}_{\Gamma\Gamma}\right] u = \mathcal{Q}^{\delta}_{\Gamma H}\rho_H \tag{55}$$

If the spectral rational approximation of $\mathcal{D}\mathcal{N}$ is selected such that

$$[\mathcal{Q}^{\delta}_{\Gamma\Gamma} - \mathcal{Q}^{\delta}_{\Gamma\Gamma}\mathcal{K}^N\mathcal{Q}^{\delta}_{\Gamma\Gamma}]^{-1} =: V \tag{56}$$

exists, see (50), then the solution of the above equation is reduced to the inversion of a finite matrix. Indeed, introduce a new variable

$$v = \sum_{l=1}^{N}\varphi_l\frac{\langle\mathcal{Q}^{\delta}_{\Gamma\Gamma}\frac{\partial\varphi_l}{\partial n}, u\rangle}{\lambda-\lambda_l} = (\lambda I^N - L^N)^{-1}Tu.$$

In terms of v the above equation (55) can be re-written as

$$(\lambda I^N - L^N)v - TVT^{+}v = TV\mathcal{Q}^{\delta}_{\Gamma H}\rho, \tag{57}$$

hence

$$v = \frac{I}{\lambda I^N - L^N - TVT^{+}}TV\mathcal{Q}^{\delta}_{\Gamma H}\rho.$$

Then

$$u = VT^{+}v + V\mathcal{Q}^{\delta}_{\Gamma H}\rho =$$

$$VT^{+}\frac{I}{\lambda I^N - L^N - TVT^{+}}TV\mathcal{Q}^{\delta}_{\Gamma H}\rho + V\mathcal{Q}^{\delta}_{\Gamma H}\rho,$$

and

$$\mathcal{N}\mathcal{D}_H\rho = d^{\perp}\rho + \frac{\tan\sqrt{\lambda}H}{\sqrt{\lambda}}\, P_0\rho +$$

$$\mathcal{Q}^{\delta}_{H\Gamma}\left[VT^{+}\frac{I}{\lambda I^N - L^N - TVT^{+}}TV + V\right]\mathcal{Q}^{\delta}_{\Gamma H}\rho \tag{58}$$

Note that $\mathcal{Q}^{\delta}_{H\Gamma} = \frac{I}{\sqrt{\lambda}\sin\sqrt{\lambda}H}P_0 = -\mathcal{Q}^{\delta}_{\Gamma H}$. This implies the announced result.

The end of the proof

Remark 5 When deriving the expression for $\mathcal{DN}_H$ we neglected the exponentially small non-diagonal terms $O(e^{-H/\delta})$ of the component $\mathcal{Q}_{\perp}$ and obtained the diagonal expression (58) for $\mathcal{DN}_H$:

$$\mathcal{ND}_H \approx d^{\perp} + \mathbf{M}P_0. \tag{59}$$

with the scalar function

$$\mathbf{M} = \frac{\tan\sqrt{\lambda}H}{\sqrt{\lambda}} - \frac{I}{\lambda\sin^2\sqrt{\lambda}H}\text{Trace}\left[VT^{+}\frac{I}{\lambda I^N - L^N - TVT^{+}}TV + V\right]P_0 =:$$

$$\frac{\tan\sqrt{\lambda}H}{\sqrt{\lambda}} - \frac{\mathbf{D}}{\lambda\sin^2\sqrt{\lambda}H} - \frac{\text{Trace}P_0V}{\lambda\sin^2\sqrt{\lambda}H}$$

where

$$\mathbf{D} = \text{Trace}\left[VT^{+}\frac{I}{\lambda I^N - L^N - TVT^{+}}TV\right]P_0.$$

It is easy to see, that the diagonal expression (59) differs from the exact value of $\mathcal{ND}_H$ by the exponentially small error estimated as

$$\parallel \mathcal{ND}_H - d^{\perp} - \mathbf{M}P_0 \parallel \leq C\mathbf{D}e^{-H/\delta}, \tag{60}$$

which is small outside of a small neighborhood of the poles of $\mathbf{D}$.

