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Pohlmeyer reduction of $AdS_5 \times S^5$ superstring sigma model

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Abstract

Motivated by a desire to find a useful 2d Lorentz-invariant reformulation of the $AdS_5 \times S^5$ superstring world-sheet theory in terms of physical degrees of freedom we construct the "Pohlmeyer-reduced" version of the corresponding sigma model. The Pohlmeyer reduction procedure involves several steps. Starting with a coset space string sigma model in the conformal gauge and writing the classical equations in terms of currents one can fix the residual conformal diffeomorphism symmetry and kappa-symmetry and introduce a new set of variables (related locally to currents but non-locally to the original string coordinate fields) so that the Virasoro constraints are automatically satisfied. The resulting equations can be obtained from a Lagrangian of a non-abelian Toda type: a gauged WZW model with an integrable potential coupled also to a set of 2d fermionic fields. A gauge-fixed form of the Pohlmeyer-reduced theory can be found by integrating out the 2d gauge field of the gauged WZW model. The small-fluctuation spectrum near the trivial vacuum contains 8 bosonic and 8 fermionic degrees of freedom with equal mass. We conjecture that the reduced model has world-sheet supersymmetry and is ultraviolet-finite. We show that in the special case of the $AdS_2 \times S^2$ superstring model the reduced theory is indeed supersymmetric: it is equivalent to the N=2 supersymmetric extension of the sine-Gordon model.

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Contents

1 Introduction

String theory in $AdS_5 \times S^5$ is represented by a Green-Schwarz-type [1] action on a supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ [2]. It is classically integrable [3] and has an involved solitonic spectrum (see, e.g., [4, 5]). To quantize it one may attempt to eliminate first unphysical degrees of freedom by choosing a kind of light-cone gauge, i.e. an analog of $x^+ = p^+\tau,\ \Gamma^+\theta = 0$. One natural option is to expand near the null geodesic parallel to the boundary in the Poincare patch; the resulting gauge-fixed action is then quartic in fermions [6]. An alternative is to use the null geodesic wrapping S^5 [7]; the resulting action [8, 9, 10] has a rather complicated structure with many non-linear interaction terms.

An apparent disadvantage of the light-cone gauge choices is that the gauge-fixed action lacks manifest 2d Lorentz invariance (beyond the quadratic level in the fields). This makes it hard to apply familiar methods of integrable quantum field theories; in particular, the S-matrix for the elementary excitations has apparently less restricted form [11, 12] than in a Lorentz-invariant case (cf. [13]).

An alternative approach which we shall explore here is to impose the conformal gauge condition and to perform a non-local transformation of variables (from coodinates to currents) that solves the Virasoro constraints at the classical level while preserving the integrable structure. This generalizes the Pohlmeyer "reduction" (or better "reformulation") relating the classical S^2 sigma model to the sine-Gordon model [14] (see also [15, 16, 17, 18, 19]). A related work in this direction appeared in [20, 21]. One is then left with the right number of physical ("transverse") degrees of freedom. In a certain sense, this reduction approach may be viewed as a "covariant analog" of a light-cone gauge fixing.

The resulting "reduced" model should have closely related solitonic spectrum to the original one, and one may then raise the question if the classical correspondence between the two models may extend to the quantum level. This is not what happens in the case of the S^2 sigma model and the sine-Gordon model (one reason is that in the reduction procedure one uses conformal symmetry of the $SO(3)/SO(2)$ model which does not survive beyond the classical level) but we may conjecture that the relation may still hold in the very special case of the full $AdS_5 \times S^5$ superstring model which should be conformal at the quantum level.

Below we shall first discuss the Pohlmeyer-type reduction for the bosonic part of the classical $AdS_5 \times S^5$ sigma model and then consider the full supercoset superstring theory. As we shall see, the application of this procedure to the bosonic part of the $AdS_5 \times S^5$ string action leads to a *2d relativistically invariant* "reduced" theory represented by a sigma model with a potential term which has an equivalent integrable structure. It generalizes the sine-Gordon [14] and the complex sine-Gordon [14, 22] models to the case of the 4+4 dimensional target space.

We shall explain how to obtain a local Lorentz-invariant action for this reduced theory in terms of "physical" (gauge-fixed) degrees of freedom.[1] We shall follow the approach of [23, 24] (see also [25]), in which the reduced theory is interpreted as a gauge-fixed version of a gauged WZW theory with a potential representing a relevant integrable deformation, i.e. as a special case of a non-abelian

[1]This was not done explicitly in the past for the S^n models with $n > 3$. The existence of a local Lagrangian is an important issue. At the level of equations for the currents or the Lax pair equations there is a large freedom [15] in how one can choose a local field representation – many classically equivalent models have same-looking Lax equations and yet very different local field representations (and thus inequivalent quantum structure). When one addresses the issue of existence of a local action the choice of the fundamental fields becomes relevant.

Toda theory [27].

The reduced model for the full $AdS_5 \times S^5$ superstring (found after an appropriate kappa-symmetry gauge fixing) turns out to be a 2d Lorentz-covariant fermionic generalisation of a non-abelian Toda theory for $\frac{G}{H} = \frac{Sp(2,2)}{SU(2)\times SU(2)} \times \frac{Sp(4)}{SU(2)\times SU(2)}$ with $4+4$ dimensional bosonic target space. Its simple structure (and the matching of the numbers of the bosonic and the fermionic degrees of freedom) suggests that it may possess 2d supersymmetry. Indeed, the existence of the supersymmetry can be seen directly in the special case of the $AdS_2 \times S^2$ superstring theory for which the reduced model happens to be the same as the $\mathrm{N}=2$ supersymmetric sine-Gordon theory.

Though the relation of the reduced model to the original conformal superstring model involves a non-local transformation, we may still expect that it should define a UV finite 2d theory. Its conformal invariance is then only "spontaneously" broken by a scale μ (entering the potential term and its fermionic counterpart) that appears after fixing the residual conformal diffeomorphism freedom in the conformal gauge (the same happens in the plane-wave light-cone gauge case [7]). If this is indeed the case, the reduced model may serve as a starting point for understanding the corresponding quantum $AdS_5 \times S^5$ superstring theory.[2] Its small-fluctuation spectrum near a natural vacuum state contains 8 bosonic and 8 fermionic dynamical degrees of freedom of equal mass μ, and the corresponding relativistic (and 2d supersymmetric) S-matrix should have the $[SU(2)]^4$ global symmetry.

Let us now describe the contents of the paper. We shall start in section **2** with a review of the Pohlmeyer reduction in the case of the bosonic string models on $R_t \times S^2$ and $R_t \times S^3$ with sine-Gordon and complex sine-Gordon models as the corresponding reduced theories.

