Sufficiency of Favard's condition for a class of band-dominated operators on the axis

Simon N Chandler-Wilde [a,1] and Marko Lindner [a,2,*]

[a] *Department of Mathematics, University of Reading, RG6 6AX, UK.*

Abstract

The purpose of this paper is to show that, for a large class of band-dominated operators on $\ell^\infty(\mathbb{Z}, U)$, with U being a complex Banach space, the injectivity of all limit operators of A already implies their invertibility and the uniform boundedness of their inverses. The latter property is known to be equivalent to the invertibility at infinity of A, which, on the other hand, is often equivalent to the Fredholmness of A. As a consequence, for operators A in the Wiener algebra, we can characterize the essential spectrum of A on $\ell^p(\mathbb{Z}, U)$, regardless of $p \in [1, \infty]$, as the union of point spectra of its limit operators considered as acting on $\ell^\infty(\mathbb{Z}, U)$.

Key words: limit operator, Favard condition, Fredholm operator, Wiener algebra

1 Introduction

We study linear operators on the space $Y^\infty = \ell^\infty(\mathbb{Z}, U)$ of all bounded two-sided infinite sequences with values in a complex Banach space U. If M is a two-sided infinite band matrix, with entries m_{ij} in the space $L(U)$ of all bounded linear operators on U and $\sup \|m_{ij}\| < \infty$, then, after identifying elements of Y^∞ with infinite column vectors, M acts on Y^∞ as what we call a *band operator*. The closure of the set of all band operators in $L(Y^\infty)$ is denoted by $BDO(Y^\infty)$; we call its elements *band-dominated operators*.

* Corresponding author: `M.Lindner@reading.ac.uk`

[1] The first author thanks Peter Kuchment for helpful discussions and gratefully acknowledges the support of a Leverhulme Trust Fellowship and a Visiting Fellowship of the Isaac Newton Institute, Cambridge.

[2] The second author was kindly supported by a Marie Curie Fellowship of the European Commission (MEIF-CT-2005-009758).

Let $K(Y^\infty, \mathcal{P})$ denote the closure in $L(Y^\infty)$ of the set of all operators $A \in L(Y^\infty)$ which are induced by a matrix $M = [m_{ij}]$ with only finitely many non-zero entries. It is not hard to see that $K(Y^\infty, \mathcal{P})$ is a closed two-sided ideal in the Banach algebra $BDO(Y^\infty)$, and we say that a band-dominated operator A is *invertible at infinity* if its coset $A+K(Y^\infty, \mathcal{P})$ is invertible in the factor algebra $BDO(Y^\infty)/K(Y^\infty, \mathcal{P})$. Clearly, the coset $A + K(Y^\infty, \mathcal{P})$ only depends on the asymptotic behaviour at infinity of the matrix entries of (the matrix that induces) A. The study of this asymptotic behaviour requires the study of the so-called limit operators of A. The idea is to associate A with a family, denoted by $\sigma^{\mathsf{op}}(A)$, of linear operators on Y^∞, where each member of the family represents part of the behaviour of A at infinity. The elements of $\sigma^{\mathsf{op}}(A)$ are called *the limit operators* of A. It is known [7] that, for a fairly large class of band-dominated operators A, invertibility at infinity of A is equivalent to what we call uniform invertibility of $\sigma^{\mathsf{op}}(A)$, which means

- (C1) All limit operators of A are injective;
- (C2) All limit operators of A are surjective;
- (C3) The inverses of the limit operators of A are uniformly bounded.

By looking at the structure of $\sigma^{\mathsf{op}}(A)$, in particular using its compactness properties, it is now possible to reduce the set of conditions $\{(C1), (C2), (C3)\}$ to an equivalent subset. In [7] it is shown that (C3) always follows from $\{(C1), (C2)\}$, so that $\{(C1), (C2), (C3)\} = \{(C1), (C2)\}$. In [1] we then went on and partially removed (C2) under the additional assumption that $A = I + K$ with an operator K whose matrix entries form a collectively compact set in $L(U)$. Note that all results mentioned so far are shown for operators on $\ell^\infty(\mathbb{Z}^N, U)$ with $N \in \mathbb{N}$ and U a complex Banach space. The aim of this paper is to show that, under the same assumption of $A = I + K$ as was made in [1] but now for operators on the axis, i.e. for $N = 1$, condition (C2) can be fully removed so that $\{(C1), (C2), (C3)\} = \{(C1)\}$ then. The remaining condition (C1) is commonly known as *Favard's condition* in the literature [18,19,4].

Historic remarks. The story of limit operators and Favard's condition starts in spaces of functions on a continuous rather than discrete domain. The typical setting was originally that of a (ordinary or partial) differential operator with almost periodic coefficients. First of all, Favard [3] showed that the condition that was subsequently named after him guarantees the existence of almost periodic solutions to a system of ODE's with almost periodic coefficients and an almost periodic right-hand side. Later, Muhamadiev [10] proved that Favard's condition implies the invertibility of Favard's almost periodic differential operator considered as operator from $BC^1(\mathbb{R}, \mathbb{R}^n)$ to $BC(\mathbb{R}, \mathbb{R}^n)$. Extensions of Muhamadiev's result to wider classes of almost periodic operators can be found in [11,12,18,19,4], for example. For operators A with almost periodic coefficients, the connection between A and its limit operators is a lot stronger than in more general settings. In particular, all limit operators of A are norm-limits of translates of A, including the operator A itself.

In [10], Muhamadiev went on to study matrix ordinary differential operators on the real line with merely bounded and uniformly continuous coefficients which lead him to define limit operators as limits of translates of the operator A with respect to what we call $\mathcal{P}-$convergence now (see §2.2). In this wider setting he states the theorem that injectivity of all limit operators, that is Favard's condition, implies their invertibility as operator from $BC^1(\mathbb{R}, \mathbb{R}^n)$ to $BC(\mathbb{R}, \mathbb{R}^n)$. We remark that this result is very much in the spirit of our paper; it can, in fact, via reduction to an equivalent matrix integral operator, be shown to follow from our Proposition 4.1. (We note that Muhamadiev provided no proof of his result in [10] so that we do not know whether our methods of argument are a generalization of what he had in mind.) Later on, Muhamadiev [11] and Shubin [19] studied elliptic differential operators A with almost periodic coefficients. For infinitely smooth coefficients, Shubin provides a proof of Muhamadiev's result [11] that the Favard condition is equivalent to the invertibility of A on $BC^\infty(\mathbb{R}^N, \mathbb{R})$. In [12], Muhamadiev showed that, for Hölder continuous coefficients, Favard's condition is equivalent to A being Φ_+-semi Fredholm between an appropriate pair of spaces of bounded Hölder continuous functions. Similarly and much more recently, Volpert and Volpert show that, for a general class of scalar elliptic partial differential operators A on an unbounded domain but also for systems of such, the Favard condition is equivalent to the Φ_+-semi Fredholmness of A on appropriate Hölder [21,22] or Sobolev [20,22] spaces. Lange and Rabinovich [6] state a corresponding result about semi Fredholmness of band-dominated operators in the discrete scalar-valued $\ell^\infty(\mathbb{Z}^N, \mathbb{C})$ setting.

