

# Electron fractionalization for two-dimensional Dirac fermions

Claudio Chamon,<sup>1</sup> Chang-Yu Hou,<sup>1</sup> Roman Jackiw,<sup>2</sup> Christopher Mudry,<sup>3</sup> So-Young Pi,<sup>1</sup> and Gordon Semenoff<sup>4</sup>

<sup>1</sup> *Physics Department, Boston University, Boston, MA 02215, USA*

<sup>2</sup> *Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

<sup>3</sup> *Condensed Matter Theory Group, Paul Scherrer Institut, CH-5232 Villigen PSI, Switzerland*

<sup>4</sup> *Department of Physics and Astronomy, University of British Columbia, Vancouver, BC, Canada V6T 1Z1*

(Dated: December 14, 2007)

Fermion-number fractionalization without breaking of time-reversal symmetry was recently demonstrated for a field theory in  $(2+1)$ -dimensional space and time that describes the couplings between massive Dirac fermions, a complex-valued Higgs field carrying an axial gauge charge of 2, and a  $U(1)$  axial gauge field. Charge fractionalization occurs whenever the Higgs field either supports vortices by itself, or when these vortices are accompanied by half-vortices in the axial gauge field. The fractional charge is computed by three different techniques. A formula for the fractional charge is given as a function of a parameter in the Dirac Hamiltonian that breaks the spectral energy-reflection symmetry. In the presence of a charge  $\pm 1$  vortex in the Higgs field only, the fractional charge varies continuously and thus can take irrational values. The simultaneous presence of a half-vortex in the axial gauge field and a charge  $\pm 1$  vortex in the Higgs field re-rationalizes the fractional charge to the value  $1/2$ .

## I. INTRODUCTION

The concept of fractional charge emerged from quantum field theory in 1976 when Jackiw and Rebbi showed that Bose fields can induce a fractional fermion number  $1/2$  for the relativistic fermions to which they couple.<sup>1</sup> The proper conditions for this mechanism of fractionalization are the following. First, the very notion of a fractional charge demands that the fermion number is a good quantum number. Second, the Bose fields must trigger the spontaneous breaking of a symmetry that opens up a gap in the single-particle fermionic spectrum within the Born-Oppenheimer approximation. Third, the Bose fields must support local topological defects that nucleate single-particle fermionic bound states in the close vicinity to the defects. Fourth, this many-body quantum state is a finite energy eigenstate.

The first requirement rules out mean-field descriptions of superconductors that can otherwise satisfy the remaining requirements.<sup>2,3,4</sup> The last requirement implies that the fractionalization of the fermionic charge is a long-distance and low-energy property of the many-body system, while the second and third ones insure a degree of robustness against local perturbations. This, in turn, suggests that the lessons learned from the quantum field theories in Ref. 1 could apply more generally to microscopic models encountered in solid state physics, thereby opening the possibility of a “table-top” measurement of the fractional charge.

In fact, the work of Su, Schrieffer, and Heeger implies that the one-dimensional example of Ref. 1 can be thought of as an effective field theory that captures the relevant interactions between phonons and electrons in polyacetylene.<sup>5,6</sup> Excitations with exotic quantum numbers (in relation to the fundamental electron constituents of the system), such as neutral objects carrying spin  $1/2$  or charge  $\pm 1$  objects carrying zero spin, localize around a domain wall in the dimerization pattern of polyacetylene at the cost of a finite energy. Subsequent to this work, it was shown that exotic fermionic quantum numbers in one-dimension are not restricted to fractional values,<sup>7,8,9,10</sup> but can be tuned continuously by a small breaking of an energy-reflection symmetry assumed in Refs. 1 and 5, and defined below.

With the discovery of the fractional quantum Hall effect, a different paradigm for charge fractionalization, one in which spontaneous symmetry breaking plays no role, was proposed by Laughlin for two-dimensional systems with strong breaking of time-reversal symmetry.<sup>11,12</sup> Central to this paradigm is the notion of topological order, a global property that characterizes an otherwise featureless incompressible liquid state of matter by the finite degeneracy of the ground state if the system is defined on a surface of non-trivial topology, with the degeneracy depending on the genus of the surface.<sup>13</sup> The fractional charge is intimately connected to this ground-state degeneracy, leaving no room for a continuously varying fractional charge and, in particular, for an irrational charge. Since then, the preferred route towards charge fractionalization without time-reversal symmetry in two and higher dimensions has occulted any mechanism based on spontaneous symmetry breaking, presumably because it is believed that the energy cost for fractional charges is prohibitive in all but one dimension.

However, as a matter of principle, this need not be so as was already shown by Jackiw and Rebbi in three-dimensional space when coupling Yang-Mills fields through the minimal coupling to Higgs fields and to Dirac fermions. Quantization of the Dirac fermions in the static background of a t’Hooft-Polyakov monopole nucleates a fermionic bound states with the fractional charge  $1/2$  at a finite cost in energy.<sup>1</sup>

Of course, one might object that this three-dimensional example of charge fractionalization is unlikely to be realized

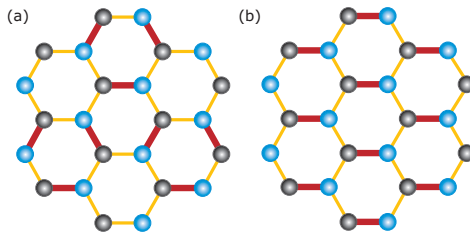


FIG. 1: (color online). The honeycomb lattice is shown in (a) and (b). The honeycomb lattice has 2 interpenetrating Bravais sublattices colored in blue and black, respectively. The electronic hopping amplitude is enhanced on thick red bonds while it is reduced on the thin yellow bonds relative to the magnitude  $t$  of the nearest-neighbor hopping amplitude. The so-called Kékulé pattern dimerization pattern in (a) opens an energy gap for the single-particle fermionic levels and it maps to the complex-valued Higgs field of Ref. 20 in the continuum limit. A dimerization pattern that shifts the relative separation of the Dirac points is shown in (b). In the continuum limit, it maps to the axial vector potential introduced in Ref. 21.

on the energy scale of the electron volt that governs solid state physics, a prerequisite for a table-top measurement of charge fractionalization.<sup>14</sup> We do not know of a realistic three-dimensional model for band electrons coupled to bosonic collective modes that mimics Dirac fermions and Higgs fields coupled with each others and coupled minimally to Yang-Mills in the continuum limit. In two and three dimensions band-theory generically predicts an insulating or a metallic state of matter. In one-dimension the Fermi surface is generically realized by an even number of discrete points, thus providing the low-energy and long-wave-length limit of the tight-binding model with a Dirac structure for free.

Semimetals, the most famous example of which is graphite, are exceptions to the hegemony of the band-insulating and of the metallic states of matter. Graphite is made of sheets of graphene, a honeycomb lattice made of carbon ions bound through  $sp^2$  orbitals, and where the fourth valence electron of each atom lazily revels predominantly between planar nearest-neighbor sites. The Fermi surface at half-filling for an isolated graphene sheet is made of two isolated points.<sup>15</sup> The excitation spectrum around these two Fermi points endows the band electrons with a four-component Dirac structure owing to the Nielsen-Ninomiya theorem.<sup>16</sup>

Although this example of fermion-doubling is often viewed as a curse for the realization of quantum anomalies,<sup>17,18,19</sup> it is this very property that opens the door to charge fractionalization without the breaking of time-reversal symmetry through spontaneous symmetry breaking, as shown by Hou, Chamon, and Mudry.<sup>20</sup> The real-valued static fluctuations depicted in Fig. 1(a) about the uniform nearest-neighbor hopping amplitudes of graphene are, in the continuum limit, represented by a complex-valued Higgs field that interacts with the four-component Dirac fermions. In the Born-Oppenheimer approximation, a constant value of this complex-valued Higgs field breaks spontaneously an effective axial  $U(1)$  symmetry of the continuum limit and opens up a gap in the single-particle fermion spectrum. If the phase of this complex-valued Higgs field is defective in that it carries a vortex, it nucleates single-particle mid-gap states that carry the fractional charge  $\pm 1/2$  per state. The energy cost is not finite, however. In the continuum approximation, it grows logarithmically with the separation between the vortices. On the lattice, it even grows linearly with the vortex separation if the wave vector of the fluctuation of the hopping amplitude is commensurate with the reciprocal lattice.

Jackiw and Pi showed that the energy cost of a vortex in the complex-valued Higgs field can be made finite if the complex-valued Higgs field and the Dirac fermions couple minimally to two real-valued Bose fields that realize the vector components of an axial gauge field and if this axial vector gauge field also supports vortices.<sup>21</sup> A honeycomb-lattice regularization of an axial gauge field without a vortex is shown in Fig. 1(b).<sup>22</sup> Alternative realizations of an axial vector potential also arise when the graphene sheet is curved<sup>23,24,25</sup> or wrapped into fullerenes,<sup>26,27</sup> into nanotubes,<sup>28,29</sup> and about a cone.<sup>30,31</sup>

Charge fractionalization in one dimension can be tuned continuously by breaking an energy-reflection symmetry, which is defined below. This property survives in two dimensions and gives a mechanism for charge fractionalization that is fundamentally different (and thus potentially observable) from the mechanism for charge fractionalization that relies on topological order. By relaxing the condition of a finite energy to that of a logarithmically diverging energy, it was argued in Ref. 20 that a small staggered chemical potential that distinguishes carbon ions sitting on the now nonequivalent triangular sublattices of the honeycomb lattice can make the fractional charge irrational. This irrational charge was calculated analytically in the continuum limit and numerically for a lattice regularization in Ref. 22. Remarkably, it was also found that the condition for finite energy in the continuum limit, i.e., the presence of a vortex in the axial vector potential, removed any dependence of the fractional charge on the staggered chemical potential.

The purpose of this paper is to give three detailed and alternative derivations of the fractional charge that supplement the derivation from Ref. 22. Graphene is of course not the only road to a semimetal in two dimensions. Threading the

elementary plaquettes of a square lattice with half a flux quantum<sup>32</sup> also realizes two nonequivalent Dirac points at half-filling. The Higgs field is then realized by a columnar pattern of dimerization whereas the axial vector gauge field is realized by a staggered pattern of dimerization.<sup>22</sup> It was shown numerically in Ref. 33 that the  $\mathbb{Z}_4$  vortex defined by the four possible columnar patterns occupying the four quadrants of the square lattice pins the fractional charge  $\pm 1/2$  at the site where the four columnar patterns meet. The  $\mathbb{Z}_4$  vortex is a discontinuous version of the vortices studied numerically in Ref. 22. The fact that different lattice regularization of vortices carry the same fractional charge illustrates the fact that the fractional charge is independent of the short-distance regularization. This property will become obvious in the analytical calculations of the fractional charge that we are going to present.

