THE FOCUSING NLS EQUATION ON THE HALF-LINE WITH PERIODIC BOUNDARY CONDITIONS

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Abstract. We consider the Dirichlet problem for the focusing NLS equation on the half-line, with Schwartz initial data and with the periodic boundary data $ae^{2i\omega t+i\epsilon}$ at $x=0$. It is known from PDE theory that there exists a unique classical solution to this problem. On the other hand, the associated inverse scattering transform formalism involves the Neumann boundary value for $x=0$. Thus the implementation of this formalism requires the understanding of the "Dirichlet-to-Neumann" map which characterises the associated Neumann boundary value.

We consider this map in an indirect way: we postulate a certain Riemann-Hilbert problem and then prove that the solution of the initial-boundary value problem for the focusing NLS constructed through this Riemann-Hilbert problem satisfies all the required properties. By the results of the PDE theory this solution is the unique solution of the Dirichlet problem.

1. Introduction

We are interested in the following initial-boundary value problem

$$\begin{aligned} iq_t(x,t)+q_{xx}(x,t)+2|q(x,t)|^2q(x,t)=0,\ x>0,\ t>0,\\ q(x,0)=q_0(x),\ 0<x<\infty,\\ q(0,t)=g_0(t),\ 0<t<\infty, \end{aligned} \tag{1.1}$$

where the function $q_0(x)$ belongs to the Schwartz class and $g_0(t)=ae^{2i\omega t+i\epsilon}$, where $a>0,\omega,\epsilon$ are real and the compatibility condition $q_0(0)=g_0(0)$ is satisfied. We will assume here that $-3a^2<\omega<a^2$.

It is known [3] that there exists a unique classical solution of this problem. On the other hand, the inverse scattering transform formalism developed in ([7], [8], [1]), in addition to $q_0(x)$ and $g_0(t)$ also requires the function $g_1(t)=q_x(0,t)$ for $0<t<\infty$. The general

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methodology of [6] is applied to the problem (1.1) in [1], where it is *assumed* that the *unknown* function g_1 is the sum of $2iabe^{2i\omega t+i\epsilon}$ (where $\omega = a^2 - 2b^2,\ b > 0$) and a Schwartz function. The aim of this paper is to prove that this assumption is indeed correct. Then, combining the results of [1] with our proof it is possible to establish rigorously that the solution of the initial boundary value problem (1.1) can be reduced to the solution of a well defined Riemann-Hilbert problem.

In this paper, we proceed as follows. We first consider the problem (1.1) up to a fixed frozen time T. The inverse scattering transform formalism now holds without any extra assumption on g_1. Following [6] we define the functions $a(k), b(k)$ corresponding to the "x-problem" specified by $q_0(x)$ and following [1] we define the functions $A^T(k), B^T(k)$ corresponding to the "t-problem" specified by g_0^T and g_1^T, where $g_0^T = g_0$ for $t \leq T$ and g_1^T is the associated Neumann boundary value whose existence is guaranteed by [3]. For each T one can easily prove that a, b, A^T, B^T satisfy the so-called *global relation* [6].

We next let T vary and define A and B as the limits of A^T, B^T when $T \to \infty$. It is easy to show that the limits exist and that a, b, A, B satisfy the associated global relation.

We then consider the corresponding Riemann-Hilbert problem for $T = \infty$. This Riemann-Hilbert problem was derived in the paper [1] under the assumption that g_1 is the sum of a periodic function and a Schwartz function; however we do *not* make this assumption here. We first prove existence and uniqueness of a solution of this Riemann-Hilbert problem using an appropriate vanishing lemma. We then prove that this solution gives rise to a solution of (1.1) with Dirichlet data g_0 and q_0. Finally, using the asymptotic method of Deift-Zhou [5], we prove the crucial property that the boundary value $q_x(0,t)$ is indeed the sum of a periodic function and a Schwartz function.