In the last 10 years, limit operators of band-dominated operators on discrete ℓ^p spaces with values in an arbitrary complex Banach space U and $p \in (1, \infty)$ have been extensively studied by Rabinovich, Roch and Silbermann [15,16]. The second author [7,8] then extended some of their results to $p \in \{1, \infty\}$. The reformulation of the so-called 'richness' property of a band-dominated operator A in terms of a particular compactness property of the operator spectrum $\sigma^{\mathsf{op}}(A)$ of A in [7] then sparked a symbiosis of the limit operator method with the generalised collectively compact operator theory that was introduced by the first author and Zhang in [2]. The first outcomes of this symbiosis are [1] and the current paper.

Contents of the paper. In §2 we introduce the classes of operators that we are interested in. We then define what a limit operator is and quote the result that connects the set of all limit operators to invertibility at infinity. Concluding surjectivity from injectivity whilst working with a family of operators (rather than just a single operator) is one of the main threads of the generalised collectively compact operator theory introduced by the first author and Zhang in [2]. Here we quote a slightly weakened version of a theorem from [2] that will do most of the work for us in §3. Roughly speaking, the strategy to conclude surjectivity of a given operator T from its injectivity is to embed it into a set of injective operators, $\mathcal{B}$, that enjoys a type of collective compactness condition and to approximate T by a sequence of operators, for example periodic operators, for which injectivity does imply surjectivity, this sequence being such that its 'limit operators' (in a certain sense) are in the set $\mathcal{B}$.

In §3 we state and prove the main theorem of this paper. In a nutshell, the plot of the proof is as follows. Let A be subject to (C1). Then we prove (C2) in these three steps:

a) If $B \in \sigma^{\mathsf{op}}(A)$ and B has a surjective limit operator C, then B is surjective itself.

b) Every $B \in \sigma^{\mathsf{op}}(A)$ has a self-similar limit operator C.

c) Self-similar limit operators (of A, including those of B) are surjective.

By a self-similar operator we mean an operator $C \in L(Y^\infty)$ with $C \in \sigma^{\mathsf{op}}(C)$.

Finally, in §4 we study a class of operators which are band-dominated on all spaces $Y^p := \ell^p(\mathbb{Z}, U)$ with $p \in [1, \infty]$ simultaneously. For this particular class of operators, the so-called Wiener algebra $\mathcal{W}$, we demonstrate how the study of Fredholmness and the essential spectrum of $A \in \mathcal{W}$ with respect to any of the spaces Y^p profits from our new results in Y^∞.

2 Preliminaries

Let $p \in [1, \infty]$ and U be a complex Banach space. By $Y^p := \ell^p(\mathbb{Z}, U)$ we denote the usual ℓ^p−space of two-sided infinite sequences $(..., x(-1), x(0), x(1), ...)$ with values $x(i)$ in the Banach space U. If we only write the letter Y then the corresponding statement holds with any space Y^p, $p \in [1, \infty]$, in place of Y.

2.1 Operators on Y and corresponding matrices

By $L(Y)$ we denote the set of bounded linear operators on Y. To every operator $A \in L(Y)$ we will associate a two-sided infinite matrix $[A] = [a_{ij}]$ in the canonical way; that is, by the following construction. For $k \in \mathbb{Z}$ let E_k :

$U \to Y$ and $R_k : Y \to U$ be extension and restriction operators, defined by $E_k y = (..., 0, y, 0, ...)$, for $y \in U$, with the y standing at the kth place in the sequence, and by $R_k x = x(k)$, for $x = (x(j))_{j\in\mathbb{Z}^N} \in Y$. Then the matrix entries of $[A]$ are defined as

$$a_{ij} \;:=\; R_i A E_j \;\in\; L(U), \qquad i, j \in \mathbb{Z}, \tag{1}$$

and $[A]$ is called the *matrix representation* of A. Conversely, given a matrix $M = [m_{ij}]_{i,j\in\mathbb{Z}}$ with entries in $L(U)$, we will say that M *induces* the operator

$$(Bx)(i) \;=\; \sum_{j=-\infty}^{\infty} m_{ij}\, x(j), \qquad i \in \mathbb{Z} \tag{2}$$

if the sum converges in U for every $i \in \mathbb{Z}$ and every $x = ((x(j))_{j\in\mathbb{Z}} \in Y$ and if the resulting operator B is a bounded mapping $Y \to Y$.

It is not hard to see that if M is an infinite matrix and B is induced, via (2), by M then the matrix representation $[B]$ from (1) is equal to M. It does not work quite like that the other way round: For $p = \infty$, there are operators $A \in L(Y^p)$ (e.g. see Example 1.26 c in [8]) for which the matrix representation $M := [A]$ induces an operator B that is different from A. However, for every $A \in L(Y^p)$ with $p \in [1, \infty)$, the matrix $M := [A]$ with entries (1) induces the operator $B = A$.

We say that $A \in L(Y)$ is a *band operator* and write $A \in BO(Y)$ if it is induced by a matrix $[m_{ij}]$ with only finitely many non-zero diagonals, and we write $A \in BDO(Y)$ and say that A is *band-dominated* if A can be approximated in the operator norm by band operators.

2.2 Invertibility at infinity and limit operators

For an arbitrary set $S \subseteq \mathbb{Z}$, let $P_S \in L(Y)$ denote the operator of multiplication by the characteristic function of S. Some frequently used special cases are $P := P_{\{0,1,...\}}$, $Q := I - P$, $P_n := P_{\{-n,...,n\}}$ and $Q_n := I - P_n$ for $n \in \mathbb{N}$. We then put $\mathcal{P} := \{P_1, P_2, ...\}$, define

$$K(Y, \mathcal{P}) \;:=\; \{T \in L(Y) : \|Q_n T\| \to 0, \|T Q_n\| \to 0 \text{ as } n \to \infty\}$$

and say that a sequence $A_1, A_2, ... \in L(Y)$ is *$\mathcal{P}-$convergent* to $A \in L(Y)$ if $\|T(A_n - A)\| \to 0$ and $\|(A_n - A)T\| \to 0$ as $n \to \infty$ for every $T \in K(Y, \mathcal{P})$. From [8, Proposition 1.65] we know that $A_n \stackrel{\mathcal{P}}{\to} A$ if and only if the sequence

(A_n) is bounded and $\|P_k(A_n - A)\| \to 0$ and $\|(A_n - A)P_k\| \to 0$ as $n \to \infty$ for every $k \in \mathbb{N}$.

