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Strings on conifolds from strong coupling dynamics: quantitative results

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Abstract

Three quantitative features of string theory on $AdS_5 \times X_5$, for any (quasi)regular Sasaki-Einstein X_5, are recovered exactly from an expansion of field theory at strong coupling around configurations in the moduli space of vacua. These configurations can be thought of as a generalized matrix model of (local) commuting matrices. First, we reproduce the spectrum of scalar Kaluza-Klein modes on X_5. Secondly, we recover the precise spectrum of BMN string states, including a nontrivial dependence on the volume of X_5. Finally, we show how the radial direction in global AdS_5 emerges universally in these theories by exhibiting states dual to AdS giant gravitons.

1 Introduction

The AdS/CFT correspondence [1] provides, in principle, a nonperturbative approach to quantum gravity in asymptotically Anti-de Sitter space. A traditionally thorny issue in quantum gravity is the emergence of spacetime and gravitons in a semiclassical limit. In AdS/CFT, addressing this question requires us to directly tackle the dual strongly coupled conformal field theory, in the large N limit. This is a different sort to problem to much work that has been done in AdS/CFT, in which protected quantities, or integrable sectors of the theory, are computed at weak and strong coupling and compared directly.

A program aimed at understanding the emergence of semiclassical quantum gravity from field theory was initiated in [2]. The starting point is a guess concerning the effective, semiclassical, degrees of freedom which characterize the ground state and dominate the low energy physics of the strongly coupled theory, together with a proposal for their dynamics. We will review aspects of this proposal below. Using this effective low energy theory, various non protected quantities were computed and successfully compared with the dual string theory [3, 4, 5]. Furthermore, the proposal was extended from the original case of $\mathcal{N} = 4$ Super Yang-Mills theory to orbifolds of this theory in [6, 7].

It was recently argued [8] that the original proposal, which was for the $\mathcal{N} = 4$ theory, can be generalized to a large class of conformal field theories with only $\mathcal{N} = 1$ supersymmetry. In particular to the theories arising on N D3 branes at the tip of a Calabi-Yau cone. This is a substantial generalization, as there are many such theories. In fact, these theories are in one to one correspondence with the space of five dimensional Sasaki-Einstein metrics [9].

In this paper we will use and extend the recent proposal [8] to derive various quantities in the strongly coupled $\mathcal{N} = 1$ theories. These will be non-BPS quantities and they will reproduce in detail the dual, spacetime, AdS gravity results. We start in sections 3 and 4 by obtaining the ground state of the effective theory and showing that it describes the emergence of the dual Sasaki-Einstein geometry. In sections 5, 6 and 7 we study fluctuations about the ground state, reproducing the spacetime spectrum of scalar Kaluza-Klein harmonics and of BMN string states. Finally, in section 8 we show, by considering excitations dual to giant gravitons, that the radial direction of AdS_5 emerges universally, i.e. orthogonally to and independently of the internal Sasaki-Einstein manifold. In the concluding discussion we emphasize the many computations that remain to be done in order to flesh out this framework in detail.

2 Summary of the computational framework

The upshot of the detailed arguments in [8] will now be summarized, together with some new statements which we will expand upon in later sections. The objective is to describe the low energy physics of the strongly coupled superconformal theories arising on N D3 branes at the tip of a Calabi-Yau cone. The field theory is on a spatial S^3 and hence dual to global AdS_5 space.

- The degrees of freedom which dominate the large N low energy dynamics are configurations of scalar fields that explore the moduli space of vacua of the field theory. The scalar fields are uniform on the S^3, that is, only the s-wave modes are excited. Locally on the moduli space this is very similar to $\mathcal{N} = 4$ SYM, where the configurations are given by six commuting $N \times N$ matrices. Thus in the first instance we have integrated out the higher harmonics on the spatial S^3, all the gauge fields and fermions, and all the (generalized) off diagonal modes.

- The N eigenvalues of these matrices, $\{x_i\}$, are valued on a Calabi-Yau cone over a Sasaki-Einstein manifold X_5. This would be the moduli space of the theory on $\mathbb{R}^3$. Placing the theory on S^3 lifts the moduli space. Firstly because of the conformal coupling mass term. Secondly because there is an enhanced symmetry $U(1)^2 \to U(2)$ when two eigenvalues coincide, the measure terms arising from this degeneration induces a repulsion between eigenvalues. The competition between these two effects is captured by the Hamiltonian

$$H = \sum_i \left[-\frac{1}{2\mu^2} \nabla_i \cdot (\mu^2 \nabla_i) + K_i \right] . \tag{1}$$

  This expression is a sum of single particle Hamiltonians, labeled by the subscript i, except for the measure factor μ^2 which depends on the locations of all the eigenvalues. Here K is the Kähler potential of the Calabi-Yau.

- The measure factor μ^2 requires an inspired guess. In the $\mathcal{N} = 4$ theory we have access to a weakly coupled regime in which the measure factor can be directly determined [2]. This is also possible in the case of orbifolded theories [6, 7]. One can then use nonrenormalization theorems for the BPS sector of the theory to argue that the expression remains valid at strong coupling. The conjectured form we use here, generalizing a property of the measure in the $\mathcal{N} = 4$ case, is that

$$\mu^2 = e^{-\sum_{i \neq j} s_{ij}} , \tag{2}$$

where s_{ij} is the Green's function of a sixth order differential operator on the Calabi-Yau cone

$$-\nabla^6 s(x,x') = 64\pi^3 \delta^{(6)}(x,x')\,. \tag{3}$$

As we will discuss below, this expression has the virtue of automatically localizing the large N eigenvalue distribution on a hypersurface in the Calabi-Yau cone and thus leading to an emergent geometry.

- Given the Hamiltonian (1), one can find the ground state. We do this in section 3 below. The answer is simply

$$\psi_0 = e^{-\sum_i K_i}\,. \tag{4}$$

In section 4 below we show that in this state in the large N limit the eigenvalues form an X_5 at fixed radius r in the cone, which we compute. This is to be interpreted as the X_5 of the dual geometry, which has emerged from the matrix quantum mechanics.

- Given the ground state (4), one can find the spectrum of low lying excitations. There are three types of excitations to consider. The first are those given by operators made from the six matrices that appear in the matrix quantum mechanics. The energies of these states are given by the spectrum of the Hamiltonian (1). An important set of eigenstates, that we will consider, are of the form

$$\psi = \psi_0 \mathrm{Tr} f(x)\,, \tag{5}$$

where $f(x)$ is some function of the six matrices that has polynomial growth.

Secondly, there are excitations of the off diagonal modes of the six matrices, which commute in the ground state. These require additional input. It was argued in [8] that the physics of off diagonal modes connecting nearby eigenvalues is the same as that of the $\mathcal{N}=4$ theory, with an effective, $\mathcal{N}=4$ coupling $g_{\text{eff.}}$. In particular, this implies that the energy of the mode connecting nearby eigenvalues x_i and x_j is [2, 3]

$$E_{ij} = \sqrt{1 + \frac{g^2_{\text{eff.}}}{2\pi^2}|x_i - x_j|^2}\,, \tag{6}$$

where the distance is given by the metric on the Calabi-Yau cone. We require moreover that $g^2_{\text{eff.}}N$ is large in the sense of 't Hooft.

Thirdly, there are excitations of the fields that have been set to zero in the quantum mechanics: the higher harmonics of fields on S^3, the gauge fields and the fermions. These modes remain largely unexplored, although see [4, 10].

With this framework, the strategy for computing quantities is as follows. Firstly we compute the ground state wavefunction of the Hamiltonian. We can then compute the energies of excitations about the ground state. These will not in general be BPS. We show that the spectrum of various excitations matches that computed in supergravity and string theory, providing evidence for the calculational recipe just presented.