The paper is organized as follows. The quantum field theory is defined in Section II. The charge induced by the vortices in the Higgs or axial gauge fields is related to the spectral asymmetry in Section III. The spectral asymmetry is computed in Section IV. The fractional charge is computed a second time by a perturbative expansion of the Dirac propagator in Section V. Finally, the fractional charge is computed on the basis of symmetry arguments in Section VI. A summary comprises the last Section VII

## II. DEFINITIONS

In recent Letters,<sup>20,21,22</sup> we have developed the theory of charge fractionalization in planar models with topological defects encoded by vortices. The models are elaborations on graphene, with dynamics linearized around two Dirac points (the two inequivalent points in the first Brillouin zone at which the conduction and valence bonds of graphene meet). In a familiar fashion, the Schrödinger equation for the Bloch states at low energies and long wave length measured relative to the Dirac points takes a Dirac-like  $4 \times 4$  matrix form, for a four-component “spinor”, which interacts with further scalar and gauge fields. The scalar field and the gauge fields are induced by fluctuations in the hopping amplitudes of the underlying microscopic tight-binding model. In this Section, we start with definitions.

In second quantization, the planar Hamilton density reads

$$\mathcal{H} = \psi^\dagger [\boldsymbol{\alpha} \cdot (\mathbf{p} - \gamma_5 \mathbf{A}_5) + \beta (\varphi_1 - i\gamma_5 \varphi_2) + R\mu] \psi \equiv \psi^\dagger H \psi. \quad (2.1)$$

Here,  $\psi^\dagger$  and  $\psi$  are creation and annihilation operators for four-components Dirac fermions, respectively,  $\mathbf{p} = -i(\partial_x, \partial_y)$ ;  $\mathbf{A}_5$  is an axial vector gauge potential  $(A_5^x, A_5^y)$ ;  $\varphi_1$  and  $\varphi_2$  are the real and imaginary parts of a complex scalar field  $\varphi = \varphi_1 + i\varphi_2$ ; and  $\mu$  is a field that acts like a staggered chemical potential – the staggering is governed by the matrix  $R$ . All fields depend on the three-vector  $x^\mu = (t, \mathbf{r}) = (t, x, y)$ . The matrices in  $H$  are conventional  $4 \times 4$  Dirac matrices:

$$\boldsymbol{\alpha} = (\alpha^x, \alpha^y) \equiv \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}, \quad \beta \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 \equiv -i\alpha^x \alpha^y \alpha^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.2a)$$

where the “third”  $\alpha$ -matrix,

$$\alpha^z \equiv \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad (2.2b)$$

participates in the definition of  $\gamma_5$  and also coincides with the matrix  $R \equiv \alpha^z$ . (The matrices  $\sigma_{1,2,3}$  are the standard Pauli matrices.)

The Lagrange density corresponding to Eq. (2.1),

$$\mathcal{L} = i\psi^\dagger \partial_t \psi - \mathcal{H}, \quad (2.3a)$$

is presented in covariant notation as

$$\mathcal{L} = \bar{\psi} [\gamma^\nu (i\partial_\nu + \gamma_5 A_{5\nu}) - (\varphi_1 - i\gamma_5 \varphi_2) - \gamma^3 \mu] \psi, \quad (2.3b)$$

with

$$\gamma^0 \equiv \beta, \quad \bar{\psi} \equiv \psi^\dagger \gamma^0, \quad \boldsymbol{\gamma} \equiv \beta \boldsymbol{\alpha} = \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \gamma^3 \equiv \beta R = \beta \alpha^z = \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}. \quad (2.3c)$$

[In Eq. (2.1), we set the axial scalar gauge potential  $A_5^0$  to zero.]

Two gauge transformations leave the model unchanged. There is the local axial gauge symmetry

$$\psi \rightarrow e^{i\omega \gamma_5} \psi, \quad A_{5\nu} \rightarrow A_{5\nu} + \partial_\nu \omega, \quad \varphi \rightarrow e^{2i\omega} \varphi, \quad \mu \rightarrow \mu, \quad (2.4)$$

where  $\omega$  is a real-valued field and the index  $\nu = t, x, y$ . Also, there is the global phase symmetry,

$$\psi \rightarrow e^{i\varpi}\psi, \quad A_{5\nu} \rightarrow A_{5\nu} \quad \varphi \rightarrow \varphi, \quad \mu \rightarrow \mu, \quad (2.5)$$

where  $\varpi$  is a real-valued number, which acts on the four components of the spinors. The latter leads to the conserved fermion (charge) number current.

$$J^\nu \equiv \bar{\psi}\gamma^\nu\psi = (\rho, \mathbf{j}) = (\psi^\dagger\psi, \psi^\dagger\boldsymbol{\alpha}\psi), \quad \partial_\nu J^\nu = 0. \quad (2.6)$$

We shall show that the charge

$$Q = \int d^2r \rho(\mathbf{r}) \quad (2.7)$$

fractionalizes when the background bose fields are topologically nontrivial.

The model possesses the usual discrete symmetries under the parity transformation  $\mathcal{P}$  defined by

$$\mathcal{P} : \begin{cases} (t, x, y) & \rightarrow (t, -x, y), \\ \psi(t, x, y) & \rightarrow i\gamma^3\gamma_1\psi(t, -x, y), \\ A_5^{t,y}(t, x, y) & \rightarrow A_5^{t,y}(t, -x, y), \\ A_5^x(t, x, y) & \rightarrow -A_5^x(t, -x, y), \\ \varphi(t, x, y) & \rightarrow \varphi(t, -x, y), \\ \mu(t, x, y) & \rightarrow -\mu(t, -x, y), \end{cases} \quad (2.8)$$

the charge conjugate transformation  $\mathcal{C}$  defined by

$$\mathcal{C} : \begin{cases} \psi_i & \rightarrow \gamma_{ij}^1\bar{\psi}_j, \\ A_{5\nu} & \rightarrow -A_{5\nu}, \\ \varphi & \rightarrow \varphi^*, \\ \mu & \rightarrow \mu, \end{cases} \quad (2.9)$$

and the time-reversal transformation  $\mathcal{T}$  defined by

$$\mathcal{T} : \begin{cases} (t, x, y) & \rightarrow (-t, x, y), \\ \psi(t, x, y) & \rightarrow \gamma^1\gamma^5\psi^\dagger(-t, x, y), \\ A_5^x(t, x, y) & \rightarrow A_5^x(-t, x, y), \\ \varphi(t, x, y) & \rightarrow \varphi^*(-t, x, y), \\ \mu(t, x, y) & \rightarrow \mu(-t, x, y), \end{cases} \quad (2.10)$$

where one should remember that  $\mathcal{T}$  is antiunitary so that complex conjugation of coefficients is implied.

The theory possesses another discrete symmetry

$$\begin{cases} \psi & \rightarrow i\gamma^3\gamma^5\psi, \\ A_{5\nu} & \rightarrow -A_{5\nu}, \\ \varphi & \rightarrow \varphi^*, \\ \mu & \rightarrow -\mu. \end{cases} \quad (2.11)$$

In the lattice (the honeycomb lattice relevant to graphene, for example), the definition of parity depends on the axis used for the reflection; the transformation in Eq. (2.8) corresponds to a reflection with respect to an axis that cuts through the bonds of the honeycomb lattice.

When the staggered chemical potential  $\mu$  is dropped, i.e., the last term in the square brackets of Eq. (2.1) or Eq. (2.3b) is absent, the matrix  $R$  anticommutes with the remaining matrices in the single-particle Hamiltonian  $H$  of Eq. (2.1). Therefore,  $R$  maps positive energy eigenfunctions  $\Psi_E$  to negative energy eigenfunctions  $\Psi_{-E}$  and vice-versa,

$$H|_{\mu=0}\Psi_E = E\Psi_E, \quad R\Psi_E = \Psi_{-E}. \quad (2.12)$$

We call this an ‘‘energy-reflection symmetry’’.

We shall examine the Dirac theory with a specific vortex configuration for the Bose field  $\varphi$ , taken as a static background, and with another specific vortex configuration for the axial gauge field  $A_5^x$ , also taken as a static background. The polar decomposition of the scalar field  $\varphi$  is

$$\varphi(\mathbf{r}) = \phi(r) e^{in\theta}, \quad r = \sqrt{x^2 + y^2}, \quad \theta = \arctan \frac{y}{x}, \quad (2.13)$$

where the magnitude  $\phi$  of  $\varphi$  vanishes at the origin  $\phi(r=0)$  and tends to a nonvanishing  $\phi(\infty)$  for large  $r$ . The integer  $n$  measures the vorticity encoded by the singular nature of the phase of the complex field  $\varphi$  at the origin. The axial gauge potential vanishes in the time component

$$A_5^0(\mathbf{r}) = 0, \quad (2.14a)$$

while the spatial component reads

$$A_5^i(\mathbf{r}) = -n\epsilon_{ij}\frac{r^j}{r^2}a_5(r), \quad (2.14b)$$

where  $a_5(r)$  vanishes at the origin and tends to  $1/2$  at large  $r$ . The line integral over Eq. (2.14b) along any closed curve that encircles once the origin yields the same number, proportional to the vorticity  $n$ . Finally, the chemical potential  $\mu$ , also taken as a static background, is without topological structure and achieves a nonvanishing value  $\mu(\infty)$  at infinity. We shall take  $\mu$  to depend only on  $r$ , but it could also be constant.

In the absence of the staggered chemical potential, the Dirac equation possesses  $|n|$  zero-energy, normalizable solutions. These are the mid-gap states, eigenstates of  $R$ . Mostly, we consider the  $n = -1$  case, with a single mid-gap state,  $\Psi_0$ , which remains bound even in the presence of the axial vector potential; turning on the axial vector potential changes the wave function profile, but the zero eigenvalue remains. We assume that there are no other bound states. When the staggered chemical potential is present, but never very large, the mid-gap state migrates to a shifted eigenvalue; however it still remains isolated in the gap.