Let $K_0(Y, \mathcal{P})$ denote the set of all operators $T \in L(Y)$ which are induced by a matrix $[m_{ij}]$ that has only finitely many non-zero entries. Clearly, $K_0(Y, \mathcal{P})$ is a dense subset of $K(Y, \mathcal{P})$ since $\|T - P_nTP_n\| \to 0$ as $n \to \infty$ for all $T \in K(Y, \mathcal{P})$. The set $K(Y, \mathcal{P})$ is a closed two-sided ideal in the Banach algebra $BDO(Y)$. We say that an operator $A \in BDO(Y)$ is *invertible at infinity* if its coset $A + K(Y, \mathcal{P})$ is invertible in the factor algebra $BDO(Y)/K(Y, \mathcal{P})$. The property of invertibility at infinity is of interest for different reasons. On the one hand, it is sufficiently close to Fredholmness to be useful for the study of Fredholmness. On the other hand it is relevant to determining stability of approximation methods in numerical analysis.

For the study of invertibility at infinity, we introduce so-called limit operators. To do this, let $V_k \in L(Y)$ denote the operator of shift by $k \in \mathbb{Z}$ acting by $(V_kx)(i) = x(i - k)$ for every $x \in Y$ and $i \in \mathbb{Z}$. Given $A \in L(Y)$, we say that $B \in L(Y)$ is a *limit operator* of A if there exists a sequence $h = (h(n))_{n\in\mathbb{N}} \subseteq \mathbb{Z}$ with $|h(n)| \to \infty$ and

$$V_{-h(n)}AV_{h(n)} \xrightarrow{\mathcal{P}} B$$

as $n \to \infty$. In this case we also write A_h for B. The set of limit operators A_h of A with respect to all sequences h going to $\pm\infty$ is denoted by $\sigma_\pm^{\mathsf{op}}(A)$, respectively. We also put $\sigma^{\mathsf{op}}(A) := \sigma_+^{\mathsf{op}}(A) \cup \sigma_-^{\mathsf{op}}(A)$ and call it the *operator spectrum* of A. An operator $A \in L(Y)$ is called *rich* if every sequence h of integers going to infinity has a subsequence g such that the limit operator A_g exists. Here is the statement that connects invertibility at infinity with the study of limit operators.

Proposition 2.1 *[8, Theorem 1] A rich operator $A \in BDO(Y^\infty)$ with a preadjoint (meaning that A is the adjoint of another operator that acts on a predual space of Y^∞) is invertible at infinity if and only if the following conditions hold:*

- *(C1) All limit operators of A are injective;*
- *(C2) All limit operators of A are surjective;*
- *(C3) The inverses of the limit operators of A are uniformly bounded.*

Remark 2.2 It is well-known that, for $A \in L(X)$ with a Banach space X in the case that X is the dual of another space Z, the statements

(i) A is the adjoint of an operator $B \in L(Z)$.

(ii) The adjoint A^* maps Z, understood as a subspace of its second dual $Z^{**} = X^*$, into itself.

(iii) A is continuous in the weak* topology on X.

are equivalent. ■

The statement of Proposition 2.1 also holds with Y^∞ replaced by Y^p for $p \in [1, \infty)$ in which case the condition about the existence of a preadjoint is even unnecessary. We will however focus on the case when $p = \infty$ because then it is possible to slim the set of conditions $\{(C1), (C2), C(3)\}$ down quite considerably. More precisely, in Theorem 2 of [8] it was shown that (C3) always follows from (C1)+(C2), which is why we can delete (C3) in the formulation of Proposition 2.1. The purpose of this paper is to show that, for a large class of operators $A \in BDO(Y^\infty)$, already condition (C2), and hence (C3), follows from (C1). For such operators, even both conditions (C2) and (C3) can be removed in Proposition 2.1. The remaining condition, (C1), is often [18,19,5,4] referred to as Favard's condition after Jean Aimé Favard's work [3].

Definition 2.3 *We say that an operator $A \in L(Y^\infty)$ is subject to Favard's condition, (FC), if every limit operator of A is injective on Y^∞.*

2.3 Collective compactness

A family $\mathcal{K}$ of bounded linear operators on a Banach space Z is called *collectively compact* if, for any sequences $(K_n) \subseteq \mathcal{K}$ and $(z_n) \subseteq Z$ with $\|z_n\| \leq 1$, there is always a subsequence of $(K_n z_n)$ that converges in the norm of Z. It is immediate that every collectively compact family $\mathcal{K}$ is bounded and that all of its members are compact operators.

Definition 2.4 *For $A \in BDO(Y)$, let $\mathcal{M}(A) \subseteq L(U)$ refer to the set of all matrix entries (1) of $[A]$. Now let $UM(Y)$ denote the set of all $K \in BDO(Y)$ for which $\mathcal{M}(K)$ is collectively compact in $L(U)$. Moreover, by $UM_\$(Y)$ denote the set of all rich operators $K \in UM(Y)$ and put*

$$I + UM_\$(Y) \;:=\; \{I + K : K \in UM_\$(Y)\}.$$

Remark 2.5 a) Rabinovich and Roch study Fredholmness and the Fredholm index for operators in the class $I + \mathcal{C}_E^\$$ in [13], where $\mathcal{C}_E^\$$ denotes the set of all rich band-dominated operators (on $E = \ell^p(\mathbb{Z}, U)$ with a complex Banach space U) which are induced by infinite matrices with compact entries in $L(U)$.

This is clearly a superclass, precisely: a proper superclass iff $\dim U = \infty$, of $I + UM_\$(Y)$.

b) It should be mentioned that, if $A \in I + UM(Y)$, the invertibility at infinity of A implies its Fredholmness [8, Proposition 2.15]. Together with Proposition 2.1 and the main result of our paper, Theorem 3.1, this shows that, for $A \in I + UM_\$(Y^\infty)$, the Favard condition (FC) implies Fredholmness of A. ■

Lemma 2.6 *If U is a finite-dimensional space then*

$$I + UM_\$(Y) \;=\; UM_\$(Y) \;=\; UM(Y) \;=\; BDO(Y).$$

Proof. Let U be finite-dimensional. From Corollary 3.24 in [8] we know that then every band-dominated operator is rich. Since $L(U)$ is finite-dimensional and $\mathcal{M}(K) \subseteq L(U)$ is bounded for every $K \in BDO(Y)$, we get that $\mathcal{M}(K)$ is collectively compact, i.e. $K \in UM(Y)$. ■

We now present our main tool from the collectively compact operator theory developed in §4 of [2]. Precisely, we give an adapted version of Proposition 5.17 in [1] that is a bit weaker but still sufficient for our purposes here.