2. A Riemann-Hilbert problem

The focusing NLS equation admits the Lax pair

$$\mu_x + ik[\sigma_3, \mu] = Q(x,t)\mu, \tag{2.1a}$$

$$\mu_t + 2ik^2[\sigma_3, \mu] = \tilde{Q}(x,t,k)\mu, \tag{2.1b}$$

where $\sigma_3 = \mathrm{diag}(1, -1)$,

(2.2)

$$Q(x,t) = \begin{bmatrix} 0 & q(x,t) \\ -\bar{q}(x,t) & 0 \end{bmatrix}, \qquad \tilde{Q}(x,t,k) = 2kQ - iQ_x\sigma_3 + i|q|^2\sigma_3.$$

A novel method for analysing initial boundary value problems for integrable nonlinear PDEs was introduced in [6]. This method, which is based on the *simultaneous* spectral analysis of both the x-problem and the t-problem in the Lax pair, was rigorously implemented to the NLS on the half-line with Schwartz initial and boundary conditions in [8]. In the problem (1.1) the initial data are of Schwartz class, thus the scattering and inverse scattering of the x-problem is classical and goes back to the original investigations of Gelfand, Levitan and Marchenko (see [8]). On the other hand, the boundary values at $x = 0$ are perturbations of finite-zone functions, thus the spectral analysis of the t-problem involves aspects of the finite-zone theory. In this paper we will consider the simplest possible case of zero-zone data.

The zero-zone solution of NLS, namely $q(x,t) = q_p(x,t) = ae^{2ibx+2i\omega t+i\epsilon}$ gives rise to the Dirichlet data $ae^{2i\omega t+i\epsilon}$ and also yields $q_x(0,t) = 2iabe^{2i\omega t+i\epsilon}$.

Let us first restrict t to be less than a fixed time T. We define $g_0^T = g_0$, for $0 \leq t \leq T$, and we know that g_0^T is smooth. We need no extra assumptions on $q_x(0,t)$, apart from a minimal regularity. The existence of such a smooth function $g_1^T = q_x(0,t)$, $0 \leq t \leq T$, is guaranteed by the existence and regularity theorem of [3].

We stress that at this point we cannot set $T = \infty$, because we do not know yet that g_1 is a Schwartz perturbation of $2iabe^{2i\omega t+i\epsilon}$.

Now, let b be defined by $\omega = a^2 - 2b^2$, $b > 0$. We will assume here that $a^2 - \omega > 0$ and $b^2 < 2a^2$. Let $\Omega(k)$ be the function defined as

$$\Omega(k) = 2(k-b)X(k), \; X(k) = \sqrt{(k+b)^2 + a^2}. \tag{2.3}$$

Following [1] we consider the two-sheeted Riemann surface X defined by the function $\Omega(k)$. Our Riemann-Hilbert problem will be defined on X. We also consider the oriented contour Σ defined by $Im\Omega(k) = 0$, see Figure 1.

One easily sees that the curve Σ consists of two copies of the real line and an analytic arc $\Gamma \cup \bar{\Gamma}$ connecting the two branch points $E = -b+ia, \bar{E} = b-ia$ and the two infinities ∞_1 and ∞_2 (on the two sheets of X).

Σ defines a partition of the sphere X into D_1, D_2, D_3, D_4, where

$$\begin{aligned} D_1 &= \{Imk > 0, Im\Omega(k) > 0\}, \\ D_2 &= \{Imk > 0, Im\Omega(k) < 0\}, \\ D_3 &= \{Imk < 0, Im\Omega(k) > 0\}, \\ D_4 &= \{Imk < 0, Im\Omega(k) < 0\}. \end{aligned} \tag{2.4}$$

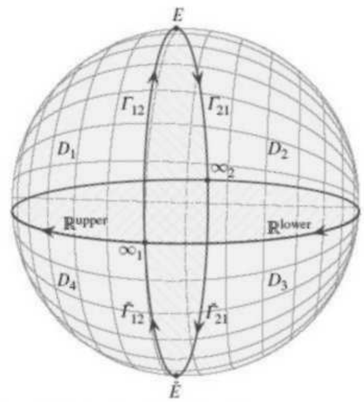

FIGURE 1. The two-sheeted Riemann surface X.

