An anomalous flow: everywhere locally absolutely unstable yet globally stable

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In the fluid-dynamical linear instability theory of slowly spatially developing flows, a considerable amount of effort has been put into attempts to connect the instability characteristics of the locally almost parallel flow at each spatial station x with the global characteristics of the flow. Other authors have shown that if the flow is locally stable everywhere, or if it has a region of at worst local *convective* instability, then it is necessarily globally stable. Furthermore, it may remain globally stable even in the case where there is a bounded region of local *absolute* instability, as long as that region is small enough. It has generally been implicitly assumed, however, that when the region of local absolute instability becomes sufficiently large, the flow will undergo a bifurcation and become globally unstable. This paper shows that this intuitively 'obvious' fact is incorrect, by exhibiting a flow based on the linearized Ginzburg–Landau equation which is *everywhere* locally absolutely unstable, and yet is globally stable.

An appendix discusses the mathematical definitions of global stability and instability in spatially developing flows and demonstrates that existing definitions are inadequate (in particular, by presenting a globally convectively unstable flow). New definitions are proposed which overcome the deficiencies.

1. Introduction: Local and Global Concepts

The linear stability of steady two-dimensional parallel shear flows may be investigated by studying the spatio-temporal evolution of infinitesimal perturbations to the basic flow. Denoting the streamwise spatial coordinate by x and the transverse (cross-stream) coordinate by z, we suppose that there is some basic flow $V(z)$ entirely in the x-direction, and introduce perturbations in the form of elementary instability waves via a stream-function $\psi(x, z, t) = \phi(z) \exp(\mathrm{i}kx - \mathrm{i}\omega t)$. The mode shape $\phi(z)$ can then be shown to satisfy an Orr–Sommerfeld differential equation (Drazin & Reid 1981), with appropriate boundary conditions; these boundary conditions can only be satisfied for particular values of k and ω, connected by a dispersion relation

$$\mathscr{D}(k, \omega) = 0. \tag{1.1}$$

We can therefore model the evolution of instability waves in the (x, t)-plane, completely suppressing any variations in the z-direction, by associating a differential operator with the dispersion relation (1.1) so that perturbations $A(x, t)$ satisfy

$$\mathscr{D}\left(-\mathrm{i}\frac{\partial}{\partial x}, \mathrm{i}\frac{\partial}{\partial t}\right) A(x, t) = F(x, t). \tag{1.2}$$

Here $F(x, t)$ is a forcing term which we introduce to enable us to model external excitation of the flow. Note that, because we are considering parallel flows where the properties of the system do not depend on the streamwise coordinate x, the differential operator $\mathscr{D}$ has constant coefficients.

Of course, a model equation such as (1.2) may also arise in many other situations: for example, in §2 we shall concentrate on the Ginzburg–Landau equation, which (as well as being pertinent to shear flows) may be derived for Rayleigh-Bénard convection using a multiple-scales analysis (Newell & Whitehead 1969). The details of the circumstances under which (1.2) was obtained are unimportant as far as what follows is concerned; we require only that the basic flow under consideration is *open*, so that fluid particles leaving the physical domain of interest are not recycled into it (thus excluding *closed* systems such as Taylor–Couette flow), and that the flow is not dominated by physical boundaries, with any downstream waves reaching them being convected through rather than being reflected. These considerations ensure that we may model the flow using an essentially infinite spatial domain $x \in (-\infty, \infty)$.

The traditional method of investigating the stability of (1.2) was to look for time-periodic solutions proportional to $\exp(-\mathrm{i}\omega t)$; the existence of such a solution with $\operatorname{Im}\omega > 0$ implied instability. However, during the 1950s and subsequent decades it became increasingly apparent that this approach failed to distinguish between two important different types of instability: *convective* and *absolute* (Sturrock 1961; Briggs 1964; Bers 1983; Huerre 1987; Landau & Lifshitz 1987†; Huerre & Monkewitz 1990). A different technique was therefore required, and various methods of classifying stable and unstable flows have since been developed; for definiteness, we shall use the generally accepted definitions given in the review article by Huerre & Monkewitz (1990).