Proposition 2.7 *Let $T \in BDO(Y^\infty)$ and take a sequence $T_n \in BDO(Y^\infty)$, $n \in \mathbb{N}$, such that:*

- *(a)* $T_n \overset{\mathcal{P}}{\to} T$;
- *(b)* T_n *injective* $\Rightarrow$ T_n *surjective, for each* $n \in \mathbb{N}$;
- *(c)* $\bigcup_{n=1}^\infty \mathcal{M}(T_n - I)$ *is collectively compact;*
- *(d) there exists a set $\mathcal{B} \subset L(Y^\infty)$, such that, for every sequence $(k(m)) \subset \mathbb{Z}$ and increasing sequence $(n(m)) \subset \mathbb{N}$, there exist subsequences, denoted again by $(k(m))$ and $(n(m))$, and $S \in \mathcal{B}$ such that*

$$V_{-k(m)} T_{n(m)} V_{k(m)} \overset{\mathcal{P}}{\to} S \in \mathcal{B} \quad as \quad m \to \infty;$$

- *(e) every $S \in \mathcal{B}$ is injective.*

Then T is invertible and, for some $n_0 \in \mathbb{N}$, T_n is invertible for all $n \geq n_0$, and

$$\|T^{-1}\| \leq \sup_{n \geq n_0} \|T_n^{-1}\| < \infty.$$

3 Main result

Theorem 3.1 *If (FC) holds for $A \in I + UM_\$(Y^\infty)$ then all limit operators of A are invertible on Y^∞ and their inverses are uniformly bounded.*

The rest of this section is devoted to the proof of Theorem 3.1. Since we know from [8, Theorem 2] that condition (C3) of Proposition 2.1 follows from (C1) and (C2), it remains to show that (C2) follows from (C1) alias (FC) if $A \in I + UM_\$(Y^\infty)$. We break the proof of this fact down into the following three propositions. But first we need two lemmas.

Lemma 3.2 *[8, Proposition 3.104] If $A \in L(Y)$ is rich then $\sigma^{\mathsf{op}}_\pm(A)$ is sequentially compact with respect to $\mathcal{P}-$convergence.*

Lemma 3.3 *Let $A \in L(Y)$ and B be an arbitrary limit operator of A.*

a) *If $B \in \sigma^{\mathsf{op}}_\pm(A)$ then $\sigma^{\mathsf{op}}(B) \subseteq \sigma^{\mathsf{op}}_\pm(A)$, for σ^{op}_+ and σ^{op}_- respectively.*

b) *If A is rich then B is rich.*

c) *If $A \in UM(Y)$ then $B \in UM(Y)$.*

Proof. **a)** This is Corollary 3.97 of [8].

b) Let $A \in L(Y)$ be rich and $B \in \sigma^{\mathsf{op}}(A)$. From Lemma 3.3 a) and [8, Proposition 3.94] we know that $\{V_{-k}BV_k : k \in \mathbb{Z}\} \subseteq \sigma^{\mathsf{op}}(A)$. By Lemma 3.2, we get that $\{V_{-k}BV_k\}$ is relatively $\mathcal{P}-$sequentially compact. Together with [8, Proposition 3.102] this shows that B is rich.

c) By the definition of a limit operator, the set $\mathcal{M}(B)$ is contained in the closure of $\mathcal{M}(A)$. Consequently, $\mathcal{M}(B)$ is collectively compact if $\mathcal{M}(A)$ is collectively compact. ■

Proposition 3.4 *Let $A \in I+UM_\$(Y^\infty)$ and $B \in \sigma^{\mathsf{op}}_\pm(A)$. If (FC) holds for A and if B has one surjective limit operator, $C \in \sigma^{\mathsf{op}}_\pm(B)$ (with the same choice of $+$ or $-$ as for B), then B is surjective itself.*

Proof. Suppose, without loss of generality, that $B \in \sigma^{\mathsf{op}}_+(A)$. Then $B = A_h$ for some sequence h of integers $h(1), h(2), ... \to +\infty$. By our assumption, there exists a surjective $C \in \sigma^{\mathsf{op}}_+(B)$. By Lemma 3.3 a), we have that $C = A_{\tilde{h}}$ with some integer sequence $\tilde{h}(1), \tilde{h}(2), ... \to +\infty$, and by Lemma 3.3 b) and c) we know that $C \in I + UM_\$(Y^\infty)$.

By passing to subsequences, if necessary, we can always arrange that $\tilde{h}(n-1) < h(n) < \tilde{h}(n)$ for all $n \geq 2$, with $\tilde{h}(n) - h(n) \to +\infty$ and $h(n) - \tilde{h}(n-1) \to +\infty$ as $n \to \infty$. Now, for every $n \in \mathbb{N}$, define $g_+(n) := \tilde{h}(n) - h(n) > 0$ and

$g_-(n) := \tilde{h}(n-1) - h(n) < 0$, and put

$$\begin{aligned} A_n \;:=\; & V_{g_-(n)} QCV_{-g_-(n)} \;+\; V_{g_+(n)} PCV_{-g_+(n)} \\ & \;+\; V_{-h(n)} P_{\{\tilde{h}(n-1),\dots,\tilde{h}(n)-1\}} AV_{h(n)}. \end{aligned}$$

Our plan is now to check the conditions (a)–(e) of Proposition 2.7 with $B = A_h$ in place of T and with $\mathcal{B} = \sigma^{\mathsf{op}}(A)$, in order to conclude that B is surjective.

(a) It is easy to see that $A_n \overset{\mathcal{P}}{\to} A_h = B$ since $V_{-h(n)} AV_{h(n)} \overset{\mathcal{P}}{\to} A_h$.