### III. QUANTUM MECHANICAL ANALYSIS

The following argument shows that without the staggered chemical potential  $\mu$  the charge is  $-1/2$  when there is a single normalizable mid-gap state  $\Psi_0$  that is unoccupied. (When this mid-gap state is occupied, the charge is  $-1/2 + 1 = +1/2$ .) The charge density arises from filling the negative energy continuum states of the Dirac equation,

$$\begin{aligned} \rho(\mathbf{r}) &= \int_{-\infty}^0 dE \left[ \Psi_E^\dagger(\mathbf{r})\Psi_E(\mathbf{r}) - \Upsilon_E^\dagger(\mathbf{r})\Upsilon_E(\mathbf{r}) \right] \\ &= \frac{1}{2} \int_{-\infty}^{\infty} dE \left[ \Psi_E^\dagger(\mathbf{r})\Psi_E(\mathbf{r}) - \Upsilon_E^\dagger(\mathbf{r})\Upsilon_E(\mathbf{r}) \right], \end{aligned} \quad (3.1)$$

where the second equality follows from the first due to the energy-reflection symmetry present in the problem at  $\mu = 0$ . The quantity  $\Upsilon_E^\dagger\Upsilon_E$  is constructed from reference states which solve a Dirac equation with a topologically trivial background and also possess the energy-reflection symmetry. In other words, the topologically determined charges that we compute are measured relative to a reference charge of a system with a topologically trivial background and possessing the energy-reflection symmetry. This procedure is needed to remove infinities. The reference wave functions  $\Upsilon_E$  form a complete set. The continuum wave functions  $\Psi_E$  in the presence of the vortex – we call them the vortex states – are not complete; the mid-gap state is missing

$$\delta(\mathbf{r} - \mathbf{r}') = \int_{-\infty}^{\infty} dE \Upsilon_E^\dagger(\mathbf{r})\Upsilon_E(\mathbf{r}') = \int_{-\infty}^{\infty} dE \Psi_E^\dagger(\mathbf{r})\Psi_E(\mathbf{r}') + \Psi_0^\dagger(\mathbf{r})\Psi_0(\mathbf{r}'). \quad (3.2)$$

It therefore follows from combining Eq. (3.1) with Eq. (3.2) that

$$\rho(\mathbf{r}) = -\frac{1}{2}\Psi_0^\dagger(\mathbf{r})\Psi_0(\mathbf{r}) \quad (3.3)$$

and

$$Q = \int d^2r \rho(\mathbf{r}) = -\frac{1}{2}. \quad (3.4)$$

In the presence of the staggered chemical potential  $\mu$ , the energy-reflection symmetry is no longer available to pass from the first to the second equality of Eq. (3.1). However, we may proceed as follows. We suppose that the reference

states still possess the energy-reflection symmetry, so in Eq. (3.1) we may still use this symmetry for them.<sup>34</sup>

$$\int_{-\infty}^0 dE \Upsilon_E^\dagger \Upsilon_E = \frac{1}{2} \int_{-\infty}^{\infty} dE \Upsilon_E^\dagger \Upsilon_E = \frac{1}{2} \int_{-\infty}^{\infty} dE \Psi_E^\dagger \Psi_E + \frac{1}{2} \Psi_b^\dagger \Psi_b \quad (3.5)$$

The last equality is again the statement of completeness of the continuum reference states and the continuum vortex states supplemented by the isolated bound state  $\Psi_b$ , which is no longer at zero energy but has migrated to some other value in the gap between the continuum states. Using Eq. (3.5) in the first equality of Eq. (3.1) leaves

$$\rho(\mathbf{r}) = -\frac{1}{2} \Psi_b^\dagger(\mathbf{r}) \Psi_b(\mathbf{r}) - \frac{1}{2} \int_{-\infty}^{\infty} dE \text{sign}(E) \Psi_E^\dagger(\mathbf{r}) \Psi_E(\mathbf{r}). \quad (3.6)$$

It remains to evaluate the remaining integral, which is recognized as the “ $\eta$ -invariant”, also called “spectral asymmetry”. Note that with energy-reflection symmetry the integral vanishes, leading to the previous result (3.3). In the above derivations, it is assumed that the “vacuum” is defined with the mid gap state  $\Psi_0$  unoccupied, and furthermore that the migrated state  $\Phi_b$  has positive energy so that it remains unoccupied in the definition of the vacuum. If the mid-gap state is occupied and/or the migrated state has negative energy, there occurs a sign change in Eq. (3.3) that affects to the first term of Eq. (3.6).

#### IV. SPECTRAL ASYMMETRY AND FRACTIONAL CHARGE

We begin by putting the Hamiltonian Eq. (2.1) in a more convenient form. This is done with the following unitary transformation

$$H \rightarrow \hat{H} = THT^{-1}, \quad (4.1a)$$

where

$$T \equiv \begin{pmatrix} i\sigma_- & \sigma_+ \\ -i\sigma_+ & \sigma_- \end{pmatrix}, \quad T^{-1} = T^\dagger, \quad \sigma_\pm \equiv \frac{1}{2}(1 \pm \sigma_3). \quad (4.1b)$$

The result for  $\hat{H}$  is

$$\hat{H} = \begin{pmatrix} -\mu & \mathcal{D} \\ \mathcal{D}^\dagger & \mu \end{pmatrix} \quad (4.2a)$$

where the differential operator  $\mathcal{D}$  and its adjoint  $\mathcal{D}^\dagger$  are given by

$$\mathcal{D} = i\sigma^i(\partial_i + \epsilon_{ij}A_5^j) + i\varphi_1 + \sigma_3\varphi_2, \quad (4.2b)$$

$$\mathcal{D}^\dagger = i\sigma^i(\partial_i - \epsilon_{ij}A_5^j) - i\varphi_1 + \sigma_3\varphi_2. \quad (4.2c)$$

With the factorization of the time dependence  $(u, v) = e^{-iEt}(u_E, v_E)$ , the stationary Dirac equation reads

$$\begin{pmatrix} -\mu & \mathcal{D} \\ \mathcal{D}^\dagger & \mu \end{pmatrix} \begin{pmatrix} u_E \\ v_E \end{pmatrix} = E \begin{pmatrix} u_E \\ v_E \end{pmatrix} \quad (4.3a)$$

or in terms of components,

$$\mathcal{D}v_E = (E + \mu)u_E, \quad (4.3b)$$

$$\mathcal{D}^\dagger u_E = (E - \mu)v_E. \quad (4.3c)$$

Generally,  $\mu$  can be a function on space-time. In the remainder of this Section, we shall set it to be a positive constant; the other background fields,  $\varphi(\mathbf{r})$  and  $\mathbf{A}_5(\mathbf{r})$  are position dependent and static, with asymptotes quoted in Eqs. (2.14) and (2.13).

### A. Zero-mode solutions

It is convenient to begin by considering two special cases, particular solutions of Eqs. (4.3b) and (4.3c) where either  $E = \mu$  or  $E = -\mu$ . These solutions would become zero modes of the Hamiltonian when  $\mu = 0$  and they play a special role even when  $\mu \neq 0$ .

Let us begin with the case where the energy eigenvalue  $E = \mu$ . Then, from Eqs. (4.3b) and (4.3c), it follows that

$$\mathcal{D}^\dagger u_\mu(\mathbf{r}) = 0, \quad (4.4a)$$

$$u_\mu(\mathbf{r}) = \frac{1}{2\mu} \mathcal{D}v_\mu(\mathbf{r}). \quad (4.4b)$$

First we observe that if  $v_\mu(\mathbf{r})$  were identically zero,  $u_\mu(\mathbf{r})$  would also vanish and there is no solution. So we assume that  $v_\mu(\mathbf{r}) \neq 0$ . Then, operating with  $\mathcal{D}^\dagger$  on Eq. (4.4b) and using Eq. (4.4a) yields  $\mathcal{D}^\dagger \mathcal{D}v_\mu(\mathbf{r}) = 0$ . The following argument implies that  $\mathcal{D}v_\mu(\mathbf{r}) = 0$ . Consider

$$0 = \int d^2r v_\mu^\dagger(\mathbf{r}) \mathcal{D}^\dagger \mathcal{D}v_\mu(\mathbf{r}) = \int d^2r |\mathcal{D}v_\mu(\mathbf{r})|^2. \quad (4.5)$$

Here, we are assuming that the spinor  $v_\mu(\mathbf{r})$  obeys boundary conditions so that the differential operator  $\mathcal{D}^\dagger$  is indeed the adjoint of  $\mathcal{D}$ , i.e., surface terms produced by partial integrations in the intermediate steps in (4.5) vanish. Since the last integral vanishes, its positive semi-definite integrand must also vanish and we conclude that

$$\mathcal{D}v_\mu(\mathbf{r}) = 0 \quad (4.6)$$

while Eq.(4.4a) implies that  $u_\mu(\mathbf{r}) = 0$ .

Thus, we find that, when Eq.(4.6) possesses a normalizable solution, there exists a positive energy bound state with  $E = \mu$ ,

$$\begin{pmatrix} -\mu & \mathcal{D} \\ \mathcal{D}^\dagger & \mu \end{pmatrix} \begin{pmatrix} 0 \\ v_\mu(\mathbf{r}) \end{pmatrix} = \mu \begin{pmatrix} 0 \\ v_\mu(\mathbf{r}) \end{pmatrix}, \quad \int d^2r |v_\mu(\mathbf{r})|^2 = 1. \quad (4.7)$$

Similar reasoning establishes the occurrence of a negative energy bound state with  $E = -\mu$  when there exists a normalizable solution of the equation  $\mathcal{D}^\dagger u_{-\mu}(\mathbf{r}) = 0$ ,

$$\begin{pmatrix} -\mu & \mathcal{D} \\ \mathcal{D}^\dagger & \mu \end{pmatrix} \begin{pmatrix} u_{-\mu}(\mathbf{r}) \\ 0 \end{pmatrix} = -\mu \begin{pmatrix} u_{-\mu}(\mathbf{r}) \\ 0 \end{pmatrix}, \quad \int d^2r |u_{-\mu}(\mathbf{r})|^2 = 1. \quad (4.8)$$

The existence of solutions of the equations  $\mathcal{D}^\dagger u(\mathbf{r}) = 0$  and  $\mathcal{D}v(\mathbf{r}) = 0$  and the number of solutions of each kind are determined by the topological properties of the background fields,  $\mathbf{A}_5(\mathbf{r})$  and  $\varphi(\mathbf{r})$ . An index theorem implies

$$\text{Index}(H) = \dim \ker \mathcal{D} - \dim \ker \mathcal{D}^\dagger = n \quad (4.9)$$

where  $\ker$  denotes kernel and  $n$  is the vorticity defined in Eq. (2.13). The implication of this index theorem was seen explicitly in Ref. 3 where solutions of  $\mathcal{D}^\dagger u(\mathbf{r}) = 0$  and  $\mathcal{D}v(\mathbf{r}) = 0$  were constructed for the case of the highly symmetric profile of the vector and Higgs fields given in Eqs. (2.14) and (2.13). It was found that, for a given vorticity, either one or the other of these equations has solutions, not both. Which equation had solutions depended on the sign of  $n$ . It was argued that the number of solutions is given by  $|n|$  and, when  $n = \pm 1$ , the solutions were found explicitly.