3 The ground state wavefunction

The eigenvalue dynamics takes place on the six dimensional cone over a five dimensional compact manifold X_5

$$ds_6^2 = dr^2 + r^2 ds_5^2 \,. \tag{7}$$

Denote the coordinates on the five dimensional manifold by θ. As we have mentioned, an important ingredient for writing down the Hamiltonian for these eigenvalues is the Green's function on the cone satisfying

$$-\nabla_6^6 s(r,r',\theta,\theta') = -\left(\frac{1}{r^5}\frac{d}{dr}r^5\frac{d}{dr} + \frac{1}{r^2}\nabla_5^2\right)^3 s(r,r',\theta,\theta') = 64\pi^3\delta^{(6)}(r,r',\theta,\theta')\,. \tag{8}$$

This Green's function appears in the measure that is necessary to write the Hamiltonian as a differential operator. See equations (1) and (2) above. We will now motivate the use of this Green's function.

In the case of $\mathcal{N}=4$ SYM the measure arising in going to an eigenvalue description can be calculated, and is given by a generalized Vandermonde determinant

$$\mu^2 = \prod_{i<j} |\vec{x}_i - \vec{x}_j|^2 \,, \tag{9}$$

where we use vector notation to indicate a point in $\mathbb{R}^6$ (which is the cone over S^5). We notice that this function is factorized over pairs of eigenvalues, and that if we take (minus one times) the logarithm of $|\vec{x}_i - \vec{x}_j|^2$, then it satisfies the differential equation (8).

A similar calculation was done for abelian orbifolds by a group Γ of $\mathcal{N}=4$ SYM, and the corresponding measure was also factorized [6, 7]. It was found that

$$\mu^2 = \prod_{i<j}\prod_{\gamma\in\Gamma} |\vec{x}_i - \gamma(\vec{x}_j)|^2 \,. \tag{10}$$

That is, in the logarithm of the measure, we need to take a sum over images to obtain the correct answer. Because we are summing over images, the measure function obtained this way naturally satisfies the Green's function equation (8) for $X_5 = S^5/\Gamma$.

These measures were calculated by doing a one loop calculation in the gauge theory: the volume of the gauge orbit of the configurations. If we take a general conformal field theory, the field theories will usually not have a weak coupling description. This is because the fundamental fields have large anomalous dimensions. We need some substitute for the one loop calculation that preserves the spirit of the problem.

In [8] it was argued that the measure in the general case should also be pairwise factorized and that in the limit of coinciding points, the degeneration should be identical to the case of $\mathcal{N} = 4$ SYM. This was argued by an effective field theory reasoning and is true exactly in the case of the orbifold measure. This suggests that the singularity in the logarithm of the measure should be reproduced for all cases. Choosing the Green's function above guarantees this behavior. In principle, there could be other choices.

In the theories that admitted a Gaussian fixed point, the origin of the measure was the volume of a gauge orbit. One might have anticipated that this is the correct property to generalize. However, there can be many different theories in the UV that can give rise to the same conformal fixed point. This observation is due to Seiberg [11], and the different theories are related by Seiberg dualities. If we examine various examples of these theories, we find that generically the dimension of the manifold associated to a single gauge orbit changes between different dual theories and this would indicate that the measure factor changes its scaling properties. However, we expect that the effective dynamics should not change at all. Considering that theories at strong coupling could behave very differently than at weak coupling, calculating the effective measure by just measuring the volume of the gauge orbit is suspect. Instead, it seems more natural that whatever the effective dynamics is, it should depend only on properties of the moduli space of vacua, as these are automatically invariant under Seiberg dualities. Solving a differential equation on the moduli space has this property.

In the end, we have to make a guess. The one we have made, equation (2), seems the simplest guess for the measure term that matches all known cases. Our final justification will perforce come a posteriori, after we have shown that this measure gives many desirable features, especially at the level of calculability of various properties of the strong coupling dynamics.

Given the Hamiltonian, we now want to find wave functions that solve it. In particular, to determine the ground state of the Hamiltonian, we will need to know how the Green's function scales under $r, r' \to \alpha r, \alpha r'$. In appendix A we show that the Green's function

obeys a logarithmic scaling

$$s(\alpha r, \alpha r', \theta, \theta') = s(r, r', \theta, \theta') - \frac{\pi^3 \log \alpha}{\mathrm{Vol}(X_5)} \,. \tag{11}$$

It is interesting to note that the appearance of a nontrivial scaling is intimately tied up with the need to regularize the Green's function. In this sense, the scaling symmetry might be called 'anomalous'.

We now assume that X_5 is a Sasaki-Einstein manifold (see e.g. [9, 12, 13]) so that the six dimensional cone is Calabi-Yau. Let $\{z_i^a, \bar{z}_i^{\bar{a}}\}$ be the complex coordinates of the ith eigenvalue on the cone, $a, \bar{a} = 1..3$. Let K be the Kähler potential of the Calabi-Yau.

The conjectured Hamiltonian [8] is

$$H = \sum_i \left(-\frac{g^{a\bar{b}}(z_i, \bar{z}_i)}{2\mu^2} \left[\nabla_{z_i^a} \left(\mu^2 \nabla_{\bar{z}_i^{\bar{b}}} \right) + \nabla_{\bar{z}_i^{\bar{b}}} \left(\mu^2 \nabla_{z_i^a} \right) \right] + K(z_i, \bar{z}_i) \right) \,, \tag{12}$$

where the measure factor is

$$\mu^2 = e^{-\sum_{i \neq j} s_{ij}} \,. \tag{13}$$

Here $s_{ij} = s(z_i, \bar{z}_i, z_j, \bar{z}_j)$ is the Green's function. We have suppressed the a index in places.

The ground state wavefunction for the Hamiltonian (12) will now be shown to be

$$\psi_0 = e^{-\sum_i K_i} \,, \tag{14}$$

where $K_i = K(z_i, \bar{z}_i)$. Acting on this state with the Hamiltonian (12) gives

$$H\psi_0 = \sum_j \left(K_j + 3 - \frac{g_j^{a\bar{b}}}{2} \left[(\nabla_{z_j^a} K_j) \nabla_{\bar{z}_j^{\bar{b}}} + (\nabla_{\bar{z}_j^{\bar{b}}} K_j) \nabla_{z_j^a} \right] (K_j + 2 \sum_{k \neq j} s_{kj}) \right) \psi_0 \,. \tag{15}$$

Now, for an arbitrary Calabi-Yau cone with metric (7) we have that

$$K = \frac{r^2}{2} \,. \tag{16}$$

This can be derived from a short argument starting with the observation that the Kähler form is homogeneous with degree two in r, see e.g. [14]. It follows that the vector appearing in (15) is the Euler vector of the cone

$$g_j^{a\bar{b}} \left[(\nabla_{z_j^a} K_j) \nabla_{\bar{z}_j^{\bar{b}}} + (\nabla_{\bar{z}_j^{\bar{b}}} K_j) \nabla_{z_j^a} \right] = r \frac{\partial}{\partial r} \,. \tag{17}$$

The scaling (11) then implies that

$$\sum_i r_i \frac{\partial}{\partial r_i} \sum_{j \neq i} s_{ij} = -\frac{N(N-1)}{2} \frac{\pi^3}{\mathrm{Vol}(X_5)} \,. \tag{18}$$

Putting the above statements together we obtain

$$H\psi_0 = \left(3N + \frac{N(N-1)}{2}\frac{\pi^3}{\mathrm{Vol}(X_5)}\right)\psi_0 \equiv E_0\psi_0\,. \tag{19}$$

Thus ψ_0 is an eigenstate as claimed. The lack of dependence on the angular coordinates θ suggests that it is the ground state. The two key ingredients here were the relation between the Kähler potential and the Euler vector (17), and the scaling behaviour of the Green's function (11). Any scaling function would have given the same results.

4 The emergent geometry

In the large N limit, the ground state wavefunction (14) describes an emergent semiclassical geometry [2]. This occurs because a specific configuration of eigenvalues dominates the matrix integral.