We define the linear stability of a parallel flow in terms of the behaviour of the Green's function $G(x,t)$ (i.e., the response of the system to a point-source impulse $\delta(x)\delta(t)$, where δ denotes the Dirac delta-function) in the limit $t \to \infty$. We say that the flow is *stable* if

$$\left.\begin{array}{l}
\displaystyle\lim_{t\to\infty} G(x,t) = 0 \qquad \text{along all rays } x/t = \text{const.} \\[2ex]
\text{and } \textit{unstable} \text{ if} \\[1ex]
\displaystyle\lim_{t\to\infty} G(x,t) = \infty \qquad \text{along at least one ray } x/t = \text{const.} \\[2ex]
\text{Furthermore, an unstable flow is } \textit{convectively unstable} \text{ if} \\[1ex]
\displaystyle\lim_{t\to\infty} G(x,t) = 0 \qquad \text{for all fixed } x \\[2ex]
\text{and } \textit{absolutely unstable} \text{ if} \\[1ex]
\displaystyle\lim_{t\to\infty} G(x,t) = \infty \qquad \text{for all fixed } x.
\end{array}\right\} \qquad (1.3)$$

Qualitatively, in a convectively unstable flow the response to a point-source impulse grows continuously but is simultaneously convected away, so that an observer at any fixed location eventually sees the medium return to its undisturbed state; whereas in an absolutely unstable flow, the entire medium is eventually contaminated by the response to such an impulse. In practice, the stability of a flow can usually be determined directly from the dispersion relation (1.1) by looking for saddle points of ω in the complex k plane; see Huerre & Monkewitz (1990) for a description.

Now consider a non-parallel flow where the properties of the system vary *slowly* with the spatial coordinate, over a length scale of order ε^{-1} where $\varepsilon \ll 1$ is a small parameter. Typically, the behaviour of the flow is governed at each point by one or more physical quantities (such as the local momentum thickness, etc.) whose values change over the slow spatial scale εx; such a flow may therefore be modelled by allowing each coefficient μ, say, of the differential operator $\mathscr{D}$ in (1.2) to vary as a function of εx. Correspondingly, at each streamwise location we can consider the flow to have a local dispersion relation $\mathscr{D}(k, \omega; \varepsilon x) = 0$.

† An unfortunate difference in terminology arises in Landau & Lifshitz's book (in its second edition), which was originally written in Russian. The term 'convect*ed* instability' is used throughout the English translation for the phenomenon which we shall consistently refer to here as 'convect*ive* instability'; the translation confusingly uses the latter term to instead describe the convection currents which arise in a fluid at large Rayleigh number. This is not in accordance with common modern usage.

At each station $X = \varepsilon x$ the flow is locally almost parallel, and the response of this almost parallel flow to an impulse at the location X is initially identical to that of a completely parallel flow in which each coefficient μ is frozen at the particular value $\mu(X)$ everywhere. Since the above definitions are directly applicable to this fictitious parallel flow, we can thus find, for each station X, the local stability or instability characteristics of the non-parallel flow. Hence, a flow is said to be *locally stable* (respectively *locally convectively unstable* or *locally absolutely unstable*) at a spatial location X if the fictitious parallel flow corresponding to the local coefficients $\mu(X)$ is stable (respectively convectively unstable or absolutely unstable). This is equivalent to saying that the local stability of a non-parallel flow at the point $x = \varepsilon^{-1}X$, for each fixed X, is the same as the stability of a parallel flow which has dispersion relation $\mathscr{D}(k, \omega; X) = 0$.

The *global* stability or instability of a non-parallel flow is in theory determined directly from the definitions (1.3) above, as it would be for a parallel flow. (It is usually impossible to use the dispersion relation in this case.) However, it is no longer sufficient to consider only forcing at the origin, for in a non-parallel flow the response to forcing at one spatial location may differ drastically from the response to forcing at some other location. Therefore, we must consider the response $G_{x_0}(x,t)$ to impulsive forcing at a general spatial location $x = x_0$, i.e., we take $F(x,t) = \delta(x - x_0)\delta(t)$ in (1.2); the flow is *globally stable* if $G_{x_0}(x,t) \to 0$ along all rays $(x - x_0)/t =$ const., for all values of x_0, and *globally unstable* if $G_{x_0}(x,t) \to \infty$ along at least one ray, for at least one x_0. Global convective and absolute instability are defined similarly. This topic is discussed in more detail in the Appendix (so as not to distract the reader from the main result of this paper), where it will be seen that these definitions are still insufficient for some purposes and need yet further refinement; however, this need not concern us for the moment.