The proof of the index theorem (4.9) was given in Ref. 35. The index theorem counts the difference indicated in (4.9). It proves that this is so, independent of the details of the profile of the vector and Higgs fields but with the assumption that, whatever they are, they are obtained by smooth deformations of the symmetric configurations in Eqs. (2.13) and (2.14). Ref. 35 also presented a proof of a vanishing theorem, that *either*  $\dim \ker \mathcal{D} = 0$  *or*  $\dim \ker \mathcal{D}^\dagger = 0$ . Combined with the index theorem, it implies that

$$n > 0 : \quad \dim \ker \mathcal{D}^\dagger = 0, \quad \dim \ker \mathcal{D} = n, \quad (4.10a)$$

$$n < 0 : \quad \dim \ker \mathcal{D}^\dagger = |n|, \quad \dim \ker \mathcal{D} = 0. \quad (4.10b)$$

A computation of the spectral asymmetry of the Hamiltonian in a spirit similar to the one that will be given in the remainder of this Section was originally presented in Ref. 36. Equation (6.29) of that paper contains a result for the spectral asymmetry from which the index can be deduced by taking the parameter  $\kappa$  (our  $\mu$ ) to zero and which agrees with Eq. (4.9) above. The general formula for the spectral asymmetry in their equation (6.29) also agrees with what we shall find in the following.

## B. Non-zero mode spectrum

Now, we shall look for eigenspinors of the Dirac Hamiltonian which do not have eigenvalues  $E = \pm\mu$ . From Eq. (4.3c) we can solve for the lower components of the spinor in terms of the upper components

$$v_E(\mathbf{r}) = \frac{1}{E - \mu} \mathcal{D}^\dagger u_E(\mathbf{r}). \quad (4.11)$$

Then, using (4.3b) we see that the upper components must obey the Schrödinger equation  $\mathcal{D}\mathcal{D}^\dagger u_E(\mathbf{r}) = (E^2 - \mu^2) u_E(\mathbf{r})$ . To find solutions, we begin with the eigenvalue problem

$$\mathcal{D}\mathcal{D}^\dagger u_\lambda(\mathbf{r}) = \lambda u_\lambda(\mathbf{r}), \quad \lambda \geq 0. \quad (4.12)$$

We assume that we can find a complete orthonormal set of solutions of this equation,

$$\int d^2r u_\lambda^\dagger(\mathbf{r}) u_{\lambda'}(\mathbf{r}) = \delta_{\lambda\lambda'}, \quad \sum_\lambda u_\lambda(\mathbf{r}) u_\lambda^\dagger(\mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \mathbb{1}. \quad (4.13)$$

Generally, the spectrum will contain both bound and continuum states. For continuous spectra, the right-hand side of the first equation above should be replaced with a Dirac delta function and the summation on the left of the second equation should be replaced by an integral. We shall assume that these replacements, where needed, are understood in Eq. (4.13). We can use the two-component spinor  $u_\lambda(\mathbf{r})$  to construct a normalized four-component spinor which solves the stationary Dirac equation. For each eigenvalue  $\lambda$ ,

$$\Psi_E(\mathbf{r}) = \left( \frac{\sqrt{\lambda + \mu^2} - \mu}{2\sqrt{\lambda + \mu^2}} \right)^{\frac{1}{2}} \begin{pmatrix} u_\lambda(\mathbf{r}) \\ \frac{\mathcal{D}^\dagger}{\sqrt{\lambda + \mu^2 - \mu}} u_\lambda(\mathbf{r}) \end{pmatrix}, \quad E = \sqrt{\lambda + \mu^2}, \quad (4.14a)$$

$$\Psi_E(\mathbf{r}) = \left( \frac{\sqrt{\lambda + \mu^2} + \mu}{2\sqrt{\lambda + \mu^2}} \right)^{\frac{1}{2}} \begin{pmatrix} u_\lambda(\mathbf{r}) \\ \frac{-\mathcal{D}^\dagger}{\sqrt{\lambda + \mu^2 + \mu}} u_\lambda(\mathbf{r}) \end{pmatrix}, \quad E = -\sqrt{\lambda + \mu^2}. \quad (4.14b)$$

For every  $u_\lambda$  which is a solution of the Schrödinger equation (4.12) with positive eigenvalue  $\lambda > 0$ , we obtain two solutions of the Dirac equation, one with positive energy,  $E = \sqrt{\lambda + \mu^2}$  and one with negative energy  $E = -\sqrt{\lambda + \mu^2}$ . Unlike the zero modes that we discussed in the previous Subsection, where there was either a positive or a negative energy solution, here, the positive and negative energy solutions of the Dirac equation are paired: for each positive energy solution there is a negative energy solution and vice-versa. This implies that, if there are bound states other than the zero modes, they must occur in positive and negative energy pairs. Thus, bound states, other than the zero modes, will not contribute to the spectral asymmetry. We will see this explicitly in the following. However, for states in the continuum spectrum, the pairing tells us only that the spectrum occurs symmetrically about zero: for example, there is continuum spectrum in the symmetrically placed regions  $E > \sqrt{m^2 + \mu^2}$  and  $E < -\sqrt{m^2 + \mu^2}$ . It does not tell us about the density of states in these regions, which can still be asymmetric.

## C. Charge density

Let us examine the charge density of the ground state of the system that we are considering. The charge density is given in Eq. (3.6)

$$\rho(\mathbf{r}) = \frac{1}{2} u_{-\mu}^\dagger(\mathbf{r}) u_{-\mu}(\mathbf{r}) - \frac{1}{2} v_\mu^\dagger(\mathbf{r}) v_\mu(\mathbf{r}) - \frac{1}{2} \sum_{E \neq \pm\mu} \text{sign}(E) \Psi_E^\dagger(\mathbf{r}) \Psi_E(\mathbf{r}). \quad (4.15)$$

Here, we have included both types of zero modes. Depending on the sign of the vorticity, only one of them will be non-zero and will have multiplicity given by the magnitude of the vorticity. A sum over these degenerate wave functions is implied in the first two terms on the right-hand side of (4.15). We have also assumed that  $\mu$  is positive, so that  $v_\mu$  is a positive energy state and  $u_{-\mu}$  is a negative energy state. We shall restore the possibility that  $\mu$  could have a negative sign later, where it will simply lead to a flip in sign from the contribution of the zero modes.

Now, using Eqs. (4.14a) and (4.14b), we find that the third term in the right-hand side of (4.15) is

$$\begin{aligned} \rho(\mathbf{r}) &= \frac{1}{2} u_{-\mu}^\dagger(\mathbf{r}) u_{-\mu}(\mathbf{r}) - \frac{1}{2} v_\mu^\dagger(\mathbf{r}) v_\mu(\mathbf{r}) \\ &+ \sum_{\lambda > 0} \frac{\mu}{2\sqrt{\lambda + \mu^2}} \left( u_\lambda^\dagger(\mathbf{r}) u_\lambda(\mathbf{r}) - \frac{1}{\lambda} (\mathcal{D}^\dagger u_\lambda(\mathbf{r}))^\dagger \mathcal{D}^\dagger u_\lambda(\mathbf{r}) \right). \end{aligned} \quad (4.16)$$



Using the fact that  $u_\lambda$  satisfies the Schrödinger equation (4.12) leads to

$$\begin{aligned} \rho(\mathbf{r}) &= \frac{1}{2}u_{-\mu}^\dagger(\mathbf{r})u_{-\mu}(\mathbf{r}) - \frac{1}{2}v_\mu^\dagger(\mathbf{r})v_\mu(\mathbf{r}) \\ &+ \sum_{\lambda>0} \frac{\mu}{2\lambda\sqrt{\lambda+\mu^2}} \left( u_\lambda^\dagger(\mathbf{r})\mathcal{D}\mathcal{D}^\dagger u_\lambda(\mathbf{r}) - (\mathcal{D}^\dagger u_\lambda(\mathbf{r}))^\dagger \mathcal{D}^\dagger u_\lambda(\mathbf{r}) \right). \end{aligned} \quad (4.17)$$

The last terms in this expression are a total derivative

$$\begin{aligned} \rho(\mathbf{r}) &= \frac{1}{2}u_{-\mu}^\dagger(\mathbf{r})u_{-\mu}(\mathbf{r}) - \frac{1}{2}v_\mu^\dagger(\mathbf{r})v_\mu(\mathbf{r}) \\ &+ \boldsymbol{\partial} \cdot \sum_{\lambda>0} \frac{\mu}{2\lambda\sqrt{\lambda+\mu^2}} \left( u_\lambda^\dagger(\mathbf{r}) \mathbf{i}\boldsymbol{\sigma} \mathcal{D}^\dagger u_\lambda(\mathbf{r}) \right). \end{aligned} \quad (4.18)$$

The total charge is a volume integral of the charge density. If we volume integrate the last term in the equation above and use Gauss' theorem, it will be expressed as a line integral on the circle at infinity of the quantity which is to the right of the derivative operator. Thus we see that the charge will depend on the asymptotic form of the wave-functions. We observe that, consistent with our discussion after Eqs. (4.14a) and (4.14b), since the wave-functions of bound states fall off exponentially at large distances, bound states will not contribute to the charge. Only continuum states are important. Further, studying the asymptotics of the continuum states will allow us to compute the total charge. What will make the task easy is the fact that the volume integral of the part of the last term in (4.18) will pick up contributions which go like  $1/r$ .

Before we do that, we re-organize the expression for the charge density. We use the identity

$$\frac{\mu}{2\sqrt{\lambda+\mu^2}} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\mu}{\lambda+\mu^2+\omega^2} \quad (4.19)$$

and the Schrödinger equation (4.12) to re-write Eq. (4.17) as

$$\begin{aligned} \rho(\mathbf{r}) &= \frac{1}{2}u_{-\mu}^\dagger(\mathbf{r})u_{-\mu}(\mathbf{r}) - \frac{1}{2}v_\mu^\dagger(\mathbf{r})v_\mu(\mathbf{r}) \\ &+ \boldsymbol{\partial} \cdot \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left( \sum_{\lambda>0} u_\lambda^\dagger(\mathbf{r}) \frac{1}{\mathcal{D}\mathcal{D}^\dagger} \frac{\mu}{\mathcal{D}\mathcal{D}^\dagger + \mu^2 + \omega^2} \mathbf{i}\boldsymbol{\sigma} \mathcal{D}^\dagger u_\lambda(\mathbf{r}) \right), \end{aligned} \quad (4.20)$$

or, as the basis-independent expression

$$\begin{aligned} \rho(\mathbf{r}) &= \frac{1}{2}u_{-\mu}^\dagger(\mathbf{r})u_{-\mu}(\mathbf{r}) - \frac{1}{2}v_\mu^\dagger(\mathbf{r})v_\mu(\mathbf{r}) \\ &+ \boldsymbol{\partial} \cdot \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \text{tr} \left\langle \mathbf{r} \left| \frac{P}{\mathcal{D}\mathcal{D}^\dagger} \frac{\mu}{\mathcal{D}\mathcal{D}^\dagger + \mu^2 + \omega^2} \mathbf{i}\boldsymbol{\sigma} \mathcal{D}^\dagger \right| \mathbf{r} \right\rangle, \end{aligned} \quad (4.21)$$

where “tr” denotes a trace over Dirac matrices and  $P$  is a projection operator onto states orthogonal to the zero mode wave-functions. (We shall make use of the expression only where  $|\mathbf{r}| \rightarrow \infty$  and the zero-mode wave-functions have vanishing contribution. For this reason, explicit use of this projection will never be needed.)