Next, define the following matrices

$$E(k) = \begin{pmatrix} (\frac{k+b+X(k)}{2X(k)})^{1/2} & ie^{i\epsilon}(\frac{X(k)-k-b}{2X(k)})^{1/2} \\ ie^{-i\epsilon}(\frac{X(k)-k-b}{2X(k)})^{1/2} & (\frac{k+b+X(k)}{2X(k)})^{1/2} \end{pmatrix}, \tag{2.5}$$

$$H(t,k) = exp(i\omega\sigma_3 t)E(k)exp(-i\omega\sigma_3 t),$$
$$\Psi(t,k) = H(t,k)exp(i[\omega - \Omega(k)]\sigma_3 t).$$

Let the functions $a(k)$ and $b(k)$ be the (classical) scattering data for the function $q_0(x)$ defined in [8]. All we need to know here is that $a(k)$ is smooth for k real and can be analytically extended in the upper half-plane, with $a(k) = 1 + O(1/k)$ as $k \to \infty$. Similarly, $b(k)$ is a Schwartz function for k real which can be extended to the upper half-plane such that $b(k) = O(1/k)$ as $k \to \infty$. Furthermore, $|a^2| + |b^2| = 1$ for k real and a can have at most a finite number of simple zeros in the complex plane, say $k_1, k_2, ..., k_n$, with $Im(k_j) > 0, j = 1, ..., n$.

Let the functions A^T, B^T be the scattering data for the t-problem, defined in [1]:

$$\begin{pmatrix} -B^T(k) \\ \bar{A}^T(\bar{k}) \end{pmatrix} = \hat{\Phi}(T,k), \tag{2.6}$$

where $\hat{\Phi}(t,k)$ is the vector-valued function satisfying the integral equation

$$\hat{\Phi}(t,k) = [second\ column\ of\ E^{-1}(k)] + \int_0^t \Psi^{-1}(\tau,k)Q_0(\tau,k)\hat{\Phi}(\tau,k)d\tau,\ 0 < t < \infty, \tag{2.7}$$

with Ψ defined in (2.5) and $\tilde{Q}_0(\tau,k) = Q(0,\tau,k) - \tilde{Q}_p(0,\tau,k)$, where $\tilde{Q}$ is given in (2.2) and $\tilde{Q}_p$ is obtained from $\tilde{Q}$ by replacing q with $ae^{2i\omega t+i\epsilon}$ and q_x with $2iabe^{2i\omega t+i\epsilon}$.

All we need here are the following important properties, see [2] (these properties are a priori valid in the complex plane but they can be trivially extended to X).

(i) The functions $A^T(k), B^T(k)$ are entire functions, bounded in $\bar{D}_1 \cup \bar{D}_3$. Furthermore

$$A^T(k) = 1 + O\left(\frac{exp[i(\Omega(k)+2k^2)T]}{k}\right), \quad B^T(k) = O\left(\frac{exp[i(\Omega(k)+2k^2)T]}{k}\right), \tag{2.8}$$

as $k \to \infty$.

(ii) There is a function $c(k,T)$ which is analytic and bounded in $D_1 \cup D_2$ and also of order $O(1/k)$ as $k \to \infty$ such that

$$b(k)A^T(k) - a(k)B^T(k) = c(k,T)exp[i(\Omega(k)+2k^2)T]. \tag{2.9}$$

This is the so-called *global relation.*

(iii) $A^T(k)$ may have a finite number of zeros, say $\kappa_1^T, \ldots.., \kappa_{m(T)}^T$.

3. Taking $T \to \infty$

Since the family (A^T, B^T), $0 \le T \le \infty$ is bounded in $\bar{D}_1 \cup \bar{D}_3$, a subfamily of it must have a uniform limit as $T \to \infty$, which we denote by (A, B).

It is easily seen that the following are valid:

(i) The functions $A(k), B(k)$ are analytic, bounded in $\bar{D}_1 \cup \bar{D}_3$, and

$$A(k) = 1 + O(1/k), \ B(k) = O(1/k), \tag{3.1}$$

as $k \to \infty$.

(ii) $b(k)A(k) - a(k)B(k) = 0$ in D_1. This follows from the global relation (2.9) for A^T, B^T, taking $T \to \infty$.