The probability of the eigenvalues being in some particular distribution is given by the square of the wavefunction multiplied by the measure factor (13) needed to make the Hamiltonian (12) self-adjoint. That is

$$\mu^2|\psi_0|^2 = e^{-\sum_i r_i^2 - \sum_{j\neq i} s_{ij}} \equiv e^{-S}\,. \tag{20}$$

In the large N limit, we expect a particular configuration to dominate. This will be given by minimizing the effective action

$$S = \int d^6x\rho(x)r_x^2 + \int d^6x d^6y\rho(x)\rho(y)s(x,y)\,, \tag{21}$$

where we have introduced the large N eigenvalue density, $\rho(x)$, which satisfies

$$\int d^6x\rho(x) = N\,. \tag{22}$$

The notation we are using here is that x runs over the six coordinates on the cone, which we denote r_x and θ_x.

The saddle point equations are

$$r_x = -\int dr_y d\theta_y r_y^5\sqrt{g_5(\theta_y)}\rho(r_y,\theta_y)\frac{\partial s(r_x,r_y,\theta_x,\theta_y)}{\partial r_x}\,, \tag{23}$$

$$0 = \int dr_y d\theta_y r_y^5\sqrt{g_5(\theta_y)}\rho(r_y,\theta_y)\frac{\partial s(r_x,r_y,\theta_x,\theta_y)}{\partial \theta_x}\,, \tag{24}$$

where we have explicitly separated the dependence on the r and θ coordinates.

With the Green's function discussed above, one can prove that the density of eigenvalues is not smooth. This is a generalization of an argument found in [2]. The basic idea is that we can also write the saddle point equations as

$$K(x) + \int d^6y \rho(y) s(y,x) = C\,, \tag{25}$$

where C is a Lagrange multiplier enforcing the constraint in equation (22) and K is the Kähler potential. From here, if ρ is smooth, the operation of differentiating with respect to x commutes with the integral. We can act with the Laplacian associated to the metric (7) three times on both sides of the equation. On the left hand side we find that $\nabla^2 K$ is constant, and further applications of ∇^2 give zero. Inside the integral, we would act three times with ∇^2 on the Green's function, and then we would use the defining equation of the Green's function itself to find that $\rho = 0$. This is incompatible with the constraint, so the assumption that ρ has smooth support is wrong. The simplest solution that one could imagine with singular support will have some δ function distribution in it.

Using the formulae in Appendix A for s, equations (92) - (94), and the fact that $\int d\theta \sqrt{g_5}\Theta_\nu(\theta) = 0$ for $\nu > 0$, i.e. that the higher harmonics on X_5 integrate to zero, it is straightforward to see that if $\rho(r,\theta)$ has no θ dependence, then all the θ_x dependence drops out of the equations of motion once the θ_y integrals are done. Solving the saddle point equations reduces to a purely radial problem. Moreover, since we know that the density of eigenvalues has singular support, we can make a simple guess to solve the problem.

The eigenvalues are found to fill out an X_5:

$$\rho(x) = \frac{N\delta(r_x - r)}{r_x^5 \mathrm{Vol}(X_5)}\,, \tag{26}$$

at the constant radius

$$r = \sqrt{\frac{N}{2}}\sqrt{\frac{\pi^3}{\mathrm{Vol}(X_5)}}\,. \tag{27}$$

This expression reduces to the previously known S^5 of radius $r = \sqrt{N/2}$ when the cone is over a sphere, as $\mathrm{Vol}(S^5) = \pi^3$. It is a solution to the equations of motion for all base manifolds X_5, it does not depend on the manifold being homogeneous. Thus we see that part of the $AdS_5 \times X_5$ geometry has emerged from the eigenvalue quantum mechanics. We obtain X_5 together with its Sasaki-Einstein metric, because of the requirement that the metric on the conical target space of the eigenvalues is Calabi-Yau [8].

This X_5 eigenvalue distribution is to be understood as the large N ground state of the theory, where quantum mechanical measure effects have repelled the eigenvalues away from their classical origin at $r = 0$. It is a self consistent starting point for studying the

low energy dynamics. All off diagonal fluctuations are massive [2, 3] as are all the higher harmonics on S^3 (this second point follows from the analysis in [15, 10]). In the remainder of this paper we study three particular excitations about this ground state. We will see that they reproduce quantitative features of strings and D branes in the dual spacetime.

5 The spectrum of scalar Kaluza-Klein harmonics

5.1 The spectrum for $\mathcal{N} = 4$

In the $\mathcal{N} = 4$ SYM theory, the spectrum of gravity multiplets can be deduced from the half BPS states. The half BPS primary fields corresponding to single graviton states are given by single-trace operators of the form $\mathrm{Tr} z^n$, with $z = x^1 + ix^2$. These are holomorphic highest weight states of $SO(6)$, for a symmetric traceless tensor representation of $SO(6)$.

In the commuting matrix model of strong coupling, as we reviewed in section 2 above, the wave functions of these states are conjectured to be

$$\psi = \psi_0 \mathrm{Tr} z^n \,, \tag{28}$$

where ψ_0 is the ground state wave function of the matrix model. In the $\mathcal{N} = 4$ matrix model, on $\mathbb{R}^6$, the $SO(6)$ symmetry is manifestly part of the dynamics, and ψ_0 is an $SO(6)$ singlet. It is natural to expect that the wave functions of other states that are not half BPS with respect to the same half of the supersymmetries as z^n are given by

$$\psi = c_{i_1 \dots i_n} \mathrm{Tr}(x^{i_1} \dots x^{i_n}) \psi_0 \,, \tag{29}$$

where $c_{i_1 \dots i_n}$ is symmetric and traceless.

Since the matrices commute, the trace is just a sum over eigenvalues, and we find ourselves with a one-particle wave function problem. The resulting symmetric traceless polynomials of six variables are characterized by the property that

$$\nabla^2 (c_{i_1 \dots i_n} x^{i_1} \dots x^{i_n}) \sim c_{i_1 i_1 i_3 \dots i_n} x^{i_3} \dots x^{i_n} = 0 \,, \tag{30}$$

this is, they are harmonic functions on $\mathbb{R}^6$.

These states have energy n, and thus the dual operators have dimension n. We can recover this result by considering the one-particle wave function problem for a six-dimensional harmonic oscillator

$$H = -\frac{1}{2} \nabla^2 + \frac{1}{2} \vec{x}^2 \,. \tag{31}$$

This Hamiltonian differs from the full multi matrix model Hamiltonian (12) for the $\mathcal{N}=4$ problem by the absence of the measure, which mixes the eigenvalues. We show in Appendix B, for the general conifold, that the measure may be neglected for these states to leading order at large N. Thus in this limit it is sufficient to investigate the spectrum of (31). The absence of mixing between eigenvalues allows us to focus on a single eigenvalue, hence we have dropped the i index in (31).

We take our wave function to be

$$\psi = e^{-\vec{x}^2/2} c_{i_1 \dots i_n} x^{i_1} \dots x^{i_n} \,. \tag{32}$$

When we calculate $\nabla^2 \psi$, there are three types of terms that appear. The terms with two derivatives acting on the exponential are cancelled by the term $\frac{1}{2}x^2$ in the Hamiltonian. The terms with two derivatives acting on the polynomial vanish because of equation (30). We are left with terms with one derivative acting on the exponential and one derivative acting on the polynomial. If we write the Laplacian in spherical coordinates, we find that these terms give

$$c_{i_1 \dots i_n} e^{-r^2/2} \left(\frac{1}{2r^5} \frac{d}{dr} \left(r^6 x^{i_1} \dots x^{i_n} \right) + \frac{r}{2} \frac{d}{dr} \left(x^{i_1} \dots x^{i_n} \right) \right) \,. \tag{33}$$

Now, $x^i = r f^i(\theta)$, for some $f^i(\theta)$, so we need to evaluate

$$\frac{1}{2r^5} \frac{d}{dr} r^{n+6} + \frac{r}{2} \frac{d}{dr} r^n = (n+3) r^n \,. \tag{34}$$

Via this exercise, we find that the wavefunction written down in equation (32) is an eigenfunction of the one particle Hamiltonian (31), and that its energy is n units greater than the energy of the vacuum state. The same value of n is the dimension of the corresponding operator in the conformal field theory. This calculation provides a link between the energy of a state in the harmonic oscillator problem, and the dimension of the corresponding state in supergravity. We should also notice that what matters for this computation is that the polynomial we considered was a homogeneous function (it is a scaling function under the vector $r\partial_r$), and that the energy obtained is exactly this scaling dimension.