Remark 6 We consider also another, less accurate, but more convenient for fitting, approximation of $\mathcal{ND}_H$ for short thin channel $kH << \pi/2$, $\delta/H << 1$. We obtain it via replacement of $\mathbf{D}$ by

$$\mathcal{D} = \text{Trace}\left[P_0\sum_{l,m=1}^{N}\frac{\partial\varphi_l}{\partial n}\rangle\,\langle\varphi_l\frac{I}{\lambda I^N - L^N - TVT^{+}}\varphi_m\rangle\,\langle\frac{\partial\varphi_m}{\partial n}\right].$$

Due to the estimation, for small $\mathcal{Q}^{\delta}_{\Gamma\Gamma}\mathcal{K}^N$,

$$\text{Trace}\left[TV - \sum_{l}\varphi_l\rangle\,\langle\frac{\partial\varphi_l}{\partial n}\right]P_0 \approx \text{Trace}\,\mathcal{K}^{\mathcal{N}}P_0,$$

we see, that the difference $\mathbf{D} - \mathcal{D}$ is small on the complement of a neighborhood of poles of $\mathcal{D}$

$$|\mathbf{D} - \mathcal{D}| \leq |\mathcal{D}| \,\text{Trace}\, \mathcal{K}^N P_0. \tag{61}$$

Under the same condition we estimate

$$\text{Trace}\,[P_0 V - P_0 (\mathcal{Q}^\delta_{\Gamma\Gamma})^{-1}] \approx \text{Trace}\mathcal{K}^N P_0. \tag{62}$$

Summarizing above observations (61,62) we see that for small Trace$\mathcal{K}^N \mathcal{D}^\delta_{\Gamma\Gamma}$) and $\sqrt{\lambda} H << 1$, the function $\mathbf{M}$ can be substituted by

$$H - \frac{\mathcal{D}}{\lambda^2 H^2} - \frac{I}{\lambda H} =: \mathcal{M},$$

with a relatively minor error estimated by

$$C \frac{\text{Trace} P_0 \mathcal{K}^N}{\lambda^2 H^2} \,[|\mathcal{D}| + \lambda H], \tag{63}$$

with come constant C (independent on H, λ). It is important that the structure of functions $\mathbf{M}$ and $\mathcal{M}$ is the same as one of the function $\mathcal{M}_H$, which appeared in (23) as a parameter of the solvable model, see remark 1:

Lemma 4.1 *For thin short channel the structure of* $\mathbf{M}$ *is described by the expression:*

$$\mathcal{M} = H - \frac{\mathcal{D}}{\lambda^2 H^2} - \frac{1}{\lambda H}. \tag{64}$$

If $\mathcal{M}_\Gamma$ *is selected such that*

$$\frac{1}{\mathcal{M}_\Gamma} + \frac{1}{\lambda H} + \frac{1}{\mathcal{D}} = 0,$$

then

$$\mathcal{M} = \mathcal{M}_H.$$

Proof The formula (64) is already proved above. The second statement of the lemma is derived based on (64) and (45). Indeed, if

$$H - \mathcal{M}_H = \frac{1}{\lambda H + \mathcal{M}_\Gamma} = \frac{\mathcal{D} + \lambda H}{\lambda^2 H^2} = H - \mathcal{M},$$

then

$$\mathcal{M}_\Gamma = \frac{\lambda^2 H^2}{\mathcal{D} + \lambda H} - \lambda H = -\frac{\mathcal{D} \lambda H}{\mathcal{D} + \lambda H}.$$

The end of the proof

Remark 7 Inserting the expression for $\mathbf{M}$ into (59) we obtain, with $\mathcal{Q}_H(\lambda) =: \mathcal{Q}^*_{int}(\lambda)$ a convenient approximate expression for the denominator of the additional term of the scattering amplitude (36):

$$\delta a := \frac{1}{8\pi^3}\psi_{out}(*,\,\omega,\lambda)\,[\mathcal{Q}^*_{int}(\lambda)+\mathcal{Q}_{out}(\lambda)]^{-1}\,\psi_{out}(*,\,\nu,\lambda) \approx$$

$$\frac{1}{8\pi^3}\psi_{out}(*,\,\omega,\lambda)\left[d^{\perp}+\mathbf{M}\,P_0+\mathcal{Q}_{out}(\lambda)\right]^{-1}\psi_{out}(*,\,\nu,\lambda) =: \delta a_M, \qquad (65)$$

with an exponential estimate of deviation of $\mathcal{Q}^*_{int}$ from $d^{\perp}+\mathbf{M}\,P_0$, on the complement of some neighborhood of poles of $\mathbf{M}$, and, more rough estimate (5) of the deviation from $d^{\perp}+\mathcal{M}$ on a complement of a small neighborhood of poles of $\mathcal{D}$.

