HOMOGENIZATION OF SPECTRAL PROBLEMS IN BOUNDED DOMAINS WITH DOUBLY HIGH CONTRASTS

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ABSTRACT. Homogenization of a spectral problem in a bounded domain with a high contrast in both stiffness and density is considered. For a special critical scaling, two-scale asymptotic expansions for eigenvalues and eigenfunctions are constructed. Two-scale limit equations are derived and relate to certain non-standard self-adjoint operators. In particular they explicitly display the first two terms in the asymptotic expansion for the eigenvalues, with a surprising bound for the error of order $\varepsilon^{5/4}$ proved.

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1. INTRODUCTION

Homogenization for problems with physical properties which are not only highly oscillatory but also highly heterogeneous has long been documented to display unusual effects, for example the memory effects observed by E. Ya. Khruslov [9, 13, 14]. Of particular interest in this context are the double-porosity models where the parameter of high-contrast δ is critically scaled again the periodicity size ε, $\delta \sim \varepsilon^2$, e.g. [2, 4]. Those have been treated both by a high-contrast version of the classical method of asymptotic expansions, e.g. [16, 17, 7, 12] and using the techniques of two-scale convergence, e.g. [19, 20, 5]. In particular, for spectral problems in bounded [19] and unbounded [20] periodic domains V.V. Zhikov studied the spectral convergence, introduced two-scale limit operator, developed the techniques of two-scale resolvent convergence and two-scale compactness. In [12] the spectral convergence of eigenvalues in the gaps of Floquet-Bloch spectrum due to defects in double-porosity type media were studied, and [5] supplemented this by the analysis of eigenfunction convergence based on an analysis of a uniform exponential decay.

In this work we study spectral problems of double-porosity type in a bounded domain Ω where the high contrast might occur not only in the "stiffness" coefficient but also in the "density", and argue that this leads to some interesting new effects. Namely, referring to the next section for precise technical formulations, for the spectral problem

$$-\operatorname{div}\left(a_\varepsilon(x)\nabla u_\varepsilon\right) = \lambda^\varepsilon \rho_\varepsilon(x)\, u_\varepsilon, \tag{1}$$

with Dirichlet boundary conditions on the exterior boundary, most generally, both a_ε and ρ_ε are ε-periodic, $a_\varepsilon = \rho_\varepsilon = 1$ in the connected matrix and $a_\varepsilon \sim \varepsilon^\alpha$, $\rho_\varepsilon \sim \varepsilon^\beta$ in the disconnected inclusions. (Outside homogenization, the above resembles problems of vibrations with high contrasts in both density and stiffness, e.g. [3].) The double-porosity corresponds to $\alpha = 2$ and $\beta = 0$. For $\beta \neq 0$, it is not hard to see that it is $\alpha = \beta + 2$ when the spectral problems at the macro and micro-scales are coupled in a non-trivial way. To explore this, we choose $\beta = -1$ and $\alpha = 1$

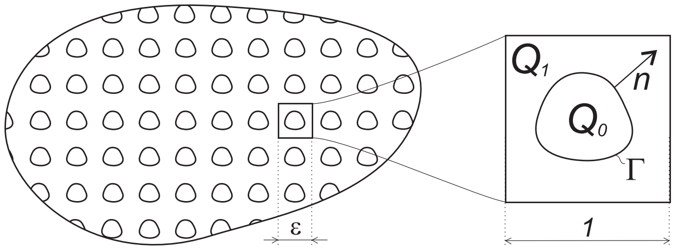

FIGURE 1. The geometry and the periodicity cell

and show that this leads to some unusually coupled two-scale limit behaviors of the eigenfunctions and the eigenvalues.

Namely, although the limit behavior of the eigenfunctions is still somewhat similar to that of double porosity, i.e. the two-scale limit is a function of only slow variable x in the matrix and a function of both x and the fast variable y in the inclusions, the limit equations themselves are quite different. We show that there exist asymptotic series of eigenvalues $\lambda^\varepsilon \sim \lambda_0 + \varepsilon\lambda_1$ with λ_0 being any eigenvalue of a non-standard self-adjoint "microscopic" inclusion problem, Theorem 3.1, whose eigenfunctions are directly related to the two-scale limit $w_0(x, y)$ in the matrix. In fact, λ_0 is either a solution of $\beta(\lambda_0) = |Q_1|\lambda_0$, where $\beta(\lambda)$ is a function introduced by Zhikov [19], or is an eigenvalue of the Dirichlet Laplacian in the inclusion Q_0 with a zero mean eigenfunction. In the matrix, $u_\varepsilon \sim v^0(x)$, where v^0 is an eigenfunction of the homogenized operator in Ω, whose eigenvalue ν determines the second term λ_1 in the asymptotics of λ^ε, see (57). This is first derived via formal asymptotic expansions, but then we prove a non-standard error bound:

$$|\lambda^\varepsilon - \lambda_0 - \varepsilon\lambda_1| \le C\varepsilon^{5/4},$$

see Theorems 4.6 & 4.7. The proof employs a combination of a high contrast boundary layer analysis with maximum principle and estimates in Hilbert spaces with ε-dependent weights. We finally briefly discuss further refinement of the results via the technique of two-scale convergence. Namely, some version of the compactness result holds, cf. [19], indicating at the presence of gaps in the spectrum for small enough ε, see Theorem 5.1.

The paper is organized as follows. The next section formulates the problem and introduces necessary notation, Section 3 executes formal asymptotic expansion and derives associated homogenized equations. Section 4 proves the error bounds and Section 5 discusses the two-scale convergence approach. Some technical details are assembled in the appendices.

2. Problem statement and notations

We consider a model of eigenvibrations for a body occupying a bounded domain Ω in $\mathbb{R}^n$ ($n = 2, 3, \dots$) containing a periodic array of small inclusions, see Figure 1. The size of inclusions is controlled by a small positive parameter ε, $\varepsilon \to 0$. First we introduce necessary notation.

Let $Q = [0, 1]^n$ be a reference periodicity cell in $\mathbb{R}^n$. Let $\widetilde{Q}_0$ be a periodic set of "inclusions", i.e. $\widetilde{Q}_0 + m = \widetilde{Q}_0$, $\forall m \in \mathbb{Z}^n$, and $Q_0 = \widetilde{Q}_0 \cap Q$ is a reference inclusion lying inside Q with C^2-smooth boundary Γ, see Figure 1. Let $Q_1 = Q\backslash\overline{Q_0}$,

$\widetilde{Q}_1 = \mathbb{R}^n \backslash \overline{\widetilde{Q}_0}$, $\widetilde{\Gamma} = \partial \widetilde{Q}_0 = \partial \widetilde{Q}_1$. Introducing $y = x/\varepsilon$ we refer to y as to a fast variable, as opposes to the slow variable x. In the x-variable the periodicity cell is $\varepsilon Q = [0, \varepsilon)^n$. If $y \in Q_j$ then $x = \varepsilon y \in \varepsilon Q_j$, $j = 0, 1$. We denote $\Omega_0^\varepsilon := \Omega \cap \varepsilon \tilde{Q}_0$, $\Omega_1^\varepsilon := \Omega \cap \varepsilon \tilde{Q}_1 = \Omega \backslash \overline{\Omega_0^\varepsilon}$, $\Gamma^\varepsilon := \varepsilon \tilde{\Gamma} \cap \Omega$, see Figure 1. The trace on Γ^ε of function $f : \Omega_j^\varepsilon \to \mathbb{R}^n$ is denoted by $f|_j$. Let n_y be the outer unit normal to Q_0 on its boundary Γ and let n_x denote the similar normal on Γ^ε.

Let stiffness a_ε and density ρ_ε be as follows

$$a_\varepsilon(x) = \begin{cases} 1, & x \in \Omega_1^\varepsilon \\ \varepsilon, & x \in \Omega_0^\varepsilon \end{cases} \quad \text{and} \quad \rho_\varepsilon(x) = \begin{cases} 1, & x \in \Omega_1^\varepsilon \\ \varepsilon^{-1}, & x \in \Omega_0^\varepsilon \end{cases}$$

with a small positive ε.

We study the asymptotic behaviour of self-adjoint spectral problem

$$\int_\Omega a_\varepsilon(x) \nabla u_\varepsilon \nabla \phi \, dx - \lambda^\varepsilon \int_\Omega \rho_\varepsilon(x) u_\varepsilon \phi \, dx = 0, \quad \forall \phi \in H_0^1(\Omega) \tag{2}$$

as $\varepsilon \to 0$. If Γ and $\partial\Omega$ are smooth enough then variational problem (2) can be equivalently represented in a classical formulation

$$-\operatorname{div}(a_\varepsilon(x) \nabla u_\varepsilon) = \lambda^\varepsilon \rho_\varepsilon(x) u_\varepsilon, \quad x \in \Omega, \tag{3}$$

$$u_\varepsilon|_{\partial\Omega} = 0, \tag{4}$$

implying that at the interfaces the transmission conditions are satisfied

$$u_\varepsilon\Big|_1 = u_\varepsilon\Big|_0, \quad \frac{\partial u_\varepsilon}{\partial n_x}\Big|_1 = \varepsilon \frac{\partial u_\varepsilon}{\partial n_x}\Big|_0. \tag{5}$$

3. Formal asymptotic expansions

We seek formal asymptotic expansions for the eigenvalues λ^ε and eigenfunctions u_ε in the form

$$\lambda^\varepsilon \sim \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \dots, \tag{6}$$

$$u_\varepsilon(x) \sim \begin{cases} v_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon v_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 v_2\left(x, \frac{x}{\varepsilon}\right) + \dots, & x \in \Omega_1^\varepsilon, \\ w_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon w_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 w_2\left(x, \frac{x}{\varepsilon}\right) + \dots, & x \in \Omega_0^\varepsilon. \end{cases} \tag{7}$$

Here all the functions $v_j(x, y)$, $w_j(x, y)$, $j \geq 0$, are required to be periodic in the "fast" variable y; v_0 and w_0 are not simultaneously identically zero

$$v_0^2 + w_0^2 \not\equiv 0. \tag{8}$$

In a standard way, the ansatz (6), (7) is then formally substituted into (3)–(5). In particular, from (3), for $(x, y) \in \Omega \times Q_1$, we obtain

$$-\Delta_y v_0 = 0, \tag{9}$$

$$-\Delta_y v_1 = 2\frac{\partial^2 v_0}{\partial x_j \partial y_j}, \tag{10}$$

$$-\Delta_y v_2 = 2\frac{\partial^2 v_1}{\partial x_j \partial y_j} + \Delta_x v_0 + \lambda_0 v_0, \tag{11}$$

(with Δ_y and Δ_x denoting the Laplace operators in y and x, respectively, and summation henceforth implied with respect to repeated indices), and for $(x, y) \in$

$\Omega\times Q_0$ we have

$$-\Delta_y w_0 = \lambda_0 w_0, \tag{12}$$

$$-\Delta_y w_1 = 2\frac{\partial^2 w_0}{\partial x_j \partial y_j} + \lambda_1 w_0 + \lambda_0 w_1, \tag{13}$$

$$-\Delta_y w_2 = 2\frac{\partial^2 w_1}{\partial x_j \partial y_j} + \Delta_x w_0 + \lambda_2 w_0 + \lambda_1 w_1 + \lambda_0 w_2. \tag{14}$$

Further, the first of conditions (5) transforms to

$$v_j(x,y)\Big|_{y\in\Gamma} = w_j(x,y)\Big|_{y\in\Gamma}, \quad x\in\Omega, \quad j=0,1\dots. \tag{15}$$

Similarly, the other transmission condition (5) yield

$$\frac{\partial v_0}{\partial n_y}\Big|_{y\in\Gamma} = 0, \tag{16}$$

$$\frac{\partial v_1}{\partial n_y}\Big|_{y\in\Gamma} = -\frac{\partial v_0}{\partial n_x}\Big|_{y\in\Gamma} + \frac{\partial w_0}{\partial n_y}\Big|_{y\in\Gamma}, \tag{17}$$

$$\frac{\partial v_2}{\partial n_y}\Big|_{y\in\Gamma} = -\frac{\partial v_1}{\partial n_x}\Big|_{y\in\Gamma} + \frac{\partial w_1}{\partial n_y}\Big|_{y\in\Gamma} + \frac{\partial w_0}{\partial n_x}\Big|_{y\in\Gamma}. \tag{18}$$

The above has employed the identity

$$\frac{\partial u}{\partial n_x}\left(x,\frac{x}{\varepsilon}\right) = \varepsilon^{-1}\frac{\partial u}{\partial n_y}(x,y) + \frac{\partial u}{\partial n_x}(x,y), \quad y=\frac{x}{\varepsilon}, \tag{19}$$

where $\frac{\partial}{\partial n_y} := n_y\cdot\nabla_y$, $\frac{\partial}{\partial n_x} := n_y\cdot\nabla_x$, with ∇_y and ∇_x standing for gradients in y and x, respectively.