To systematically construct the Lagrangians of reduced models for higher-dimensional bosonic $SO(n,m)/SO(n-1,m)$ examples we shall first explain the relation between the equations of motion of geometrical ("right") F/G coset model written in terms of currents and the G/H ("left-right") gauged WZW model (gWZW) with an integrable potential. As a preparation, we shall review the classical equations of the F/G symmetric-space sigma model (sect. **3.1**) and the equations of the G/H gWZW model with a potential, i.e. of a special case of the non-abelian Toda theory (sect. **3.2**). The potential is determined by a choice of an element $T_+ = T_- = T$ in the abelian subspace in the complement of the algebra $\mathfrak{g}$ of G in the algebra $\mathfrak{f}$ of F, and H is such that its algebra $\mathfrak{h}$ is a centralizer of T in $\mathfrak{g}$.

In sect. **4** we shall show how to relate the equations of motion of the F/G coset model to those of the G/H gWZW model by (i) imposing the so called reduction gauge in the equations of the F/G model written in terms of the current components, and by (ii) making use of the residual 2d conformal diffeomorphism symmetry to eliminate an additional degree of freedom (setting components of the stress tensor to be constant and thus satisfying the conformal gauge constraints of the string theory on $R_t \times F/G$). This will allow us to solve part of the gauge-fixed equations of motion explicitly in terms

[2] While the transformation used to arrive at the reduced model is non-local (e.g., the Poisson structures of the original and reduced models are different [20, 21]), one may hope that in an integrable finite field theory the solitonic spectrum should be determined essentially by the semiclassical approximation [28] and it may then be the same in a pair of theories with classically equivalent integrable structures. Having obtained the reduced model via the classical procedure and using it as a starting point for quantization one would still need to understand how to compute the "observables" of the original theory in terms of the quantum reduced theory (at the classical level one can do this by solving the linear Lax system). In particular, one would need to compute the global charges of the $PSU(2,2|4)$ symmetry group as these are relevant for comparison with the gauge theory side.

of a new field g taking values in G and the $\mathfrak{h}$-valued gauge field $A_\pm$ (sect. **4.2**). The resulting system will turn out to be invariant under the both left and right H gauge symmetries. After imposing a special gauge condition under which the gauge symmetry reduces to that of the G/H gWZW model these equations of motion become equivalent to the ones following from the gWZW action with a special integrable potential described in sect. **3.2**. That the reduced equations of motion of the F/G coset model can be related to those of the gWZW model with an integrable potential was first suggested (and checked on several examples) in [24, 25]. Here we shall explain why this correspondence should work in general and specify the necessary conditions on the groups and the algebras involved. We shall also note that the potential term is equal to the original F/G coset Lagrangian in the reduction gauge.

In sect. **4.3** we shall mention the equivalence of the Lax representations for the F/G coset and the G/H gWZW models and in sect. **4.4** we shall consider the reduced equations for the $S^n = SO(n+1)/SO(n)$ coset model in the $A_\pm = 0$ [24] H-gauge. These equations, are, however, non-Lagrangean on physical subspace.[3]

As we shall discuss in sect. **5**, to get the Lagrangean equations for the independent n–1 degrees of freedom of the reduced counterpart of the S^n model (that generalizes the sine-Gordon and the complex sine-Gordon cases) one should start with the gWZW action, impose the H-gauge on the group element $g \in G$ and integrate out the gauge field components $A_\pm$. The resulting reduced action is that of a sigma model with a curved target space metric (but no antisymmetric tensor coupling) combined with a relevant integrable potential term given universally by a cosine of one of the n–1 angles. We describe few explicit examples of reduced models for strings on $R_t \times S^4$ and $R_t \times S^5$ in sect. **5.2**. The generalisation to $AdS_n \times S^n$ models is then straightforward (sect. **5.3**).

In sect. **6** we shall turn to the $AdS_5 \times S^5$ superstring starting with the equations of motion for the $\frac{\widehat{F}}{G} = \frac{PSU(2,2|4)}{Sp(2,2)\times Sp(4)}$ supercoset model (with the bosonic part $\frac{F}{G} = AdS_5 \times S^5 = \frac{SU(2,2)}{Sp(2,2)} \times \frac{SU(4)}{Sp(4)}$). We choose conformal gauge and write them in terms of the components of the left-invariant current of $PSU(2,2|4)$. We use the formulation based on Z_4 grading property [54, 3] of the superalgebra $psu(2,2|4)$. Fixing a particular kappa-symmetry gauge we perform the analog of the Pohlmeyer reduction discussed earlier for the similar bosonic cosets. An important ingredient is a generalization to the $psu(2,2|4)$ superalgebra case of the Lie algebra decomposition originally used in [18] in the bosonic coset case.

Introducing the new set of fermionic variables directly related to the odd components of the supercoset current we show in sect. **6.4** that the reduced system of equations follows from a 2d Lorentz-invariant Lagrangian (6.49). Its bosonic part is that of $\frac{G}{H} = \frac{Sp(2,2)}{SU(2)\times SU(2)} \times \frac{Sp(4)}{SU(2)\times SU(2)}$ gWZW model with an integrable potential determined by a special diagonal matrix $T = T_\pm$ in the even part of the $psu(2,2|4)$ superalgebra. In addition, the Lagrangian contains a quadratic fermionic part with a standard first-derivative kinetic term. The fermions interact "minimally" with the H gauge field $A_\pm$ and are also coupled (by a "Yukawa-type" term) to the bosonic field $g \in G$. We mention that as in the bosonic case, the sum of the μ-dependent potential and "Yukawa" interaction terms in the reduced Lagrangian is equal to the original superstring Lagrangian written in terms of currents.

The vacua of the theory are described by constant g taking values in H; in the $A_\pm = 0$ gauge the

[3]The original observation of [24] that the gWZW model with an integrable potential provides a Lagrangean formulation of the reduced equations of motion of the F/G coset model applied on the extended configuration space involving the "auxiliary" $A_\pm$ fields. Similar construction was discussed in a string context in [26].

small-fluctuation spectrum near the trivial vacuum consists of 8 bosonic and 8 fermionic dynamical modes of the same mass μ. We comment on the interpretation of the parameter μ and mention that the corresponding scattering matrix should have a global $H = [SU(2)]^4$ symmetry.

The structure of the reduced action suggests the presence of a 2d supersymmetry. Its existence is indeed confirmed in sect. **7** on the example of a similar $AdS_2 \times S^2$ superstring model based on the $psu(1,1|2)$ superalgebra. The corresponding reduced Lagrangian is found to be the same as that of the $\mathrm{N} = 2$ supersymmetric extension of the sine-Gordon model.

There are also several Appendices containing some technical details and definitions.