**Case  $\mathbf{A}_5 = 0$ :** Let us first consider the case where the axial vector gauge field is absent. In the asymptotic region, the Higgs field is

$$\varphi(\mathbf{r}) \equiv \varphi_1(\mathbf{r}) + \mathbf{i}\varphi_2(\mathbf{r}) = \phi e^{in\theta} + \mathcal{O}(r^{-2}), \quad (4.22a)$$

$$\partial_i \varphi(\mathbf{r}) = -in\epsilon_{ij} \frac{r^j}{r^2} \phi e^{in\theta} + \mathcal{O}(r^{-3}). \quad (4.22b)$$

With

$$\mathcal{D}\mathcal{D}^\dagger = -\partial_1^2 - \partial_2^2 + |\varphi|^2 + \boldsymbol{\sigma} \cdot \boldsymbol{\partial}(\varphi_1 + \mathbf{i}\sigma_3\varphi_2), \quad (4.23)$$

we see that the derivatives of the Higgs field provide a long-ranged potential  $\sim 1/r$  in the Schrödinger equation (4.12).

It is easy to find the asymptotic behavior of the propagators in Eq. (4.21) by perturbative expansion in the deviation of the operator in Eq. (4.23) from the free operator  $\mathcal{D}\mathcal{D}^\dagger|_{\varphi=m} = -\partial_1^2 - \partial_2^2 + m^2$ . For example,

$$\begin{aligned} \left\langle \mathbf{r} \left| \frac{1}{\mathcal{D}\mathcal{D}^\dagger + \mu^2 + \omega^2} \right| \mathbf{r}' \right\rangle = & \\ & \int \frac{d^2p}{(2\pi)^2} e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')} \frac{1}{p^2 + \phi^2 + \mu^2 + \omega^2} \\ & - \boldsymbol{\sigma} \cdot \boldsymbol{\partial} \left( \varphi_1(\mathbf{r}) + i\sigma_3 \varphi_2(\mathbf{r}) \right) \int \frac{d^2p}{(2\pi)^2} e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')} \frac{1}{[p^2 + \phi^2 + \mu^2 + \omega^2]^2} + \dots \end{aligned} \quad (4.24)$$

The right-hand side in this equation has support in the region where  $\phi|\mathbf{r} - \mathbf{r}'| < 1$  as it falls off exponentially with the distance  $|\mathbf{r} - \mathbf{r}'|$  when this distance is greater than  $1/\phi$ . The second-term on the right-hand side of (4.24) goes like  $1/r$  and the corrections to it, represented by  $\dots$ , fall off faster and will not be needed.

Using the asymptotic expression (4.24) in Eq. (4.21), we obtain

$$\begin{aligned} \rho(\mathbf{r}) = & \frac{1}{2} u_{-\mu}^\dagger(\mathbf{r}) u_{-\mu}(\mathbf{r}) - \frac{1}{2} v_\mu^\dagger(\mathbf{r}) v_\mu(\mathbf{r}) \\ & - \boldsymbol{\partial} \wedge \left( \frac{\mu}{8\pi\phi^2 \sqrt{m^2 + \mu^2}} (\varphi^* i\boldsymbol{\partial}\varphi - i\boldsymbol{\partial}\varphi^* \varphi) + \dots \right). \end{aligned} \quad (4.25)$$

Upon integrating this expression, we obtain

$$Q = -\frac{1}{2} \text{Index}(H) - \frac{\mu}{2\sqrt{\phi^2 + \mu^2}} \frac{1}{4\pi m^2} \oint d\ell \cdot (\varphi^* i\boldsymbol{\partial}\varphi - i\boldsymbol{\partial}\varphi^* \varphi) \quad (4.26)$$

where, in the first term in the right-hand side, we have remembered that the number of zero modes is determined by the index and the line integral in the second term is taken on the circle at infinite radius. Using the index theorem (4.9) and the asymptotic expression for  $\varphi$  in Eq. (4.22a), we obtain

$$Q = \left( -\frac{1}{2} \text{sign}(\mu) + \frac{\mu}{2\sqrt{\phi^2 + \mu^2}} \right) n. \quad (4.27)$$

We have restored the possibility that  $\mu$  could be negative in the first term by recalling that the sign of the energy of the zero modes is determined by  $\mu$ .

**Case  $\mathbf{A}_5 \neq 0$ :** The second case is when there is also an axial vector gauge field with asymptotic form

$$A_5^i(\mathbf{r}) = -n\epsilon^{ij} \frac{r^j}{2r^2} + \mathcal{O}(r^{-2}) \quad (4.28)$$

so that the covariant derivative of the Higgs field falls off at infinity

$$(\boldsymbol{\partial} + 2i\mathbf{A}_5) \varphi(\mathbf{r}) = \mathcal{O}(r^{-2}). \quad (4.29)$$

Here, we have assumed a power-law fall-off that is sufficiently fast for our purposes. In fact, for a classical field theory with a vortex solution, the covariant derivative falls off exponentially with distance from the vortex. Then,

$$\left\langle \mathbf{r} \left| \frac{1}{\mathcal{D}\mathcal{D}^\dagger + \mu^2 + \omega^2} \right| \mathbf{r}' \right\rangle = e^{-i\sigma_3 \int_r^{r'} d\ell \cdot \mathbf{A}_5} \int \frac{d^2p}{(2\pi)^2} e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')} \frac{1}{p^2 + \phi^2 + \mu^2 + \omega^2} + \dots \quad (4.30)$$

Corrections represented by  $\dots$  fall off at least as fast as  $1/r^2$  as  $r \rightarrow \infty$ . The line integral in the phase factor is to be taken along a straight line between  $\mathbf{r}$  and  $\mathbf{r}'$ . (Since the axial magnetic field also goes to zero at least as fast as  $r^{-2}$  as  $r \rightarrow \infty$ , the path is not important for our purposes.)

The trivial asymptotic form of (4.30) means that the background fields do not contribute to the relevant asymptotic of the last term in (4.21) and the volume integral of that term vanishes. It therefore does not contribute to the total charge. We find that the charge in this case is entirely determined by the zero modes,

$$Q = -\frac{1}{2} \text{Index}(H) \text{sign}(\mu) = -\frac{n}{2} \text{sign}(\mu). \quad (4.31)$$

This is dramatically different from the result for the case without an axial vector gauge field quoted in Eq. (4.27). As we have seen, the difference can be attributed to the asymptotics of the background field configurations. Another way to understand it is to realize that, when the Higgs field approaches its asymptotic form its covariant derivatives as well as the axial magnetic field fall off quickly enough at  $r \rightarrow \infty$ , so that stereographic projection can be used to map the problem of solving the Dirac equation on the plane to the problem of solving it on the sphere (where the vector field is a connection on a Wu-Yang monopole bundle). Then, the entire spectrum is discrete and, by the arguments following Eqs. (4.14a) and (4.14b) we can see that all non-zero-mode solutions of the Schrödinger equation with eigenvalue  $\lambda$  result in pairs of solutions of the Dirac equation: one positive  $E = \sqrt{\lambda + \mu^2}$  and one negative energy  $E = -\sqrt{\lambda + \mu^2}$  state. For this reason, only the zero modes can contribute to the spectral asymmetry and the contribution must be proportional to the index.

As is well known, the axial gauge field is needed to render the vortex energy finite; it screens the infinite energy coming from the scalar field. Evidently, it also screens the irrational charge which arises from the staggered chemical potential. Some further insight into this phenomenon is given below.

## V. FIELD THEORY ANALYSIS

An alternative method for finding the charge induced by the vortex background makes use of a field theoretic evaluation of the expectation value of the current in the “vacuum” state for the Dirac field operators in the vortex background,

$$J^\nu(x) = \langle \bar{\psi}(x) \gamma^\nu \psi(x) \rangle = -\text{Tr} [\gamma^\nu S(x, x)], \quad (5.1)$$

where  $S(x, y)$  is the Dirac field propagator for the Lagrange density (2.3b).

We consider first the theory without the axial gauge field and present Eq. (2.3b) as

$$\mathcal{L}_0 = \bar{\psi}(i\gamma^\nu \partial_\nu - \Phi)\psi \quad (5.2a)$$

where

$$\Phi = \varphi_1 - i\gamma^5 \varphi_2 + \gamma^3 \mu \equiv \varphi_1 - i\gamma^5 \varphi_2 + \gamma^3 \varphi_3. \quad (5.2b)$$

Evidently, we need to invert

$$S^{-1}(x, y) = -i(i\gamma^\nu \partial_\nu - \Phi) \delta(x - y). \quad (5.3)$$

This can be done perturbatively in a gradient expansion for  $\Phi$ . We set

$$\Phi(x) = M + \delta\Phi(x), \quad M = \Phi(0), \quad \delta\Phi(x) = x^\nu \partial_\nu \Phi(0). \quad (5.4)$$

In a graphical representation, a thick line denotes  $S(x, y)$  while a thin line represents the free propagator.

$$S_0(x, y) = \int \frac{d^3 p}{(2\pi)^3} \frac{i}{\not{p} - M} e^{ip(x-y)}, \quad \not{p} \equiv \gamma^\nu p_\nu. \quad (5.5)$$

Hence we have

$$\begin{aligned} \text{---} &= \text{---} - i \frac{\delta\Phi(z)}{\text{---}} \\ \text{x} \quad \text{y} \quad \text{x} \quad \text{y} \quad \text{x} \quad \text{z} \quad \text{y} \\ &- \frac{\delta\Phi(z)}{\text{x}} \frac{\delta\Phi(z')}{\text{z}} \frac{\delta\Phi(z'')}{\text{z}''} \text{y} + i \frac{\delta\Phi(z)}{\text{x}} \frac{\delta\Phi(z')}{\text{z}} \frac{\delta\Phi(z'')}{\text{z}''} \text{y} \end{aligned} \quad (5.6)$$

with integration over the  $z$ -variables understood. The result of the calculation is

$$J^\nu = \frac{-1}{8\pi m^3} \epsilon^{\nu\alpha\beta} \epsilon^{abc} \varphi_a \partial_\alpha \varphi_b \partial_\beta \varphi_c = \frac{-1}{8\pi} \epsilon^{\nu\alpha\beta} \epsilon^{abc} n_a \partial_\alpha n_b \partial_\beta n_c \quad (5.7a)$$

where

$$m^2 = \sum_{a=1}^3 \varphi_a^2 = \phi^2 + \mu^2, \quad n_a = \frac{\varphi_a}{m}, \quad a = 1, 2, 3. \quad (5.7b)$$

The second equality in Eq. (5.7a) shows that  $J^\nu$  is manifestly divergence-free. [The overall sign is set by the requirement that the isolated gap state is filled.]