(b) Since C is invertible it is Fredholm of index zero. So also $D_1 := PCP + QCQ = C - PCQ - QCP$ is Fredholm of index zero since PCQ and QCP are compact for $C \in I + UM(Y^\infty)$ (note that all entries of $C - I$ are compact operators and that C can be norm-approximated by band operators C' in which case both $PC'Q$ and $QC'P$ have only finitely many non-zero entries). We claim that the same is true for $D_2 := V_{g_-(n)} QCQV_{-g_-(n)} + V_{g_+(n)} PCPV_{-g_+(n)} + P_{\{g_-(n),\dots,g_+(n)-1\}}$ and every $n \in \mathbb{N}$. Indeed, since

$$\begin{aligned} \ker D_2 &= \{(\dots, x_{-2}, x_{-1}, 0, \dots, 0, x_0, x_1, \dots) \;:\; (x_i) \in \ker D_1\}, \\ \operatorname{im} D_2 &= \{(\dots, x_{-2}, x_{-1}, y_{g_-(n)}, \dots, y_{g_+(n)-1}, x_0, x_1, \dots) \\ & \qquad\qquad :\; (x_i) \in \operatorname{im} D_1,\; y_j \in U\} \end{aligned}$$

hold with the zeros and y_j's in the positions $\{g_-(n), \dots, g_+(n) - 1\}$ of the sequence, respectively, we get that

$$\dim \ker D_2 \;=\; \dim \ker D_1 \;<\; \infty, \quad \operatorname{codim} \operatorname{im} D_2 \;=\; \operatorname{codim} \operatorname{im} D_1 \;<\; \infty$$

and hence D_2 is also Fredholm with the same index (namely zero) as D_1. But this proves that

$$\begin{aligned} A_n \;=\; D_2 \;+\; & V_{g_-(n)} QCPV_{-g_-(n)} \;+\; V_{g_+(n)} PCQV_{-g_+(n)} \\ & \;+\; V_{-h(n)} P_{\{\tilde{h}(n-1),\dots,\tilde{h}(n)-1\}} (A - I) V_{h(n)} \end{aligned}$$

is Fredholm of index zero since all of QCP, PCQ and $P_{\{\tilde{h}(n-1),\dots,\tilde{h}(n)-1\}}(A - I)$ are compact. So each A_n is surjective if injective.

(c) Clearly,

$$\bigcup_{n=1}^{\infty} \mathcal{M}(A_n - I) \;\subseteq\; \mathcal{M}(A - I) \cup \mathcal{M}(C - I)$$

is collectively compact in $L(U)$ since $A - I \in UM(Y^\infty)$ by our premise and $C - I \in UM(Y^\infty)$ by Lemma 3.3 c).

(d) Moreover, if $(k(m)) \subseteq \mathbb{Z}$ is arbitrary and $(n(m)) \subseteq \mathbb{N}$ is increasing then, since A and C are rich, there exist subsequences, denoted again by $(k(m))$ and $(n(m))$, and an operator D such that

$$V_{-k(m)} A_{n(m)} V_{k(m)} \xrightarrow{\mathcal{P}} D.$$

It is an easy exercise to check that D is either a translate of B or a limit operator of B (in particular it may be a translate or limit operator of C). In each of these cases D is a limit operator of A, and so $D \in \mathcal{B}$.

(e) Every $D \in \mathcal{B}$ is injective by assumption (FC).

We have seen that conditions (a)–(e) of Proposition 2.7 are satisfied with $\mathcal{B} := \sigma^{\mathsf{op}}(A)$ and we therefore conclude that B is surjective. ■

Definition 3.5 *We call $C \in L(Y)$ a self-similar operator if $C \in \sigma^{\mathsf{op}}(C)$.*

Roughly speaking, we think of self-similar operators as containing a copy of themselves, at infinity.

Remark 3.6 A concept that is related to self-similar operators is that of a recurrent operator. An operator $C \in L(Y)$ is called *recurrent* [11] if, for every limit operator D of C, it holds that $\sigma^{\mathsf{op}}(D) = \sigma^{\mathsf{op}}(C)$. It is easy to see that, if C is recurrent, then

a) All limit operators of C are self-similar.

b) All limit operators of C are recurrent.

c) The local operator spectra $\sigma^{\mathsf{op}}_{+}(C)$ and $\sigma^{\mathsf{op}}_{-}(C)$ coincide with $\sigma^{\mathsf{op}}(C)$.

We also remark that, in the proof of the following proposition, we even show the slightly stronger result that every rich operator has a recurrent limit operator (namely the operator denoted by B' in the proof). It is not difficult to see that an element $\sigma^{\mathsf{op}}(B)$ of the partially ordered set $(\mathcal{A}, \supseteq)$ in the proof below is maximal iff B is recurrent. ■

Proposition 3.7 *Every rich operator $B \in L(Y)$ has a self-similar limit operator C.*

Proof. Let

$$\mathcal{A} \ := \ \{ \ \sigma^{\mathsf{op}}(B) \ : \ B \in \sigma^{\mathsf{op}}(A) \ \}$$

which is a partially ordered set, equipped with the order '$\supseteq$'. To be able to apply Zorn's lemma to $\mathcal{A}$, we have to check that its conditions are satisfied.

So let $\mathcal{B}$ be a totally ordered subset of $\mathcal{A}$, i.e.

$$\mathcal{B} \; := \; \{ \; \sigma^{\mathsf{op}}(B) \; : \; B \in \sigma \; \}$$

for a subset $\sigma \subseteq \sigma^{\mathsf{op}}(A)$, such that for any two $B_1, B_2 \in \sigma$, we either have $\sigma^{\mathsf{op}}(B_1) \supseteq \sigma^{\mathsf{op}}(B_2)$ or $\sigma^{\mathsf{op}}(B_2) \supseteq \sigma^{\mathsf{op}}(B_1)$.

On $X := \sigma^{\mathsf{op}}(A)$ we define the following family of seminorms. Let

$$\varrho_{2n-1}(T) \; := \; \|P_n T\|, \qquad \varrho_{2n}(T) \; := \; \|T P_n\|$$

for $n = 1, 2, ...$ and every $T \in X$, and denote the topology that is generated on X by $\{\varrho_1, \varrho_2, ...\}$ by $\mathcal{T}$. By [8, Proposition 1.65] and since $\|T\| \leq \|A\|$ for every $T \in X$, convergence in $(X, \mathcal{T})$ is equivalent to $\mathcal{P}-$convergence on X. Also, since $\mathcal{T}$ is generated by a countable family of seminorms, the topological space $(X, \mathcal{T})$ is metrizable. Therefore, the $\mathcal{P}-$sequential compactness mentioned in Lemma 3.2 is in fact $\mathcal{P}-$compactness, by which we mean compactness in $(X, \mathcal{T})$. In particular, X itself and all elements of $\mathcal{B}$ are compact sets in $(X, \mathcal{T})$.