2 Examples of reduced models: strings in $R_t \times S^2$ and $R_t \times S^3$

Let us begin with a review of the prototypical example: reduction of the S^2 sigma model to the sine-Gordon model [14]. Starting with the action of the sigma model on the sphere written in terms of the embedding coordinates $S = \frac{1}{4\pi\alpha'} \int d^2\sigma \, L$ where $(\partial_\pm = \partial_0 \pm \partial_1)$

$$L = \partial_+ X^m \partial_- X^m - \Lambda(X^m X^m - 1) \, , \qquad m = 1,2,3 \, , \tag{2.1}$$

we get for the classical equations of motion

$$\partial_+\partial_- X^m + \Lambda X^m = 0 \, , \qquad \Lambda = \partial_+ X^m \partial_- X^m \, , \qquad X^m X^m = 1 \, . \tag{2.2}$$

Then the stress tensor satisfies

$$\mathrm{T}_{+-} = 0 \, , \quad \partial_+ \mathrm{T}_{--} = 0 \, , \quad \partial_- \mathrm{T}_{++} = 0 \, , \quad \mathrm{T}_{\pm\pm} = \partial_\pm X^m \partial_\pm X^m, \tag{2.3}$$

so that $\mathrm{T}_{++} = f(\sigma_+)$, $\mathrm{T}_{--} = h(\sigma_-)$. Since the theory is classically conformally invariant one can apply conformal transformations to put $\mathrm{T}_{\pm\pm}$ into the special constant form

$$\partial_+ X^m \partial_+ X^m = \mu^2 \, , \qquad \partial_- X^m \partial_- X^m = \mu^2 \, , \qquad \mu = \text{const} \, . \tag{2.4}$$

This effectively fixes one of the two fields of S^2 leaving us with a one-dimensional "reduced" theory. Indeed, one can introduce a new field variable φ via the following non-local transformation $X_m \to \varphi$

$$\mu^2 \cos 2\varphi = \partial_+ X^m \partial_- X^m \, . \tag{2.5}$$

Then the equations for X^m (2.2) and the conditions (2.4) are solved provided φ is subject to the sine-Gordon (SG) equation $\partial_+\partial_-\varphi + \frac{\mu^2}{2} \sin 2\varphi = 0$. The latter follows from

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \frac{\mu^2}{2} \cos 2\varphi \, , \tag{2.6}$$

which is thus the Lagrangian of the "reduced" theory. The classical solutions and integrable structure (Lax pair, etc.) of the original sigma model and its reduced counterpart are then directly related.

This reduction from sigma model on S^2 to the SG theory has also an equivalent interpretation as a classical equivalence between the bosonic string theory in $R_t \times S^2$ in a special gauge and the SG theory. Indeed, starting with the Polyakov string action containing the time direction term $-\partial_+ t \partial_- t$

in addition to the S^2 term (2.1) and choosing the *conformal gauge* combined with $t = \mu\tau$ (to fix the residual conformal reparametrisation symmetry) we end up with the same conditions (2.4), now interpreted as the conformal gauge (Virasoro) constraints. Then the classical string equations on $R_t \times S^2$ become equivalent to the SG equation for the one remaining "transverse" degree of freedom parametrized by φ (the gauge conditions eliminate 1+1 out of 1+2 string degrees of freedom).

One interesting outcome of the above reduction is that while the conditions (2.4) obviously violate the 2d Lorentz invariance of the original theory ($t = \mu\tau$ "spontaneously breaks" the 2d Lorentz invariance in the string-theory version of the reduction), the resulting SG theory is still Lorentz invariant. Note also that the $SO(3)$ global symmetry of the original model (2.1) becomes trivial in the reduced model: φ defined in (2.5) is $SO(3)$ invariant. Given a SG solution for φ and thus a specific value of the Lagrange multiplier function $\Lambda = \mu^2 \cos 2\varphi = \partial_+ X^m \partial_- X^m$ in (2.2) one can reconstruct the corresponding solution for X_m by solving the linear equation $\partial_+\partial_- X^m + \Lambda X^m = 0$.[4] For a given solution for X_m one can then find the corresponding $SO(3)$ conserved charges. Thus the classical solitonic spectra of the two models should be in direct correspondence (see [30, 31, 32] for some specific examples).

This classical equivalence relation obviously breaks down in quantum theory where there are UV divergences and mass generation in the S^2 sigma model so that the classical conformal invariance is broken (invalidating, in particular, the argument leading to (2.4)). Still, one may hope that an analog of this reduction may extend to the quantum level in the case of a theory like $AdS_5 \times S^5$ superstring which remains conformally invariant upon quantisation.

The above reduction has a straightforward generalisation to the case when S^2 is replaced by S^3 [14, 22]. The reduced model corresponding to the string on $R_t \times S^3$ is the complex sine-Gordon (CSG) model

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \tan^2\varphi \, \partial_+\theta\partial_-\theta + \frac{\mu^2}{2}\cos 2\varphi \, . \tag{2.7}$$

The variables φ and θ are expressed in terms of the $SO(4)$ invariant combinations of derivatives of the original variables X_m ($m = 1, 2, 3, 4$)

$$\mu^2 \cos 2\varphi = \partial_+ X^m \partial_- X^m \, , \qquad \mu^3 \sin^2\varphi \, \partial_\pm\theta = \mp\frac{1}{2}\epsilon_{mnkl} X^m \partial_+ X^n \partial_- X^k \partial_\pm^2 X^l \, . \tag{2.8}$$

Again, the integrable structures and the soliton solutions of the two models are closely related (see [31, 32]). The CSG model can be interpreted as a special case of a non-abelian Toda theory [27] – a massive integrable perturbation of a gauged (coset) WZW model (here $\frac{SO(3)}{SO(2)}$ model) [51].[5]

Reduced equations of motion for sigma models on higher spheres S^n ($n = 4, 5, ...$) involve field variables related to $SO(n+1)$ invariants built out of X_m and its higher derivatives $\partial_\pm X_m$, $\partial_\pm^2 X_m$, $\partial_\pm^3 X_m, ...$ (with indices contracted using δ_{mk} and $\epsilon_{m_1...m_{n+1}}$); they were found in [17] (see also [16, 19]). The resulting equations were not, however, derivable from a local Lagrangian.

It was later shown in [24] that they can be obtained as a particular gauge-fixed version of the classical equations of the $\frac{SO(n)}{SO(n-1)}$ gauged WZW model with an integrable potential term. This provided

[4] To find periodic solutions on $R \times S^1$ one would need to start with a periodic solution of SG model and also impose periodicity on X_m in solving the linear system.

[5] The corresponding quantum S-matrix was discussed in [52].

a Lagrangean formulation of these equations on the *extended* field space including the 2d gauge field $A_\pm$ of the gWZW model.

This construction gives a strong indication that there should exist an alternative version of the classical reduced equations of motion which is *manifestly* Lagrangean, i.e. that can be derived from an action containing only physical "reduced" set of fields as was found in the previous cases of the SG and CSG models.

The reason for this expectation is that the classical equations written in the Lax-pair form admit different "gauge-equivalent" [15] versions related by (non-local) field redefinitions.[6] This was already noticed in [19] in the S^3 case where the field variables corresponding to the CSG model were related by a non-local transformation to the variables of the reduced model of [17].