Finally, (4) suggests

$$v_0\Big|_{x\in\partial\Omega} = w_0\Big|_{x\in\partial\Omega} = 0. \tag{20}$$

(The boundary layer problem does not generally permit satisfying (4) by v_j and w_j for $j\ge 1$, as also clarified later.)

Combining (9) and (16), together with the periodicity conditions in y, implies that v_0 is a constant with respect to y, i.e.

$$v_0(x,y)\equiv v^0(x).$$

Then, (10) and (17) form the following boundary value problem for v_1

$$-\Delta_y v_1(x,y) = 0 \quad \text{in} \quad \Omega\times Q_1, \qquad \frac{\partial v_1}{\partial n_y}\Big|_{y\in\Gamma} = -\frac{\partial v^0}{\partial n_x}\Big|_{y\in\Gamma} + \frac{\partial w_0}{\partial n_y}\Big|_{y\in\Gamma}. \tag{21}$$

The latter is solvable if and only if

$$\int_\Gamma \frac{\partial w_0}{\partial n_y}\,dy = 0. \tag{22}$$

Considering next (12) and (15) gives

$$-\Delta_y w_0 = \lambda_0 w_0 \quad \text{in} \quad \Omega\times Q_0, \qquad w_0(x,y)\Big|_{y\in\Gamma} = v^0(x). \tag{23}$$

Since

$$\int_\Gamma \frac{\partial w_0}{\partial n_y}\,dy = \int_{Q_0}\Delta_y w_0\,dy = -\lambda_0\int_{Q_0} w_0\,dy,$$

condition (22) is equivalent to

$$\lambda_0\langle w_0\rangle = 0, \tag{24}$$

where

$$\langle u\rangle := \int_{Q_0} u(y)\,dy.$$

We notice that (23)–(24) together with (8) constitutes restrictions on possible values of λ_0. Those are described by Theorem 3.1 below. Before, let us consider an auxiliary Dirichlet problem

$$-\Delta_y\phi = \lambda^D\phi \quad \text{in} \quad Q_0, \qquad \phi\Big|_\Gamma = 0. \tag{25}$$

Let $\{\lambda_j^D\}_{j=1}^\infty$ be eigenvalues for (25), labelled in the ascending order counting for the multiplicities, and let $\{\phi_j\}_{j=1}^\infty$ be the corresponding eigenfunctions, orthonormal in $L_2(Q_0)$, i.e.

$$\int_{Q_0} \phi_j\phi_k\,dy = \delta_{jk},$$

where δ_{jk} is Kronecker's delta. Denote by σ_D the spectrum of (25): $\sigma_D = \bigcup_{j=1}^\infty \lambda_j^D$.

We additionally introduce the following auxiliary problem:

$$-\Delta_y\eta = \lambda_0\eta \quad \text{in} \quad Q_0, \qquad \eta(y)\Big|_{y\in\Gamma} = 1. \tag{26}$$

Notice that (26) is solvable if and only if $\lambda_0 \notin \sigma_D$ or $\lambda_0 = \lambda_j^D$ with all the associated eigenfunctions ϕ_j having zero mean, $\langle\phi_j\rangle = 0$[1]. In the former case η is determined uniquely and (23) implies $w_0(x,y) = v^0(x)\eta(y)$. In the latter case η is determined up to an arbitrary eigenfunction ϕ_j associated with λ_j^D, however $\langle\eta\rangle$ is determined uniquely.

By direct inspection, (23), (24) has a non-trivial solution (v^0, w_0), i.e. with (8) holding, if and only if λ_0 is an eigenvalue of following problem:

$$-\Delta_y\zeta = \lambda_0\zeta \quad \text{in} \quad Q_0, \qquad \zeta(y)\Big|_{y\in\Gamma} = \text{constant}, \qquad \lambda_0\langle\zeta\rangle = 0. \tag{27}$$

Theorem 3.1. *The problem* (27) *is equivalent to an eigenvalue problem for a self-adjoint operator in* $L_2(Q_0)$ *with a compact resolvent. Therefore the spectrum of* (27) *is a countable set of real non-negative eigenvalues (of finite multiplicity) with the only accumulation point at* $+\infty$*, with the eigenfunctions complete in* $L_2(Q_0)$ *and those corresponding to different* λ_0 *mutually orthogonal.*

The spectrum consists of all the eigenvalues λ^D *of problem* (25) *with a zero mean eigenfunction and all the solutions of the equation*

$$B(\lambda_0) := \lambda_0\langle\eta\rangle = \lambda_0\left(|Q_0| + \lambda_0\sum_{j=1}^\infty \frac{\langle\phi_j\rangle^2}{\lambda_j^D - \lambda_0}\right) = 0 \tag{28}$$

(which are hence all real non-negative). In (28) *the summation is with respect to only those* λ_j^D *for which there exists an eigenfunction with a non-zero mean.*

The associated eigenfunctions ζ *are either proportional to* η *as in* (26) *or are eigenfunctions of* (25) *with zero mean.*

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[1] We remark that the case of eigenvalues with zero mean is known to be not a "generic" case, i.e. unstable via a small perturbation of the shape of Q_0, see e.g. discussion in [10] and further references therein.

Proof. We claim that (27) corresponds to a self-adjoint operator associated with the (symmetric, closed, densely defined, bounded from below) Dirichlet form

$$\alpha(\zeta, h) := \int_{Q_0} \nabla\zeta \cdot \nabla h \; dy \tag{29}$$

with domain

$$D(\alpha) := \{h \in H^1(Q_0) : \quad h\Big|_{y\in\Gamma} = \text{ constant}\}. \tag{30}$$

To see this, in the weak formulation of the eigenvalue problem associated with (29)–(30)

$$\int_{Q_0} \nabla\zeta \cdot \nabla h \; dy = \lambda_0 \int_{Q_0} \zeta \, h \; dy, \qquad \forall h \in D(\alpha), \tag{31}$$

we first set h to be an arbitrary function from $C_0^\infty(Q_0)$ which implies $-\Delta_y\zeta = \lambda_0\zeta$ in Q_0, and then set $h \equiv 1$ yielding $\lambda_0\langle\zeta\rangle = 0$. Further, since the resolvent is obviously compact, each eigenvalue has a finite multiplicity, the set of all eigenfunctions ζ is complete in $L_2(Q_0)$ and those corresponding to different λ_0 are mutually orthogonal.

Obviously, the spectrum of (27) includes those and only those eigenvalues of (25) which have an eigenfunction ϕ_j with zero mean. In this case corresponding eigenfunctions of (27) are given by $\zeta_j = C\phi_j$, $C \neq 0$. If λ_j^D does not have a zero-mean eigenfunction, then the solvability of (27) requires $\zeta\Big|_{y\in\Gamma} = 0$ implying $\zeta \equiv 0$. Considering other possibilities, fix λ_0 outside σ_D and let η be the unique solution of (26). Then λ_0 is an eigenvalue of (27) if and only if

$$\lambda_0\langle\eta\rangle = 0, \tag{32}$$

with corresponding eigenfunction given by $\zeta(y) = C\eta(y)$, $C \neq 0$.

Via the spectral decomposition, the solution to (26) is found to be, cf. [19]:

$$\eta(y) = 1 + \lambda_0 \sum_{j=1}^{\infty} \frac{\langle\phi_j\rangle}{\lambda_j^D - \lambda_0}\phi_j(y). \tag{33}$$

Substituting (33) further into (32) yields (28). □

The formula (28) can be transformed to read

$$B(\lambda_0) = \beta(\lambda_0) - |Q_1|\lambda_0 = 0, \tag{34}$$

where function $\beta(\lambda)$ has been introduced by Zhikov [19]:

$$\beta(\lambda) = \lambda + \lambda^2 \sum_{j=1}^{\infty} \frac{\langle\phi_j\rangle^2}{\lambda_j^D - \lambda}, \tag{35}$$

see Figure 2. This implies that λ_0 is either a solution to the nonlinear equation

$$\beta(\lambda) = |Q_1|\lambda, \tag{36}$$

as visualized on Figure 2, or is an eigenvalue of (25) with a zero mean eigenfunction.

Remark 1. *If Q_0 is a ball of radius $0 < a < 1/2$, i.e $Q_0 = B_a = \{y : |y| < a\} + y_0$, then we have an explicit representation for $\beta(\lambda)$. Indeed, for $\lambda_0 \notin \sigma_D$ the solution of (26) is radially symmetric and (placing the origin in the ball's centre) reads*

$$\eta(y) = |y|^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(\lambda_0^{1/2}|y|) \left(|a|^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(\lambda_0^{1/2}a)\right)^{-1},$$

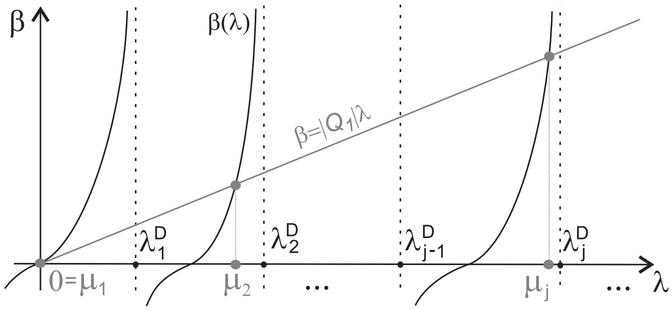

FIGURE 2. The limit eigenvalues $\lambda_0 = \mu_j$

where $J_{\frac{n-2}{2}}(|y|)$ *is Bessel function. Further, we have*

$$B(\lambda_0) = \lambda_0 \langle \eta \rangle = -\int_{\partial B_a} \frac{\partial \eta}{\partial n_y} dy =$$

$$= -\frac{1}{a}|\Gamma| \left(1 - n/2 + a\lambda_0^{1/2} J'_{\frac{n-2}{2}}(\lambda_0^{1/2} a)/J_{\frac{n-2}{2}}(\lambda_0^{1/2} a)\right).$$

Using (35), (33) *we obtain*

$$\beta(\lambda) = \lambda(1 - |B_a|) - \frac{1}{a}|\Gamma| \left(1 - n/2 + a\lambda^{1/2} J'_{\frac{n-2}{2}}(\lambda^{1/2} a)/J_{\frac{n-2}{2}}(\lambda^{1/2} a)\right).$$

In particular, for $n = 3$ *we have,*

$$B(\lambda_0) = \lambda_0 \langle \eta \rangle = 4\pi a \left(1 - a\lambda_0^{1/2} \operatorname{cotan}(\lambda_0^{1/2} a)\right),$$

$$\beta(\lambda) = \lambda(1 - 4\pi a^3/3) + 4\pi a \left(1 - a\lambda^{1/2} \operatorname{cotan}(\lambda^{1/2} a)\right).$$

We next explore in detail the further steps in the method of asymptotic expansions, to determine v^0, etc. Let us consider a K-dimensional eigenspace ($K \geq 1$) for a given eigenvalue λ_0 of (27), and let $\zeta_1, \ldots, \zeta_K$ be associated linearly independent eigenfunctions. Then, (23) and (24) imply

$$w_0(x, y) = \sum_{k=1}^{K} c_k(x) \zeta_k(y). \tag{37}$$

Following Theorem 3.1 we distinguish two cases:

(a) $\lambda_0 \notin \sigma_D$. In this case (26) and (23) suggest

$$w_0(x, y) = v^0(x) \eta(y), \tag{38}$$

and (8) implies $v^0 \not\equiv 0$.