In the case of $\mathcal{N}=4$ SYM, all of these symmetric traceless functions are obtained by acting with rotations on z^n, and therefore they are in some sense locally holomorphic with respect to a suitable choice of complex coordinates. This is characterized exactly by having a harmonic function. We will now extend this calculation on the moduli space of a 'single brane' in $\mathcal{N}=4$ SYM to the case of a 'single brane' in the case of a conformal field theory associated to a general conifold. As we noted, the term mixing the eigenvalues in

the Hamiltonian, the measure, will again not be important to leading order at large N for this problem.

5.2 The spectrum for general conifolds

As we have discussed, for the general conifold the eigenvalue dynamics is locally given by $\mathcal{N} = 4$ SYM. The wave function is a global object, but the property of being a harmonic function is something that one can check locally, as it is governed by a second order differential equation. It seems natural to take the same ansatz for this more involved case as we did for $\mathcal{N} = 4$ SYM. We consider wave functions of the form

$$\psi = \psi_0 \mathrm{Tr} h(x)\,, \tag{35}$$

where $h(x)$ is a harmonic function on the Calabi-Yau cone over X_5 and $\psi_0 = e^{-r^2/2}$, as we found above.

The one particle problem (i.e. without the measure, see Appendix B) now corresponds to the Hamiltonian

$$H = -\frac{1}{2r^5}\frac{d}{dr}r^5\frac{d}{dr} - \frac{1}{2r^2}\nabla_5^2 + \frac{r^2}{2}\,. \tag{36}$$

One can separate variables in θ (the coordinates on X_5) and r, and hence consider harmonic functions of the form $h(x) = h(r)\Theta(\theta)$, where Θ is an eigenfunction of the Laplacian on the Sasaki-Einstein space. That is

$$-\nabla_5^2\Theta(\theta) = \nu^2\Theta(\theta)\,. \tag{37}$$

Harmonicity now requires solving the following differential equation for $h(r)$

$$\left(-\frac{1}{r^5}\frac{d}{dr}r^5\frac{d}{dr} + \frac{\nu^2}{r^2}\right)h(r) = 0\,. \tag{38}$$

This is solved by $h(r) = r^\lambda$, where λ satisfies

$$\lambda(\lambda+4) - \nu^2 = 0\,, \tag{39}$$

or equivalently

$$\lambda = -2 + \sqrt{4+\nu^2}\,, \tag{40}$$

where we chose the root that makes the wavefunction nonsingular at the origin. The same manipulations that told us in the case of $\mathcal{N} = 4$ SYM that the energy of the harmonic function of weight n multiplying the ground state wave function has energy n, now show us that the energy of the single-particle wave function (35) is given by λ.

The equation (39) is familiar from supergravity in AdS_5 [16, 17] (see also [18]), where one associates a scaling dimension λ to a scalar particle in five dimensions that originates from perturbations mixing the graviton and the self-dual five-form field strength. We see that the scaling dimensions of the operators are controlled by harmonic analysis on the Sasaki-Einstein space, recovering exactly the spectrum of some of the scalar fluctuations in the dual gravity theory. In particular, for all holomorphic wave functions we recover the exact scaling dimension predicted by the chiral ring. Most of these harmonic functions are not holomorphic, however, so we are recovering universally the spectrum of a large family of non-BPS Kaluza-Klein harmonics of the dual supergravity theory.

In Appendix C we discuss the possibility of building coherent states using these single trace states. These appear to be dual to classical geometries, as one would expect for coherent states of gravitons.

6 Spectrum of off-diagonal fluctuations

The off diagonal modes connect pairs of eigenvalues. For small separations, $\Delta z_{ij} = z_j - z_i$, the energies of these modes are given by their mass term plus the distance between the two eigenvalues, see [3, 8] and section 2 above,

$$E_{ij}^2 = 1 + \frac{g_{\text{eff.}}^2}{2\pi^2} g_{i\,a\bar{b}} \Delta z_{ij}^a \Delta \bar{z}_{ij}^{\bar{b}} \,. \tag{41}$$

Recall that $g_{\text{eff.}}$ is the effective $\mathcal{N} = 4$ coupling which controls the masses of off diagonal modes connecting nearby eigenvalues. The z_i are all at constant radius r given by (27). Thus we have

$$E_{ij}^2 = 1 + \frac{\lambda_{\text{eff.}}}{4\pi^2} \frac{\pi^3}{\text{Vol}(X_5)} |\Delta\theta_{ij}|^2_{g_{5,i}} \,, \tag{42}$$

where $\lambda_{\text{eff.}} = g_{\text{eff.}}^2 N$, $\Delta\theta_{ij}$ is the separation in X_5, and g_5 is the metric on X_5.

We would like to write down an operator that describes these off diagonal fluctuations. The operators that do the trick [2, 3, 4] are strong coupling realisations of the BMN [19] operators

$$\mathcal{O}_{k,J} = \sum_{n=1}^{J} \text{Tr}\left[z^n \beta^\dagger z^{J-n} \tilde{\beta}^\dagger \right] e^{2\pi i n k/J} \,. \tag{43}$$

In this expression k is an integer, J is the R charge of the operator, $\beta^\dagger$ and $\tilde{\beta}^\dagger$ are creation operators for off diagonal modes, and z is a complex coordinate on the conical moduli space with a fixed scaling dimension c.

The wavefunction corresponding to this operator is

$$\psi_{k,J} = \mathcal{O}_{k,J}\psi_0 \,. \tag{44}$$

In principle, the inclusion of the operator $\mathcal{O}_{k,J}$ will backreact on the dominant eigenvalue distribution, in a way similar to that described below in probing the radial direction. However, here we wish to take J large, but not of order N. In this case the effect of the insertion of $\mathcal{O}_{k,J}$ in (44) on the eigenvalue distribution is subleading in $1/N$. Thus we can take the eigenvalues z_i to lie on the ground state solution (26).

Invariance under the unbroken $U(1)^N$ symmetry requires that $\beta^\dagger$ and $\tilde{\beta}^\dagger$ carry opposite charges. Thus if we take $\beta^\dagger$ to connect the ith and jth eigenvalues, then $\tilde{\beta}^\dagger$ must connect the jth to the ith. This is implemented automatically by the trace in (43). The operator (43) may be written

$$\mathcal{O}_{k,J} = \sum_{i,j}\sum_{n=1}^{J} z_i^n z_j^{J-n}\beta_{ij}^\dagger \tilde{\beta}_{ji}^\dagger e^{2\pi i n k/J} \,. \tag{45}$$

At large J, there is a dominant contribution to this sum [3, 4]. Firstly, the dominantly contributing eigenvalues maximize $|z|$. This does not fix the location along the angle ψ dual to the R charge, as z is a chiral operator and hence $|z|$ is invariant under R charge rotations. More specifically, on the locus where $|z|$ is maximized we may write

$$z_i \propto r^c e^{ic\psi_i} \,. \tag{46}$$

The exponent follows from two observations. Firstly, because z has conformal dimension c, we have $r\partial_r z = cz$. Secondly, see for instance [12, 13], $\partial_\psi = \mathcal{J}(r\partial_r)$, where $\mathcal{J}$ is the complex structure on the Calabi-Yau. Therefore $\partial_\psi z = icz$, as implied by (46). Now doing a saddle point approximation to the sum over n in (45) we find

$$\psi_i - \psi_j = -\frac{2\pi k}{cJ} \,. \tag{47}$$

This is a crucial relation which says that for given k and J, the dominant contribution to the operator $\mathcal{O}_{k,J}$ comes from two off diagonal modes connecting a pair of eigenvalues separated according to (47). It follows from our previous expression (42) for the off diagonal energies that

$$E_{\mathcal{O}_{k,J}} - cJ = 2E_{ij} = 2\sqrt{1 + \frac{\lambda_{\text{eff.}}\pi^3}{\text{Vol}(X_5)}\frac{k^2}{c^2J^2}} \,, \tag{48}$$

where we included the contribution to the energy from z^J in (43). Conveniently, we did not need to find the point on the remaining directions in X_5 at which $|z|$ is maximized, as

$g_5(\partial_\psi, \partial_\psi) = 1$ is in fact constant over the Sasaki-Einstein space, see for instance [12, 13]. We will now see that this result is precisely the spectrum of excitations about a rapidly rotating BMN string in the dual spacetime.