Below we shall present an explicit form of the reduced Lagrangian models for the string on $R_t \times S^4$ and $R_t \times S^5$; the AdS_n versions can be found by an analytic continuation. One is then able to write down the reduced Lagrangian for the bosonic part of the $AdS_5 \times S^5$ theory. The basic idea is to follow [24] and start with the $\frac{SO(n)}{SO(n-1)}$ gWZW model with a relevant integrable perturbation term but instead of fixing the gauge field $A_\pm = 0$ as in [24] fix the gauge on the group element and integrate out the gauge field $A_\pm$ as in [36, 37, 38, 40] (see also [25, 29]). In the case of the $\frac{SO(3)}{SO(2)}$ (or equivalently $\frac{SU(2)}{U(1)}$) model that procedure immediately explains the appearance of the familiar $D = 2$ target space metric in the CSG action (2.7) as was originally observed in [39].

The construction of the reduced models based on the conformal gauge and fixing the remaining conformal transformations by $t = \mu\tau$ condition was applied above to a string on $R_t \times S^n$. The same can be done for the bosonic string model on $AdS_n \times S^1$ in conformal gauge and with fixing the residual conformal symmetry choosing the S^1 angle α equal to $\mu\tau$. Denoting the embedding coordinates of AdS_n as Y_s (with $Y^sY_s = -Y_0^2 - Y_{-1}^2 + Y_1^2 + ... + Y_n^2 = -1$) the AdS_n Lagrangian is then the analog of (2.1)

$$L = \partial_+ Y^s \partial_- Y_s - \widetilde{\Lambda}(Y^s Y_s + 1) \ , \tag{2.9}$$

with the equations of motion and conformal gauge constraints being

$$\partial_+\partial_- Y_s + \widetilde{\Lambda} Y_s = 0, \qquad \widetilde{\Lambda} = -\partial_+ Y^s \partial_- Y_s \ , \qquad Y^s Y_s = -1 \ , \tag{2.10}$$

$$\partial_+ Y_s \partial_+ Y^s = -\mu^2 \ , \qquad \partial_- Y_s \partial_- Y^s = -\mu^2 \ . \tag{2.11}$$

By concentrating on the plane formed by the normalized vectors $\partial_+ Y^s$ and $\partial_- Y^s$ (orthogonal to Y^s) one can see that their scalar product can be set equal to

$$\partial_+ Y^s \partial_- Y_s = -\mu^2 \cosh 2\phi \ , \tag{2.12}$$

where ϕ is a new variable (cf. (2.5)). Then in the AdS_2 case we get $\partial_+\partial_-\phi + \frac{\mu^2}{2}\sinh 2\phi = 0$ which follows from the reduced Lagrangian (cf. (2.6))

$$\widetilde{L} = \partial_+\phi\partial_-\phi - \frac{\mu^2}{2}\cosh 2\phi \ . \tag{2.13}$$

[6] This is a classical gauge equivalence when gauge transformations at the level of Lax equations lead to equivalent integrable systems. The resulting non-local relation at the level of field theory models does not, in general, extend to the quantum level, cf. [34, 35].

Let us now explain how the above special examples can be generalized to the case of the bosonic string on $AdS_n \times S^n$. Denoting the embedding coordinates of AdS_n as Y_s and the coordinates of S^n as X_m the conformal gauge condition means the vanishing of the total stress tensor,

$$\mathrm{T}_{++}(Y) + \mathrm{T}_{++}(X) = 0 \,, \qquad \mathrm{T}_{--}(Y) + \mathrm{T}_{--}(X) = 0 \,. \tag{2.14}$$

Since in the conformal gauge the equations of motion for Y_s and X_m factorize, the corresponding stress tensors are separately traceless and conserved. Then instead of using $t = \mu\tau$ or $\alpha = \mu\tau$ conditions (t is now part of AdS_n and α – part of S^n) we can fix the residual conformal transformation freedom "implicitly" by following [14] and demanding as in (2.4) that $\mathrm{T}_{\pm\pm}(X) = \mu^2 = \mathrm{const}$. Then (2.14) implies that

$$\mathrm{T}_{\pm\pm}(X) = \mu^2 \,, \qquad \mathrm{T}_{\pm\pm}(Y) = -\mu^2 \,. \tag{2.15}$$

We thus get two decoupled AdS_n and S^n sigma models with the constraints (2.15), to which we can separately apply the Pohlmeyer's reduction procedure. That eliminates 1+1 out of $n+n$ degrees of freedom, leaving us with an action for only the $(n-1)+(n-1)$ physical degrees of freedom.

Later in section 6 we shall discuss a generalisation of this reduction procedure to the presence of the superstring fermions when the AdS_n and S^n parts are no longer decoupled.

3 Coset sigma model and the corresponding gauged WZW model with an integrable potential

Let us give a short review of a coset sigma model (of which S^n model is a special case) and the associated gauged WZW model. This will set up the notation for section **4** where we are going to construct an explicit change of variables which relates the F/G coset sigma model to certain G/H gauged WZW model with a potential, giving an explicit realisation of the relationship originally proposed in [24].

3.1 F/G coset sigma model

Let G be a subgroup of a Lie group F and $M = F/G$ be a coset space. Let us assume that the Lie algebra $\mathfrak{f}$ of F is equipped with a positive-definite invariant bilinear form $\langle\ ,\ \rangle$; explicitly, let F be a matrix group and $\langle a, b\rangle = \mathrm{Tr}(ab)$. In addition let F/G be a symmetric space which is the case when

$$\mathfrak{f} = \mathfrak{p} \oplus \mathfrak{g} \,, \qquad [\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g} \,, \qquad [\mathfrak{g},\mathfrak{p}] \subset \mathfrak{p} \,, \qquad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{g} \,, \tag{3.1}$$

where $\mathfrak{p}$ denotes the orthogonal complement of the algebra $\mathfrak{g}$ of G in $\mathfrak{f}$.

The action of the sigma model on F/G is given by

$$S = -\frac{1}{2}\int d^2\sigma\, \eta^{ab}\, \mathrm{Tr}(P_a P_b) \,, \qquad P_a = (f^{-1}\partial_a f)_{\mathfrak{p}} \,, \tag{3.2}$$

where $(\ldots)_{\mathfrak{p}}$ denotes the orthogonal projection to $\mathfrak{p}$, i.e.

$$J = f^{-1}df = \mathcal{A} + P \,, \qquad \mathcal{A} = J_{\mathfrak{g}} \in \mathfrak{g} \,, \quad P = J_{\mathfrak{p}} \in \mathfrak{p} \,. \tag{3.3}$$

The action is invariant under the G gauge transformation $f \to fg$ for an arbitrary G valued function g. Indeed, under this transformation $J = f^{-1}df \to g^{-1}(f^{-1}df)g + g^{-1}dg$ so that P transforms into $g^{-1}Pg$ ensuring the invariance of the Lagrangian. The current J and therefore the action is also invariant under the global F symmetry $f \to f_0 f$ for any constant $f_0 \in F$. Furthermore, the classical coset sigma model action is invariant under the 2d conformal transformations.