5 Fitting of the solvable model

It was noticed in section 2, that the solvable model of the Helmholtz resonator may be considered as"fitted" on an essential spectral interval if the model scattering matrix serves an approximation of the original scattering matrix of the Helmholtz resonator. One may relax this non-formal definition when comparing the basic formula for the additional terms of the scattering amplitudes of the model problem (26) with the additional term of the scattering amplitude of the original problem for the Helmholtz resonator (36). Notice, first of all, that for zero-range model the values of scattered waves $\psi_{out}(x,\,\omega,\lambda)$, $\psi_{out}(x,\,\nu,\lambda)$ should be taken at the single point $x_H = a$. According to Remark 3 at the end of section 3, the solution u of the basic equation (37), obtained with the corresponding regularization, is unique and smooth, $u \in W_2^{3/2}(\Gamma)$, for non-singular λ. Now we evaluate the solution u of (37) replacing $\mathcal{Q}^*_{int}$ by the approximate expression (59), see Remark 4 above. We also substitute $\psi_{out}(*,\,\omega,\lambda)$ and $\psi_{out}(*,\,\nu,\lambda)$ on Γ_H by their values at the center x_H of the upper lid, multiplied by the indicator $\chi_{\Gamma_H} =: \chi_H$ of the upper lid:

$$\psi_{out}(x),\,\omega,\lambda) \longrightarrow \psi_{out}(x_H,\,\omega,\lambda)\chi_H(x),\; x\in\Gamma_H,$$

$$\psi_{out}(x),\,\nu,\lambda) \longrightarrow \psi_{out}(x_H,\,\nu,\lambda)\chi_H(x),\; x\in\Gamma_H,.$$

Due to $\psi_{out} \in W_2^2(\Omega)$ the trace of ψ_{out} on Γ_H is smooth, $\psi_{out} \in W_2^{3/2}(\Gamma_H)$,hence, due to embedding theorem $W_2^{3/2}(\Gamma_H) \subset \text{Lip}\,_1(\Gamma_H)$ we have

$$|\psi_{out}(x),\, \omega, \lambda) - \psi_{out}(x_H,\, \omega, \lambda)\chi(x)| \leq \delta|\lambda|C,\, x \in \Gamma_H, \qquad (66)$$

with an absolute constant C, because $\parallel \psi_{out} \parallel_{W_2^{3/2}}(\Gamma_H) \leq C|\lambda|$. The resulting expression for the approximate correcting term of the amplitude coincides with

$$\frac{1}{8\pi^3}\ \psi_{out}(x_H,\, \omega, \lambda)\ \langle \chi_H, \frac{1}{\mathcal{Q}^*_{int} + \mathcal{Q}_{out}} \chi_H \rangle\ \psi_{out}(x_H,\, \nu, \lambda) =: \delta\hat{a} \qquad (67)$$

We estimate first the difference between $\delta\hat{a}$ and the original correcting term

$$\frac{1}{8\pi^3}\psi_{out}(*,\, \omega, \lambda)\, [\mathcal{Q}^*_{int}(\lambda) + \mathcal{Q}_{out}(\lambda)]^{-1}\, \psi_{out}(*,\, \nu, \lambda =: \delta a. \qquad (68)$$

Denote by $u, \hat{u}$ solutions of the equations

$$[\mathcal{Q}^*_{int} + \mathcal{Q}_{out}]u = \psi_{out}(x),\ \ [\mathcal{Q}^*_{int} + \mathcal{Q}_{out}]\hat{u} = \chi_H \psi_{out}(x_H)(x_H,\, \nu, \lambda).$$

Applying to this equation the procedure described in remark 4, we reduce it to the finite-dimensional linear system, with the determinant

$$\det\left[I + \left(I + \mathcal{DN}_{out}\mathbf{K}^N\right)^{-1} \mathcal{DN}_{out}\mathcal{Q}^N_{int}\right] =: \det B(\Lambda).$$

If $\det B(\Lambda) \neq 0$, then for thin channel, $\delta/H << 1$,

$$\parallel u - \hat{u} \parallel \leq \frac{C\delta|\lambda|}{\det B(\lambda)},$$

with a constant C non depending on λ. This implies an estimate of the difference $\delta a - \delta\hat{a}$ between the original and approximate correcting term by the following expression:

$$\sup_x |\psi_{out}(x) - \psi_{out}(x_H)| \sup_{\Gamma_H} |u| + \sup |u - \hat{u}| \sup_{\Gamma_H} |\psi_{out}| \leq \frac{C\delta|\lambda|}{\det B(\lambda)},$$

again with some constant C which does not depend on λ.