7 Comparison with the plane wave limit

The spacetime dual to the superconformal field theory is $AdS_5 \times X_5$. The metric may be written as

$$ds^2 = L^2 ds^2_{AdS_5} + L^2 \left[(d\psi + \sigma)^2 + ds^2_{KE} \right] . \tag{49}$$

Here ds^2_{KE} is a four dimensional Kähler-Einstein metric and $d\sigma$ is proportional to the Kähler two form corresponding to this metric. We restrict ourselves here to (quasi)regular Sasaki-Einstein manifolds, in which the fibre coordinate ψ has a finite periodicity.

States with large angular momentum about the ψ direction, corresponding to large R charge, are captured by the Penrose limit of this background [19]. This limit was computed in [20] – Penrose limits of the special case of $X_5 = T^{1,1}$ were also computed in [21, 22] – to give

$$ds^2 = -4dx^+ dx^- - |x|^2_8 (dx^+)^2 + dx^2_8 \,, \tag{50}$$

where

$$x^+ = \frac{1}{2}(t + \psi) \,, \qquad x^- = \frac{L^2}{2}(t - \psi) \,. \tag{51}$$

Note that (50) is just the maximally supersymmetric plane wave background [19, 23].

The conformal dimension and R charge are given by

$$\Delta = i\partial_t \,, \qquad cJ = -i\partial_\psi \,. \tag{52}$$

Where for ease of comparison with the previous subsection, we denote the total R charge by cJ. Therefore from (51)

$$2p^- = i\partial_{x^+} = i\left(\partial_t + \partial_\psi\right) = \Delta - cJ \,, \tag{53}$$

$$2p^+ = i\partial_{x^-} = \frac{i}{L^2}\left(\partial_t - \partial_\psi\right) = \frac{1}{L^2}\left(\Delta + cJ\right) \,. \tag{54}$$

Quantising the string excitations [19, 24] in the plane wave background (50) gives the spectrum of excitations

$$2\delta p^- = \sqrt{1 + \frac{k^2}{\alpha'^2 (p^+)^2}} \,. \tag{55}$$

Using the expressions for the momenta (53) and working to leading order at large J, but with $L^2/\alpha' J$ fixed, one obtains

$$\Delta - cJ = \sqrt{1 + \frac{L^4 k^2}{\alpha'^2 c^2 J^2}} \,. \tag{56}$$

The supergravity background has a Ramond-Ramond five form

$$F^{(5)} = \frac{N\sqrt{\pi}}{2\mathrm{Vol}(X_5)} \left(\mathrm{vol}_{AdS_5} + \mathrm{vol}_{X_5}\right) \,. \tag{57}$$

The solution to the supergravity equations specifies a relation between the units of flux, N, and the AdS radius, L, in string units

$$\frac{L^4}{\alpha'^2} = \frac{4\pi g_s N \pi^3}{\mathrm{Vol}(X_5)} \,, \tag{58}$$

To further relate this expression to our previous results, note that the local effective $\mathcal{N} = 4$ coupling, $g_{\text{eff.}}$, must be related to the expectation value of the dilaton in the usual way

$$g^2_{\text{eff.}} = 4\pi g_s \,. \tag{59}$$

This follows, for instance, by noting that these quantities transform in the correct way under S duality. Hence we obtain from (56)

$$\Delta - cJ = \sqrt{1 + \frac{\lambda_{\text{eff.}} \pi^3}{\mathrm{Vol}(X_5)} \frac{k^2}{c^2 J^2}} \,. \tag{60}$$

Recalling that the eigenvalue Hamiltonian is in fact the conformal dimension, $H = \Delta$, we have precisely reproduced the matrix quantum mechanics result (48). We need to multiply (60) by two because we are considering two excitations. Note that we nontrivially match the appearance of the volume factor $\mathrm{Vol}(X_5)$.

8 Exciting an eigenvalue: the radial direction

In the previous two sections we have shown how off diagonal modes connecting ground state eigenvalues are dual to string excitations about the $AdS_5 \times X_5$ background in the BMN limit. In this section we return to purely eigenvalue excitations, no off-diagonal modes, but with a larger R charge, $J \sim N$. We will see how these excitations move a single eigenvalue into the radial direction, and are dual to AdS giant gravitons. To familiarise ourselves with the procedure, we will consider the $\mathcal{N} = 4$ case first.

8.1 Probing the radial direction in $\mathcal{N} = 4$

In the $\mathcal{N} = 4$ case, the cone is over S^5, i.e. the total space is just $\mathbb{R}^6$. We will use cartesian coodinates $\vec{x}$ to denote the matrices, rather than the 'polar' coordinates r, θ.

Consider the wavefunction

$$\psi = \psi_0 \mathrm{Tr} z^J \,, \tag{61}$$

where $z = x^1 + ix^2$ and ψ_0 is the ground state wavefunction. In Appendix B we show that in the large N limit, this is an eigenfunction of the Hamiltonian (12) with eigenvalue $E = E_0 + J$. The probability density is

$$\mu^2 |\psi|^2 = e^{-\sum_i \vec{x}_i^2 + \frac{1}{2}\sum_{i\neq j} \log|\vec{x}_i - \vec{x}_j|^2 + \log \sum_{i,j} \left(x_i^1 + ix_i^2\right)^J \left(x_j^1 - ix_j^2\right)^J} \,, \tag{62}$$

where we used the explicit Green's function on $\mathbb{R}^6$ to write $\mu^2 = \prod_{i<j} |\vec{x}_i - \vec{x}_j|^2$.

We will make the assumption, to be verified a posteriori, that $|x_N| > |x_i|$ for all $i \neq N$ and that $J \gg 1$. We may thus approximate the last term

$$\log \sum_{i,j} \left(x_i^1 + ix_i^2\right)^J \left(x_j^1 - ix_j^2\right)^J \to J \log\left[(x_N^1)^2 + (x_N^2)^2\right] \,. \tag{63}$$

This is the assumption that one eigenvalue will be moved away from the others.