To evaluate the induced charge, we observe that the charge density for static fields is

$$\rho = \frac{-1}{8\pi m^3} \epsilon^{ij} \epsilon^{abc} \varphi_a \partial_i \varphi_b \partial_j \varphi_c. \quad (5.8)$$

With our profile Eq. (2.13), this becomes

$$\rho(r) = \frac{n}{4\pi r} \frac{d}{dr} \frac{\mu(r)}{m(r)} \quad (5.9)$$

whose spatial integral yields

$$Q = \int d^2r \rho(r) = \frac{n}{2} \frac{\mu(r)}{m(r)} \Big|_0^\infty = \left(-\frac{1}{2} \text{sign}(\mu(0)) + \frac{1}{2} \frac{\mu(\infty)}{m(\infty)}\right) n, \quad (5.10)$$

since the amplitude  $\phi$  vanishes at the origin. This charge can be an irrational quantity, reducing to  $\pm n/2$  as the staggered chemical potential tends to  $\pm 0$ .

It is noteworthy that the induced current (5.7a) exhibits an  $SO(3)$  algebraic structure, even though neither the Lagrange density in Eq. (5.2a) nor the propagator in Eq. (5.3) put such structure into evidence. We shall explain below how this comes about.

Another interesting point is that the current can take a simpler form after fields are redefined. First, we rewrite Eq. (5.7a) in terms of  $\varphi$  and  $\varphi^*$

$$J^\nu = \frac{i}{8\pi m^3} \epsilon^{\nu\alpha\beta} \left[ \mu (\partial_\alpha \varphi)^* (\partial_\beta \varphi) - \partial_\alpha \mu (\varphi^* \partial_\beta \varphi - \varphi \partial_\beta \varphi^*) \right]. \quad (5.11)$$

Next, we define

$$\varphi = 2m\chi\sqrt{1-|\chi|^2}, \quad \mu = m(1-2|\chi|^2), \quad (5.12)$$

thereby expressing the current (5.11) as

$$J^\nu = \frac{i}{2\pi} \epsilon^{\nu\alpha\beta} \partial_\alpha \chi^* \partial_\beta \chi = \frac{i}{4\pi} \epsilon^{\nu\alpha\beta} \partial_\alpha (\chi^* \partial_\beta \chi - \chi \partial_\beta \chi^*). \quad (5.13)$$

This shows that  $J^\nu$  is a total divergence and is manifestly conserved.

Next we write the current when the axial gauge potential is present. Rather than calculating from first principles, we appeal to local axial gauge invariance, and promote all the derivatives in Eq. (5.11) to covariant derivatives

$$D_\nu \equiv \partial_\nu + 2iA_{5\nu}. \quad (5.14)$$

However, the resulting expression is not conserved, but it can be made conserved by adding an axial gauge invariant term, which is linear in the axial gauge field. In this way we arrive at

$$J^\nu = \frac{i}{8\pi m^3} \epsilon^{\nu\alpha\beta} \left\{ \mu (D_\alpha \varphi)^* (D_\beta \varphi) - \partial_\alpha \mu [\varphi^* (D_\beta \varphi) - \varphi (D_\beta \varphi^*)] \right\} + \frac{1}{2\pi} \frac{\mu}{m} F_5^\nu, \quad (5.15)$$

where  $F_5^\nu$  is the axial dual field strength

$$F_5^\nu \equiv \frac{1}{2} \epsilon^{\nu\alpha\beta} F_{5\alpha\beta} = \epsilon^{\nu\alpha\beta} \partial_\alpha A_{5\beta}. \quad (5.16)$$

As a check, the coefficient of the last term in Eq. (5.15) can be computed from the relevant graph. When the axial gauge potential contribution to Eq. (5.15) is separated, Eq. (5.15) equals

$$J^\nu = \frac{i}{8\pi m^3} \epsilon^{\nu\alpha\beta} \left\{ \mu (\partial_\alpha \varphi)^* (\partial_\beta \varphi) - \partial_\alpha \mu [\varphi^* (\partial_\beta \varphi) - \varphi (\partial_\beta \varphi^*)] \right\} + \frac{\epsilon^{\nu\alpha\beta}}{2\pi} \partial_\alpha \left( \frac{\mu}{m} A_{5\beta} \right). \quad (5.17)$$

Therefore, the axial gauge potential's contribution to the charge density is

$$\Delta\rho = -\frac{\epsilon^{ij}}{2\pi} \partial_i \left( \frac{\mu}{m} A_5^j \right), \quad (5.18)$$

which, for  $A_5^j$  as in Eq. (2.14), equals

$$\Delta\rho(r) = -\frac{n}{2\pi} \frac{1}{r} \frac{d}{dr} \left( \frac{\mu(r)}{m(r)} a_5(r) \right), \quad (5.19)$$

and its contribution to the total charge is

$$\Delta Q = \int d^2r \Delta\rho(r) = -\frac{n}{2} \frac{\mu(\infty)}{m(\infty)} \quad (5.20)$$

since  $a_5(\infty) = 1/2$  while  $a_5(0) = 0$ . This cancels the continuous dependence on  $\mu(\infty)$  found in Eq. (5.10), leaving the same rational result obtained in the absence of the staggered chemical potential.

Note that with variables defined as in Eq. (5.12), the current in the presence of the axial gauge field reads

$$\begin{aligned} J^\nu &= \frac{i}{2\pi} \epsilon^{\nu\alpha\beta} (D_\alpha \chi)^* (D_\beta \chi) + \frac{1}{2\pi} (1 - 2|\chi|^2) F_5^\nu \\ &= \frac{i}{4\pi} \epsilon^{\nu\alpha\beta} \partial_\alpha [\chi^* D_\beta \chi - \chi D_\beta \chi^* - 2i A_{5\beta}], \end{aligned} \quad (5.21)$$

or when the gauge field is separated

$$\begin{aligned} J^\nu &= \frac{i}{2\pi} \epsilon^{\nu\alpha\beta} (\partial_\alpha \chi)^* (\partial_\beta \chi) + \frac{1}{2\pi} \epsilon^{\nu\alpha\beta} \partial_\alpha [(1 - 2|\chi|^2) A_{5\beta}] \\ &= \frac{i}{4\pi} \epsilon^{\nu\alpha\beta} \partial_\alpha [\chi^* \partial_\beta \chi - \chi \partial_\beta \chi^* - 2i (1 - 2|\chi|^2) A_{5\beta}]. \end{aligned} \quad (5.22)$$

Therefore, the total divergence feature and the conservation of the current are again explicitly exhibited also in the presence of the axial gauge field. In particular for the charge density, we have

$$\rho = \frac{i}{4\pi} \epsilon^{ij} \partial_i [\chi^* (D_j \chi) - \chi (D_j \chi)^*] + \frac{1}{2\pi} \epsilon^{ij} \partial_i A_{5j}. \quad (5.23)$$

Upon integration over space, the first term is cast on the circle at infinity, where the covariant derivatives of  $\chi$  vanish. The second term shows that the induced charge is exactly the vortex flux, equal to  $n/2$  for  $\mathbf{A}_5$  as in Eq. (2.14).

## VI. INDUCED CHARGE FROM SYMMETRY ARGUMENTS

As we observed in Sec. V, the form of the induced current (5.7a) without an axial gauge field  $A_5^j$  exhibits an  $SO(3)$  algebraic structure despite the absence of any such symmetry in the Lagrange density (5.2a).<sup>37</sup> In this section we explain why this is so. Also we obtain expressions for the induced fractional charge from symmetry arguments.

Our starting point is the Lagrange density (2.3b) with the mass terms collected into  $\Phi$ , as in (5.2b), but written as

$$\mathcal{L}_\psi = \bar{\psi} [\gamma^\nu (i\partial_\nu + \gamma_5 A_{5\nu}) - m M_a n_a] \psi \quad (6.1a)$$

where

$$M_1 = \mathbb{1}, \quad M_2 = -i\gamma^5, \quad M_3 = \gamma^3. \quad (6.1b)$$

The background fields  $mn_a$ , functions on two-dimensional  $(2+1)$  dimensional space-time, are given by

$$n_1 = \frac{\varphi_1}{m}, \quad n_2 = \frac{\varphi_2}{m}, \quad n_3 = \frac{\mu}{m}, \quad (6.2)$$

which, since  $m^2 = |\varphi|^2 + \mu^2$ , satisfy the local constraint

$$1 = n_1^2 + n_2^2 + n_3^2 \equiv \mathbf{n}^2. \quad (6.3)$$

Despite the suggestive form in which the 3-dimensional vector  $\mathbf{n}$  is written above, the Lagrange density (6.1a) is *not*  $SU(2)$  symmetric because the  $M_a$  matrices do not satisfy the  $su(2)$  algebra.

However, it is the induced current (5.7a) and not the starting Lagrange density (6.1a) that exhibits the symmetry. Thus, let us turn our attention to the  $U(1)$  charge current induced by the background field  $m\mathbf{n}$ ,

$$J^\nu(x) = \frac{\int D[\bar{\psi}, \psi] e^{i \int d^3x \mathcal{L}_\psi} (\bar{\psi} \gamma^\nu \psi)(x)}{\int D[\bar{\psi}, \psi] e^{i \int d^3x \mathcal{L}_\psi}}. \quad (6.4)$$

In Eq. (6.4), we are free to change integration variables as long as this transformation leaves the current unchanged. This we do through the nonunitary transformation

$$\bar{\psi} = \bar{\chi} \gamma_5 \gamma^3, \quad \psi = \chi, \quad (6.5)$$

for some arbitrarily chosen constant unit vector  $\mathbf{N}$ . Thus, the induced current (6.4) is now given by

$$J^\nu = \frac{\int D[\bar{\chi}, \chi] e^{i \int d^3x \mathcal{L}_\chi} (\bar{\chi} \Gamma^\nu \chi)}{\int D[\bar{\chi}, \chi] e^{i \int d^3x \mathcal{L}_\chi}} \quad (6.6a)$$

where the transformed Lagrange density reads

$$\mathcal{L}_\chi = \bar{\chi} [\Gamma^\nu (i\partial_\nu + \gamma_5 A_{5\nu}) - m \Sigma_a N_a] \chi, \quad (6.6b)$$

with  $\Gamma^\mu = \gamma_5 \gamma^3 \gamma^\mu$  and  $\Sigma_a = \gamma_5 \gamma^3 M_a$  satisfying

$$\{\Gamma_\nu, \Gamma_{\nu'}\} = 2g_{\nu\nu'}, \quad [\Sigma_a, \Sigma_b] = 2i\epsilon_{abc} \Sigma_c, \quad [\Gamma_\nu, \Sigma_b] = 0. \quad (6.6c)$$

Since  $\gamma_5$  does not commute with all  $\Sigma_a$ ,

$$\{\Sigma_1, \gamma_5\} = \{\Sigma_1, \gamma_5\} = 0, \quad \{\Sigma_3, \gamma_5\} = 2\mathbb{1}, \quad [\Sigma_3, \gamma_5] = 0, \quad (6.7)$$

$\mathcal{L}_\chi$  is an  $SU(2)$  singlet at  $A_5^\nu = 0$  only.