It is clearly of great interest to connect the local instability characteristics, as a function of x, to the global characteristics of the flow, and much work has been done in this area. (An excellent review of fundamental developments can be found in Huerre & Monkewitz 1990.) In particular, it has been shown that if the flow is locally stable everywhere, or everywhere except for a bounded region of local convective instability, then it must be globally stable also. Moreover, it has been demonstrated that the flow may remain globally stable even when it exhibits a small bounded region of local *absolute* instability. Once this region exceeds a certain critical size, however, the flow undergoes a bifurcation to global absolute instability via a self-excited global mode (Chomaz *et al.* 1988; Hunt & Crighton 1991; Huerre & Monkewitz 1990). The major problem is in estimating this critical size; in the next section we show that it may in fact be infinite.

2. The Ginzburg–Landau Model

The linearized Ginzburg–Landau equation

$$\frac{\partial A}{\partial t} + U\frac{\partial A}{\partial x} = \mu A + \gamma\frac{\partial^2 A}{\partial x^2} + F(x,t) \tag{2.1}$$

arises (together with a cubic nonlinear term) in perturbation analyses of instability waves in many fluid-dynamical systems close to marginal instability. (Early seminal derivations pertinent in this context include those of Newell & Whitehead 1969 and Stewartson & Stuart 1971.) Here A is the complex amplitude of the marginal wavepacket $A(x,t)\exp(\mathrm{i}k_c x - \mathrm{i}\omega_c t)$, where the suffix c indicates the critical neutral condition, and μ is a (possibly complex) constant representing the degree of supercriticality of the system. U is a real convection velocity and γ is a complex constant with $\operatorname{Re}\gamma > 0$; $F(x,t)$ represents external forcing of the system. This flow is strictly parallel, and its stability properties depend on the real part of the constant control parameter μ: it is stable when $\operatorname{Re}\mu < 0$, convectively unstable when $0 < \operatorname{Re}\mu < U^2 \operatorname{Re}\gamma/4|\gamma|^2$, and absolutely unstable when $\operatorname{Re}\mu > U^2 \operatorname{Re}\gamma/4|\gamma|^2$. See Huerre (1987) or Hunt & Crighton (1991), for example, for methods of deriving these results.

The non-parallel flow which results when μ is allowed to vary with spatial location according to $\mu = \mu(\varepsilon x)$ has been used as a general model for the growth of instabilities and global modes in arbitrary slowly spatially developing flows (Huerre & Monkewitz 1990; Monkewitz 1990; Chomaz *et al.* 1988; Hunt & Crighton 1991; Monkewitz *et al.* 1993; Hunt 1995). The case of linear variation of μ with respect to the spatial coordinate, i.e., $\mu(\varepsilon x) = \mu_0 + \varepsilon\mu_1 x$ where μ_0 and μ_1 are complex constants, will interest us here. At each streamwise location x, the local instability characteristics are governed by the real part of the local value of $\mu(\varepsilon x)$, viz. $\operatorname{Re}\mu_0 + \varepsilon \operatorname{Re}\mu_1 x$. It is clear that whenever $\operatorname{Re}\mu_1 \neq 0$, there are three distinct regions of space: a locally stable region (where $\operatorname{Re}\mu_0 + \varepsilon\operatorname{Re}\mu_1 x < 0$), a locally convectively unstable region (where $0 < \operatorname{Re}\mu_0 + \varepsilon\operatorname{Re}\mu_1 x < U^2 \operatorname{Re}\gamma/4|\gamma|^2$), and a locally absolutely unstable region (where $\operatorname{Re}\mu_0 + \varepsilon\operatorname{Re}\mu_1 x > U^2\operatorname{Re}\gamma/4|\gamma|^2$). However, when $\operatorname{Re}\mu_1 = 0$ the situation is radically different: the flow has *exactly* the same local instability characteristics at every spatial location, governed by $\operatorname{Re}\mu_0$ alone and independent of the exact value of $\operatorname{Im}\mu_1$. In particular, if $\operatorname{Re}\mu_0 > U^2\operatorname{Re}\gamma/4|\gamma|^2$ then the flow is uniformly locally absolutely unstable everywhere.