The equations of motion take the form

$$D_a P^a = 0\,, \qquad D_a = \partial_a + [\mathcal{A}_a,\ ]\,, \qquad \mathcal{A}_a = (f^{-1}\partial_a f)_{\mathfrak{g}}\,. \tag{3.4}$$

Using the light-cone coordinates σ^+, σ^- they can also be written as

$$D_+ P_- = 0\,, \qquad D_- P_+ = 0\,. \tag{3.5}$$

Indeed, the zero curvature condition for the current J projected to $\mathfrak{p}$ implies

$$(\partial_+ J_- - \partial_- J_+ + [J_+, J_-])_{\mathfrak{p}} = \partial_+ P_- - \partial_- P_+ + [\mathcal{A}_+, P_-] + [P_+, \mathcal{A}_-] = 0\,, \tag{3.6}$$

i.e. $D_+ P_- - D_- P_+ = 0$. This together with (3.4), i.e. $D_+ P_- + D_- P_+ = 0$, then leads to (3.5).[7]

The nonvanishing components of the stress-tensor are

$$\mathrm{T}_{++} = -\frac{1}{2}\mathrm{Tr}(P_+ P_+)\,, \qquad \mathrm{T}_{--} = -\frac{1}{2}\mathrm{Tr}(P_- P_-)\,. \tag{3.7}$$

Equations of motion imply the conservation law $\partial_- \mathrm{T}_{++} = 0\,,\ \partial_+ \mathrm{T}_{--} = 0$. Then making an appropriate conformal transformations one can always set as in (2.4) $\mathrm{T}_{\pm\pm} = \mu^2$.

The Lax representation for the coset sigma model is found from the zero curvature condition $d\omega + \omega \wedge \omega = 0$ for the Lax connection

$$\omega = d\sigma^+(\mathcal{A}_+ + \ell P_+) + d\sigma^-(\mathcal{A}_- + \ell^{-1} P_-)\,, \tag{3.8}$$

i.e.

$$[\partial_+ + \mathcal{A}_+ + \ell P_+, \partial_- + \mathcal{A}_- + \ell^{-1} P_-] = 0\,, \tag{3.9}$$

where ℓ is a spectral parameter. The equations of motion (3.5) follow from (3.9) as the coefficients of order ℓ^{-1} and ℓ terms. The coefficient of the order λ^0 term is the $\mathfrak{g}$-component of the zero curvature condition for the connection $J = \mathcal{A} + P$.

Let us recall also two representations of the Lagrangian of the F/G sigma model. One is to introduce an explicit parametrisation of the coset $M = F/G$ as embedded into F. If x^i are coordinates on M, let $dx^i J_i^*$ be a pullback of J to M. Then the Lagrangian in (3.2) takes the form

$$L = -\frac{1}{2}\eta^{ab}\partial_a x^i \partial_b x^j\, G_{ij}(x)\,, \qquad G_{ij}(x) = \mathrm{Tr}(J_i^*(x) J_j^*(x))\,, \tag{3.10}$$

where G_{ij} is the metric on the coset space. Note that by choosing a particular parametrisation of the coset we have fixed the G gauge symmetry. An alternative form of L is found by introducing a gauge field $\mathrm{A}_a \in \mathfrak{g}$ which serves to implement the projection of the $\mathfrak{f}$-current on $\mathfrak{p}$

$$L = -\frac{1}{2}\eta^{ab}\mathrm{Tr}[f(\partial_a + \mathrm{A}_a)f^{-1}\ f(\partial_b + \mathrm{A}_b)f^{-1}]\,, \tag{3.11}$$

[7]Note that the global right F-symmetry is not seen at the level of equations of motion written in terms of currents because all the currents are explicitly invariant.

or, equivalently,

$$L = -\frac{1}{2}\eta^{ab}\mathrm{Tr}[(f^{-1}\partial_a f - \mathrm{A}_a)\,(f^{-1}\partial_b f - \mathrm{A}_b)]\ . \tag{3.12}$$

Substituting the equation of motion for A

$$\mathrm{A} = \mathcal{A} = (f^{-1}df)_{\mathfrak{g}} \tag{3.13}$$

into (3.11) one returns back to the original Lagrangian in (3.2).

3.2 G/H gauged WZW model with an integrable potential

As was suggested in [24] (see also [25]), a sigma model on a symmetric space F/G can be reduced to a "symmetric space sine-Gordon" model with a Lagrangean formulation in terms of the $\frac{G}{H}$ left-right symmetrically gauged WZW model with a gauge-invariant integrable potential.[8]

The potential is determined by a choice of two elements T_+, T_- in the maximal abelian subspace $\mathfrak{a}$ in the complement $\mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of G in the algebra $\mathfrak{f}$ of F. The algebra $\mathfrak{h}$ of the subgroup H of G should be the centralizer of $T_\pm$ in $\mathfrak{g}$: $[\mathfrak{h}, T_\pm] = 0$. Then the action is

$$S_\mu(g, A) = S_{\mathrm{gWZW}}(g, A) - \mu^2 \int \frac{d^2\sigma}{2\pi}\ \mathrm{Tr}(T_+ g^{-1} T_- g)\ , \tag{3.14}$$

where S_{gWZW} is the action of the left-right symmetrically gauged WZW model [41] (we omit an overall level k factor)

$$\begin{aligned} S_{\mathrm{gWZW}} = &-\int \frac{d^2\sigma}{4\pi}\mathrm{Tr}(g^{-1}\partial_+ g g^{-1}\partial_- g) + \int \frac{d^3\sigma}{12\pi}\mathrm{Tr}(g^{-1}dgg^{-1}dgg^{-1}dg) \\ &- \int \frac{d^2\sigma}{2\pi}\mathrm{Tr}\big(A_+\,\partial_- g g^{-1} - A_-\, g^{-1}\partial_+ g - g^{-1}A_+ g A_- + A_+ A_-\big)\ . \end{aligned} \tag{3.15}$$

Here $g \in G$ and $A_\pm \in \mathfrak{h}$ (all the fields are assumed to be matrices in a given representation of F or of its Lie algebra $\mathfrak{f}$).