(b) $\lambda_0 \in \sigma_D$. The latter means $\lambda_0 = \lambda_j^D$ for some j. This includes two further possibilities:

(i) The eigenspace of (25) has an eigenfunction ϕ_j^* with a non-zero mean. Since the solvability conditions for (23) include

$$v^0(x) \langle \phi_j^* \rangle \; = \; 0, \tag{39}$$

necessarily $v^0 \equiv 0$. Moreover, with K_D denoting the multiplicity of λ_j^D as of the eigenvalue of the Dirichlet problem (25), necessarily $K_D \geq 2$:

if $K_D = 1$ then $w_0 = C(x)\phi_j^*$ and thus (24) implies $C(x) \equiv 0$ and $w_0 \equiv 0$ contradicting to (8). Hence w_0 is given by (37) with $K = K_D - 1$, with ζ_k, $k = 1, ..., K$ being linearly independent eigenfunctions of (25) with zero mean (such K eigenfunctions exist).

(ii) All of the eigenfunctions corresponding λ_j^D have a zero mean. In this case w_0 is again given by (37), with $K = K_D$ if $\langle\eta\rangle \neq 0$ i.e. $B(\lambda_0) \neq 0$ and $K = K_D + 1$ if $B(\lambda_0) = 0$ with $\zeta_{K_D+1} = \eta$ where $\eta(y)$ is any solution of (26).

3.1. **Case (a):** $\lambda_0 \notin \sigma_D$**.** In this case λ_0 are solutions of (36). There is a countable set of $\lambda_0 = \mu_j$, $j = 1, 2, \dots$ as Figure 2 illustrates. Note that this includes $\lambda_0 = 0$. Function β blows up at the points λ_j^D, which are eigenvalues of (25) having an eigenvalue with a non-zero mean, monotonically increasing between such points. It also directly follows from (35) that $\beta(\lambda) > |Q_1|\lambda$ for $\lambda \in (0, \lambda_1^D)$, implying $\lambda_1^D < \mu_2 < \lambda_2^D$. Let λ_0 satisfying (36) be fixed.

We consider problem (21) taking into account (38), i.e.

$$(40)\quad -\Delta_y v_1(x,y) = 0 \quad \text{in} \quad \Omega \times Q_1, \qquad \frac{\partial v_1}{\partial n_y}\Big|_{y\in\Gamma} = -\frac{\partial v^0}{\partial n_x}\Big|_{y\in\Gamma} + v^0(x)\frac{\partial \eta}{\partial n_y}\Big|_{y\in\Gamma},$$

where $\eta(y)$ solves (26) and is given by (33). Hence v_1 is a solution to a problem depending linearly on v^0 and $\nabla_x v^0$, implying

$$v_1(x,y) = v^0(x)\mathcal{N}(y) + \frac{\partial v^0}{\partial x_j}N_j(y) + v_1^*(x), \tag{41}$$

with an arbitrary function $v_1^*(x)$. The choice of v_1^* does not affect the subsequent constructions, so we set for simplicity $v_1^* \equiv 0$. In (41) functions N_j and $\mathcal{N}$ are solutions to the problems

$$\Delta_y N_j(y) = 0 \quad \text{in} \quad Q_1, \qquad \frac{\partial N_j}{\partial n_y}\Big|_{y\in\Gamma} = -n_j(y), \tag{42}$$

and

$$\Delta_y \mathcal{N}(y) = 0 \quad \text{in} \quad Q_1, \qquad \frac{\partial \mathcal{N}}{\partial n_y}\Big|_{y\in\Gamma} = \frac{\partial \eta}{\partial n_y}\Big|_{y\in\Gamma}. \tag{43}$$

Solvability of (43) requires

$$\int_\Gamma \frac{\partial \eta}{\partial n_y}\, dy = 0,$$

which is equivalent to (32) and is hence already assured. Since the solutions of (42) and (43) are unique up to an arbitrary constant, we fix those by choosing

$$\int_{Q_1} N_j(y)\, dy = \int_{Q_1} \mathcal{N}(y)\, dy = 0.$$

We next consider the problem for w_1, which from (13) and (15) combined with (38) reads

$$-\Delta_y w_1 - \lambda_0 w_1 = \lambda_1 v^0 \eta + 2\frac{\partial v^0}{\partial x_j}\frac{\partial \eta}{\partial y_j} \quad \text{in} \quad \Omega \times Q_0, \tag{44}$$

$$w_1\Big|_{y\in\Gamma} = v^0\mathcal{N}(y)\Big|_{y\in\Gamma} + \frac{\partial v^0}{\partial x_j}N_j(y)\Big|_{y\in\Gamma}. \tag{45}$$

Since the problem depends linearly on v^0, $\lambda_1 v^0$ and $\frac{\partial v^0}{\partial x_j}$, the solution admits representation

$$w_1(x,y) = \frac{\partial v^0}{\partial x_j}(x)\mathcal{M}_j(y) + v^0(x)\mathcal{P}(y) + \lambda_1 v^0(x)\mathcal{R}(y), \tag{46}$$

where functions $\mathcal{M}_j$, $\mathcal{P}$ and $\mathcal{R}$ are solutions to the problems

$$-\Delta_y \mathcal{M}_j - \lambda_0 \mathcal{M}_j = 2\frac{\partial \eta}{\partial y_j}(y) \quad \text{in} \quad Q_0, \qquad \mathcal{M}_j\Big|_\Gamma = N_j\Big|_\Gamma, \tag{47}$$

$$-\Delta_y \mathcal{P} - \lambda_0 \mathcal{P} = 0 \quad \text{in} \quad Q_0, \qquad \mathcal{P}\Big|_\Gamma = \mathcal{N}\Big|_\Gamma, \tag{48}$$

and

$$-\Delta_y \mathcal{R} - \lambda_0 \mathcal{R} = \eta(y) \quad \text{in} \quad Q_0, \qquad \mathcal{R}\Big|_\Gamma = 0. \tag{49}$$

Since by the assumption $\lambda_0 \notin \sigma_D$, all the problems (47) – (49) are uniquely solvable.

The problem for v_2 is in turn given by (11) and (18), whose solvability condition hence reads

$$\int_{Q_1}\left(\Delta_x v_0 + \lambda_0 v_0 + 2\frac{\partial^2 v_1}{\partial x_j \partial y_j}\right) dy = \int_\Gamma \left(-\frac{\partial v_1}{\partial n_x} + \frac{\partial w_1}{\partial n_y} + \frac{\partial w_0}{\partial n_x}\right) dy, \tag{50}$$

with functions v_1, w_1 and w_0 given by (41), (46) and (38) respectively.

Appendix A provides a detailed calculation showing that the above yields the following equations for v^0:

$$-\operatorname{div} A^{\text{hom}} \nabla_x v^0 = \nu(\lambda_1) v^0 \quad \text{in} \quad \Omega, \tag{51}$$

$$v^0\Big|_{\partial\Omega} = 0. \tag{52}$$

Here $A^{\text{hom}} = \left(A^{\text{hom}}_{jk}\right)_{j,k=1}^n$ is the classical homogenized matrix for periodic perforated domains, see e.g. [11]

$$A^{\text{hom}}_{jk} = |Q_1|\delta_{jk} + \int_{Q_1} \frac{\partial N_k}{\partial y_j}\, dy; \tag{53}$$

$$\nu(\lambda_1) = \mathcal{C}\lambda_1 + \lambda_0\Big(|Q_1| + \int_{Q_0} \mathcal{P}\, dy\Big), \tag{54}$$

where

$$\mathcal{C} := \int_{Q_0} \eta^2 dy > 0. \tag{55}$$

Note that the problem (51)–(52) involves $\nu = \nu(\lambda_1)$ as a spectral parameter.

The spectrum of (51)–(52) consists of a countable set of eigenvalues

$$0 < \nu_1 < \nu_2 \le \cdots \le \nu_n \le \cdots \to +\infty. \tag{56}$$

Corresponding eigenfunctions v_n form an orthonormal basis in $L_2(\Omega)$,

$$\int_\Omega v^0_n v^0_m\, dx = \delta_{nm}.$$

Fixing an eigenvalue ν of (51), (52) with corresponding eigenfunction v^0 of unit norm in $L_2(\Omega)$, according to (54) we find

$$\lambda_1 = \mathcal{C}^{-1}\left(\nu - \lambda_0\Big(|Q_1| + \int_{Q_0} \mathcal{P}\, dy\Big)\right). \tag{57}$$

The following diagram summarizes the algorithm for constructing the first terms of the asymptotic expansions (for the case $\lambda_0 \notin \sigma_D$)

$$\left.\begin{array}{ccccc} N_j & \overset{(53)}{\to} & A^{\rm hom} & \overset{(51)}{\to} & \nu, v^0 \\ \lambda_0 & \overset{(26)}{\to} & \eta & \overset{(43)}{\to} & \mathcal{N} \end{array}\right\} \left.\begin{array}{cc} \overset{(57)}{\to} & \lambda_1 \\ \overset{(41)}{\to} & v_1 \\ \overset{(47)-(49)}{\longrightarrow} & \mathcal{M}, \mathcal{P}, \mathcal{R} \\ \overset{(38)}{\to} & w_0 \end{array}\right\} \overset{(46)}{\to} w_1 \overset{(11),(18)}{\longrightarrow} v_2.$$

We can additionally construct w_2 from (14) and (15), whose unique solution exists for any choice of λ_2. For purposes of the justification of the first two terms in the asymptotics (the next section) it is sufficient to set $\lambda_2 = 0$ and fix the corresponding solution w_2.

This completes constructing a formal asymptotic approximation, which we now summarize. We introduce an approximate eigenvalue

$$\Lambda_\varepsilon = \lambda_0 + \varepsilon\lambda_1, \tag{58}$$

and corresponding approximate eigenfunction

$$W_\varepsilon(x) = \begin{cases} v^0(x) + \varepsilon v_1\left(x, \dfrac{x}{\varepsilon}\right) + \varepsilon^2 v_2\left(x, \dfrac{x}{\varepsilon}\right), & x \in \Omega_1^\varepsilon, \\ w_0\left(x, \dfrac{x}{\varepsilon}\right) + \varepsilon w_1\left(x, \dfrac{x}{\varepsilon}\right) + \varepsilon^2 w_2\left(x, \dfrac{x}{\varepsilon}\right), & x \in \Omega_0^\varepsilon. \end{cases} \tag{59}$$

The essence of the above formal asymptotic construction is that the action of differential operator $\mathcal{A}^\varepsilon$ on W_ε defined by

$$\mathcal{A}^\varepsilon W_\varepsilon := \operatorname{div}(a_\varepsilon \nabla W_\varepsilon) + \Lambda_\varepsilon \rho_\varepsilon W_\varepsilon \tag{60}$$

produces a small right-hand side in both Ω_1^ε and Ω_0^ε, and on the interface Γ^ε in the following sense.

Lemma 3.2. *(i)* $\max_{\bar\Omega_1^\varepsilon} |\operatorname{div}(a_\varepsilon \nabla W_\varepsilon) + \Lambda_\varepsilon \rho_\varepsilon W_\varepsilon| \le C\varepsilon$.