However, to determine the *global* characteristics of this flow it is necessary to know the asymptotic properties of the solution of the entire non-parallel system. This is in general an extremely difficult problem, but in the present situation we may make use of the results of Hunt & Crighton (1991) who obtained *exact* solutions for the Green's function $G(x,t)$ of (2.1) in the cases of both linear and quadratic variation of μ with the spatial coordinate x. In the current linear case, the solution is

$$G(x,t) = \tfrac{1}{2}(\pi\gamma t)^{-1/2}\exp\left\{\mu_0 t - \frac{(x-Ut)^2}{4\gamma t} + \tfrac{1}{2}\varepsilon\mu_1 x t + \tfrac{1}{12}\varepsilon^2\mu_1^2\gamma t^3\right\}\mathrm{H}(t), \tag{2.2}$$

where $\mathrm{H}(t)$ denotes the Heaviside unit step function, and the power of $-\frac{1}{2}$ signifies the principal value of the root. That this is indeed the correct Green's function can easily be verified by elementary means, by direct differentiation and substitution into (2.1); the only difficulty is in checking that the boundary conditions are satisfied. The correct conditions to apply here are that $G(x,t) = 0$ for all $t < 0$ (trivially true) and that $G(x,t) \to \delta(x)$ as $t \to 0^+$.

It is clear from (2.2) that when $\operatorname{Re}\mu_1 = 0$, the peak of the wavepacket (i.e., the maximum of $|G(x,t)|$ over all x, at a fixed time t) occurs at $x = Ut$, where it is given by

$$|G|_{\max}(t) = \tfrac{1}{2}(\pi|\gamma|t)^{-1/2}\exp\left\{\operatorname{Re}\mu_0\, t + \tfrac{1}{12}\varepsilon^2\mu_1^2\operatorname{Re}\gamma\, t^3\right\}\mathrm{H}(t). \tag{2.3}$$

As $t \to \infty$, the behaviour is dominated by the t^3 term in the exponential; since μ_1^2 is real and negative, and $\operatorname{Re}\gamma > 0$, we see that $|G|_{\max} \to 0$. Moreover, this result clearly also holds when the initial forcing impulse occurs at some location other than the origin: for if $F(x,t) = \delta(x-x_0)\delta(t)$ in (2.1), then the substitution $x' = x - x_0$ transforms the equation back to the previous case of forcing at the origin, but with μ_0 simply replaced by $\mu_0 + \varepsilon\mu_1 x_0$ (which does not affect the behaviour of (2.3) as $t \to \infty$). Hence the flow is globally stable (since on any ray, $|G(x,t)| \leqslant |G|_{\max}(t) \to 0$), even though it is locally absolutely unstable everywhere.

This result is certainly counter-intuitive, and demonstrates that it is quite possible for a flow to have an extremely large (or, as in this case, doubly-infinite) region of local absolute instability and yet still be globally stable to perturbations. A typical plot of the spatio-temporal evolution of the solution (2.2) is shown in figure 1; contour plots of the magnitude and phase of the wavepacket are shown in figure 2. It can be seen that the initial behaviour is for the wavepacket to spread out in both directions from the origin, as it must do, for the flow is locally absolutely unstable there. However, the wavepacket then reaches other spatial locations x and starts to be modified by the new values of μ there; although the real part of $\mu(\varepsilon x)$ is identical to its value at the origin, its imaginary part is different, and this affects the phase of the wave. These differing phase changes at each spatial location interact with each other, and the overall effect is that of destructive interference: the wave is cancelled out by phase interactions caused by its own dispersion.