Note that using Polyakov–Wiegmann identity the action (3.15) can be written also in the following form

$$S_{\mathrm{gWZW}} = S_{\mathrm{WZW}}(h^{-1}gh') - S_{\mathrm{WZW}}(h^{-1}h')\,, \tag{3.16}$$

$$A_+ = h^{-1}\partial_+ h\ , \qquad A_- = h'^{-1}\partial_- h'\ . \tag{3.17}$$

To define the action with $T_\pm$ belonging to the algebra of F it is assumed that $g \in G$ is trivially (diagonally) embedded into F. The action is then invariant under the vector gauge transformations with parameters taking values in H:

$$g \to hgh^{-1}\ , \qquad A_a \to h(A_a + \partial_a)h^{-1}\ , \qquad h \in H\ , \tag{3.18}$$

[8]This is a special case of a non-abelian Toda theory [27]. Non-abelian Toda models are of the two basic types – "homogeneous sine-Gordon" and "symmetric space sine-Gordon" [25]. For the first type the gWZW part of the Toda model corresponds to $\frac{G}{[U(1)]^r}$ (r is a rank of G). The models of the second type are reduced theories associated to sigma models on compact symmetric spaces. They are quantum-integrable but their S-matrix is not known, except for special cases of SG and CSG models. A review can be found in [43].

where $A_a \in \mathfrak{h}$ and $h^{-1}T_\pm h = T_\pm$ (since $[\mathfrak{a}, \mathfrak{h}] = 0$).

The equations of motion following from (3.15) are

$$\partial_-(g^{-1}\partial_+ g \ + \ g^{-1}A_+ g) - \partial_+ A_- \\ + [A_-, g^{-1}\partial_+ g + g^{-1}A_+ g] + \mu^2[g^{-1}T_- g, T_+] = 0 \,, \tag{3.19}$$

$$A_+ = (g^{-1}\partial_+ g + g^{-1}A_+ g)_\mathfrak{h} \,, \qquad A_- = (-\partial_- g g^{-1} + g A_- g^{-1})_\mathfrak{h} \,. \tag{3.20}$$

Note that $g^{-1}T_- g \in \mathfrak{p}$ so that $[T_+, g^{-1}T_- g] \in \mathfrak{m}$, where $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. In particular, the $\mathfrak{h}$-component of the first equation implies that A_a is flat,

$$\partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0 \,. \tag{3.21}$$

Let us comment on the classical integrability of the above model (3.14). It is well known that the equations of motion of the standard WZW model can be written in the Lax form. The same also applies to gauged WZW model with the above potential. More precisely, using $[A_a, T_\pm] = 0$ one can show that equation (3.19) can be written in the Lax form, i.e. it follows from $[\mathcal{L}_+, \mathcal{L}_-] = 0$ where (ℓ is a spectral parameter)

$$\mathcal{L}_+ = \partial_+ + g^{-1}\partial_+ g + g^{-1}A_+ g + \ell\mu T_+ \,, \qquad \mathcal{L}_- = \partial_- + A_- + \ell^{-1}\mu g^{-1}T_- g \,, \tag{3.22}$$

or, equivalently, from the zero curvature equation for the $\mathfrak{f}$-valued Lax connection

$$\omega = d\sigma^+(g^{-1}\partial_+ g + g^{-1}A_+ g + \ell\mu T_+) + d\sigma^-(A_- + \ell^{-1}\mu g^{-1}T_- g) \,. \tag{3.23}$$

While the remaining equations (3.20) (constraints) do not follow from this condition, they may be considered as consequences of (3.19) in the sense that given a solution to (3.19) one can find a gauge transformation such that the transformed solution satisfies (3.20).

This is possible because eq. (3.19) has a *larger gauge symmetry* than the original gWZW model (3.15): it is invariant under the $H \times H$ gauge symmetry

$$g \to h^{-1} g \bar{h} \,, \qquad A_+ \to h^{-1}A_+ h + h^{-1}\partial_+ h \,, \qquad A_- \to \bar{h}^{-1}A_- \bar{h} + \bar{h}^{-1}\partial_- \bar{h} \,, \tag{3.24}$$

where h and $\bar{h}$ are two arbitrary H-valued functions. The symmetry of (3.15) is the diagonal subgroup (with $h = \bar{h}$) of the extended "on-shell" gauge symmetry (3.24). It turns out that using this extended symmetry one can fulfil the constraints (3.20). Further details and the proof are relegated to the Appendix **A**. We shall also use this observation in section **4** below.

Let us note also that given an automorphism τ of the algebra H preserving the trace one can fix the $H \times H$ gauge symmetry of the equations of motion in a more general way so that (3.20) is replaced by

$$\tau(A_+) = (g^{-1}\partial_+ g + g^{-1}A_+ g)_\mathfrak{h} \,, \qquad A_- = (-\partial_- g g^{-1} + g\tau(A_-)g^{-1})_\mathfrak{h} \,. \tag{3.25}$$

The corresponding equations (3.19),(3.25) then follow from the Lagrangian (3.14),(3.15) with the replacement

$$A_- \ \to \ \tau(A_-) \tag{3.26}$$

in the $A_- g^{-1}\partial_+ g$ and the $g^{-1}A_+ g A_-$ terms. The corresponding gauge symmetry is then $g \to h^{-1}\, g\, \widehat{\tau}(h)$ where $\widehat{\tau}$ is a lift of τ from $\mathfrak{h}$ to H (see [25, 29]). In this case the left-right symmetrically gauged WZW model is thus replaced by a more general asymmetrically gauged WZW model [40, 33].

It was observed in [24] that since the field strength of A_a vanishes (3.21) on the equations of motion, one can choose a gauge where[9]

$$A_+ = A_- = 0 \, . \tag{3.27}$$

Then the classical equations (3.19),(3.20) reduce to

$$\partial_-(g^{-1}\partial_+ g) - \mu^2 [T_+, g^{-1}T_- g] = 0 \, , \tag{3.28}$$

$$(g^{-1}\partial_+ g)_{\mathfrak{h}} = 0 \, , \qquad (\partial_- g g^{-1})_{\mathfrak{h}} = 0 \, . \tag{3.29}$$

These equations happen to be equivalent to the equations of motion of the reduced F/G model found in [16, 18, 19].

Various special cases, structure of vacua and solitonic solutions of the equations (3.28),(3.29) were discussed in [43, 29] and refs. there.

The set of equations (3.28),(3.29) do not directly follow from a local Lagrangian. As was implied in [24], to get a local Lagrangian formulation of these equations one is to go back to the action (3.15) on a bigger configuration space involving both g and A_a with the gauge invariance (3.18).

At the same time, one would like also to have a reduced action involving only the independent degrees of freedom, i.e. generalizing the actions of the SG (2.6) and the CSG (2.7) models.

Below in section **4** we shall explain why and under which conditions the relation between the equations of the reduced theory corresponding to the F/G coset model and the equations of the G/H gWZW model proposed in [24] actually works. Then in section **5** we shall suggest how to use this correspondence to find a local Lagrangian for the physical number of degrees of freedom of the reduced model.

The main observation will be that there exists an equivalent representation for the classical equations following from (3.14) (or gauge-equivalent, in the sense of [15], representation of the Lax equations corresponding to (3.22)) in which they admit an explicit Lagrangean formulation without any residual gauge invariance, thus generalizing the SG and CSG examples. Instead of the "on-shell" gauge $A_a = 0$ used in [24] one can impose an "of-shell" H-gauge on the group element g and then solve for the gauge field A_a. "Integrating out" A_a then leads to a sigma model for the independent dim(G/H) number of parameters in g in the same way as in the examples of conformal sigma models associated to gWZW models [36, 38, 40].[10]

[9] This gauge is thus possible only on-shell; to gauge away A_a at the level of the gWZW Lagragian one would need some additional local gauge invariance.