(iii) A has a finite number of zeros, say, $\kappa_1, \kappa_2, \ldots., \kappa_m$.

We will now define a Riemann-Hilbert problem in X, with jump data given in terms of a, b, A, B, following [1].

We define the matrices

$$s(k) = \begin{pmatrix} \bar{a}(\bar{k}) & b(k) \\ -\bar{b}(\bar{k}) & a(k) \end{pmatrix}, \tag{3.2}$$

$$S(k) = \begin{pmatrix} \bar{A}(\bar{k}) & B(k) \\ -\bar{B}(\bar{k}) & A(k) \end{pmatrix} \tag{3.3}$$

and $G(k) = s^{-1}(k)S(k)$. Let

$$\rho(k) = \frac{G_{21}(k)}{G_{11}(k)},$$
$$r(k) = \frac{\bar{b}(k)}{a(k)},$$
$$c(k) = \rho(k) - r(k),$$
$$f(k) = \frac{2iX(k)}{aG_{11}(k-0)G_{11}(k+0)}.$$

Consider now the following Riemann-Hilbert problem with the jump contour Σ:

$$\begin{aligned} M_-(x,t,k) = M_+(x,t,k)J(x,t,k),\ k \in \Sigma, \\ lim_{k\to\infty_1} M(x,t,k) = I, \end{aligned} \tag{3.4}$$

where

(3.5)

$$J(x,t,k) = \begin{pmatrix} 1 & -\bar{r}(k)e^{-2i(kx+(\Omega(k)-\omega)t)} \\ r(k)e^{2i(kx+(\Omega(k)-\omega)t)} & 1+|r(k))|^2 \end{pmatrix},\ k \in \mathbb{R}^{upper},$$
$$J(x,t,k) = \begin{pmatrix} 1 & -\bar{\rho}(k)e^{-2i(kx+(\Omega(k)-\omega)t)} \\ \rho(k)e^{2i(kx+(\Omega(k)-\omega)t)} & 1+|\rho(k))|^2 \end{pmatrix},\ k \in \mathbb{R}^{lower},$$
$$J(x,t,k) = \begin{pmatrix} 1 & 0 \\ c^+(k)e^{2i(kx+(\Omega(k)-\omega)t)} & 1 \end{pmatrix},\ k \in \Gamma_{12},$$
$$J(x,t,k) = \begin{pmatrix} 1 & 0 \\ -c^-(k)e^{2i(kx+(\Omega(k)-\omega)t)} & 1 \end{pmatrix},\ k \in \Gamma_{21},$$
$$J(x,t,k) = \begin{pmatrix} 1 & -\bar{c}^+(\bar{k})e^{-2i(kx+(\Omega(k)-\omega)t)} \\ 0 & 1 \end{pmatrix},\ k \in \bar{\Gamma}_{12},$$
$$J(x,t,k) = \begin{pmatrix} 1 & -\bar{c}^-(\bar{k})e^{-2i(kx+(\Omega(k)-\omega)t)} \\ 0 & 1 \end{pmatrix},\ k \in \bar{\Gamma}_{21}.$$

Here c_+ and c_- are boundary values of the function c which is analytic in D_2.

Furthermore the following pole conditions are satisfied.

(3.6)
$$
\begin{aligned}
res_{k=k_j}[M(x,t,k)]_1 &= im_j^1 e^{2i(kx+(\Omega(k_j)-\omega)t)}[M(x,t,k_j)]_2,\ k_j \in D_1,\\
res_{k=z_j}[M(x,t,k)]_1 &= im_j^2 e^{2i(kx+(\Omega(z_j)-\omega)t)}[M(x,t,z_j)]_2,\ z_j \in D_2,\\
res_{k=\bar{z}_j}[M(x,t,k)]_2 &= -i\bar{m}_j^2 e^{-2i(\bar{k}x+(\Omega(\bar{z}_j)-\omega)t)}[M(x,t,\bar{z}_j)]_1,\ \bar{z}_j \in D_3,\\
res_{k=\bar{k}_j}[M(x,t,k)]_2 &= -i\bar{m}_j^1 e^{2i(\bar{k}x+(\Omega(\bar{k}_j)-\omega)t)}[M(x,t,\bar{k}_j)]_1,\ \bar{k}_j \in D_4,
\end{aligned}
$$