Now put $\Sigma := \cap_{B \in \sigma} \sigma^{\mathsf{op}}(B)$. We claim that Σ is nonempty. Conversely, suppose

$$\varnothing \; = \; \Sigma \; = \; \bigcap_{B \in \sigma} \sigma^{\mathsf{op}}(B).$$

Then

$$\bigcup_{B \in \sigma} (X \setminus \sigma^{\mathsf{op}}(B)) \; = \; X \setminus \bigcap_{B \in \sigma} \sigma^{\mathsf{op}}(B) \; = \; X \setminus \Sigma \; = \; X$$

is an open cover of X. Since X is compact, there is a finite subset $\{B_1, ..., B_n\}$ of σ such that

$$X \; = \; \bigcup_{i=1}^{n} (X \setminus \sigma^{\mathsf{op}}(B_i)) \; = \; X \setminus \bigcap_{i=1}^{n} \sigma^{\mathsf{op}}(B_i)$$

so that $\cap_{i=1}^{n} \sigma^{\mathsf{op}}(B_i) = \varnothing$. But that is impossible since $\{\sigma^{\mathsf{op}}(B_1), ..., \sigma^{\mathsf{op}}(B_n)\}$ is a finite subchain of $\mathcal{B}$ consisting of nonempty sets that contain one another.

So $\Sigma \neq \varnothing$. Take a

$$T \; \in \; \Sigma \; = \; \bigcap_{B \in \sigma} \sigma^{\mathsf{op}}(B) \; \subseteq \; \sigma^{\mathsf{op}}(A).$$

¿From Lemma 3.3 a) we know that $\sigma^{\mathsf{op}}(B) \supseteq \sigma^{\mathsf{op}}(T)$ for every $B \in \sigma$. So $\sigma^{\mathsf{op}}(T) \in \mathcal{A}$ is an upper bound on the chain $\mathcal{B}$.

Now we can apply Zorn's lemma to $\mathcal{A}$ and get that our partially ordered set $(\mathcal{A}, \supseteq)$ has a maximal element, say $\sigma^{\mathsf{op}}(B')$ with some $B' \in \sigma^{\mathsf{op}}(A)$. Now pick any $C \in \sigma^{\mathsf{op}}(B')$. From Lemma 3.3 a) we get $\sigma^{\mathsf{op}}(B') \supseteq \sigma^{\mathsf{op}}(C)$. But the maximality of $\sigma^{\mathsf{op}}(B')$ means that $\sigma^{\mathsf{op}}(B') = \sigma^{\mathsf{op}}(C)$. So $C \in \sigma^{\mathsf{op}}(B') = \sigma^{\mathsf{op}}(C)$ is a self-similar limit operator of A. ■

Proposition 3.8 *If $C \in I + UM_{\$}(Y^\infty)$ is self-similar and subject to (FC) then C is surjective.*

Proof. Since C is self-similar, there is a sequence $h = (h(n))_{n\in\mathbb{Z}}$ with $|h(n)| \to \infty$ and $V_{-h(n)}CV_{h(n)} \xrightarrow{\mathcal{P}} C$ as $n \to \infty$. Suppose, for simplicity of our notations, that $h(n) \to +\infty$ and $h(n) > 0$ for all $n \in \mathbb{N}$. (The argument is completely analogous if $h(n) \to -\infty$, where we can suppose that $h(n) < 0$ for all $n \in \mathbb{N}$.)

For every $n \in \mathbb{N}$, define $C_n \in BDO(Y^\infty)$ by

$$(C_n u)(i) \;:=\; (CV_{-\alpha h(n)}u)(\beta), \qquad i = \alpha h(n) + \beta,\ \alpha \in \mathbb{Z},\ \beta \in \{0, ..., h(n) - 1\},$$

so that C_n commutes with $V_{h(n)}$.

We claim that this construction is such that Proposition 2.7 applies to C (in place of T) with $\mathcal{B} = \sigma^{\mathsf{op}}(C)$ and therefore proves that C is surjective. So it remains to check that conditions (a)–(e) of Proposition 2.7 are satisfied.

(a) It holds that $C_n \xrightarrow{\mathcal{P}} C$. This can be seen as follows. Fix an arbitrary $m \in \mathbb{N}$. For every $D \in L(Y^\infty)$, it is a simple consequence of the definition of the norm in Y^∞ that

$$\|D\| \;=\; \sup_{i\in\mathbb{Z}} \|P_{\{ih(n),\ldots,(i+1)h(n)-1\}}\, D\| \qquad \text{for all} \qquad n \in \mathbb{N}.$$

Therefore, for every $n \in \mathbb{N}$, it holds that $\|P_m(C - C_n)\| = \sup_{i\in\mathbb{Z}} \gamma(m, n, i)$ with

$$\gamma(m, n, i) \;:=\; \|P_{\{ih(n),\ldots,(i+1)h(n)-1\}} P_m(C - V_{ih(n)}CV_{-ih(n)})\|, \qquad i \in \mathbb{Z}.$$

But then it is clear that $\|P_m(C - C_n)\| \to 0$ as $n \to \infty$ since $\gamma(m, n, 0) = 0$,

$$\gamma(m, n, -1) = \|P_{\{-m,\ldots,-1\}}(C - V_{-h(n)}CV_{h(n)})\| \to 0 \qquad \text{as} \qquad n \to \infty$$

and $\gamma(m, n, i) = 0$ for all $i \in \mathbb{Z} \setminus \{0, -1\}$ as soon as $|h(n)| > m$.

Analogously, for every $n \in \mathbb{N}$, we have $\|(C - C_n)P_m\| = \sup_{i\in\mathbb{Z}} \delta(m,n,i)$ with

$$\delta(m,n,i) \; := \; \|P_{\{ih(n),\dots,(i+1)h(n)-1\}}(C - V_{ih(n)}CV_{-ih(n)})P_m\|, \qquad i \in \mathbb{Z}.$$

To see that $\sup_{i\in\mathbb{Z}} \delta(m,n,i) \to 0$ as $n \to \infty$, note that $\delta(m,n,0) = 0$,

$$\delta(m,n,-1) = \|P_{\{-h(n),\dots,-1\}}(C - V_{-h(n)}CV_{h(n)})P_m\| \to 0 \quad \text{as} \quad n \to \infty$$

and, for all $i \in \mathbb{Z} \setminus \{0,-1\}$,

$$\begin{aligned} \delta(m,n,i) \; &= \; \|P_{\{ih(n),\dots,(i+1)h(n)-1\}}(C - V_{ih(n)}CV_{-ih(n)})P_m\| \\ &\leq \; 2 \sup_{S,T} \|P_T C P_S\| \; \to \; 0 \end{aligned}$$

as $n \to \infty$ by [8, Theorem 1.42] and $C \in BDO(Y^\infty)$, where the supremum in the last expression is taken over all sets $S, T \subset \mathbb{Z}$ with $\operatorname{dist}(S,T) \geq h(n) - m$.