[10] Integrating out the gauge field at the quantum level induces also a dilaton [36]; there are also quantum $\alpha' \sim 1/k$ corrections to the sigma model background fields [45, 46, 47]. These will be ignored at the classical level we are restricted to here.

4 Reduced theory for F/G coset sigma model: equations of motion

The strategy to relate the equations of motion of the F/G coset model to those of the G/H gWZW model will be to impose the so called reduction gauge in the equations of the F/G model (3.5) written in terms of the independent current components and then to make use of the 2d conformal symmetry to eliminate one additional degree of freedom. This will allow us to solve all gauge-fixed equations of motion but the Maurer-Cartan equation explicitly in terms of a new field g taking values in G and the $\mathfrak{h}$-valued gauge field $A_\pm$. The remaining system of equations (i.e. the components of the Maurer–Cartan equation in this parametrization) will turn out to be invariant under both the left and the right H gauge symmetries. We will then prove that one can impose the special gauge conditions under which the gauge symmetry reduces to that of the H-gauge invariance of the G/H gWZW model and the equations become equivalent to the ones (3.19),(3.20) following from the gWZW action with an integrable potential (3.14) described in section **3.2**.

4.1 Equation of motion in terms of currents and the reduction gauge

The relation between the reduced F/G model and the G/H gWZW model will apply under certain special conditions on the structure of the Lie algebras of the groups involved. These conditions that we will specify below will be satisfied, in particular, in the case of the $S^n = SO(n{+}1)/SO(n)$ model (and its AdS_n counterpart) which is our main interest here.

Let $\mathfrak{a}$ be a maximal Abelian subspace of the orthogonal complement $\mathfrak{p}$ of the algebra $\mathfrak{g}$ of G in the algebra $\mathfrak{f}$ of F. Let $\mathfrak{h}$ be its centralizer in $\mathfrak{g}$. Following [18] we shall assume the following conditions on the structure of these algebras (which represent a special case of (3.1))

$$\mathfrak{f} = \mathfrak{p} \oplus \mathfrak{g}\,, \qquad \mathfrak{p} = \mathfrak{a} \oplus \mathfrak{n}\,, \qquad \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}\,, \qquad [\mathfrak{a},\mathfrak{a}] = 0\,, \qquad [\mathfrak{h},\mathfrak{a}] = 0\,, \tag{4.1}$$

$$[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}\,, \qquad [\mathfrak{m},\mathfrak{h}] \subset \mathfrak{m}\,, \qquad [\mathfrak{m},\mathfrak{a}] \subset \mathfrak{n}\,, \qquad [\mathfrak{a},\mathfrak{n}] \subset \mathfrak{m}\,. \tag{4.2}$$

Starting with a left-invariant current $J = f^{-1}df$ with $f \in F$ we shall use the following notation for its $\mathfrak{h}$, $\mathfrak{m}$ and $\mathfrak{p}$ components

$$A_a = (f^{-1}\partial_a f)_{\mathfrak{h}}\,, \qquad B_a = (f^{-1}\partial_a f)_{\mathfrak{m}}\,, \qquad P_a = (f^{-1}\partial_a f)_{\mathfrak{p}}\,, \tag{4.3}$$

i.e. $\mathcal{A}_a \in \mathfrak{g}$ in (3.3) is equal to $A_a + B_a$. The equations of motion of the F/G sigma model (3.5) written in terms of the *current components* A_a, B_a, P_a viewed as *independent fields* then take the form

$$D_+ P_- = 0\,, \qquad D_- P_+ = 0\,, \tag{4.4}$$

$$\partial_+(A_- + B_-) - \partial_-(A_+ + B_+) + [A_+ + B_+, A_- + B_-] = [P_-, P_+]\,, \tag{4.5}$$

where $D_\pm = \partial_\pm + [A_\pm + B_\pm,\ ]$.

The choice of the *reduction gauge* [18] is based on the "polar decomposition" theorem which states that for any $k \in \mathfrak{p}$ there exists $g_0 \in G$ such that $g_0^{-1} k g_0 \in \mathfrak{a}$. Using the G gauge freedom of the coset model equations of motion one can therefore assume that one of the components of P_a, e.g., P_+ is $\mathfrak{a}$-valued. Then $D_- P_+ = 0$ implies

$$\partial_- P_+ = 0 \,, \qquad\qquad [B_-, P_+] = 0 \,. \tag{4.6}$$

Here we made use of the condition $[\mathfrak{m}, \mathfrak{a}] \subset \mathfrak{n}$ in (4.2). Under a certain regularity condition which we shall assume (in the case when $\mathfrak{a}$ is one-dimensional, e.g., for $F/G = SO(n+1)/SO(n)$, it is enough to require that $P_+ \neq 0$) the equation $[B_-, P_+] = 0$ implies that

$$B_- = 0 \,. \tag{4.7}$$

To summarise, by imposing the gauge in which $P_+ \in \mathfrak{a}$ and eliminating B_- by solving $[B_-, P_+] = 0$ (i.e. setting B_- to zero) one can bring the system of the F/G model equations of motion (4.4),(4.5) to the following form:

$$\partial_- P_+ = 0 \,, \qquad\qquad \partial_+ P_- + [A_+, P_-] + [B_+, P_-] = 0 \,, \tag{4.8}$$

$$\partial_- B_+ + [A_-, B_+] = [P_+, P_-] \,, \tag{4.9}$$

$$\partial_- A_+ - \partial_+ A_- + [A_-, A_+] = 0 \,, \tag{4.10}$$

where (4.9) and (4.10) are $\mathfrak{m}$ and $\mathfrak{h}$ projections of (4.5) (we are using the conditions (4.1),(4.2)).

In this reduction gauge the original G gauge symmetry is reduced to H gauge symmetry under which the current component $A_\pm$ transforms as a gauge potential while $B_\pm$ and $P_\pm$ transform covariantly, i.e. as $(...) \rightarrow h^{-1}(...)h$. In particular, P_+ is invariant because it takes values in $\mathfrak{a}$ and $[\mathfrak{a}, \mathfrak{h}] = 0$.