where

$$
\begin{aligned}
m_j^1 &= (ib(k_j)\frac{da}{dk}(k_j))^{-1},\ m_j^2 = -res_{k=z_j}c(k),\\
\bar{m}_j^1 &= (i\bar{b}(\bar{k}_j)\frac{d\bar{a}}{dk}(\bar{k}_j))^{-1},\ \bar{m}_j^2 = -res_{k=\bar{z}_j}\bar{c}(\bar{k}).
\end{aligned} \tag{3.7}
$$

Theorem 3.1. *The above Riemann-Hilbert problem admits a unique solution.*

The theorem follows immediately from the so-called vanishing lemma extended to the surface X [9] by employing the symmetries of the jump J. Although the vanishing lemma applies to holomorphic Riemann-Hilbert problems, the above meromorphic Riemann-Hilbert problem can be easily transformed to a holomorphic Riemann-Hilbert problem as in [4] by adding small loops around the poles and changing variables inside the loops (see also [7], [8]).

4. Asymptotic analysis of the Riemann-Hilbert problem

The analysis in section 3.3 of [1] shows that the Riemann-Hilbert problem above gives rise to a solution of the focusing NLS in the first quadrant. Furthermore the initial data $q(x,0)$ are equal to q_0 because of the definition of a, b. What is not a priori clear is that $q(0,t) = g_0(t)$. What is even less clear is whether $q_x(0,t)$ is the sum of $2iabe^{2i\omega t+i\epsilon}$ and a Schwartz function.

This is the main result of this paper.

Theorem 4.1. *Define $q(x,t) = 2ilim_{k\to\infty_1}kM_{12}(x,t,k)$ where M_{12} is the (12) entry of the solution of the above Riemann-Hilbert problem. Then $q(x,t)$ solves the focusing NLS equation in the first quadrant, with $q(x,0) = q_0(x),\ q(0,t) = g_0(t),\ q_x(0,t) = 2iabe^{2i\omega t+i\epsilon} + v(t)$ where $v(t)$ is a Schwartz function.*

PROOF: The fact that $q(0,t) = g_0(t)$ follows easily from a limiting argument. As shown in [8], if we replace A, B by A^T, B^T in the Riemann-Hilbert problem we will still arrive at a solution $q^T(x,t)$ of the focusing NLS equation in the first quadrant, with $q^T(x,0) = q_0(x)$

and $q(0,t) = g_0(t),\ 0 \le t \le T$. Taking $T \to \infty$ we clearly recover the boundary data $g_0(t),\ 0 \le t < \infty$. More precisely, by the standard Riemann-Hilbert deformation theory [5] which connects Riemann-Hilbert problems to singular integral operators in $L^2(\Sigma)$ and using the fact that the resolvent of the singular integral operator corresponding to the focusing NLS equation is uniformly bounded in x, t, it follows that $q^T(x,t)$ converges uniformly to $q(x,t)$ and hence $q(0,t) = g_0(t)$ for all t.

The fact that the function $v(t) = q_x(0,t) - 2iabe^{2i\omega t + i\epsilon}$ is a Schwartz function follows from the asymptotic analysis of the Riemann-Hilbert problem (for data a, b, A, B), as $t \to \infty$. From section 3.3 of [1] we have that the Riemann-Hilbert problem above reduces to the following Riemann-Hilbert problem when $x = 0$ (see p.601 of [1] and note that we have inverted the orientation of part of the contour):

$$\begin{aligned} M_-^{(t)}(t,k) = M_+^{(t)}(t,k)J^{(t)}(t,k),\ k \in \Sigma, \\ lim_{k\to\infty_1} M^{(t)}(t,k) = I, \end{aligned} \tag{4.1}$$

where

$$J^{(t)}(t,k) = \begin{pmatrix} 1 & \frac{B(k)}{A(k)} e^{-2i(\Omega-\omega)t} \\ \frac{\bar{B}(\bar{k})}{\bar{A}(\bar{k})} e^{2i(\Omega-\omega)t} & \frac{1}{A(k)\bar{A}(\bar{k})} \end{pmatrix},\ k \in \Sigma, \tag{4.2}$$

where the superscript $+$ denotes the limit from the $+$side of the contour and the superscript $-$ denotes the limit from the $-$side of the contour. For simplicity we will assume that there do not exist any poles, but our analysis can be easily generalized in the presence of poles.