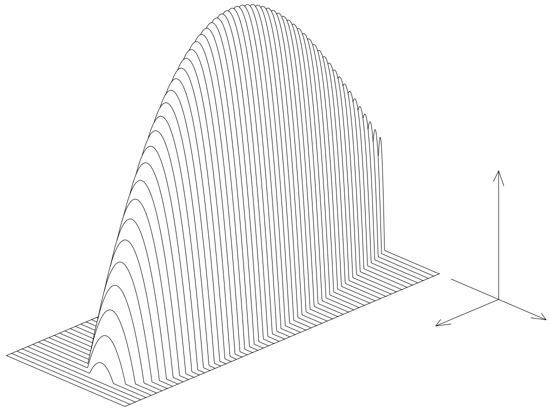

Figure 1. A plot of the spatio-temporal development of $|G(x,t)|$, where G is given by equation (2.2). The vertical axis measures $\ln|G|$, cut off below $|G| = \mathrm{e}^{-100}$; the spatial axis measures $-1700 < x < 1700$ from left to right across the plot; and $|G|$ is plotted as a function of x for 60 equal time-increments in the range $0 < t \leqslant 30$, with t increasing towards the viewer. Parameter values are $\mu_0 = 20$, $\varepsilon\mu_1 = 0.1\,\mathrm{i}$, $\gamma = 35(1-\mathrm{i})$ and $U = 15$. The wavepacket clearly grows at a linear rate (on this logarithmic scale) initially, governed by the local absolute instability; but after a short while it begins to decay at a nonlinear rate.

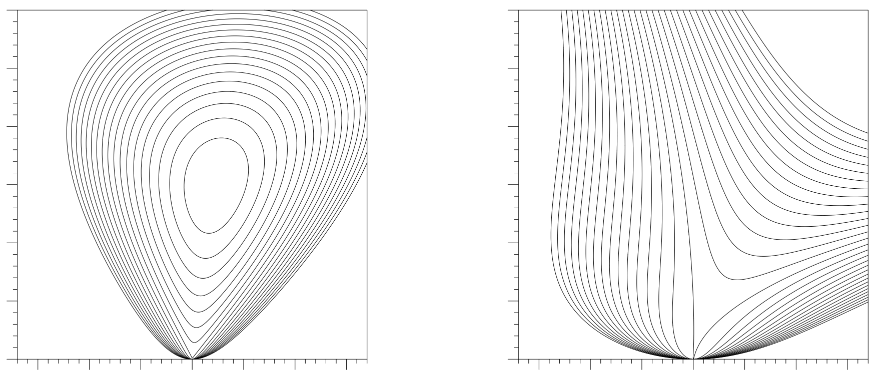

Figure 2. Contour plots of $G(x,t)$, with the same parameter values as in figure 1. Left: The magnitude $|G|$. Eighteen contours are plotted, in a geometric sequence from $|G| = \mathrm{e}^{-250}$ (outermost) to e^{+175} (innermost) in ratios of e^{25}. Right: The phase of G, calculated from the imaginary part of the argument of the exponential in (2.2) adjusted by $-\frac{1}{2}\arg\gamma$. Contour values range from $-1500+\frac{\pi}{8}$ (top left, around the outside, to bottom right) to $+1500+\frac{\pi}{8}$ (top right) in increments of 100. Note that the contour of phase $\frac{\pi}{8}$ $(= -\frac{1}{2}\arg\gamma)$ has a discontinuity of gradient at the origin. As $t \to \infty$, all phase contours converge onto the curves $x \sim 7(1 \pm \sqrt{7/6}\,)t^2$.

It has been suggested that the mechanism which is responsible here for stabilizing the flow against the underlying absolute instabilities may have a physical origin. For instance, a plane vortex sheet is inherently unstable to small perturbations, both locally and globally; but Saffman & Baker (1979) have shown that if the sheet expands longitudinally, due to its being stretched, then an increase of length at a rate faster than $t^{1/2}$ stabilizes it against the local Kelvin–Helmholtz instability. Perhaps the complex part of μ_1 in the current model models some kind of stretching of the underlying medium? This question can only be answered in any specific case by examining the origin of the μ_1 term in a particular derivation of (2.1); there is no general answer. Note, however, that if we remove the phase interactions mentioned above from the system by setting μ_1 to zero, then the flow becomes globally absolutely unstable, but *any* purely imaginary value of μ_1, however small, above or below the real axis, results in global stability.