On the second step we compare the approximate correcting term

$$\frac{1}{8\pi^3}\psi_{out}(x_H,\, \omega, \lambda)\left[d^{\perp} + \mathbf{M}\, P_0 + \mathcal{Q}_{out}(\lambda)\right]^{-1} \psi_{out}(x_H,\, \nu, \lambda). \qquad (69)$$

with the correcting term recovered from the above solvable model, see (26).

Definition We call the solvable model fitted, if the model correcting term $\delta \hat{a}$ serves an approximation of the approximate correcting term δa.

If the model is fitted in the above sense, then obviously it is also fitted in terms of amplitudes or scattering matrices. Note that the calculation of the approximate correcting term () requires solution $\hat{u}$ of the equation

$$\left[P_0 \mathbf{M} + d^{\perp} + \mathcal{Q}_{out}\right] u = \chi_H. \tag{70}$$

The difference between the approximate correcting term and the original correcting term (36) is estimated, based on the results of previous section. In particular, the term $\mathcal{Q}^*_{int}$ can be replaced, with a minor error by $P_0\mathbf{M}+d^{\perp}$. Denoting $u = (\rho_0 \chi + u_{\perp})$ with $\chi := \chi_{\Gamma_H}$ and applying $P_0 = [\pi\delta^2]^{-1}\chi\rangle\langle\chi, P_{\perp} = I - P_0$ to (70) we obtain a system of two equations for $\rho_0, u_{\perp}$:

$$P_{\perp} d^{\perp} u_{\perp} + P_{\perp}\mathcal{Q}_{out} u_{\perp} + P_{\perp}\mathcal{Q}_{out}\chi_H \rho_0 = 0,$$

$$\mathbf{M}\chi_H \rho_0 + P_0 \mathcal{Q}_{out} u_{\perp} + P_0\mathcal{Q}_{out}\chi_H\rho_0 = \chi. \tag{71}$$

The middle term of the above formula (67) is directly connected to the component ρ_0 of the solution of the system

$$\langle \chi_H, \frac{1}{\mathcal{Q}^*_{int} + \mathcal{Q}_{out}} \chi_H\rangle = \langle \rho_0 \chi_H, \chi_H\rangle = \rho_0\ \pi\,\delta^2. \tag{72}$$

The kernel of the integral operator $\mathcal{Q}_{out}$ coincides with the Green function of the Neumann Laplacian in the outer domain, and can be represented, based on resolvent identity, as

$$\mathcal{Q}_{out} u = \int_{\Gamma_H} G_{out}(x,y,\lambda) u(y) dy =$$

$$\int_{\Gamma_H} G_{out}(x,y,M) u(y) dy + \int_{\Gamma_H} \mathcal{M}_{out}(x,y,\lambda,M) u(y) dy,$$

with a large negative M and a continuous kernel $\mathcal{M}_{out}(x,y,\lambda,M) = (\lambda - M)\int_{\Omega_{out}} G_{out}(x,z,M) G_{out}(z,y,\lambda) dz$. One can show that near a smooth point $y \in \partial\Omega_{out}$ the Green function $G_{out}(x,y,M)$ has an asymptotic expansion

$$G_{out}(x,y,M) = \frac{1}{2\pi|x-y|} - \frac{\sqrt{|M|}}{2\pi} + \dots.$$

We will also use the notation

$$\gamma_1 = \delta^{-3} \int_\Gamma \frac{dxdy}{2\pi|x-y|},$$

and denote the limit of $\mathcal{M}_{out}$, see (8), at the center of the upper lid $x_H \equiv a$

$$\lim_{x,y\to a} \mathcal{M}_{out}(x,y,\lambda,M) = C(a,M) + i\pi\frac{d\mathcal{E}}{d\lambda}(a) + VP\mathcal{M}_{out} \equiv \mathcal{M}_{out}(a).$$

Theorem 5.1 *The component ρ_0 of the solution of (70) is approximately calculated, for small δ, as*

$$\rho_0 = \frac{1 - o\,(\gamma_1\delta^2\,\lambda)}{\mathbf{M} + \delta\gamma_1 + \mathcal{M}_{out}\pi\delta^2}.$$