### A. Induced charge without axial flux

In the absence of an axial gauge field, the Lagrange density (6.6b) with  $A_5^\nu = 0$  is an  $SU(2)$  singlet. The induced current and charges must therefore be  $SO(3)$  singlets given by

$$J^\nu = C \frac{1}{8\pi} \epsilon^{\nu\alpha\beta} \epsilon^{abc} n_a \partial_\alpha n_b \partial_\beta n_c, \quad (6.8a)$$

$$Q = \int d^2r J^0(t, \mathbf{r}) = C \frac{\Omega}{4\pi}, \quad (6.8b)$$

where  $\Omega$  is the spherical angle (in units of  $4\pi$ ) covered by the mapping between the base space  $\mathbf{r} \in \mathbb{R}^2$  and a closed curve on the surface of the 2-sphere (6.3), to lowest order in a gradient expansion. Thus, we arrived at our previous result, except the factor  $C$  must still be determined.

We can fix the constant  $C$  using our results for the fractional charge derived in the simple case when there is no staggered chemical potential: the charge is  $Q = -1/2$  when the midgap state is empty, and  $Q = 1/2$  when the midgap state is filled. There is an ambiguity for the charge as given in Eq. (6.8b) if the bound state is at exactly zero energy, because then it can be filled or empty, but this can be lifted by considering the case where  $\mu \rightarrow 0^+$  with  $0^+$  a positive infinitesimal. In this case, the bound state solution exists for an antivortex ( $n = -1$ ), and  $E = -\mu$  (see Refs. 20,21 and Sec. III), so that the level is filled and the charge is thus  $Q = 1/2$ .

Because  $\mu \rightarrow 0^+$  or, equivalently,  $n_3 \rightarrow 0^+$ , the spherical angle traced by the antivortex in the  $n_{1,2}$  plane is just one full hemisphere (traced in the negative orientation):  $\Omega/4\pi = -1/2$ . Hence, the constant  $C = -1$ , leading to the induced current

$$J^\nu = -\frac{1}{8\pi} \epsilon^{\nu\alpha\beta} \epsilon^{abc} n_a \partial_\alpha n_b \partial_\beta n_c \quad (6.9a)$$

and the induced charge

$$Q = -\frac{\Omega}{4\pi}. \quad (6.9b)$$

### B. Abelian formulation, including an axial flux

Here we shall show that the induced charge in the presence of vortices in the off-diagonal masses and in an axial vector gauge potential is the same as that in a problem with constant off-diagonal mass and effective Abelian gauge flux. To this end, we make a further unitary transformation on the Lagrange density (6.6b)

$$\bar{\chi} = \bar{\xi}U, \quad \chi = U^\dagger \xi. \quad (6.10a)$$

The unitary matrix  $U$  is generated by the  $4 \times 4$  matrices  $\Sigma_a$  and is fixed by demanding that it takes the space-time dependent vector  $\mathbf{n}$  in the fixed unit vector  $\mathbf{N}$ ,

$$\left(\Sigma_a N_a\right) = U(x) \left(\Sigma_a n_a(x)\right) U^\dagger(x). \quad (6.10b)$$

It follows that  $U$  commutes with  $\Gamma^\nu$ , but not with  $\gamma_5$ . With  $\mathbf{N} = (0, 0, 1)$ , this is achieved by choosing<sup>38</sup>

$$U(x) = e^{-i\frac{\beta(x)}{2}\Sigma_3} e^{+i\frac{\alpha(x)}{2}\Sigma_2} e^{+i\frac{\beta(x)}{2}\Sigma_3} \quad (6.11a)$$

where the polar angle  $\alpha$  and the azimuthal angle  $\beta$  parametrize  $\mathbf{n}$ ,

$$\mathbf{n} = (\sin \alpha \cos \beta, \sin \alpha \sin \beta, \cos \alpha). \quad (6.11b)$$

The new Lagrange density reads

$$\mathcal{L}_\xi = \bar{\xi} [\Gamma^\nu (i\partial_\nu + B_\nu) - m\Sigma_3] \xi \quad (6.12a)$$

where the matrix  $B_\nu$  is

$$B_\nu = U\gamma_5 U^\dagger A_{5\nu} + U i\partial_\nu U^\dagger. \quad (6.12b)$$

Now once the vector  $\mathbf{N}$  is fixed to  $(0, 0, 1)$ , all the information about the original mass vortex and axial vector-gauge vortex is combined in  $B_\nu$ . The induced charge we want to compute is linear in these potentials (with higher orders suppressed by powers of  $m^{-1}$ ). Indeed to linear order, the current can only depend on the component of  $B_\nu$  along the  $a = 3$  direction. To see this, consider a further rotation around the  $a = 3$  direction by a constant angle  $\delta$ ,

$$B_\nu \rightarrow e^{+i\frac{\delta}{2}\Sigma_3} B_\nu e^{-i\frac{\delta}{2}\Sigma_3}. \quad (6.13)$$

The current is invariant under this rotation, but the components of  $B_\nu$  along the  $a = 1, 2$  directions do change. Hence, the induced current, at linear order, must not be a function of these components and it must depend solely on the component along the  $a = 3$  direction

$$\begin{aligned} b_\nu &= \frac{1}{4} \text{tr} (\Sigma_3 B_\nu) \\ &= \frac{1}{4} \text{tr} [\Sigma_3 (U\gamma_5 U^\dagger A_{5\nu} + U i\partial_\nu U^\dagger)] \\ &= \frac{1}{2} \partial_\nu \beta - \frac{1}{2} (\partial_\nu \beta + 2A_{5\nu}) \cos \alpha. \end{aligned} \quad (6.14)$$

We thus arrive at the result that the induced current and charge, computed using the Lagrangian (6.12a), are the same as those computed using the simpler Lagrange density

$$\bar{\mathcal{L}}_\xi = \bar{\xi} [\Gamma^\nu (i\partial_\nu + b_\nu \Sigma_3) - m\Sigma_3] \xi. \quad (6.15)$$

Finally, one last change of variables

$$\bar{\xi} = \bar{\eta} \Sigma_3, \quad \xi = \eta, \quad (6.16a)$$

which does not affect the current, (again because of a trivial Jacobian in the path integral) and a redefinition of Dirac matrices

$$\bar{\gamma}_\nu = \Sigma_3 \Gamma_\nu, \quad (6.16b)$$

which preserves their Clifford algebra, gives the result that the induced current

$$J^\nu = \frac{\int \mathcal{D}[\bar{\eta}, \eta] e^{i \int d^3x \bar{\mathcal{L}}_\eta (\bar{\eta} \bar{\gamma}^\nu \eta)} \int \mathcal{D}[\bar{\eta}, \eta] e^{i \int d^3x \bar{\mathcal{L}}_\eta}}{\int \mathcal{D}[\bar{\eta}, \eta] e^{i \int d^3x \bar{\mathcal{L}}_\eta}} \quad (6.17a)$$

can be simply obtained from the Lagrange density with the gauge potential  $b_\nu$  and constant mass  $m$

$$\bar{\mathcal{L}}_\eta = \bar{\eta} [\bar{\gamma}^\nu (i\partial_\nu + \gamma_5 b_\nu) - m] \eta. \quad (6.17b)$$

The flux due to  $b_\nu$  is the only quantity left that retains any information on the mass and axial vortices, and thus it is the only variable controlling the value of the induced current and charge. The induced current must be an axial gauge invariant quantity and thus must be constructed from the axial flux due to  $b_\nu$ . The total charge, in particular, must be proportional to the total flux

$$\Phi_5 = \frac{1}{2\pi} \int d^2r (\partial \wedge \mathbf{b})(\mathbf{r}), \quad (6.18)$$

i.e.,

$$Q = C \Phi_5. \quad (6.19)$$

The prefactor  $C$  is determined for the special case without a staggered chemical potential ( $\mu \rightarrow 0^+$ ) and without an axial flux, in which case  $Q = +1/2$ . In this situation the polar angle  $\alpha(\mathbf{r}) \rightarrow \pi/2$ ,  $\cos \alpha(\mathbf{r}) \rightarrow 0$ , and  $b_\nu(\mathbf{r}) \rightarrow \frac{1}{2} \partial_\nu \beta(\mathbf{r})$  as  $\mathbf{r} \rightarrow \infty$ , so that for an antivortex the axial flux due to  $b_\nu$  is simply half the vorticity of the azimuthal angle  $\beta(\mathbf{r})$ :  $\Phi_5 = -\frac{1}{2}$ . This fixes the constant  $C = -1$ . We conclude that

$$Q = -\Phi_5. \quad (6.20)$$

Equation (6.20) is the expression that we seek for the charge induced by static vortices in the mass and in the axial vector-gauge fields.<sup>39</sup> We now consider the following two cases (when  $\mu > 0$ ).

- (i) *Static case with staggered chemical potential, no axial flux, and vorticity  $n$  in the mass, i.e.,  $\cos \alpha(\mathbf{r}) \rightarrow \mu/m$  as  $\mathbf{r} \rightarrow \infty$  and  $a_5 = 0$ :* The flux is  $\Phi_5 = n \frac{1}{2} [1 - \cos \alpha(\mathbf{r} \rightarrow \infty)]$  and the induced charge equals

$$Q = -\frac{n}{2} \left(1 - \frac{\mu}{m}\right). \quad (6.21)$$

- (ii) *Static case with staggered chemical potential and an axial vortex that screens the mass vortex, i.e.,  $\cos \alpha(\mathbf{r}) \rightarrow \mu/m$  and  $\partial_\nu \beta(\mathbf{r}) + 2 A_{5\nu}(\mathbf{r}) \rightarrow 0$  as  $\mathbf{r} \rightarrow \infty$ :* The last term in Eq. (6.14) drops out and so does the dependence on the polar angle  $\alpha(\mathbf{r})$ , along with the dependence on the staggered chemical potential. The flux due to  $b_\nu$  is simply half the vorticity of the azimuthal angle  $\beta(\mathbf{r})$ ,  $\Phi_5 = \frac{1}{2} n$ , and thus the charge is pinned at the rational value

$$Q = -\frac{n}{2}. \quad (6.22)$$

## VII. SUMMARY

The fractional charge induced by vortices supported by a complex-valued Higgs field carrying a  $U(1)$  axial gauge charge of 2 that couples to massive Dirac fermions in  $(2+1)$ -dimensional space-time was computed by three different techniques based on (i) the computation of a spectral asymmetry, (ii) a one-loop perturbative computation of the conserved fermion-number current, (iii) expressing the fractional charge in terms of an Abelian axial flux, respectively. The fractional charge can vary continuously as a function of an energy-reflection symmetry breaking parameter and thus can take irrational values. Remarkably, this fractional charge re-rationalizes to the value  $1/2$  taken in the presence of the spectral energy-reflection symmetry if an axial gauge field couples covariantly to both the Higgs and Dirac fields.