It is not clear to what extent this result could be replicated experimentally. Perhaps computational experiments could determine whether some similar outcome could be achieved in a finite (as opposed to doubly-infinite) flow domain. It is, however, certainly a *theoretical* anomoly worthy of note.

The author wishes to express the debt he owes to the late Prof. D. G. Crighton, FRS, for many helpful conversations on this and other topics, and for his guidance and support over many years.

Appendix A. Definitions of Global Stability and Instability

Consider again the exact solution (2.2) for the Green's function for linear spatial variation of μ, with general complex values of μ_1. At any given fixed time t, the real part of the argument of the exponential vanishes at the two spatial locations

$$x = \Big\{U \pm \frac{|\gamma|}{\operatorname{Re}\gamma}\sqrt{4\operatorname{Re}\mu_0\operatorname{Re}\gamma + 2\varepsilon U\operatorname{Re}\mu_1\operatorname{Re}\gamma\, t + \varepsilon^2\big(|\gamma|^2(\operatorname{Re}\mu_1)^2 + \tfrac{1}{3}\operatorname{Re}\mu_1^2\gamma\operatorname{Re}\gamma\big)t^2}\Big\}\, t + \frac{|\gamma|^2\operatorname{Re}\mu_1}{\operatorname{Re}\gamma}\,\varepsilon t^2. \quad \text{(A 1)}$$

As $t \to \infty$, these occur at

$$x \sim \frac{|\gamma|}{\operatorname{Re}\gamma}\Big\{|\gamma|\operatorname{Re}\mu_1 \pm \sqrt{|\gamma|^2(\operatorname{Re}\mu_1)^2 + \tfrac{1}{3}\operatorname{Re}\mu_1^2\gamma\operatorname{Re}\gamma}\Big\}\,\varepsilon t^2. \quad \text{(A 2)}$$

In the two spatial regions outside these so-called 'edges' (Hunt & Crighton 1991) the Green's function is exponentially small in t as $t \to \infty$, whilst between them it is exponentially large (subject, of course, to the square root in (A 2) being real).

Hence, if $|\gamma|^2(\operatorname{Re}\mu_1)^2 + \frac{1}{3}\operatorname{Re}\mu_1^2\gamma\operatorname{Re}\gamma > 0$ then there is a region of space in which the wavepacket grows as $t \to \infty$. If in addition $\operatorname{Re}\mu_1^2\gamma < 0$ (which can occur simultaneously), then we note from (A 2) that the two 'edges' occur at values of x with the same sign – that is, although the wavepacket is continuously growing, the region in which it is doing so is moving away from the origin, and elsewhere it is damped. (Once again, for the same reasons as in §2, this result holds independently of the spatial location x_0 of the initial impulse.) This is clearly a case which should be considered to be *global convective instability*; yet the definitions given in equations (1.3) classify it as global stability, because they consider only rays $x/t =$ const. rather than modified rays of the form $x/t^2 =$ const. which would be required here. Consequently, the definitions of §1 are inadequate to cover the range of behaviours which may be exhibited by simple non-parallel flows.

We therefore need to explore suitable alternative definitions. One approach is to consider global instabilities in terms of time-periodic self-excited global modes (Huerre & Monkewitz 1990); but, although certainly useful and of great practical and experimental importance, this approach cannot distinguish globally convectively unstable flows at all, and is restricted to differentiating between 'stability' and 'instability' alone. (In many ways, it is similar to the 'traditional method' of investigating stability mentioned in §1.) We could instead return to the original definitions used in the field of plasma physics, as described for instance by Bers (1983): he defined a convectively unstable flow to be one in which $G(x,t) \to 0$ as $t \to \infty$ for any fixed x, but $G(x_V,t) \to \infty$ where $x_V(t)$ is a point 'moving within the growing envelope of G'. However, although this definition does cover the above example correctly, it is mathematically extremely inconvenient for proving general results. It also requires modification to take account of the spatial location x_0 of the initial forcing impulse, which is of vital importance in a non-parallel flow. Therefore the present author instead suggests, as the beginning of a consistent set of definitions of terms, the following mathematically precise formulation (which will be found useful in a theoretical setting, but is likely to be of very limited practical use in an experimental situation).