The following asymptotic analysis will show that as $t \to \infty$, we recover the pure zero-zone solution.

Theorem 4.2. *Up to an exponentially small error, the Riemann-Hilbert problem for $M^{(t)}$ is asymptotically (as $t \to \infty$) equivalent to the trivial Riemann-Hilbert problem which has no jump.*

Proof. Note the factorization of $J^{(t)}$ on Σ:

$$\begin{aligned} J^{(t)}(t,k) &= J^{lo} J^{up}, \\ where\ J^{lo} &= \begin{pmatrix} D_+ & 0 \\ \frac{-\bar{B}(\bar{k})D_+}{\bar{A}(\bar{k})(1+|B/A(k)|^2)} e^{2i(\Omega-\omega)t} & 1/D_+ \end{pmatrix}, \\ and\ J^{up} &= \begin{pmatrix} 1/D_- & -\frac{-B(k)}{A(k)D_-} e^{-2i(\Omega-\omega)t} \\ 0 & D_- \end{pmatrix}, \end{aligned} \tag{4.3}$$

and where D solves the scalar problem

$$D_+ = D_-(1 + |B/A(\bar{k})|^2), \ k \in \Sigma$$

and satisfies $lim_{k\to\infty_1} D(k) = 1$. This factorization follows from the identity $A(k)\bar{A}(\bar{k}) + B(k)\bar{B}(\bar{k}) = 1$ for $k \in \Sigma$.

For the asymptotic analysis we must deform our Riemann-Hilbert problem in small lenses with boundaries consisting of the different components of $\mathbb{R} \cup \Gamma$ and slight deformations of these components.

For example we consider the oriented contours $C^{1,up}$ and $C^{1,lo}$ from ∞_1 to ∞_2 on the upper sheet of the Riemann surface slightly deforming the real line, with $C^{1,up}$ lying in D_1 and $C^{1,lo}$ lying in D_4, and denote the corresponding lenses $D_{1,up}$ and $D_{1,lo}$ in a way that $\partial D_{1,up} = C^{1,up} \cup \mathbb{R}^{upper}$ and $\partial D_{2,up} = C^{2,up} \cup \mathbb{R}^{upper}$. We construct similar lenses around $\Gamma, \bar{\Gamma}, \mathbb{R}^{lower}$.

We define O as follows:

$$\begin{aligned} O(t,k) &= M^{(t)}(t,k) J^{lo}, \ k \in D_{1,up}, \\ O(t,k) &= M^{(t)}(t,k)(J^{up})^{-1}, \ k \in D_{1,lo}. \end{aligned} \tag{4.4}$$

Similarly for the other lenses.

We now observe that the off-diagonal entries of the jump matrix for O are $uniformly$ exponentially small. On the other hand, the diagonal entries are uniformly bounded. So, according to standard asymptotic analysis of Riemann-Hilbert factorization problems [5], it follows that, up to an exponentially small error, O is given by the solution of a problem with diagonal jump, which in turn reduces to the scalar problem for D. Noting further that the residue of the solution of the diagonal problem at ∞_1 is zero we recover our result. □

The limiting Riemann-Hilbert problem is trivial and corresponds to the purely zero-zone solution of NLS. Using the formula $q_x(0,t) = lim_{k\to\infty}[4k^2 M^{(t)}_{12} + 2iq(0,t)kM^{(t)}_{22}]$ we see that $v(t) = q_x(0,t) - 2iabe^{2i\omega t + i\epsilon}$ is actually exponentially small. Using similar formulae for $\frac{\partial^j}{\partial t^j} v(t), \ i = 1, 2, 3, ...$ in terms of $M^{(t)}$ it is possible to show that v is a Schwartz function and thus Theorem 4.1 is proved.

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