(ii) $\max_{\bar\Omega_0^\varepsilon} |\operatorname{div}(a_\varepsilon \nabla W_\varepsilon) + \Lambda_\varepsilon \rho_\varepsilon W_\varepsilon| \le C\varepsilon^2$.

(iii) $\max_{\Gamma^\varepsilon} \left| a_\varepsilon \dfrac{\partial W_\varepsilon}{\partial n}\Big|_0 - a_\varepsilon \dfrac{\partial W_\varepsilon}{\partial n}\Big|_1 \right| \le C\varepsilon^2$.

Proof. *(i)* Since the function W_ε is two-scale by the construction, in Ω_1^ε

$$\begin{aligned} &\operatorname{div}(a_\varepsilon \nabla W_\varepsilon) + \Lambda_\varepsilon \rho_\varepsilon W_\varepsilon = \\ &= \left(\varepsilon^{-2}\Delta_y + \varepsilon^{-1} 2\frac{\partial^2}{\partial x_j \partial y_j} + \Delta_x + \lambda_0 + \varepsilon\lambda_1\right)\left(v^0(x) + \varepsilon v_1(x,y) + \varepsilon^2 v_2(x,y)\right)\big|_{y=\frac{x}{\varepsilon}} = \\ &= \left\{\varepsilon^{-1}\Delta_y v_1(x,y) + \varepsilon^0\left(\Delta_y v_2 + 2\frac{\partial^2 v_1}{\partial x_j \partial y_j} + \Delta_x v^0 + \lambda_0 v^0\right) + \right. \\ &\left. + \ \varepsilon^1\left(2\frac{\partial^2 v_2}{\partial x_j \partial y_j} + \Delta_x v_1 + \lambda_1 v^0 + \lambda_0 v_1\right) + \varepsilon^2(\Delta_x v_2 + \lambda_1 v_1 + \lambda_0 v_2) + \varepsilon^3 \lambda_1 v_2 \right\}\Big|_{y=\frac{x}{\varepsilon}}. \end{aligned} \tag{61}$$

Since v_1 is a solution to (40), the coefficient of ε^{-1} vanishes. The same is with the coefficient of ε^0 since v_2 satisfies (11). Functions v^0, v_1 and v_2 are solutions of elliptic problems with smooth enough coefficients to guarantee belonging solutions to C^2. Thus, maxima for coefficients of ε^1, ε^2 and ε^3 in (61) exist.

(*ii*) Similarly, in Ω_0^ε

$$\begin{aligned}
&\operatorname{div}(a_\varepsilon \nabla W_\varepsilon) + \Lambda_\varepsilon \rho_\varepsilon W_\varepsilon = \\
&= \left(\varepsilon^{-1}\Delta_y + 2\frac{\partial^2}{\partial x_j \partial y_j} + \varepsilon\Delta_x + \varepsilon^{-1}\lambda_0 + \lambda_1\right)(w_0(x,y) + \varepsilon w_1(x,y) + \varepsilon^2 w_2(x,y))|_{y=\frac{x}{\varepsilon}} = \\
&= \Bigg\{\varepsilon^{-1}(\Delta_y w_0 + \lambda_0 w_0) + \varepsilon^0\left(\Delta_y w_1 + 2\frac{\partial^2 w_0}{\partial x_j \partial y_j} + \lambda_0 w_1 + \lambda_1 w_0\right) + \\
&+ \varepsilon^1\left(\Delta_y w_2 + 2\frac{\partial^2 w_1}{\partial x_j \partial y_j} + \Delta_x w_0 + \lambda_0 w_2 + \lambda_1 w_1\right) + \\
&+ \varepsilon^2\left(2\frac{\partial^2 w_2}{\partial x_j \partial y_j} + \Delta_x w_1 + \lambda_1 w_2\right) + \varepsilon^3 \Delta_x w_2\Bigg\}\Bigg|_{y=\frac{x}{\varepsilon}}.
\end{aligned}$$

Since $w_0(x,y) = v^0(x)\eta(y)$ is chosen according to (26), the coefficient of ε^{-1} vanishes. The coefficient of ε^0 vanishes due to (44). Further, w_2 satisfies (14) with $\lambda_2 = 0$ and thus the coefficient of ε^1 is zero as well. Since w_1 and w_2 are solutions of elliptic problems with smooth enough coefficients, the maxima of the coefficients of ε^2 and ε^3 exist.

(*iii*) Using (19), we obtain

$$\begin{aligned}
a_\varepsilon \frac{\partial W_\varepsilon}{\partial n}\Big|_0 - a_\varepsilon \frac{\partial W_\varepsilon}{\partial n}\Big|_1 &= \left(\frac{\partial}{\partial n_y} + \varepsilon\frac{\partial}{\partial n_x}\right)(w_0(x,y) + \varepsilon w_1(x,y) + \varepsilon^2 w_2(x,y))\Big|_{\substack{x\in\Gamma^\varepsilon\\ y\in\Gamma}} - \\
&- \left(\varepsilon^{-1}\frac{\partial}{\partial n_y} + \frac{\partial}{\partial n_x}\right)(v^0(x) + \varepsilon v_1(x,y) + \varepsilon^2 v_2(x,y))\Big|_{\substack{x\in\Gamma^\varepsilon\\ y\in\Gamma}} = \\
&= \varepsilon^0\left(\frac{\partial w_0}{\partial n_y} - \frac{\partial v_1}{\partial n_y} - \frac{\partial v^0}{\partial n_x}\right)\Big|_{\substack{x\in\Gamma^\varepsilon\\ y\in\Gamma}} + \varepsilon^1\left(\frac{\partial w_1}{\partial n_y} + \frac{\partial w_0}{\partial n_x} - \frac{\partial v_2}{\partial n_y} - \frac{\partial v_1}{\partial n_x}\right)\Big|_{\substack{x\in\Gamma^\varepsilon\\ y\in\Gamma}} + \\
&+ \varepsilon^2\left(\frac{\partial w_2}{\partial n_y} + \frac{\partial w_1}{\partial n_x} - \frac{\partial v_2}{\partial n_x}\right)\Big|_{\substack{x\in\Gamma^\varepsilon\\ y\in\Gamma}} + \varepsilon^3\frac{\partial w_2}{\partial n_x}\Big|_{\substack{x\in\Gamma^\varepsilon\\ y\in\Gamma}}.
\end{aligned} \tag{62}$$

The coefficients of ε^0 and ε^1 vanish because of (21) and (18) respectively. The rest of the coefficients are smooth enough to guarantee that their maxima for $x \in \Gamma^\varepsilon$ and $y \in \Gamma$ exist. □

3.2. **Case (b):** $\lambda_0 = \lambda_j^D$**.** For simplicity, we consider here only the case of eigenvalues of multiplicity $K = 1$ with zero mean eigenfunction ($\phi = \phi_j$), assuming additionally λ_0 is *not* a solution of (34). All other degenerate cases, see page 7, could be considered similarly.

In this case we can introduce a refined approximation for the eigenfunction

$$W_\varepsilon^*(x) = \begin{cases} \varepsilon v_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 v_2\left(x, \frac{x}{\varepsilon}\right), & x \in \Omega_1^\varepsilon, \\ w_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon w_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 w_2\left(x, \frac{x}{\varepsilon}\right), & x \in \Omega_0^\varepsilon. \end{cases} \tag{63}$$

where

$$w_0(x,y) = c(x)\phi(y). \tag{64}$$

Lemma 3.3. *Let $c \in C^3(\Omega)$, then there exist smooth functions v_1, w_1, v_2, w_2 and a constant λ_1 such that $\Lambda_\varepsilon = \lambda_j^D + \varepsilon\lambda_1$ and W_ε^* defined by (63) satisfy*

(*i*) $W_\varepsilon^*(x) \in C(\Omega)$,

(*ii*) $\max_{\bar\Omega_1^\varepsilon} |\operatorname{div}(a_\varepsilon \nabla W_\varepsilon^*) + \Lambda_\varepsilon \rho_\varepsilon W_\varepsilon^*| \le C\varepsilon$,

(*iii*) $\max_{\bar\Omega_0^\varepsilon} |\operatorname{div}(a_\varepsilon \nabla W_\varepsilon^*) + \Lambda_\varepsilon \rho_\varepsilon W_\varepsilon^*| \le C\varepsilon^2$, *(iv)* $$\max_{\Gamma^\varepsilon}\left| a_\varepsilon \frac{\partial W_\varepsilon^*}{\partial n}\Big|_0 - a_\varepsilon \frac{\partial W_\varepsilon^*}{\partial n}\Big|_1 \right| \leq C\varepsilon^2.$$

Proof. See Appendix B. □

4. Justification of asymptotics

4.1. **Operator formulation.** We use a standard notation for Lebesgue and Sobolev spaces: $L^2_p(\Omega)$ is a p-weighted L^2-space of square-integrable functions in Ω. Notation $(\cdot,\cdot)_H$ is used for a scalar product in a Hilbert space H.

Let $\mathcal{L}^\varepsilon = L^2_{\rho_\varepsilon}(\Omega)$ and $\mathcal{H}^\varepsilon$ be $H^1_0(\Omega)$ Sobolev space with a scalar product

$$(u,v)_{\mathcal{H}^\varepsilon} = \int_\Omega a_\varepsilon(x)\nabla u\cdot\nabla v\,dx + \int_\Omega \rho_\varepsilon(x)uv\,dx.$$

Following a standard procedure, see e.g. [11], we introduce a bounded operator $\mathcal{B}_\varepsilon : \mathcal{L}^\varepsilon \to \mathcal{L}^\varepsilon$ such that

$$(\mathcal{B}_\varepsilon f, v)_{\mathcal{H}^\varepsilon} = (f,v)_{\mathcal{L}^\varepsilon}, \quad \forall v\in\mathcal{H}^\varepsilon. \tag{65}$$

In other words $\mathcal{B}_\varepsilon f = u_\varepsilon$, where u_ε is the solution of the problem

$$-\operatorname{div}(a_\varepsilon\nabla u_\varepsilon) + \rho_\varepsilon u_\varepsilon = \rho_\varepsilon f, \quad x\in\Omega, \tag{66}$$

$$u_\varepsilon|_{\partial\Omega} = 0, \tag{67}$$

$$u_\varepsilon\Big|_1 = u_\varepsilon\Big|_0, \quad \frac{\partial u_\varepsilon}{\partial n_x}\Big|_1 = \varepsilon\frac{\partial u_\varepsilon}{\partial n_x}\Big|_0. \tag{68}$$

Note that operator $\mathcal{B}_\varepsilon$ is positive, self-adjoint and compact for any fixed $\varepsilon>0$ (since its image is in $\mathcal{H}^\varepsilon$). Eigenvalue problem (2) is equivalent to

$$\mathcal{B}_\varepsilon u_\varepsilon = (\lambda^\varepsilon+1)^{-1}u_\varepsilon \quad \text{in} \quad \mathcal{L}^\varepsilon. \tag{69}$$

Hence the spectrum of the problem consists of a countable set of eigenvalues

$$0<\lambda_1^\varepsilon<\lambda_2^\varepsilon\leq\cdots\leq\lambda_k^\varepsilon\leq\cdots\to+\infty,$$

with the only accumulation point at $+\infty$. Moreover, the set of corresponding eigenfunctions is complete in $\mathcal{L}^\varepsilon$.

4.2. **Case (a).** In this Section we justify the leading terms of asymptotic expansions constructed above in case $\lambda_0\notin\sigma_D$ and thus $v^0\not\equiv 0$, see Section 3.1. Let λ_0 be a solution to equation (36). All the functions (η, N_j, $\mathcal{N}$, $\mathcal{M}$, $\mathcal{P}$, $\mathcal{R}$, w_0, w_1, v_1 and w_2, v_2) are as defined in Section 3.1. We also fix λ_1 according to (57). The approximate eigenvalue Λ_ε and eigenfunction W_ε are given by (58) and (59) respectively.