(b) By Corollary 6.8 in [1] and $C_n V_{h(n)} = V_{h(n)} C_n$ we get that C_n is surjective if injective.

(c) Clearly,

$$\bigcup_{n=1}^{\infty} \mathcal{M}(C_n - I) \; \subseteq \; \mathcal{M}(C - I)$$

is collectively compact in $L(U)$ since $C - I \in UM(Y^\infty)$.

(d) Let $(k(m)) \subseteq \mathbb{Z}$ be arbitrary and $(m(n)) \subseteq \mathbb{N}$ be monotonically increasing. Write each $k(m)$ as $\alpha(m)h(n(m)) + \beta(m)$ with $\alpha(m) \in \mathbb{Z}$ and $\beta(m) \in \{0, ..., h(n(m)) - 1\}$. Then

$$\begin{aligned} D_m \; &:= \; V_{-k(m)} C_{n(m)} V_{k(m)} \; = \; V_{-\beta(m)} V_{-h(n(m))}^{\alpha(m)} C_{n(m)} V_{h(n(m))}^{\alpha(m)} V_{\beta(m)} \\ &= \; V_{-\beta(m)} C_{n(m)} V_{\beta(m)} \end{aligned}$$

holds for each $m \in \mathbb{N}$. If $(\beta(m))_{m\in\mathbb{N}}$ has a bounded subsequence, then it even has a constant subsequence, of value $\gamma \in \mathbb{Z}$ say, and the corresponding subsequence of (D_m) converges to $V_{-\gamma}CV_\gamma$. Being a translate of $C \in \sigma^{\mathsf{op}}(C) = \mathcal{B}$, this operator is also in $\sigma^{\mathsf{op}}(C) = \mathcal{B}$. If $(\beta(m))_{m\in\mathbb{N}}$ goes to infinity, then, since C is rich, it has a subsequence for which the corresponding subsequence of (D_m) is $\mathcal{P}-$convergent to a limit operator of C, clearly also being an element of $\mathcal{B}$.

(e) All operators in $\mathcal{B} = \sigma^{\mathsf{op}}(C)$ are injective by our assumption that (FC) holds for C. ■

4 The essential spectrum of operators in the Wiener algebra

Our main result from §3 is only valid in Y^∞. By this we mean that there are examples of band-dominated operators all limit operators of which are injective on Y^p without all of them being surjective. But in this section we study a class of operators, the so-called Wiener algebra, which are bounded on all spaces Y^p with $p \in [1, \infty]$ and for which it is possible to profit from our Y^∞ results in the general Y^p setting.

Let $p \in [1, \infty]$ and recall that an operator $A \in L(Y)$ is called a band operator if it is induced by a banded matrix M. From the boundedness of A we get that every diagonal d_k of M is a bounded sequence of elements in $L(U)$. We then put

$$\|A\|_{\mathcal{W}} := \sum_{k=-\infty}^{+\infty} \|d_k\|_\infty = \sum_{k=-\infty}^{+\infty} \sup_{j \in \mathbb{Z}} \|a_{j+k,j}\|_{L(U)}$$

and denote by $\mathcal{W}$ the closure of $BO(Y)$ in the norm $\|.\|_{\mathcal{W}}$. The set $\mathcal{W}$, equipped with the norm $\|.\|_{\mathcal{W}}$, turns out to be a Banach algebra and is called *the Wiener algebra*.

It is easy to see that $\|A\|_{L(Y)} \le \|A\|_{\mathcal{W}}$ for all band operators A, so that the closure of $BO(Y)$ in the $\|.\|_{\mathcal{W}}$ norm, i.e. $\mathcal{W}$, is contained in the closure of $BO(Y)$ in the operator norm, i.e. $BDO(Y)$. Not only are operators $A \in \mathcal{W}$ bounded and band-dominated on all Y^p, $p \in [1, \infty]$ simultaneously, one can also show that if A is invertible on one of the spaces Y, its inverse A^{-1} is automatically in $\mathcal{W}$ again and therefore acts as the inverse of A on all spaces Y. Another important result is that all limit operators of $A \in \mathcal{W}$, with respect to any of the spaces Y, are also contained in $\mathcal{W}$ so that $\sigma^{\mathsf{op}}(A)$ is contained in $\mathcal{W}$ and does not depend on the space Y under consideration.

The following two results follow immediately from Corollary 6.43 and 6.44 in [1] and our Theorem 3.1. For illustrations of these results in the particular case of a discrete Schrödinger operator and for a class of integral operators on the axis, see the final two chapters of [1].

Proposition 4.1 *Suppose $A \in I + UM_\$(Y)$ is in the Wiener algebra $\mathcal{W}$ and A, if considered on Y^∞, has a preadjoint (acting on a predual space of Y^∞). Then the following statements are equivalent:*

(FC) *All limit operators of* A *are injective on* Y^∞*.*

(i) *All limit operators of* A *are invertible on* ***one*** *of the spaces* Y*.*

(ii) *All limit operators of* A *are invertible on* ***all*** *the spaces* Y *and*

$$\sup_{p\in[1,\infty]} \sup_{B\in\sigma^{\mathsf{op}}(A)} \|B^{-1}\|_{L(Y^p)} < \infty.$$

(iii) A *is invertible at infinity on* ***one*** *of the spaces* Y*.*

(iv) A *is invertible at infinity on* ***all*** *the spaces* Y*.*

(v) A *is Fredholm on* ***one*** *of the spaces* Y*.*

(vi) A *is Fredholm on* ***all*** *the spaces* Y*.*

Further, on every space Y *it holds that*

$$\mathrm{spec}_{\mathrm{ess}}(A) \;=\; \bigcup_{B\in\sigma^{\mathsf{op}}(A)} \mathrm{spec}\,(B) \;=\; \bigcup_{B\in\sigma^{\mathsf{op}}(A)} \mathrm{spec}^\infty_{\mathrm{point}}(B). \tag{3}$$

In equality (3) we denote by

$$\mathrm{spec}\,(B) \;=\; \{\lambda\in\mathbb{C} : \lambda I - B \text{ is not invertible on } Y\}$$

the (invertibility) spectrum of B, by

$$\mathrm{spec}_{\mathrm{ess}}(A) \;=\; \{\lambda\in\mathbb{C} : \lambda I - A \text{ is not Fredholm on } Y\}$$

the essential spectrum of A, and by

$$\mathrm{spec}^\infty_{\mathrm{point}}(B) \;=\; \{\lambda\in\mathbb{C} : \lambda I - B \text{ is not injective on } Y^\infty\}$$

the point spectrum of B on Y^∞.