The large N semiclassical support of the wavefunction is found by extremising the exponent in (62). The equations of motion are

$$x_i^A = \sum_{i\neq j} \frac{x_i^A - x_j^A}{|\vec{x}_i - \vec{x}_j|^2} + \frac{J\delta_{iN}\left[\delta^{1A} x_N^1 + \delta^{2A} x_N^2\right]}{(x_N^1)^2 + (x_N^2)^2} \,. \tag{64}$$

We look for a solution to these equations which is given by the ground state before the insertion, an S^5 of radius squared $r^2 = N/2$ and density $N/\pi^3 r^5$, together with a single eigenvalue $\vec{x}_N$ separated from the sphere. There will be an S^1 worth of such solutions, where the S^1 lies in the $x^1 - x^2$ plane. Without loss of generality, we can take the eigenvalue to move off in the x^1 direction

$$x_N^A = x_N \delta^{1A} \,. \tag{65}$$

For $i \neq N$, the equation of motion (64) is satisfied to leading order in N, because the equation of motion is just that corresponding to the ground state wavefunction which is solved by the S^5. The effect of the extra eigenvalue is subleading. The equation for $i = N$, however, gives a nontrivial equation for x_N. Using the integral

$$\frac{8N}{3\pi} \int_0^\pi d\theta \sin^4\theta \frac{x_N - r\cos\theta}{x_N^2 + r^2 - 2x_N r\cos\theta} = \frac{N(6x_N^4 - 4x_N^2 r^2 + r^4)}{6x_N^5} \,, \tag{66}$$

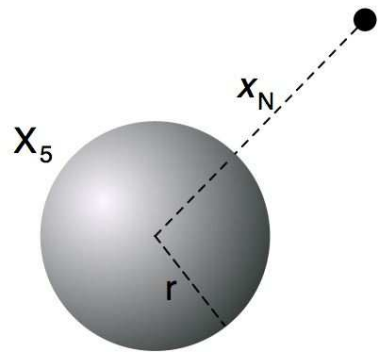

Figure 1: A single eigenvalue is removed from the X_5 to a distance $x_N \sim \sqrt{J}$.

the equation of motion becomes, using $r^2 = N/2$,

$$x_N^6 - (J+N)x_N^4 + \frac{N^2}{3}x_N^2 - \frac{N^3}{24} = 0\,. \tag{67}$$

If we set

$$x_N^2 = d^2 N\,, \qquad J = jN\,, \tag{68}$$

then the solution to the (cubic) equation (67) is

$$d^2 = \frac{1+j}{3} + \frac{2j(j+2)}{3p(j)^{1/3}} + \frac{p(j)^{1/3}}{6}\,, \tag{69}$$

where

$$p(j) = \frac{1}{2}\left[1 + 24j + 48j^2 + 16j^3 + \sqrt{1 + 48j + 672j^2 + 288j^3}\right]\,. \tag{70}$$

This gives us the distance of the x_N eigenvalue from the origin as a function of the angular momentum J. We see that $J \sim N$ is indeed large as required. Figure 1 illustrates the configuration we have just obtained.

Taking the further limit that the eigenvalue is far away from the sphere in $\sqrt{N}$ units, i.e. $j \gg 1$, gives the result

$$x_N = \sqrt{J} + \cdots \qquad (J/N \gg 1)\,. \tag{71}$$

The association of an object with large R charge to a radial motion is strongly reminiscent of AdS giant gravitons. This will shortly lead us to identify the radial direction of the Calabi-Yau cone outside of the X_5 occupied by the ground state with the radial direction of global AdS_5.

8.2 Probing the radial direction for general conifolds

The argument goes through essentially unchanged for the case of a general cone over X_5. We make the assumption that one eigenvalue will have a larger modulus than the others

$|z_N| > |z_i|$, for all $i \neq N$. Thus in the limit $J \gg 1$ we may write the probability density as

$$\mu^2|\psi|^2 = \mu^2|\psi_0 \mathrm{Tr} z^J|^2 = e^{-\sum_i r_i^2 - \sum_{i\neq j} s_{ij} + J\log|z_N|^2}\,. \tag{72}$$

As we note in Appendix B, the holomorphic coordinate z must be a power of r multiplied by a harmonic function on X_5

$$z = r^c F_c(\theta)\,. \tag{73}$$

The large N semiclassical equations of motion following from (72) are therefore

$$r_i + \sum_{j\neq i} \frac{\partial s_{ij}}{\partial r_i} = \frac{cJ\delta_{iN}}{r_N}\,, \qquad \sum_{j\neq i} \frac{\partial s_{ij}}{\partial \theta_i} = \frac{J\delta_{iN}\partial_{\theta_N}|F_c(\theta_N)|}{|F_c(\theta_N)|}\,. \tag{74}$$

In the large N limit, as for the case of S^5 above, the equations of motion for the eigenvalues $i \neq N$ are unaffected by the insertion of $\mathrm{Tr} z^J$, as the motion of the single eigenvalue z_N away from the ground state configuration is a subleading effect. The equations of motion for r_N and θ_N however are nontrivial. Recall the observation we made in section 4: that the independence of the ground state eigenvalue density on θ implies that any integral of the form $\int d\theta_x \rho(\theta_x) s(\theta_x, \theta_y)$ kills the θ_y dependence. This fact, together with the expression for s in equation (92) and the integral

$$\int_0^\infty \frac{d\lambda}{\lambda^3} \frac{J_2(\sqrt{\lambda}r)}{r^2} \frac{\partial}{\partial r_N} \frac{J_2(\sqrt{\lambda}r_N)}{r_N^2} = \frac{-4r_N^2 r^2 + 6r_N^4 + r^4}{192 r_N^5}\,, \tag{75}$$

leads to the following equations

$$r_N^6 - \left(cJ + \tfrac{N\pi^3}{\mathrm{Vol}(X_5)}\right) r_N^4 + \tfrac{1}{3}\left(\tfrac{N\pi^3}{\mathrm{Vol}(X_5)}\right)^2 r_N^2 - \tfrac{1}{24}\left(\tfrac{N\pi^3}{\mathrm{Vol}(X_5)}\right)^3 = 0\,, \tag{76}$$

$$\partial_{\theta_N}|F_c(\theta_N)| = 0\,, \tag{77}$$

where we also used the radius of the ground state X_5 in (27).

The immediate observation we can make from these equations is that the radial and angular parts have completely decoupled. We can interpret this as the fact that the radial direction in the bulk geometries emerges universally. It does not depend on where the eigenvalue is sitting in X_5. This reflects the direct product structure of the dual geometry: $AdS_5 \times X_5$.

The equation (77) for θ says that $|z_N|$ is maximized given its fixed radius r_N. This, together with the fact that we will find $r_N > r$, guarantees that our assumption that $|z_N| > |z_i|$ for $i \neq N$ is consistent. As in the case for S^5, there will not be a unique solution to (77). Rather there will be an S^1 worth of solutions, corresponding to the R symmetry circle.

If we make the definitions

$$r_N^2 = d^2 \frac{N\pi^3}{\mathrm{Vol}(X_5)}\,, \qquad cJ = j\frac{N\pi^3}{\mathrm{Vol}(X_5)}\,, \tag{78}$$

then we find that the radial equation (76) is exactly the same as the one we found in the case of S^5, with solution (69). Thus (69) describes how the eigenvalue z_N moves out in the radial direction as a function of the R charge cJ. From the equation (76) we see that the general relation between r_N and cJ depends on the volume of X_5. However, in the limit $j \gg 1$ we again find

$$r_N = \sqrt{cJ} + \cdots \quad (J/N \gg 1)\,. \tag{79}$$

8.3 Comparison with AdS giant gravitons

In the $\mathcal{N} = 4$ case, at weak coupling, the wavefunction dual to an AdS giant graviton with angular momentum J along the equator of the S^5 is [25]

$$\psi = \psi_0 \chi_{S_J}(z)\,, \tag{80}$$

where χ_{S_J} is the Schur polynomial corresponding to the totally symmetric representation of rank J. In terms of the eigenvalues of z

$$\chi_{S_J}(z) = \sum_{1\le i_1\le\cdots\le i_J\le N} z_{i_1}\cdots z_{i_J}\,. \tag{81}$$

We would like to approximate $\chi_{S_J}(z)$ with $\mathrm{Tr}z^J$, so that we can use the results of the previous section to evaluate the semiclassical wavefunction. This will be valid provided that the largest eigenvalue $|x_N/x_p| \gg 1$ for all $p \neq N$, which requires $j \gg 1$. In this limit $d^2 = j$ in (69). Furthermore, it is unclear that the Schur polynomials (81) will be orthogonal at strong coupling. On the other hand, we have shown in Appendix B that the states $\psi_0 \mathrm{Tr}z^J$ are eigenstates to leading order at large N, with different eigenvalues, and therefore are orthogonal.