Notice that although $W_\varepsilon\in H^1(\Omega)$ since $W_\varepsilon\Big|_1 = W_\varepsilon\Big|_0$, it does not satisfy the zero Dirichlet boundary conditions on $\partial\Omega$. To fix this we introduce the following boundary-layer corrector to our approximation.

Lemma 4.1. *There exists a corrector V_ε solving the problem*

$$-\operatorname{div}(a_\varepsilon\nabla V_\varepsilon) + \rho_\varepsilon V_\varepsilon = 0 \quad in \quad \Omega, \tag{70}$$

$$V_\varepsilon|_{\partial\Omega} = -W_\varepsilon|_{\partial\Omega}, \quad V_\varepsilon\Big|_1 = V_\varepsilon\Big|_0, \quad \frac{\partial V_\varepsilon}{\partial n}\Big|_1 = \varepsilon\frac{\partial V_\varepsilon}{\partial n}\Big|_0, \tag{71}$$

such that $U_\varepsilon = W_\varepsilon + V_\varepsilon \in H^1_0(\Omega)$ and $\max_{\bar\Omega}|V_\varepsilon|\leq C\varepsilon$.

Proof. Clearly such solution of (70), (71) does exist. On each of the subsets Ω_1^ε and Ω_0^ε the coefficients of (70) are smooth. Then the function V_ε can reach its positive maximum or negative minimum only at the boundaries Γ^ε or $\partial\Omega$. Let us prove that this cannot be Γ^ε. Suppose to the contrary the existence of $x_* \in \Gamma^\varepsilon$ such that $\max\limits_{\bar\Omega} |V_\varepsilon| = |V_\varepsilon(x_*)|$. The strong maximum principle yields that there is no more point inside Ω_1^ε or Ω_0^ε where the maximum is reached. Without loss of generality we assume $V_\varepsilon(x_*) > V_\varepsilon(x)$ for any $x \in \Omega\backslash\Gamma^\varepsilon$ and $V_\varepsilon(x_*) \geq 0$ (otherwise the point would be a positive maximum for $-V_\varepsilon$ and we would then consider $-V_\varepsilon$). Then by the virtue of Hopf's Lemma [8, p.330] applied in the relevant component of Ω_0^ε we have

$$\left.\frac{\partial V_\varepsilon}{\partial n}\right|_0 (x_*) > 0.$$

From transmission conditions (71) we have that the normal derivative on the Ω_1^ε side of domain is also positive. Therefore the value of V_ε increases from the point x_* inside Ω_1^ε in the n-direction and hence x_* is not a point of maximum of V_ε in Ω_1^ε. The contradiction proves that $|V_\varepsilon|$ reaches it's maximum at $\partial\Omega$. Then, from boundary conditions (71),

$$\max_{\bar\Omega} |V_\varepsilon| = \max_{\partial\Omega} |V_\varepsilon| = \max_{\partial\Omega} |W_\varepsilon| \leq \varepsilon \max_{\partial\Omega} \left| v_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon v_2\left(x, \frac{x}{\varepsilon}\right)\right| +$$
$$\varepsilon \max_{\partial\Omega} \left| w_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon w_2\left(x, \frac{x}{\varepsilon}\right)\right| \leq C\varepsilon.$$

Obviously $U_\varepsilon = W_\varepsilon + V_\varepsilon$ satisfies zero boundary condition on $\partial\Omega$ and thus belongs to $H_0^1(\Omega)$. □

Lemma 4.2. *The constructed corrector V_ε satisfies the estimate $\|V_\varepsilon\|_{\mathcal{L}^\varepsilon} \leq C\varepsilon^{3/4}$.*

Proof. Let $\chi \in C^\infty(\mathbb{R})$ and $\chi(t) = 0,\ t < 1$ and $\chi(t) = 1,\ t > 2$. Let us define a family of cut-off functions:

$$\chi_\varepsilon(x) = \chi\left(\varepsilon^{-1/2}\,\mathrm{dist}\,(x, \partial\Omega)\right), \quad x \in \Omega.$$

Then $\chi_\varepsilon : \Omega \to \mathbb{R}$ satisfies the properties

- $\chi_\varepsilon(x) = 0$ if $\mathrm{dist}\,(x, \partial\Omega) \leq \varepsilon^{1/2}$,
- $\chi_\varepsilon(x) = 1$ if $\mathrm{dist}\,(x, \partial\Omega) \geq 2\varepsilon^{1/2}$,
- $|\nabla\chi_\varepsilon| \leq C\varepsilon^{-1/2}$ and $|\mathrm{supp}\,\nabla\chi_\varepsilon| \leq C\varepsilon^{1/2}$,

where "supp" denotes a function support, and $|\mathrm{supp}\,\cdot|$ is the measure of the corresponding support. Multiplying (70) by $\chi_\varepsilon^2 V_\varepsilon$ and integrating by parts, we obtain

$$\int_\Omega a_\varepsilon \nabla V_\varepsilon \cdot \nabla(\chi_\varepsilon^2 V_\varepsilon)\, dx + \int_\Omega \rho_\varepsilon \chi_\varepsilon^2 V_\varepsilon^2\, dx = 0. \tag{72}$$

Then using the identity

$$\nabla V_\varepsilon \cdot \nabla(\chi_\varepsilon^2 V_\varepsilon) = |\nabla(\chi_\varepsilon V_\varepsilon)|^2 - V_\varepsilon^2 |\nabla\chi_\varepsilon|^2,$$

we get from (72)

$$\int_\Omega a_\varepsilon |\nabla(\chi_\varepsilon V_\varepsilon)|^2\, dx + \int_\Omega \rho_\varepsilon \chi_\varepsilon^2 V_\varepsilon^2\, dx = \int_\Omega a_\varepsilon V_\varepsilon^2 |\nabla\chi_\varepsilon|^2\, dx, \tag{73}$$

implying

$$\int_\Omega \rho_\varepsilon \chi_\varepsilon^2 V_\varepsilon^2\, dx \leq \int_\Omega a_\varepsilon V_\varepsilon^2 |\nabla\chi_\varepsilon|^2\, dx. \tag{74}$$

Lemma 4.1 provides the estimate $V_\varepsilon^2 \le C\varepsilon^2$. Moreover, $|\mathrm{supp}\,\nabla\chi_\varepsilon| \le C\varepsilon^{1/2}$ and $|\nabla\chi_\varepsilon|^2 \le C\varepsilon^{-1}$. Therefore estimate (74) yields

$$\|\chi_\varepsilon V_\varepsilon\|^2_{\mathcal{L}^\varepsilon} = \int_\Omega \rho_\varepsilon \chi_\varepsilon^2 V_\varepsilon^2\,dx \le C\varepsilon^{3/2}. \tag{75}$$

Similarly we estimate

$$\|(1-\chi_\varepsilon)V_\varepsilon\|^2_{\mathcal{L}^\varepsilon} = \int_\Omega \rho_\varepsilon (1-\chi_\varepsilon)^2 V_\varepsilon^2\,dx \le C\varepsilon^{3/2}, \tag{76}$$

since $|\mathrm{supp}\,(1-\chi_\varepsilon)| \le C\varepsilon^{1/2}$ and $|\rho_\varepsilon| \le C\varepsilon^{-1}$. Combining (75) and (76), we obtain $\|V_\varepsilon\|_{\mathcal{L}^\varepsilon} = \|(1-\chi_\varepsilon)V_\varepsilon + \chi_\varepsilon V_\varepsilon\|_{\mathcal{L}^\varepsilon} \le C\varepsilon^{3/4}$. □

Lemma 4.3. *If $\varphi \in H_0^1(\Omega)$ then*

$$\Big(\int_{\Gamma^\varepsilon} |\varphi|^2\,dx\Big)^{1/2} \le C\|\varphi\|_{\mathcal{H}^\varepsilon}. \tag{77}$$

Proof. Extend function φ by zero to whole of $\mathbb{R}^n$. Then (77) follows upon rescaling $y = x/\varepsilon$ from the standard trace estimates applied to each connected component of $\widetilde{Q}_0$ (which are shifts of Q_0). □

Lemma 4.4. *The corrected approximation U_ε satisfies the estimate*

$$\|\mathcal{B}_\varepsilon U_\varepsilon - (\Lambda_\varepsilon+1)^{-1}U_\varepsilon\|_{\mathcal{L}^\varepsilon} \le \|\mathcal{B}_\varepsilon U_\varepsilon - (\Lambda_\varepsilon+1)^{-1}U_\varepsilon\|_{\mathcal{H}^\varepsilon} \le C\varepsilon^{3/4}.$$

Proof. For an arbitrary $\varphi \in \mathcal{H}^\varepsilon$ consider

$$\begin{aligned}
&|(\mathcal{B}_\varepsilon U_\varepsilon - (\Lambda_\varepsilon+1)^{-1}U_\varepsilon,\varphi)_{\mathcal{H}^\varepsilon}| = |\Lambda_\varepsilon+1|^{-1}|(U_\varepsilon - (\Lambda_\varepsilon+1)\mathcal{B}_\varepsilon U_\varepsilon,\varphi)_{\mathcal{H}^\varepsilon}| \le \\
&\quad \le C|(U_\varepsilon,\varphi)_{\mathcal{H}^\varepsilon} - (\Lambda_\varepsilon+1)(\mathcal{B}_\varepsilon U_\varepsilon,\varphi)_{\mathcal{H}^\varepsilon}| = C|(U_\varepsilon,\varphi)_{\mathcal{H}^\varepsilon} - (\Lambda_\varepsilon+1)(U_\varepsilon,\varphi)_{\mathcal{L}^\varepsilon}| = \\
&\quad = C\left|\int_\Omega a_\varepsilon \nabla U_\varepsilon \nabla\varphi\,dx - \Lambda_\varepsilon \int_\Omega \rho_\varepsilon U_\varepsilon \varphi\,dx\right| \le \\
&\quad \le C\left|\int_{\Omega_0^\varepsilon\cup\Omega_1^\varepsilon} \left(\mathrm{div}\,(a_\varepsilon\nabla U_\varepsilon) + \Lambda_\varepsilon\rho_\varepsilon U_\varepsilon\right)\varphi\,dx\right| + C\int_{\Gamma^\varepsilon}\left|\left. a_\varepsilon\frac{\partial U_\varepsilon}{\partial n}\right|_0 - \left. a_\varepsilon\frac{\partial U_\varepsilon}{\partial n}\right|_1\right| |\varphi|\,dx.
\end{aligned} \tag{78}$$

Denote the right-hand side of (78) by $F_\varepsilon(U_\varepsilon,\varphi)$. Substituting $U_\varepsilon = W_\varepsilon + V_\varepsilon$ and taking into account (70) and (71),

$$F_\varepsilon(U_\varepsilon,\varphi) \le F_\varepsilon(W_\varepsilon,\varphi) + C\left|(\Lambda_\varepsilon+1)\int_\Omega \rho_\varepsilon V_\varepsilon\varphi\,dx\right| \le F_\varepsilon(W_\varepsilon,\varphi) + C\|V_\varepsilon\|_{\mathcal{L}^\varepsilon}\|\varphi\|_{\mathcal{L}^\varepsilon}.$$

By Lemma 4.2 and obvious inequality $\|\varphi\|_{\mathcal{L}^\varepsilon} \le \|\varphi\|_{\mathcal{H}^\varepsilon}$,

$$F_\varepsilon(U_\varepsilon,\varphi) \le F_\varepsilon(W_\varepsilon,\varphi) + C\varepsilon^{3/4}\|\varphi\|_{\mathcal{H}^\varepsilon}. \tag{79}$$

According to Lemma 3.2 (i) and (ii),

$$F_\varepsilon(W_\varepsilon,\varphi) \le C\varepsilon\|\varphi\|_{L^2(\Omega)} + C\int_{\Gamma^\varepsilon}\left|\left. a_\varepsilon\frac{\partial W_\varepsilon}{\partial n}\right|_0 - \left. a_\varepsilon\frac{\partial W_\varepsilon}{\partial n}\right|_1\right| |\varphi|\,dx.$$

Due to Lemmas 3.2 (iii) and 4.3 the latter yields

$$F_\varepsilon(W_\varepsilon,\varphi) \le C\varepsilon\|\varphi\|_{\mathcal{L}^\varepsilon} + C\varepsilon^{3/2}\Big(\int_{\Gamma^\varepsilon}|\varphi|^2\,dx\Big)^{1/2} \le C\varepsilon\|\varphi\|_{\mathcal{H}^\varepsilon}. \tag{80}$$

Using (80), (79) in (78) yields

$$|(\mathcal{B}_\varepsilon U_\varepsilon - (\Lambda_\varepsilon+1)^{-1}U_\varepsilon,\varphi)_{\mathcal{H}^\varepsilon}| \le C\varepsilon^{3/4}\|\varphi\|_{\mathcal{H}^\varepsilon}$$

for all $\varphi \in \mathcal{H}^\varepsilon$. Hence, $\|\mathcal{B}_\varepsilon U_\varepsilon - (\Lambda_\varepsilon+1)^{-1}U_\varepsilon\|_{\mathcal{H}^\varepsilon} \le C\varepsilon^{3/4}$. □

Lemma 4.5. $\|U_\varepsilon\|_{\mathcal{L}^\varepsilon} \geq C\varepsilon^{-1/2}$.