Remark 4.2 a) In [13], the Fredholm index of A (see our Remark 2.5 a for the class of operators studied in [13]) is shown to be subject to

$$\mathrm{ind}\, A \;=\; \mathrm{ind}(PB_+P+Q) \;+\; \mathrm{ind}(QB_-Q+P) \tag{4}$$

for an arbitrary choice of operators $B_\pm\in\sigma^{\mathsf{op}}_\pm(A)$, respectively. The arguments there are made for operators on $\ell^p(\mathbb{Z},U)$ with $p\in(1,\infty)$ but inspection of the proofs shows that the result carries over to $p\in[1,\infty]$. The other condition in [13] is that the Banach space U has to have what Rabinovich and Roch call the *symmetric approximation property (sap)*. This means that there is a sequence $\Pi_1, \Pi_2, ...$ of finite rank projections on U such that $\Pi_n \to I$ and $\Pi_n^* \to I^*$ pointwise on U and its dual space U^*, respectively. Note that [13] extends results, in particular formula (4), from [14,17], where band-dominated operators on $\ell^p(\mathbb{Z}, \mathbb{C})$ are studied with $p = 2$ and $p \in (1, \infty)$, respectively.

b) In [9], Fredholmness and index of operators on $\ell^p(\mathbb{Z}^N, U)$ are studied for (almost) arbitrary Banach spaces U and arbitrary operators $A \in \mathcal{W}$. In particular, it is shown that if $A \in \mathcal{W}$ is Fredholm on one of the spaces $\ell^p(\mathbb{Z}^N, U)$ with $p \in [1, \infty]$, then it is Fredholm on all of these spaces and its index does not depend on p. The key observation here is that A has a Fredholm regularizer in the Wiener algebra that acts as its regularizer on all spaces $\ell^p(\mathbb{Z}^N, U)$. ■

In the particularly simple case of a finite-dimensional space U we know, by Lemma 2.6, that $I + UM_\$(Y) \cap \mathcal{W} = \mathcal{W}$ and that the predual of $Y^\infty = \ell^\infty(\mathbb{Z}, U)$ exists and is isomorphic to $\ell^1(\mathbb{Z}, U^*)$ and the preadjoint operator of $A \in L(Y^\infty)$ always exists and is induced by $[a_{ji}^*]$ on $\ell^1(\mathbb{Z}, U^*)$. Consequently, the conditions of Proposition 4.1 simplify, and we can even make a statement on the Fredholm index.

Corollary 4.3 *Suppose $A \in \mathcal{W}$ and U is finite-dimensional. Then statements (FC) and (i)–(vi) of Proposition 4.1 are all equivalent. Moreover, if A is subject to all these equivalent statements then the Fredholm index of A is the same on each space Y and is given by (4). Further, on every space Y, (3) holds.*

References

[1] S. N. Chandler-Wilde and M. Lindner: *Limit Operators, Collective Compactness, and the Spectral Theory of Infinite Matrices*, submitted for publication.

[2] S. N. Chandler-Wilde and B. Zhang: A generalised collectively compact operator theory with an application to second kind integral equations on unbounded domains, *J. Integral Equations Appl.* **14** (2002), 11–52.

[3] J. A. Favard: Sur les equations differentiales a coefficients presque periodiques, *Acta Math.* **51** (1927), 31–81.

[4] V. G. Kurbatov: *Functional Differential Operators and Equations*, Kluwer Academic Publishers, Dordrecht, Boston, London 1999.

[5] V. G. Kurbatov: On the invertibility of almost periodic operators, *Math USSR Sbornik* **67** (1990), 367–377.

[6] B. V. Lange and V. S. Rabinovich: On the Noether property of multidimensional discrete convolutions, *Mat. Zametki* **37** (1985), 407–421 (Russian original, English translation: *Math. Notes* **37** (1985), 228–237).

[7] M. Lindner: Limit Operators and Applications on the Space of Essentially Bounded Functions, *PhD Dissertation, TU Chemnitz,* 2003.

[8] M. Lindner: *Infinite Matrices and their Finite Sections: An Introduction to the Limit Operator Method*, Frontiers in Mathematics, Birkhäuser 2006.

[9] M. Lindner: Independence of Fredholmness and index on the underlying space for operators in the Wiener algebra, *submitted for publication.*

[10] E.M. Muhamadiev: On the invertibility of differential operators in the space of continuous functions bounded on the axis, *Soviet math Dokl.* **12** (1971), 49–52.

[11] E.M. Muhamadiev: On the invertibility of elliptic partial differential operators, *Soviet Math Dokl.* **13** (1972), 1122–1126.

[12] E. M. Muhamadiev: On normal solvability and Noether property of elliptic operators in spaces of functions on $\mathbb{R}^n$, Part I: *Zapiski nauchnih sem. LOMI* **110** (1981), 120–140 (Russian original, English translation: *J. Soviet Math.* **25** (1984), 884–901).

[13] V. S. Rabinovich and S. Roch: The Fredholm index of locally compact band-dominated operators on $L^p(\mathbb{R})$, *Integral Equations and Operator Theory*, **57** (2007), 263–281.

[14] V. S. Rabinovich, S. Roch and J. Roe: Fredholm indices of band-dominated operators, *Integral Equations Operator Theory* **49** (2004), 221–238.

[15] V. S. Rabinovich, S. Roch and B. Silbermann: Fredholm Theory and Finite Section Method for Band-dominated operators, *Integral Equations Operator Theory* **30** (1998), 452–495.

[16] V. S. Rabinovich, S. Roch and B. Silbermann: *Limit Operators and Their Applications in Operator Theory*, Birkhäuser 2004.

[17] S. Roch: Band-dominated operators on ℓ^p-spaces: Fredholm indices and finite sections, *Acta Sci. Math.* **70** (2004), 783–797.

[18] M. A. Shubin: The Favard-Muhamadiev theory and pseudodifferential operators, *Soviet Math Dokl*, **16** (1975), 1646–1649.

[19] M. A. Shubin: Almost periodic functions and partial differential operators, *Russian Math. Surv.* **33** (1978), 1–52.

[20] V. Volpert and A. Volpert: Normal solvability and properness of elliptic operators, Partial differential equations, *Amer. Math. Soc. Transl. Ser. 2* 2002, 193-237.

[21] V. Volpert and A. Volpert: Properness and topological degree for general elliptic operators, *Abstract Applied Analysis* 2003, 129-181.

[22] A. Volpert and V. Volpert: Normal solvability of general linear elliptic problems, *Abstract Applied Analysis* 2005, 733-756.