Proof We already noticed at the end of section 3 that, due to local smoothness of eigenfunctions of the Laplacian, see the Remark 4 in the end of section 3, the solution u of the above equation (70) is smooth, in particular $u \in W_2^{3/2}(\Gamma) \subset \text{Lip}\ _1(\Gamma)$. More precise, due to the corresponding embedding theorem on a small lid we have:

$$\sup_\Gamma |u_\perp| = \sup_\Gamma |u - P_0 u| \le \delta \parallel u \parallel_{W_2^{3/2}(\Gamma)} \approx \delta\lambda.$$

Then the second equation (71) implies

$$\mathbf{M}\rho_0 + [\pi\delta^2]^{-1} \int_\Gamma \int_\Gamma \frac{dxdy}{2\pi|x-y|}\rho_0 + \mathcal{M}_{out}(a)\pi\delta^2\,\rho_0 =$$

$$1 - \int_\Gamma \int_\Gamma \frac{u_\perp}{2\pi|x-y|}dxdy = 1 - o(\gamma_1\lambda\delta^2). \qquad (73)$$

This gives the announced approximate expression for ρ_0.

The end of the proof

Remark 7 Combining the above result with (72) we obtain:

$$\langle \chi_{\Gamma_H}, \frac{1}{\mathcal{Q}^*_{int} + \mathcal{Q}_{out}} \chi_{\Gamma_H} \rangle = \frac{1}{\frac{\mathbf{M}}{\pi\delta^2} + \frac{\gamma_1}{\pi\delta} + \mathcal{M}_{out}}[1 + 0(\gamma_1\ \delta^2)] \qquad (74)$$

Note that, due to Lemma 4.1 the expression $\frac{\mathbf{M}}{\pi\delta^2} + \frac{\gamma_1}{\pi\delta} + \mathcal{M}_{out}$ can be substituted, with a minor error, by $\frac{\mathcal{M}}{\pi\delta^2} + \frac{\gamma_1}{\pi\delta} + \mathcal{M}_{out}$, that is $\mathbf{M} \to \mathcal{M}$. Comparing the obtained expression with the denominator of the additional term of the model scattering amplitude (26), we conclude that they coincide with each other if the inner structure is selected as prescribed in Lemma 4.1, and the boundary parameters are chosen as

$$|\beta_{01}|^2 = \pi\delta^2,\ \beta_{11} = \gamma_1\,\delta.$$

Hence the following statement is proved:

Theorem 5.2 *There exist a real function with the asymptotic behavior near the origin,*

$$\gamma(\delta) = \frac{\gamma_1}{\pi\delta}$$

such that the scattering amplitude of the Helmholtz resonator with short thin channel is defined by the approximate formula

$$a(\omega,\nu,\lambda) \approx a_{out}(\omega,\nu,\lambda) + \frac{1}{8\pi^3}\ \frac{\psi_{out}(x_{\Gamma_H},\omega)\ \psi_{out}(x_{\Gamma_H},\nu)}{\mathcal{M}_{out} + \mathcal{M}(\pi\delta^2)^{-1} + \gamma(\delta)}, \tag{75}$$

with minor error estimated as (5), on a complement of a small neighborhood of poles of $\mathcal{D}$. The corresponding Kirchhoff constant

$$A_H = \frac{1}{\mathcal{M}_{out} + \mathcal{M}(\pi\delta^2)^{-1} + \frac{\gamma_1}{\pi\delta}}\psi_{out}(x_{\Gamma_H}). \tag{76}$$

defines the model scattered wave in the outer domain.

Note, that the role of the term $\mathcal{M}_{int}$ in [9] now is played by the corresponding re-normalized term $\mathcal{M}(\pi\delta^2)^{-1}$, and the approximate scattering amplitude is obtained as a scattering amplitude of some solvable model with "inner structure", defined from $\mathcal{M}(\pi\delta^2)^{-1}$ based on Lemma 4.1.

Remark 8 To construct the fitted solvable model of the Helmholtz resonator with thin short channel, $\sqrt{\lambda}H << 1, \delta/H << 1$, we have to make the following steps:

1. Select an essential spectral interval $\Delta \subset [0, \pi^2/4H^2$, on which, we expect, the model scattering matrix should approximate the scattering matrix of the resonator.