Proof. By the triangle inequality we have

$$\|U_\varepsilon\|_{\mathcal{H}^\varepsilon} \geq \|U_\varepsilon\|_{\mathcal{L}^\varepsilon} \geq \|W_\varepsilon\|_{\mathcal{L}^\varepsilon} - \|V_\varepsilon\|_{\mathcal{L}^\varepsilon}. \tag{81}$$

We consider

$$\|W_\varepsilon\|^2_{\mathcal{L}^\varepsilon} = \varepsilon^{-1}\int_{\Omega_0^\varepsilon}\left|w_0\left(x,\frac{x}{\varepsilon}\right)+\varepsilon w_1\left(x,\frac{x}{\varepsilon}\right)+\varepsilon^2 w_2\left(x,\frac{x}{\varepsilon}\right)\right|^2 dx+$$
$$+\int_{\Omega_0^\varepsilon}\left|v^0(x)+\varepsilon v_1\left(x,\frac{x}{\varepsilon}\right)+\varepsilon^2 v_2\left(x,\frac{x}{\varepsilon}\right)\right|^2 dx=$$
$$=\varepsilon^{-1}\int_{\Omega_0^\varepsilon}\left|w_0\left(x,\frac{x}{\varepsilon}\right)\right|^2 dx+O(1)=\varepsilon^{-1}\int_{\Omega_0^\varepsilon}\left|v^0(x)\eta\left(\frac{x}{\varepsilon}\right)\right|^2 dx+O(1),\quad \varepsilon\to 0. \tag{82}$$

Extending $\eta : Q_0 \to \mathbb{R}$ by zero onto entire periodicity cell Q, by the mean value property we obtain

$$\int_{\Omega_0^\varepsilon}\left|v^0(x)\eta\left(\frac{x}{\varepsilon}\right)\right|^2 dx \quad\to\quad C_* := \langle\eta^2\rangle\int_\Omega\left|v^0(x)\right|^2 dx \quad\text{as}\quad \varepsilon\to 0, \tag{83}$$

where $\langle\eta^2\rangle$ is the mean value of function η^2 over Q, namely

$$\langle\eta^2\rangle = \int_Q \eta^2(y)\,dy = \int_{Q_0}\eta^2(y)\,dy > 0.$$

Since also $v^0(x) \not\equiv 0$, C_* is positive. Therefore (83) and (82) yield

$$\|W_\varepsilon\|_{\mathcal{L}^\varepsilon} = C_*^{1/2}\varepsilon^{-1/2} + o(\varepsilon^{-1/2}), \quad \varepsilon \to 0. \tag{84}$$

Due to Lemma 4.2 and (84), it follows from (81) that $\|U_\varepsilon\|_{\mathcal{L}^\varepsilon} \geq C_*^{1/2}\varepsilon^{-1/2}+o(\varepsilon^{-1/2})$ as $\varepsilon \to 0$. □

Theorem 4.6. *Let λ_0 be a solution to* (36) *such that $\lambda_0 \neq \lambda_j^D$ and λ_1 is defined according to* (57)*. Then*

1. For sufficiently small $\varepsilon > 0$ there exists an eigenvalue λ^ε of (2) *such that*

$$|\lambda^\varepsilon - \lambda_0 - \varepsilon\lambda_1| \leq C_1\varepsilon^{5/4}, \tag{85}$$

with constant C_1 independent of ε.

2. Let W_ε be defined by (59) *and $\widetilde{W}_\varepsilon = \|W_\varepsilon\|^{-1}_{\mathcal{L}^\varepsilon}W_\varepsilon$. Then there exist constants $c_j(\varepsilon)$ such that*

$$\Big\|\widetilde{W}_\varepsilon - \sum_{j\in J_\varepsilon} c_j(\varepsilon)u_j^\varepsilon\Big\|_{\mathcal{L}^\varepsilon} < C_2\varepsilon^{5/4}, \tag{86}$$

where $J_\varepsilon = \{j : |\lambda_j^\varepsilon - \lambda_0 - \varepsilon\lambda_1| < C\varepsilon^{5/4}\}$, and λ_j^ε, u_j^ε are eigenvalues and ($\mathcal{L}^\varepsilon$-normalized) eigenfunctions of (2)*, and the constants C and C_2 are independent of ε.*

Proof. Application of classical lemma on "approximate eigenvalues", e.g. [18], with $\widetilde{U}_\varepsilon = \|U_\varepsilon\|^{-1}_{\mathcal{L}^\varepsilon}U_\varepsilon$ as a test function and $\Lambda_\varepsilon = \lambda_0 + \varepsilon\lambda_1$ as an approximate eigenvalue, ensures, via Lemmas 4.4 & 4.5, the existence of an eigenvalue μ_ε of operator $\mathcal{B}_\varepsilon$ such that

$$|(\Lambda_\varepsilon + 1)^{-1} - \mu_\varepsilon| \leq C\varepsilon^{5/4}, \tag{87}$$

and delivers the estimate analogous to (86) with u_j^ε being eigenfunctions of $\mathcal{B}_\varepsilon$ and $\widetilde{W}_\varepsilon$ replaced by $\widetilde{U}_\varepsilon$. It suffices to notice that the eigenfunctions of the problem (2) and of operator $\mathcal{B}_\varepsilon$ coincide, their eigenvalues are related via $\mu_\varepsilon^{-1} = \lambda^\varepsilon + 1$ and that

$\mathcal{L}^\varepsilon$ norm of the difference between $\widetilde{U}_\varepsilon$ and $\widetilde{W_\varepsilon}$ can be estimated via the right hand side of (86) (see Lemmas 4.2 and 4.5). □

Remark 2. *Notice that* (86) *implies weaker but more transparent interpretations on the approximate eigenfunctions. For example, introducing*

$$u(x,y) = \begin{cases} v^0(x), & y \in Q_1, \\ w_0(x,y), & y \in Q_0, \end{cases} \tag{88}$$

we claim that

$$\left\| u\left(x, \frac{x}{\varepsilon}\right) - \sum_{j\in J_\varepsilon} d_j(\varepsilon) u_j^\varepsilon \right\|_{L^2(\Omega)} \le C\varepsilon^{3/4}, \tag{89}$$

with appropriate $d_j(\varepsilon)$. Note that $\|u(\cdot, \frac{\cdot}{\varepsilon})\|_{L^2(\Omega)} \ge C_0 > 0$. Then (89) *follows from* (86) *by splitting its left hand side into the parts corresponding to Ω_1^ε and Ω_0^ε, removing the weight, retaining only the main-order terms and then adding the inequalities up.*

We also remark that, in principle, the result (86) on the convergence of eigenfunctions could be further sharpened, e.g. using the technique of two-scale convergence, cf. Section 5 below and [5].

4.3. **Case (b).** In this section we assume that $\lambda_0 = \lambda_j^D$ for some j, its multiplicity is equal to 1 and the corresponding eigenfunction ϕ has zero mean, i.e. $\langle\phi\rangle = 0$, see Section 3.2.

Theorem 4.7. *Let $c \in C^3(\Omega)$, $c = 0$ on $\partial\Omega$, λ_0 be not a solution to* (36) *and λ_1 be defined according to* (B.17). *Then there exist $\varepsilon_0 > 0$ and constants C, C_1 independent of ε (but dependent on c) such that for any $0 < \varepsilon \le \varepsilon_0$,*

1. There exists an eigenvalue λ^ε of (2) *such that*

$$|\lambda^\varepsilon - \lambda_0 - \varepsilon\lambda_1| \le C\varepsilon^{5/4}. \tag{90}$$

2. Let W_ε^* be defined by (63) *and $\widetilde{W_\varepsilon^*} = \|W_\varepsilon^*\|_{\mathcal{L}^\varepsilon}^{-1} W_\varepsilon^*$. Then there exist constants $c_j(\varepsilon)$ such that*

$$\left\| \widetilde{W_\varepsilon^*} - \sum_{j\in J_\varepsilon} c_j(\varepsilon) u_j^\varepsilon \right\|_{\mathcal{L}^\varepsilon} < C_1\varepsilon^{5/4}, \tag{91}$$

where $J_\varepsilon = \{j : |\lambda_j^\varepsilon - \lambda_0 - \varepsilon\lambda_1| < C\varepsilon^{5/4}\}$, and $\lambda_j^\varepsilon, u_j^\varepsilon(x)$ are eigenvalues and ($\mathcal{L}^\varepsilon$-normalized) eigenfunctions of (2).

Proof. Proof of this theorem literally follows the proof of Theorem 4.6 with reference to Lemma 3.3. □

A direct analogue of Remark 2 also holds.

5. On the eigenfunction convergence

In this section we give a brief sketch of further refinement of the presented results using the technique of two-scale convergence, [15, 1, 19].

First, the inclusions intersecting or touching the boundary are "excluded", e.g. by re-defining a_ε and ρ_ε there as in the matrix phase ($a_\varepsilon(x) = \rho_\varepsilon(x) = 1$). Denoting now via $\varepsilon \to 0$ an appropriate subsequence in ε, without relabelling, let u_ε and λ^ε be eigenfunctions and eigenvalues of the original problem, with normalization

$$\int_{\Omega_1^\varepsilon} \nabla u_\varepsilon^2 + \varepsilon^2 \int_{\Omega_0^\varepsilon} \nabla u_\varepsilon^2 = 1. \tag{92}$$

The boundedness of u_ε in $L^2(\Omega)$ is then implied by (92) e.g. via the uniform positivity of the double-porosity operator whose form is given by the left hand side of (92), [19, Thm 8.1]. This implies that, up to a subsequence, $u_\varepsilon \stackrel{2}{\rightharpoonup} u(x,y)$ and $\varepsilon\nabla u_\varepsilon \stackrel{2}{\rightharpoonup} \nabla_y u(x,y)$, where $u \in L^2(\Omega, H^1_{per})$ and $\stackrel{2}{\rightharpoonup}$ denotes weak two-scale convergence. Additionally, since (92) implies $\varepsilon\|\nabla u_\varepsilon\|_{L^2(\Omega_1^\varepsilon)} \to 0$, [19, Thm 4.1] assures that the two-scale limit is independent of y in the matrix, i.e. is exactly in the form (88). Further, by [19, Thm 4.2], $v^0 \in H^1_0(\Omega)$ and

$$\theta_1^\varepsilon \nabla u_\varepsilon \stackrel{2}{\rightharpoonup} \theta_1(y)(\nabla v^0(x) + p(x,y)), \tag{93}$$

where $p \in L^2(\Omega, V_{pot})$ with θ_1^ε and $\theta_1(y)$ denoting the characteristic functions of Ω_1^ε and Q_1, respectively, and V_{pot} denoting the space of potential vector fields on Q_1, i.e. with respect to the Lebesgue measure supported on Q_1, cf. [19, §3.2].