In the bulk, the AdS giant gravitons are $D3$ branes in which an S^3 expands to a finite radius in AdS_5, due to their angular momentum about the R symmetry direction. With angular momentum J the radius is [26, 27] given by $r^2_{\mathrm{giant}} = J/N$. Here we are using global coordinates in $AdS_5 \times X_5$

$$\frac{1}{L^2}ds^2 = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2 d\Omega^2_{S^3} + d\Omega^2_{X_5}\,. \tag{82}$$

Comparing this bulk result with our matrix model result (79), and absorbing the factor of c into the definition of J, now the total R charge, we obtain

$$\frac{r_N}{\sqrt{N}} = r_{\mathrm{giant}} \qquad \text{for} \qquad r_{\mathrm{giant}} \gg 1\,. \tag{83}$$

Thus the radial direction in the space of eigenvalues, in units of $\sqrt{N}$, is exactly equal to the radial direction of AdS in the large radius limit. This is a nice result, but it is also not clear why the particular coordinate r that we chose in (82) should have been singled out in this way.

We can unambiguously draw the following conclusions from this section, however: The radial direction of the Calabi-Yau cone outside of the X_5 occupied by the eigenvalues is to be topologically identified with the radial direction in global AdS_5. This direction emerges independently of and orthogonally to the manifold X_5. The radial coordinate may be probed using operators with large R charge and identifying the dual string theory states. It is of great interest to obtain more information using this approach, such as the warping of spacetime and the redshift of AdS as a function of radius.

9 Discussion and future directions

One main objective of this paper has been to show that a framework now exists for performing precise computations in many strongly coupled $\mathcal{N} = 1$ conformal field theories. This is of interest because these theories are dual to compactifications of type IIB string theory on Sasaki-Einstein spaces. We have seen how the Sasaki-Einstein manifold emerges as the semiclassical limit of the ground state of a commuting matrix model. We have then found that the spectrum of certain non-BPS supergravity and stringy excitations may be reproduced exactly as excited states in the matrix quantum mechanics.

The basic setup has exploited a connection that all these field theories have an effective (local) $\mathcal{N} = 4$ SYM description on moduli space, and that one should copy strategies that worked in $\mathcal{N} = 4$ SYM by analogy and a use of local concepts on moduli space. In particular, the formalism used in this paper required the introduction of a measure that was determined by solving a differential equation on the moduli space. If one can find a closed form expression for the corresponding measure in various cases (let us say the conifold), it would be possible to test this proposal further.

A very important result that follows from our proposed measure is that a particular Sasaki-Einstein slice of the Calabi-Yau cone is singled out by a saddle point calculation. We checked that the volume of this manifold is properly normalized in field theory units: we had no additional free parameters in matching the BMN limits. These volumes are also related to the gravitational calculation of the conformal anomalies of the field theory.

These matchings show that the conjectured framework can precisely capture quantitative

aspects of strongly coupled theories. The ultimate objective of this research program is to provide a description of situations where no other approach seems feasible, such as when the dual spacetime develops a region of high curvature. However, before reaching that point, more computations should be done.

It is clear that the calculations that have been done here can be improved further and one might be able to go beyond BMN limits to capture more information about string motion in these geometries. Ideally, one would want to derive that the string motion should obey the equations of motion associated to a non-linear sigma model on the corresponding AdS dual geometry.

It is also important to understand more precisely to what extent the approximations that we have described are applicable, and when they break down.

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A Logarithmic scaling of the Green's function

In this appendix we show that the Green's function satisfying

$$-\nabla_6^6 s(r,r',\theta,\theta') = -\left(\frac{1}{r^5}\frac{d}{dr}r^5\frac{d}{dr}+\frac{1}{r^2}\nabla_5^2\right)^3 s(r,r',\theta,\theta') = 64\pi^3\delta^{(6)}(r,r',\theta,\theta')\,, \qquad (84)$$

has a logarithmic scaling under $r, r' \to \alpha r, \alpha r'$, as advertised in the main text. Recall that r is the radial direction in the cone (7), whereas the θ are coordinates on the five dimensional base manifold.

One could find the Green's function using a standard partial wave expansion for this Laplace-like equation. However, the symmetry $r \leftrightarrow r'$, crucial for our purposes, may be kept manifest as follows. Consider the eigenmodes of the related equation

$$-\nabla_6^2\phi_\lambda(r,\theta) = \lambda\phi_\lambda(r,\theta)\,. \qquad (85)$$

These modes give a complete basis of functions. There is an infinite degeneracy for each value of λ given by the modes

$$\phi_\lambda(r,\theta) = \Phi_{\lambda,\nu}(r)\Theta_\nu(\theta)\,, \qquad (86)$$

where

$$-\nabla_5^2\Theta_\nu(\theta)=\nu^2\Theta_\nu(\theta)\,,\qquad \int d\theta\sqrt{g_5}\Theta^*_\nu(\theta)\Theta_{\nu'}(\theta)=\delta_{\nu,\nu'}\,. \tag{87}$$

The eigenvalues ν^2 are discrete and the lowest is $\nu=0$, corresponding to a constant mode on the base of the cone. For each value of ν, the radial functions are normalised as

$$\int dr r^5\Phi_{\lambda,\nu}(r)\Phi_{\lambda',\nu}(r)=\delta(\lambda-\lambda')\,. \tag{88}$$

The delta function may now be written

$$\delta^{(6)}(r,r',\theta,\theta')=\sum_\nu\int_0^\infty d\lambda\Phi^*_{\lambda,\nu}(r)\Phi_{\lambda,\nu}(r')\Theta^*_\nu(\theta)\Theta_\nu(\theta')\,. \tag{89}$$

Solving the equation (85) for the radial part of the mode (86) and imposing the normalisation (88), one obtains the Bessel function

$$\Phi_{\lambda,\nu}(r)=\frac{J_{\sqrt{4+\nu^2}}(\sqrt{\lambda}\,r)}{\sqrt{2}r^2}\,. \tag{90}$$

Note that this expression is real. We may now use this expression to solve for the Green's function in (84). Naïvely, we would like to write the following

$$s_{\text{naïve}}(r,r',\theta,\theta')=\sum_\nu\int_0^\infty\frac{64\pi^3 d\lambda}{\lambda^3}\Phi_{\lambda,\nu}(r)\Phi_{\lambda,\nu}(r')\Theta^*_\nu(\theta)\Theta_\nu(\theta')\,. \tag{91}$$

Although this expression formally solves the equation (84), it is divergent. The divergence arises as $\lambda\to 0$ in the integrand of the $\nu=0$ term. This problem is entirely expected, due to the fact that the sixth order equation (84) has zero modes. Specifically, the six modes are: $\{1,\log r,r^{\pm 2},r^{\pm 4}\}$. We can deal with this divergence as follows.

Firstly, regularize the divergent part of the naïve expression (91):

$$s_\epsilon=\frac{32\pi^3}{\text{Vol}(X_5)}\int_\epsilon^\infty\frac{d\lambda J_2(\sqrt{\lambda}\,r)J_2(\sqrt{\lambda}\,r')}{r^2r'^2\lambda^3}+s_{\nu>0}\,, \tag{92}$$

where $s_{\nu>0}$ contains the terms in (91) with $\nu>0$:

$$s_{\nu>0}=\sum_{\nu>0}\int_0^\infty\frac{32\pi^3 d\lambda}{r^2r'^2\lambda^3}J_{\sqrt{4+\nu^2}}(\sqrt{\lambda}\,r)J_{\sqrt{4+\nu^2}}(\sqrt{\lambda}\,r')\Theta^*_\nu(\theta)\Theta_\nu(\theta')\,. \tag{93}$$

The integral over λ in this last expression can be performed to obtain a hypergeometric function. Performing the integral will break the symmetry $r\leftrightarrow r'$ however, as the result depends on which of r and r' is bigger.