2. Solve the inner spectral problem in the cavity and construct the corresponding relative DN-map $\mathcal{DN}^\Gamma$. Transfer it to the upper lid as $\mathcal{ND}^*_{int} = \mathcal{Q}^*_{int}$.

3. Select the rational approximation of the DN-map $\mathcal{Q}^*_{int} = \mathcal{Q}^N_{int} + \mathcal{K}^N$, such that $\| \mathcal{Q}_{out}\mathcal{K}^N \| << 1$.

4. Find the poles of $\mathcal{D}$ as vector zeros of the matrix $\left[L^N + TVT^+ - \lambda I^N\right] e = 0$. Choose a neighborhood of zeros such that on the complement of it inside the essential spectral interval the estimate () holds.

5. Construct the inner structure, selecting $\mathcal{M}_\Gamma$ as prescribed by Lemma 4.1, and substituting TVT^+ by

$$\sum_{m,l=1}^{N} \varphi_l\rangle\langle \frac{\partial \varphi_l}{\partial n} \mathcal{Q}^\delta_{\Gamma,\Gamma} \frac{\partial \varphi_m}{\partial n}\rangle\langle \varphi_m + : V(\lambda),$$

with a minor error estimated, for short thin channel, as $\| \mathcal{Q}^\delta_{\Gamma,\Gamma}\mathcal{K}^N \| << 1$.

6. Select the boundary parameters as prescribed in Theorem 5.2.

The constructed model is automatically fitted, because its scattering matrix serves an approximation of the scattering matrix of the Helmholtz resonator on a complement of a neighborhood of zeros of $\mathcal{D}$, which are found as spectrum of the above spectral problem

$$L^N e + T^+ V T e = \lambda e.$$

7. Find the zeros of the denominator of the additional term of the model scattering matrix, for the fitted model,

$$\mathcal{M}_{out} + \mathcal{M}(\pi\delta^2)^{-1} + \gamma_1(\pi\delta)^{-1} = 0,$$

Since the zeros are situated, for thin short channel, near the essential spectral interval $\Delta \subset R$, $\mathcal{M}_{out}$ can be substituted by the value at the center $x_H \equiv a$ of the upper lid, see (9)

$$\mathcal{M}_{out}(a) = C(a, M) + i\pi \, \frac{\partial \mathcal{E}}{\partial \lambda}(a, a) + \mathcal{M}_{VP}(a, a, \lambda).$$

One may expect that the poles of the additional term of the original scattering amplitude or, equivalently, ones of the above Kirchhoff constant, with $\mathcal{M}_{out}$ substituted by the asymptotic (9), can be found as zeros of the denominator of (76) which give a “first order approximation” for resonances of the Helmholtz resonator, with short thin channel. In fact calculation of resonances based on the above formula, as poles of the approximate amplitude or poles of the Kirchhoff constant, requires more accurate investigation of analytic properties of the amplitude in the complex plane and can be done based on the operator version of Rouche theorem, see [13].

6 Conclusion

Our technique was based on specific transport properties of the thin short channel. Considering longer channel just requires taking into account oscillating modes in the channel. The corresponding analysis can be also done in explicit form. More interesting is the problem on approximate calculation of scattering matrix of Helmholtz resonator with relatively wide opening. Recall, that we succeeded to simplify the basic equation (37) for thin channel because all exponential modes in the channel which correspond to positive cross-section eigenvalues $\lambda_{l,m} : \mathcal{J}'_m(\lambda_{m,l}) = 0$, are "filtered out" by the thin channel, that is: they do not contribute to the transfer of Dirichlet/Neumann data from the lower section to the upper section, because $\lambda^1_{l,m}\ \delta^{-2} >> \lambda$, beginning from $l = 1$. When using direct computing we are able to relax the above condition, see (47) neglecting only the transfer by the higher modes: $\sinh^{-1}\sqrt{\lambda^1_{m,l} - \delta^2\lambda}H/\delta << 1$ if $\lambda_{m,l} > \Lambda_0$, beginning from some Λ_0 which is large enough. There is only a finite number of modes below that level, which can be taken into account in explicit form, by the direct computing. The suggested procedure is actually an extension to the case of Helmholtz resonator of the perturbation procedure for the junction of a quantum network, suggested recently in [12]. Note that in the recent preprint [14] a computational procedure suggested for resonances in any resonator, without additional conditions on the diameter of the opening. The described procedure permits to extend *analysis* for Helmholtz resonator with wide opening. Intriguing applications of this extension will be discussed in [**?**].

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