Define $G_{x_0}(x,t)$ to be the (linear) response to forcing $\delta(x-x_0)\delta(t)$, i.e., to a unit impulse at $x=x_0$. We will say that the flow is *globally stable* if

$$\max_{x\in\mathbb{R}} |G_{x_0}(x,t)| \to 0 \qquad \text{as } t\to\infty\text{, uniformly in } x_0 \text{ for all } x_0\in\mathbb{R}$$

and *globally unstable* if

$$\max_{x\in\mathbb{R}} |G_{x_0}(x,t)| \to \infty \qquad \text{as } t\to\infty\text{, for at least one } x_0.$$

A globally unstable flow is *globally convectively unstable* if

$$|G_{x_0}(x,t)| \to 0 \qquad \begin{array}{l}\text{as } t\to\infty\text{, for each fixed } x \text{ and } x_0,\\ \text{locally uniformly in both } x \text{ and } x_0\end{array}$$

and *globally absolutely unstable* if

$$|G_{x_0}(x,t)| \to \infty \qquad \text{as } t\to\infty\text{, for all fixed } x\text{, for at least one } x_0. \tag{A 3}$$

We use the term 'locally uniformly' in the definition of global convective instability here to mean that the convergence is uniform in both x and x_0 whenever they are contained in any closed bounded interval in $\mathbb{R}$ (though the convergence is not necessarily uniform over the whole of $\mathbb{R}$). Note that not all flows are classified by (A 3); there are certainly flows which fall between the categories, just as has been the case with previous classifications, even (1.3).

These definitions, although at first sight complicated, give consistent results for the global response of the flow to arbitrary forcing over some given length of time. For example, suppose that $A(x,t)$ is the response to forcing $F(x,t)$, where $F=0$ for $t<0$ and $t>T$ say, subject only to the requirement that $\int_0^T\int_{-\infty}^{\infty}|F(x,t)|\,\mathrm{d}x\,\mathrm{d}t$ be finite. Then A is given by

$$A(x,t) = \int_0^T\int_{-\infty}^{\infty} G_{x_0}(x,t-t_0)F(x_0,t_0)\,\mathrm{d}x_0\,\mathrm{d}t_0. \tag{A 4}$$

Suppose also that the flow is globally stable, as defined above. We have

$$\begin{aligned}\max_{x\in\mathbb{R}}|A(x,t)| &\leqslant \max_{x\in\mathbb{R}}\int_0^T\int_{-\infty}^{\infty}|G_{x_0}(x,t-t_0)|\,|F(x_0,t_0)|\,\mathrm{d}x_0\,\mathrm{d}t_0\\ &\leqslant \int_0^T\int_{-\infty}^{\infty}\Big(\max_{x\in\mathbb{R}}|G_{x_0}(x,t-t_0)|\Big)\,|F(x_0,t_0)|\,\mathrm{d}x_0\,\mathrm{d}t_0.\end{aligned}$$

But from the definition of global stability, for any given small $\delta>0$ there exists some τ such that $\max_x|G_{x_0}(x,t)|<\delta$ for all $t>\tau$, where τ does *not* depend on x_0 (because of the uniform convergence in x_0). So whenever $t>\tau+T$, we have $\max_x|G_{x_0}(x,t-t_0)|<\delta$ for any t_0 in the range of integration $[0,T]$. Hence $\max_x|A(x,t)|\to 0$ as $t\to\infty$ as expected: the peak of the wavepacket is damped to zero in the long-time limit for a globally stable flow, under any forcing which lasts for only a finite time.

Similarly, it is easy to show that if the flow is globally convectively unstable, then for any forcing confined to a bounded region of space over a given length of time, $A(x,t)\to 0$ as $t\to\infty$ for any fixed x; so an initially bounded disturbance is eventually convected away, leaving the medium unaffected in the long-time limit.

The definitions given in (A 3) therefore provide a self-consistent mathematical basis on which to form ideas of global stability and instability in non-parallel flows, and cover cases which previous definitions have either failed to classify or have classified incorrectly. They may also, of course, be used for parallel flows, where the dependence on x_0 is trivial and follows simply from translational invariance.

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