Let $\lambda^\varepsilon \to \lambda_0$ and $(\lambda^\varepsilon - \lambda_0)/\varepsilon \to \lambda_1$. Selecting then in (2) appropriate oscillating test functions $\phi = \phi_\varepsilon$ one can pass to the limit recovering the weak forms of the equations derived in Section 3. For example, selecting $\phi_\varepsilon(x) = \varepsilon\psi(x)b(x/\varepsilon)$, $\psi \in C_0^\infty(\Omega)$, $b(y) \in C^\infty_{per}(Q)$ yields

$$\int_\Omega \int_{Q_1} (\nabla v^0(x) + p(x,y)) \cdot \nabla_y b(y)\psi(x) dy dx + \int_\Omega \int_{Q_0} \nabla_y w_0(x,y) \cdot \nabla_y b(y)\psi(x) dy dx =$$

$$= \lambda_0 \int_\Omega \int_{Q_0} w_0(x,y) b(y) \psi(x) dy dx. \tag{94}$$

This can be seen to be a weak form of (23) and (21). Selecting further $\phi_\varepsilon(x) = \psi(x)$ can be seen, after some careful technical analysis, to recover (51), (52) and (54).

The above implies that as long as $(v^0)^2 + w_0^2 \not\equiv 0$, λ_0, λ_1, v^0 and w_0 can only be those constructed in Section 3. This does not however rule out the possibility that v^0 and w_0 are both trivial (equivalently, the two-scale limit $u(x,y)$ is identically zero). Therefore additional two-scale compactness type arguments are required, cf. [19, Lemma 8.2]. In fact, following literally the argument of Zhikov one observes that the two-scale compactness of the eigenfunctions does hold, i.e. $u_\varepsilon \stackrel{2}{\to} u(x,y)$, where $\stackrel{2}{\to}$ denotes strong two-scale convergence, in particular there is a convergence of norms:

$$\|u_\varepsilon - u(x, x/\varepsilon)\|_{L^2(\Omega)} \to 0 \quad \text{as } \varepsilon \to 0. \tag{95}$$

However, this in turn does not rule out the possibility of $\|u_\varepsilon\| \to 0$ with the normalization (92), which requires a separate analysis.

We announce here a partial result with this effect, postponing detailed discussions for future.

Theorem 5.1. *Let λ_0 be not an eigenvalue of the Dirichlet problem in Q_0, i.e. $\lambda_0 \neq \lambda_j^D$, $j \geq 1$, see* (25)*. Then*

(i) *In the above setting, necessarily, $\beta(\lambda_0) \geq |Q_1|\lambda_0$, i.e. there are gaps developed for small enough ε in the spectrum, containing in the limit at least $\{\lambda : \beta(\lambda) < |Q_1|\lambda\}$.*

(ii) If $\beta(\lambda_0) = |Q_1|\lambda_0$, necessarily $u(x,y) \not\equiv 0$. Consequently, λ_1 can only be one of those described by (57). *The eigenfunctions converge strongly, in particular* (95) *holds. For fixed λ_0 and λ_1, for small enough ε the multiplicity of the eigenvalues λ^ε near $\Lambda_\varepsilon = \lambda_0 + \varepsilon\lambda_1$ coincides with the multiplicity of ν as an eigenvalue of* (51), (52).

We remark that the above statement does not provide a full analogue of Hausdorff convergence of the spectra as in the double porosity case [19, Thm 8.1]. It does ensure however the existence of the gaps (on Figure 2, (λ_j^D, μ_{j+1}), $j \geq 1$) and of the spectrum accumulation near the left ends μ_j, $j \geq 1$, of the "bands" $[\mu_j, \lambda_j^D]$. However it does not clarify whether the "rests" of the bands, $(\mu_j, \lambda_j^D]$ could be accumulation points. We conjecture that they could. For a chosen $\lambda_0 = \mu_j$ there exist infinitely many $\lambda_1 = \lambda_1^{(n)}$ according to (57), (56), and $\lambda_1^{(n)} \to +\infty$ as $n \to \infty$. On any band, for any small enough ε there exists a finite but infinitely increasing number $N(\varepsilon)$ of eigenvalues according to (85). The issue is hence, in a sense, whether $\varepsilon\lambda_1^{(n)}$ may become of order one for large n ($n \sim N(\varepsilon)$). For $\lambda_1^{(n)} \sim \varepsilon^{-1}$, according to (57) $\nu \sim \varepsilon^{-1}$, and hence, formally, the solutions v^0 of the homogenized equation (51) becomes oscillatory on the scale $x/\varepsilon^{1/2}$. One can attempt deriving asymptotic expansions similarly to those in Section 3, involving this new scale. A preliminary analysis has shown that those have formal solutions near every point inside the band. More detailed analysis is beyond the scope of the present work.

Appendix A. Derivation of the limit equation for v_0.

Since

$$-\int_\Gamma \frac{\partial v_1}{\partial n_x}\, dy = -\int_\Gamma \frac{\partial v_1}{\partial x_j} n_j\, dy = \int_{Q_1} \frac{\partial^2 v_1}{\partial x_j \partial y_j}\, dy,$$

(50) transforms to

$$(\Delta_x v_0 + \lambda_0 v_0)|Q_1| + \int_{Q_1} \frac{\partial^2 v_1}{\partial x_j \partial y_j}\, dy = \int_\Gamma \left(\frac{\partial w_1}{\partial n_y} + \frac{\partial w_0}{\partial n_x}\right) dy.$$

Taking into account (41) and (38) this becomes

$$(\Delta_x v_0 + \lambda_0 v_0)|Q_1| + \frac{\partial^2 v_0}{\partial x_j \partial x_k} \int_{Q_1} \frac{\partial N_k}{\partial y_j}\, dy =$$

$$= \frac{\partial v_0}{\partial x_j}\Big(-\int_{Q_1} \frac{\partial \mathcal{N}}{\partial y_j}\, dy + \int_\Gamma \eta n_j\, dy\Big) + \int_\Gamma \frac{\partial w_1}{\partial n_y}\, dy. \tag{A.1}$$

Since $\eta(y) = 1$ on Γ,

$$\int_\Gamma \eta n_j\, dy = \int_\Gamma n_j\, dy = 0. \tag{A.2}$$

We introduce homogenized matrix $A^{\mathrm{hom}} = (A_{jk}^{\mathrm{hom}})_{j,k=1}^n$ by (53). According to (46) we have

$$\int_\Gamma \frac{\partial w_1}{\partial n_y}\, dy = \frac{\partial v^0}{\partial x_j} \int_\Gamma \frac{\partial \mathcal{M}_j}{\partial n_y}\, dy + v^0 \int_\Gamma \frac{\partial \mathcal{P}}{\partial n_y}\, dy + \lambda_1 v^0 \int_\Gamma \frac{\partial \mathcal{R}}{\partial n_y}\, dy. \tag{A.3}$$

Substituting (A.2) – (A.3) into (A.1) yields

$$-\operatorname{div} A^{\mathrm{hom}} \nabla_x v^0 = \nu(\lambda_1) v^0 + \mathcal{K}_j \frac{\partial v^0}{\partial x_j} \quad \text{in} \quad \Omega, \tag{A.4}$$

with

$$\nu(\lambda_1) = \lambda_0|Q_1| - \lambda_1 \int_\Gamma \frac{\partial \mathcal{R}}{\partial n_y}\, dy - \int_\Gamma \frac{\partial \mathcal{P}}{\partial n_y}\, dy$$

and

$$\mathcal{K}_j = \int_{Q_1} \frac{\partial \mathcal{N}}{\partial y_j}\, dy - \int_\Gamma \frac{\partial \mathcal{M}_j}{\partial n_y}\, dy. \tag{A.5}$$

Lemma A.1. *$\nu(\lambda_1)$ depends on λ_1 with a non-zero linear coefficient.*

Proof. We estimate the linear coefficient

$$\mathcal{C} := -\int_\Gamma \frac{\partial \mathcal{R}}{\partial n_y}\, dy.$$

Note that

$$\mathcal{C} = \int_\Gamma \Big(\mathcal{R}\frac{\partial \eta}{\partial n_y} - \eta\frac{\partial \mathcal{R}}{\partial n_y}\Big)\, dy = \int_{Q_0} (\mathcal{R}\Delta_y \eta - \eta \Delta_y \mathcal{R})\, dy = \int_{Q_0} \eta^2\, dy > 0,$$

where (49) and (26) have been used. Thus

$$\nu(\lambda_1) = \mathcal{C}\lambda_1 + \lambda_0|Q_1| - \int_\Gamma \frac{\partial \mathcal{P}}{\partial n_y}\, dy$$

with positive constant $\mathcal{C}$ (depending on the choice of λ_0). □

Corollary 1. *(i) If $\lambda_0 = 0$ then $\eta(y) \equiv 1$ and hence $\mathcal{C} = |Q_0|$.*
(ii) According to (48) *we also have the representation*

$$\nu(\lambda_1) = \mathcal{C}\lambda_1 + \lambda_0\Big(|Q_1| + \int_{Q_0} \mathcal{P}\, dy\Big). \tag{A.6}$$

Lemma A.2. *All $\mathcal{K}_j$ defined by* (A.5) *equal zero.*

Proof. First we prove an auxiliary identity, namely

$$\int_\Gamma \left(\frac{\partial \mathcal{M}_j}{\partial n_y}\eta - \mathcal{M}_j\frac{\partial \eta}{\partial n_y}\right) dy = 0. \tag{A.7}$$

Notice for this that the left-hand side is

$$\int_{Q_0} (\Delta\mathcal{M}_j\eta - \mathcal{M}_j\Delta\eta)\, dy = -\int_{Q_0} 2\frac{\partial \eta}{\partial y_j}\eta\, dy = -\int_{Q_0} \frac{\partial \eta^2}{\partial y_j}\, dy,$$

where equations (47) and (26) have been used. Since $\eta(y) = 1$ on Γ,

$$\int_{Q_0} \frac{\partial \eta^2}{\partial y_j}\, dy = \int_\Gamma \eta^2 n_j\, dy = \int_\Gamma n_j\, dy = 0,$$

which finishes the proof of (A.7).

Then consider

$$\mathcal{K}_j = -\int_\Gamma \mathcal{N} n_j\, dy - \int_\Gamma \frac{\partial \mathcal{M}_j}{\partial n_y}\eta\, dy,$$

which, according to (42) and (A.7), yields

$$\mathcal{K}_j = \int_\Gamma \mathcal{N}\frac{\partial N_j}{\partial n_y}\, dy - \int_\Gamma \mathcal{M}_j\frac{\partial \eta}{\partial n_y}\, dy. \tag{A.8}$$

Since $\mathcal{N}$ and N_j are both harmonic in Q_0,

$$\int_\Gamma \mathcal{N}\frac{\partial N_j}{\partial n_y}\, dy = \int_\Gamma N_j\frac{\partial \mathcal{N}}{\partial n_y}\, dy. \tag{A.9}$$

Using (47) and (43) we obtain

$$\int_\Gamma \mathcal{M}_j \frac{\partial \eta}{\partial n_y}\, dy = \int_\Gamma N_j \frac{\partial \mathcal{N}}{\partial n_y}\, dy. \tag{A.10}$$

Substitution of (A.9) and (A.10) into (A.8) proves the lemma. □

Finally we come to the formulation of homogenized problem for the function v_0, which comes from (A.4) and boundary condition (4), resulting in (51)-(52).