We can now obtain a finite 'renormalised' Green's function via a minimal substraction

$$s=\lim_{\epsilon\to 0}\left(s_\epsilon+\frac{\pi^3\log\epsilon}{2\text{Vol}(X_5)}\right)\,. \tag{94}$$

The expression (94) solves the equation for the Green's function (84) and is manifestly symmetric in $r \leftrightarrow r'$. However, we need to check that it is regular as $r \to 0$ with r' fixed. Taking $r \ll r'$ we obtain

$$s_\epsilon(r \ll r') = \frac{4\pi^3}{\mathrm{Vol}(X_5)} \int_{\epsilon r'^2}^{\infty} \frac{d\lambda J_2(\sqrt{\lambda})}{\lambda^2} + \mathcal{O}(r/r')\,, \tag{95}$$

which via (94) leads to

$$s(r \ll r') = -\frac{\pi^3 \log r'}{\mathrm{Vol}(X_5)} + \text{const.} + \mathcal{O}(r/r')\,, \tag{96}$$

where the constant is unimportant, as the Green's function is only defined up to a constant in any case. This expression is manifestly finite as $r \to 0$.

The expression (96) provides a further nontrivial check of the result (94) as follows. Integrating over a ball of large radius r'

$$-\int_{B_{r'}} dr d\theta \sqrt{g_6}\, \nabla_6^6 s = \pi^3 \left[r^5 \frac{d}{dr} \left(\frac{1}{r^5} \frac{d}{dr} r^5 \frac{d}{dr} \right)^2 \log r \right]^{r'} = 64\pi^3\,, \tag{97}$$

as required by (84).

From (94) or (96) it is easy to see that the Green's function obeys the logarithmic scaling advertised in (11)

$$s(\alpha r, \alpha r', \theta, \theta') = s(r, r', \theta, \theta') - \frac{\pi^3 \log \alpha}{\mathrm{Vol}(X_5)}\,. \tag{98}$$

B Holomorphic polynomials are eigenfunctions at large N

In this appendix we show that wavefunctions of the form

$$\psi = \psi_0 \mathrm{Tr} P(z) = \sum_i P(z_i) e^{-\sum_j K_j}\,, \tag{99}$$

for $P(z)$ a holomorphic polynomial in z with all terms of degree J, which in turn is a holomorphic coordinate on the Calabi-Yau cone with fixed conformal dimension c, are eigenfunctions of the Hamiltonian (12) to leading order at large N. Some, but not all, of these arguments essentially appear in appendix A of [5]. These arguments go through if P is not holomorphic, but simply a harmonic function on the Calabi-Yau cone.

Holomorphy implies $\nabla^2 P(z) = 0$ and scaling dimension c of z implies $r\partial_r z = cz$. Straightforward algebra then shows that

$$H\psi = (E_0 + cJ)\psi + \psi_0 \sum_i \nabla_i P(z_i) \cdot \sum_{j \neq i} \nabla_i s(z_i, z_j)\,. \tag{100}$$

We now show that the last term vanishes to leading order at large N.

In the continuum large N limit, the last term in (100) is proportional to

$$\int d^6x\rho(x)\nabla_x P(z_x)\cdot\int d^6y\rho(y)\nabla_x s(z_x,z_y)\,. \tag{101}$$

If $P(z)$ is a polynomial with not too high degree, as in the case $P(z)=z^J$ we considered in section 8 above, then the backreaction of $P(z)$ onto the eigenvalue distribution is subleading at large N. Therefore in (101) we may take the $\rho(x)$ to be the ground state (26). In particular, this distribution adds no extra dependence on the coordinates θ of X_5. The integral over d^6y includes an integral over X_5. From the fact that $\int d\theta\sqrt{g_5}\Theta_\nu(\theta)=0$ for $\nu>0$ and from the expression (93), only the part of $s(z_x,z_y)$ that is independent of both θ_x and θ_y survives the θ_y integral. Thus (101) is proportional to

$$\int d^5\theta\sqrt{g_5}\partial_r P(z)\,. \tag{102}$$

The final step is now to show that $P(z)$, and hence also $\partial_r P(z)$, is a nontrivial eigenfunction of the Laplacian on X_5, and therefore the integral (102) vanishes. From holomorphy we have

$$\nabla^2 P(z)=\left(\frac{1}{r^5}\frac{d}{dr}r^5\frac{d}{dr}+\frac{1}{r^2}\nabla_5^2\right)P(z)=0\,. \tag{103}$$

The scaling dimension of z implies that each monomial in $P(z)$ is of the form $P_J(z)=r^{cJ}F_J(\theta)$. It is immediately seen that (103) implies that

$$-\nabla_5^2 F_J(\theta)=cJ(cJ+4)F_J(\theta)\,. \tag{104}$$

Therefore $P_J(z)$ is a harmonic of the Laplacian on X_5, as we required.

The upshot of the preceding paragraph is that (101) does indeed vanish and hence the holomorphic polynomial does give an eigenfunction, as claimed.

C Coherent states and orthogonality

In the large N limit, single trace operators are supposed to be related to single string states. To the extent that these are free, one can build coherent states of these traces. Formally, we would want to consider a coherent state as an exponential of a raising operator. In our identification, we have said that $\mathrm{Tr}h(x)$ is a single graviton state, so a coherent state of gravitons would be described formally by

$$\psi_{\rm coh}\sim e^{\alpha\mathrm{Tr}h(x)}\psi_0\,. \tag{105}$$

We can try to understand the distribution of particles on the cone that is associated to this wave function. We do this by thinking of α as a formal parameter (usually $h(x)$ will grow faster at infinity than the decay of ψ_0, which is just gaussian decay).

If we replace $\mathrm{Tr}h(x)$ by $\int \rho(x)h(x)$, as is required for the large N limit, we can repeat the arguments made in studying equation (25) to show that once again the dominant semi-classical density of eigenvalues is a singular distribution. It was suggested in [2] that having singular distributions of particles in the saddle point limit is exactly the type of situation that leads to classical gravity solutions. This supports the proposal made above for the wave functions associated to non-BPS gravitons.

Unfortunately, it seems that the corresponding wave functions are not eigenfunctions of the full effective Schrödinger equation with the measure added. This has already been seen for the case of $\mathcal{N} = 4$ SYM [5]. There is no new effect that shows up in this more general case that is not there in the case of maximal supersymmetry. Moreover, as shown in appendix B, they become eigenstates to leading order in the large N limit.

One can also show that these single trace wave functions (including the measure) are approximately orthogonal to the ground state and to each other. One would need to evaluate the overlap

$$\int e^{-r^2} \mu^2 \mathrm{Tr}h_1(x)\mathrm{Tr}h_2(x)\,. \tag{106}$$

The idea to show approximate orthogonality is that the overlap is dominated by the saddle of the ground state. Then the dependence of μ^2 on eigenvalue i, that is written as

$$\mu_i^2 = \exp(-\int \rho(x)s(x_i,x))\,, \tag{107}$$

can be approximated by a function that depends only on the radial variable r_i, but not on the angular variables. Using the product decomposition of h into a radial and angular part, we see that the orthogonality of the angular part of the wave functions makes these single trace perturbations orthogonal to each other (unless $h_1 \sim h_2^*$).

For $\mathcal{N} = 4$ SYM this statement is exact and follows from orthogonality of different unitary representations of the $SO(6)$ symmetry group. This orthogonality is also exact in $\mathcal{N} = 1$ cases when h_1 and h_2 have different R-charges. These arguments can be extended further, and suggest that the standard large N counting arguments are applicable in some generality.

The arguments show that the details of the calculations depend on various properties of harmonic analysis on the Saski-Einstein manifold and the particular saddle point we found that determines the vacuum structure. To our knowledge, the most general study of the

spectrum of the scalar Laplacian has been done in [28].

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