## Acknowledgments

This work was supported in part by Grants NSF DMR-0305482 (C. C. and C-Y. H.), DOE DE-FG02-05ER41360 (R. J.) and DOE DE-FG02-91ER40676 (S-Y. P.). G. S. acknowledges the kind hospitality of the Isaac Newton Institute, Cambridge, and the financial support of NSERC of Canada.

- 
- <sup>1</sup> R. Jackiw and C. Rebbi, “Solitons with Fermion Number 1/2,” *Phys. Rev. D* **13**, 3398 (1976).
  - <sup>2</sup> C. Caroli, P. G. De Gennes, and J. Matricon, “Bound Fermion States on a Vortex Line in a Type II Superconductor,” *Phys. Lett.* **9**, 307 (1964).
  - <sup>3</sup> R. Jackiw and P. Rossi, “Zero Modes of the Vortex-Fermion System,” *Nucl. Phys. B* **190**, 681 (1981).
  - <sup>4</sup> N. Read and D. Green, “Paired States of Fermions in Two Dimensions with Breaking of Parity and Time-Reversal Symmetries and the Fractional Quantum Hall Effect,” *Phys. Rev. B* **61**, 10 267 (2000).
  - <sup>5</sup> W. P. Su, J. R. Schrieffer, and A. J. Heeger, “Solitons in Polyacetylene,” *Phys. Rev. Lett.* **42**, 1698 (1979), “Soliton Excitations in Polyacetylene,” *Phys. Rev. B* **22**, 2099 (1980).
  - <sup>6</sup> R. Jackiw and J. R. Schrieffer, “Solitons with Fermion Number 1/2 in Condensed Matter and Relativistic Field Theories,” *Nucl. Phys. B* **190**, 253 (1981).
  - <sup>7</sup> J. Goldstone and F. Wilczek, “Fractional Quantum Numbers on Solitons,” *Phys. Rev. Lett.* **47**, 986 (1981).
  - <sup>8</sup> M. J. Rice and E. J. Mele, “Elementary Excitations of a Linearly Conjugated Diatomic Polymer,” *Phys. Rev. Lett.* **49**, 1455 (1982).
  - <sup>9</sup> R. Jackiw and G. Semenoff, “Continuum Quantum Field Theory for a Linearly Conjugated Diatomic Polymer with Fermion Fractionization,” *Phys. Rev. Lett.* **50**, 439 (1983).
  - <sup>10</sup> S. Kivelson, “Solitons with Adjustable Charge in a Commensurate Peierls Insulator,” *Phys. Rev. B* **28**, 2653 (1983).
  - <sup>11</sup> R. B. Laughlin, “Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations,” *Phys. Rev. Lett.* **50**, 1395 (1983).
  - <sup>12</sup> B. I. Halperin, “Statistics of Quasiparticles and the Hierarchy of Fractional Quantized Hall States,” *Phys. Rev. Lett.* **52**, 1583 (1984).
  - <sup>13</sup> X.-G. Wen, “Topological Orders in Rigid States”, *Int. J. Mod. Phys. B* **4**, 239 (1990); “Topological Orders and Edge Excitations in FQH States”, *Adv. Phys.* **44**, 405 (1995).
  - <sup>14</sup> Cold atoms in optical lattices could also realize Hamiltonians displaying charge fractionalization via spontaneous symmetry breaking: S.-L. Zhu, B. Wang, and L.-M. Duan, “Simulation and Detection of Dirac Fermions with Cold Atoms in an Optical Lattice,” *Phys. Rev. Lett.* **98**, 260402 (2007); J. Ruostekoski, J. Javanainen, and G. V. Dunne, “Manipulating Atoms in an Optical Lattice: Fractional Fermion Number and its Optical Quantum Measurement,” arXiv:0709.2187.
  - <sup>15</sup> P. R. Wallace, “The Band Theory of Graphite,” *Phys. Rev.* **71**, 622 (1947).
  - <sup>16</sup> H. B. Nielsen and M. Ninomiya, “Absence of Neutrinos on a Lattice (II). Intuitive Topological Proof,” *Nucl. Phys. B* **193**, 173 (1981).
  - <sup>17</sup> G. Semenoff, “Condensed-Matter Simulation of a Three-Dimensional Anomaly,” *Phys. Rev. Lett.* **53**, 2449 (1984).
  - <sup>18</sup> F. D. M. Haldane, “Model for a Quantum Hall Effect without Landau Levels: Condensed-Matter Realization of the ‘Parity Anomaly’,” *Phys. Rev. Lett.* **61**, 2015 (1988).
  - <sup>19</sup> S. Ryu, C. Mudry, H. Obuse, and A. Furusaki, “ $\mathbb{Z}_2$  topological Term, the Global Anomaly, and the Two-Dimensional Symplectic Symmetry Class of Anderson localization,” *Phys. Rev. Lett.* **99**, 116601 (2007).
  - <sup>20</sup> C.-Y. Hou, C. Chamon, and C. Mudry, “Electron Fractionalization in Two-Dimensional Graphenelike Structures,” *Phys. Rev. Lett.* **98**, 186809 (2007).
  - <sup>21</sup> R. Jackiw and S.-Y. Pi, “Chiral Gauge Theory for Graphene,” *Phys. Rev. Lett.* **98**, 266402 (2007).
  - <sup>22</sup> C. Chamon, C.-Y. Hou, R. Jackiw, C. Mudry, S.-Y. Pi, and A. P. Schnyder, “Irrational vs. Rational Charge and Statistics in Two-Dimensional Quantum Systems,” arXiv:0707.0293.
  - <sup>23</sup> S. V. Morozov, K. S. Novoselov, M. I. Katsnelson, F. Schedin, L. A. Ponomarenko, D. Jiang, and A. K. Geim, “Strong Suppression of Weak Localization in Graphene,” *Phys. Rev. Lett.* **97**, 016801 (2006).
  - <sup>24</sup> A. F. Morpurgo and F. Guinea, “Intervalley Scattering, Long-Range Disorder, and Effective Time-Reversal Symmetry Breaking in Graphene,” *Phys. Rev. Lett.* **97**, 196804 (2006).
  - <sup>25</sup> A Cortijo and M. A. H Vozmediano, “Electronic Properties of Curved Graphene Sheets,” *Europhys. Lett.* **77**, 47002 (2007).
  - <sup>26</sup> J. González, F. Guinea, and M. Angeles H. Vozmediano, “Continuum Approximation to Fullerene Molecules,” *Phys. Rev. Lett.* **69**, 172 (1992), and “The Electronic Spectrum of Fullerenes from the Dirac Equation,” *Nucl. Phys. B* **406**, 771 (1993).
  - <sup>27</sup> J. K. Pachos, M. Stone, and K. Temme, “Graphene with Geometrically Induced Vorticity,” arXiv:0710.0858.
  - <sup>28</sup> C. L. Kane and E. J. Mele, “Size, Shape, and Low Energy Electronic Structure of Carbon Nanotubes,” *Phys. Rev. Lett.* **78**, 1932 (1997).
  - <sup>29</sup> Claudio Chamon, “Solitons in Carbon Nanotubes,” *Phys. Rev. B* **62**, 2806 (2000).
  - <sup>30</sup> P. E. Lammert and V. H. Crespi, “Topological Phases in Graphitic Cones,” *Phys. Rev. Lett.* **85**, 5190 (2000) and “Graphene cones: Classification by Fictitious Flux and Electronic Properties,” *Phys. Rev. B* **69**, 035406 (2004).
  - <sup>31</sup> V. A. Osipov and E. A. Kochetov, “Dirac Fermions on Graphite Cones,” *JETP* **73**, 562 (2001).
  - <sup>32</sup> For references on the  $\pi$ -flux phase, see E. H. Lieb, “Flux Phase of the Half-Filled Band,” *Phys. Rev. Lett.* **73**, 2158 (1994).

- <sup>33</sup> B. Seradjeh, C. Weeks, and M. Franz, “Fractionalization in a Square-Lattice Model with Time-Reversal Symmetry,” arXiv:0706.1559.
- <sup>34</sup> That the reference states do possess the energy-reflection symmetry in the case of Hamiltonian Eq. (2.1) is argued as follows. In the reference state, all scalar fields ( $\varphi_1, \varphi_2$ , and  $\mu$ ) are space-time independent, and the axial gauge potential  $\mathbf{A}_5$  is absent. In this case, one can perform a chiral gauge transformation so as to gauge the phase of the scalar field and make it purely imaginary (i.e.,  $\varphi_1 = 0$ ). The corresponding Hamiltonian now satisfy  $\{\beta, H\} = 0$ , and thus eigenvalues of positive and negative energy can be easily related to one another. An argument can be also constructed for the energy-reflection symmetry on the reference state on a discrete lattice model.
- <sup>35</sup> E. J. Weinberg, “Index Calculations for the Fermion-Vortex System,” Phys. Rev. D **24**, 2669 (1981).
- <sup>36</sup> A. J. Niemi and G. W. Semenoff, “Spectral Asymmetry on an Open Space,” Phys. Rev. D **30**, 809 (1984).
- <sup>37</sup> The case of a  $4 \times 4$  Dirac equation with explicit  $SU(2)$  symmetry was treated in T. Jaroszewicz, “Induced Fermion Current in the Sigma Model in (2+1)-Dimensions,” Phys. Lett. **146B**, 337 (1984), *ibid*, “Fermion-Solenoid Interactions: Vacuum Charge and Scattering Theory,” Phys. Rev. D **34**, 3128 (1986); Y.-H. Chen and F. Wilczek, “Induced Quantum Numbers in some (2+1)-Dimensional Models,” Int. J. Mod. Phys. B **3**, 1252 (1989); Z. Hlousek et al, “Induced Hopf Term in the Nonlinear  $\sigma$  Model,” Phys. Rev. D **41**, 3773 (1990).
- <sup>38</sup> Whether one should rotate to  $\mathbf{N} = (0, 0, 1)$  or  $\mathbf{N} = (0, 0, -1)$  is determined by the sign of  $\mu$ . We choose to work with  $\mu > 0$  in the computation in Sec. VIB.
- <sup>39</sup> Notice that our expression Eq. (6.20) can also be used to tie, in a very simple way, the local charge accumulation on a graphene sheet due to ripples, which induce an axial gauge field.