Appendix B. Proof of Lemma 3.3

We look for v_1, w_1, v_2, w_2 in the form

$$v_1(x,y) = c(x)\mathcal{V}_1(y), \tag{B.1}$$

$$w_1(x,y) = c(x)\mathcal{W}_1(y) + \frac{\partial c(x)}{\partial x_k}\mathcal{Z}_1^{(k)}(y), \tag{B.2}$$

$$v_2(x,y) = c(x)\mathcal{V}_2(y) + \frac{\partial c(x)}{\partial x_k}\mathcal{P}_k(y), \tag{B.3}$$

$$w_2(x,y) = c(x)\mathcal{W}_2(y) + \frac{\partial c(x)}{\partial x_k}\mathcal{Z}_2^{(k)}(y), \tag{B.4}$$

where c is an arbitrary smooth function in Ω and $\mathcal{V}_i, \mathcal{W}_i, \mathcal{Z}_i^{(k)}, \mathcal{P}_k$, $i = 1,2$, $k = 1,..,n$, are functions to be found.

Applying differential operator $\mathcal{A}^\varepsilon$ given by (60) to (63) in Ω_1^ε we obtain

$$\begin{aligned}
&\operatorname{div}(a_\varepsilon \nabla W_\varepsilon^*) + \Lambda_\varepsilon \rho_\varepsilon W_\varepsilon^* = \\
&= \Big\{ \varepsilon^{-1} c\Delta_y \mathcal{V}_1 + \varepsilon^0 \Big(c\Delta_y \mathcal{V}_2(y) + \frac{\partial c}{\partial x_k}\Big\{\Delta_y \mathcal{P}_k(y) + 2\frac{\partial \mathcal{V}_1}{\partial y_k}\Big\}\Big) + \\
&+ \varepsilon^1 \Big(2\frac{\partial^2 v_2}{\partial x_j \partial y_j} + \Delta_x v_1 + \lambda_0 v_1 \Big) + \varepsilon^2 (\Delta_x v_2 + \lambda_1 v_1 + \lambda_0 v_2) + \varepsilon^3 \lambda_1 v_2 \Big\}\Big|_{y=\frac{x}{\varepsilon}}.
\end{aligned} \tag{B.5}$$

Applying next $\mathcal{A}^\varepsilon$ to (63) in Ω_0^ε we obtain

$$\begin{aligned}
&\operatorname{div}(a_\varepsilon \nabla W_\varepsilon^*) + \Lambda_\varepsilon \rho_\varepsilon W_\varepsilon^* = \\
&= \Big\{ \varepsilon^0 \Big(c\,[(\Delta_y + \lambda_0)\mathcal{W}_1 + \lambda_1 \phi] + \frac{\partial c}{\partial x_k}[(\Delta_y + \lambda_0)\mathcal{Z}_1^{(k)} + 2\frac{\partial \phi}{\partial y_k}] \Big) + \\
&+ \varepsilon^1 \Big((\Delta_y + \lambda_0) w_2 + 2\frac{\partial^2 w_1}{\partial x_j \partial y_j} + \Delta_x w_0 + \lambda_1 w_1 \Big) + \\
&+ \varepsilon^2 \Big(2\frac{\partial^2 w_2}{\partial x_j \partial y_j} + \Delta_x w_1 + \lambda_1 w_2 \Big) + \varepsilon^3 \Delta_x w_2 \Big\}\Big|_{y=\frac{x}{\varepsilon}}
\end{aligned} \tag{B.6}$$

Evaluating the jumps of conormal derivatives on Γ^ε, we obtain

$$\begin{aligned}
&a_\varepsilon \frac{\partial W_\varepsilon^*}{\partial n}\Big|_0 - a_\varepsilon \frac{\partial W_\varepsilon^*}{\partial n}\Big|_1 = \varepsilon^0 c \Big(\frac{\partial \phi}{\partial n_y} - \frac{\partial \mathcal{V}_1}{\partial n_y} \Big)\Big|_{\substack{x\in\Gamma^\varepsilon \\ y\in\Gamma}} + \\
&+ \varepsilon^1 \Big(c\Big\{ \frac{\partial \mathcal{W}_1}{\partial n_y} - \frac{\partial \mathcal{V}_2}{\partial n_y} \Big\} + \frac{\partial c}{\partial x_k}\Big\{ \frac{\partial \mathcal{Z}_1^{(k)}}{\partial n_y} + n_k \phi - \frac{\partial \mathcal{P}_k}{\partial n_y} - n_k \mathcal{V}_1 \Big\}\Big)\Big|_{\substack{x\in\Gamma^\varepsilon \\ y\in\Gamma}} + \\
&+ \varepsilon^2 \Big(\frac{\partial w_2}{\partial n_y} + \frac{\partial w_1}{\partial n_x} - \frac{\partial v_2}{\partial n_x} \Big)\Big|_{\substack{x\in\Gamma^\varepsilon \\ y\in\Gamma}} + \varepsilon^3 \frac{\partial w_2}{\partial n_x}\Big|_{\substack{x\in\Gamma^\varepsilon \\ y\in\Gamma}}.
\end{aligned} \tag{B.7}$$

On the other hand function W_ε^* is required to be continuous, i.e we have

$$\mathcal{W}_1 = \mathcal{V}_1 \quad \text{on} \quad \Gamma, \tag{B.8}$$

$$\mathcal{Z}_1^{(k)} = 0 \quad \text{on} \quad \Gamma, \tag{B.9}$$

$$\mathcal{W}_2 = \mathcal{V}_2 \quad \text{on} \quad \Gamma, \tag{B.10}$$

$$\mathcal{Z}_2^{(k)} = \mathcal{P}_k \quad \text{on} \quad \Gamma. \tag{B.11}$$

Equating to zero the term of order ε^{-1} in (B.5) and the term of order ε^0 in (B.7), we obtain problem for $\mathcal{V}_1$:

$$\Delta_y \mathcal{V}_1 = 0 \quad \text{in} \quad Q_1, \qquad \frac{\partial \mathcal{V}_1}{\partial n_y} = \frac{\partial \phi}{\partial n_y} \quad \text{on} \quad \Gamma. \tag{B.12}$$

A solution to this problem exists since $\langle \phi \rangle = 0$ and we can present it as:

$$\mathcal{V}_1 = \widetilde{\mathcal{V}}_1 + \widetilde{A}, \tag{B.13}$$

where $< \widetilde{\mathcal{V}}_1 >= 0$ and $\widetilde{A}$ is a constant which will be determined later.

Equating to zero the term of order ε^0 in (B.6), and using (B.8), (B.9) we obtain problems for $\mathcal{Z}_1^{(k)}$

$$(\Delta_y + \lambda_0)\mathcal{Z}_1^{(k)} = -2\frac{\partial \phi}{\partial y_k} \quad \text{in} \quad Q_0, \qquad \mathcal{Z}_1^{(k)} = 0 \quad \text{on} \quad \Gamma, \tag{B.14}$$

which admits an explicit solution

$$\mathcal{Z}_1^{(k)}(y) = -y_k \phi(y), \tag{B.15}$$

and for $\mathcal{W}_1$

$$(\Delta_y + \lambda_0)\mathcal{W}_1 = -\lambda_1 \phi \quad \text{in} \quad Q_0, \qquad \mathcal{W}_1 = \mathcal{V}_1 \quad \text{on} \quad \Gamma. \tag{B.16}$$

A solution to the latter exists if and only if

$$\lambda_1 = \int_\Gamma \mathcal{V}_1 \frac{\partial \phi}{\partial n_y}\, dy = -\int_{Q_1} |\nabla \mathcal{V}_1|^2 dy = -\int_{Q_1} |\nabla \widetilde{\mathcal{V}}_1|^2 dy, \tag{B.17}$$

and we can present it in the following way:

$$\mathcal{W}_1 = \widetilde{\mathcal{W}}_1 + \widetilde{A}\eta, \tag{B.18}$$

where $\widetilde{\mathcal{W}}_1$ solves problem (B.16) with $\mathcal{V}_1$ replaced by $\widetilde{\mathcal{V}}_1$ (a solution exists for the same reason), and η solves (26) (a solution exists since $\langle \phi \rangle = 0$). Notice that

$$\lambda_0 \langle \eta \rangle = -\int_\Gamma \frac{\partial \eta}{\partial n_y} dy \neq 0,$$

otherwise λ_0 would be a solution of (36) which contradicts to the assumptions of this section.

Equating to zero the term of order ε^0 in (B.5) and the term of order ε^1 in (B.7), we obtain problems for $\mathcal{V}_2$ and $\mathcal{P}_k$. For $\mathcal{V}_2$ we have:

$$\Delta_y \mathcal{V}_2 = 0 \quad \text{in} \quad Q_1, \qquad \frac{\partial \mathcal{V}_2}{\partial n_y} = \frac{\partial \mathcal{W}_1}{\partial n_y} \quad \text{on} \quad \Gamma. \tag{B.19}$$

A solution to this problem exists if and only if

$$0 = \int_\Gamma \frac{\partial \mathcal{W}_1}{\partial n_y}\, dy = \int_\Gamma \frac{\partial \widetilde{\mathcal{W}}_1}{\partial n_y}\, dy + \widetilde{A} \int_\Gamma \frac{\partial \eta}{\partial n_y}\, dy, \tag{B.20}$$

and consequently

$$\widetilde{A} = (\lambda_0 \langle \eta \rangle)^{-1} \int_\Gamma \frac{\partial \widetilde{\mathcal{W}}_1}{\partial n_y}\, dy. \tag{B.21}$$

The problem for $\mathcal{P}_k$ has the form:

$$\Delta_y \mathcal{P}_k(y) = -2\frac{\partial \mathcal{V}_1}{\partial y_k} \quad \text{in} \quad Q_1, \qquad \frac{\partial \mathcal{P}_k}{\partial n_y} = \frac{\partial \mathcal{Z}_1^{(k)}}{\partial n_y} - n_k \mathcal{V}_1 \quad \text{on} \quad \Gamma. \tag{B.22}$$

Solvability condition for this problem has the form

$$2\int_{Q_1} \frac{\partial \mathcal{V}_1}{\partial y_k}\, dy = \int_\Gamma \left(\frac{\partial \mathcal{Z}_1^{(k)}}{\partial n_y} - n_k \mathcal{V}_1 \right) dy. \tag{B.23}$$

The left hand side of (B.23) can be transformed as follows,

$$\int_{Q_1} 2\frac{\partial \mathcal{V}_1}{\partial y_k}\, dy = -2\int_\Gamma n_k \mathcal{V}_1\, dy. \tag{B.24}$$

On the other hand, for the right hand side of (B.23),

$$\int_\Gamma \left(\frac{\partial \mathcal{Z}_1^{(k)}}{\partial n_y} - n_k \mathcal{V}_1 \right) dy = \int_\Gamma \left(-y_k \frac{\partial \phi}{\partial n_y} - n_k \mathcal{V}_1 \right) dy =$$

$$= \int_\Gamma \left(-y_k \frac{\partial \mathcal{V}_1}{\partial n_y} - n_k \mathcal{V}_1 \right) dy = -2\int_\Gamma n_k \mathcal{V}_1\, dy. \tag{B.25}$$

Here we used (B.15), (B.22) and the integration by parts. Comparing (B.24) and (B.25) we see that solvability condition (B.23) is satisfied. Finally $\mathcal{W}_2$ and $\mathcal{Z}_2^{(k)}$ are arbitrary smooth functions satisfying (B.10) and (B.11). □

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