Let us note that (4.10) implies that we can impose the on-shell H gauge where $A_\pm = 0$. In this gauge the equations of motion (4.8),(4.9) take the form:

$$\partial_- P_+ = 0 \,, \qquad \partial_+ P_- = [P_-, B_+] \,, \qquad \partial_- B_+ = [P_+, P_-] \,. \tag{4.11}$$

4.2 Fixing conformal symmetry, field redefinition and relation to G/H gauged WZW model

The first equation $\partial_- P_+ = 0$ in (4.8) implies that $P_+ = P_+(\sigma^+)$. One can then fix one component of the matrix function P_+ using the residual conformal symmetry under which $P_+ d\sigma^+ = P'_+ d\sigma'^+$. Since in the reduction gauge P_+ belongs to the abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, then if $\dim \mathfrak{a} = 1$ (which is the case, e.g., for the $SO(n+1)/SO(n)$ coset of our interest) one can always assume that $P_+ = \mu T_+$ where $T_+ \in \mathfrak{a}$ is a constant matrix in $\mathfrak{f}$ which is a basic element of $\mathfrak{a}$ (we may also normalize it so that $\mathrm{Tr}(T_+ T_+) = -2$). This is equivalent to requiring that the corresponding component of the stress tensor in (3.7) is constant, i.e. $\mathrm{T}_{++} = \mu^2$.

Furthermore, we can use the remaining conformal symmetry $\sigma^- \rightarrow \sigma'^-(\sigma^-)$ to fix the T_{--} component in (3.7) also to be constant as in the original Pohlmeyer's argument.[11] Thus assuming that the

[11]The conservation equation $\partial_+ \mathrm{T}_{--} = 0$ can be seen directly from the second equation in (4.11).

maximal Abelian subspace $\mathfrak{a}$ of $\mathfrak{p} = \mathfrak{f} \ominus \mathfrak{g}$ is 1-dimensional and using the conformal symmetry we arrive at

$$P_+ = \ \mu \, T_+ \,, \qquad\qquad -\frac{1}{2}\,\mathrm{Tr}(P_-P_-) = \mu^2, \tag{4.12}$$

$$\mathrm{T}_{\pm\pm} = \mu^2 \,, \qquad\qquad \mu, \ T_+ = \mathrm{const} \ . \tag{4.13}$$

The first condition in (4.12) fixes one independent degree of freedom contained in P_+ in the case when $\dim \mathfrak{a} = 1$ and the second condition reduces by one the number of independent degrees of freedom in P_-. The normalization condition on P_- can be solved by

$$P_- = \ \mu \, g^{-1} T_- g \,, \qquad T_- = \mathrm{const} \,, \tag{4.14}$$

where $g \in G$ is a *new field variable* (thus non-locally related to original variable $f \in F$ in (4.3)) and T_- is a constant matrix which is a fixed element of $\mathfrak{a}$. The existence of such g follows again from the polar decomposition theorem, and the requirement of $\mathrm{T}_{--} = \mu^2$ implies that $\mathrm{Tr}(T_-T_-) = -2$. In the case of $\dim \mathfrak{a} = 1$ which we are considering here it follows that

$$T_+ = T_- \equiv T \ . \tag{4.15}$$

For generality and to indicate the Lorentz index structure, below we shall often keep the separate notation for T_+ and T_-.

The equation for P_- in (4.8) written in terms of g in (4.14) then becomes

$$\partial_+(g^{-1}T_-g) + [\mathcal{A}_+, g^{-1}T_-g] = 0 \,, \qquad \mathcal{A}_+ = A_+ + B_+ \ . \tag{4.16}$$

Considering $\mathcal{A}_+ \in \mathfrak{g}$ as an unknown, the general solution of this equation can be written as

$$\mathcal{A}_+ = g^{-1}\partial_+ g + g^{-1}A'_+ g \,, \tag{4.17}$$

where A'_+ is an arbitrary $\mathfrak{h}$-valued function. Indeed, the first term in (4.17) is obviously a particular solution of (4.16) (since $T_- = \mathrm{const}$) while the second term is a general solution of the homogeneous equation $[\mathcal{A}_+, g^{-1}T_-g] = 0$ (given that $[A'_+, T_-] = 0$ since $[\mathfrak{h}, \mathfrak{a}] = 0$). Thus

$$A_+ = (g^{-1}\partial_+ g + g^{-1}A'_+ g)_{\mathfrak{h}} \,, \qquad B_+ = (g^{-1}\partial_+ g + g^{-1}A'_+ g)_{\mathfrak{m}} \,. \tag{4.18}$$

In terms of the new variables $g, \ A'_+, \ A_-$ the first two equations of motion in (4.4) or (4.8) are solved and the remaining equation (4.5) (or (4.9),(4.10) which are its $\mathfrak{m}$ and $\mathfrak{h}$ components) then takes the form

$$\partial_-(g^{-1}\partial_+ g + g^{-1}A'_+ g) - \partial_+ A_- + [A_-, g^{-1}\partial_+ g + g^{-1}A'_+ g] = \mu^2 [T_+, g^{-1}T_- g] \,. \tag{4.19}$$

As discussed in section **3.2**, this equation is equivalent to the equations of motion of the gWZW theory (3.19),(3.20) in the sense that by an appropriate gauge transformation one can always make the following constraints satisfied:

$$A'_+ = (g^{-1}\partial_+ g + g^{-1}A'_+ g)_{\mathfrak{h}} \,, \qquad A_- = (g\partial_- g^{-1} + gA_- g^{-1})_{\mathfrak{h}} \,. \tag{4.20}$$

After renaming A'_+ as A_+ these are exactly the equation of motion (3.19) and the constraints (3.20).[12]

We have thus shown that the original system of equations of the F/G sigma model (4.8), (4.9),(4.10) is equivalent to the one described by the equation (4.19) and the constraints (4.20) with the H gauge symmetry (3.24) with $h = \bar{h}$. These are the same equations of motion (3.19), the constraints (3.20) and the gauge symmetry as corresponding to the action (3.14) of the G/H gauged WZW model (3.15) with the potential $\sim \mu^2 \mathrm{Tr}(T_+ g^{-1} T_- g)$.

That the reduced equations of motion of the F/G coset model can be related to those of the gWZW model with an integrable potential was first suggested in [24] (and checked on several examples including $SO(n+1)/SO(n)$, $SU(n+1)/U(n)$, and $SU(n)/SO(n)$ cosets). Here we explained why this correspondence should work in general and specified the necessary conditions on the groups and the algebras involved.

4.3 Gauge equivalence of Lax representations for the F/G coset and G/H gauged WZW models

Imposing the reduction gauge in terms of the Lax connections can be achieved in a directly analogous way. Let ω be an $\mathfrak{f}$-valued Lax connection defined in (3.8). The gauge equivalence transformation $\omega' = f^{-1}\omega f + f^{-1}df$ with $f \in F$ gives a new system determined by a gauge-equivalent Lax connection ω'. Decomposing $\omega = \omega_\mathfrak{p} + \omega_\mathfrak{g}$ one observes that in the special case of $f = g \in G$ the component $\omega_\mathfrak{p}$ transforms as $\omega'_\mathfrak{p} = g^{-1}\omega_\mathfrak{p} g$. Using the same polar decomposition argument as discussed above one concludes that it is always possible to find a G-valued function g such that (cf. (4.1)) $(\omega_\mathfrak{n})_+ = (A_+ + B_+ + \ell P_+)_\mathfrak{n} = 0$.

Decomposing ω' according to $\mathfrak{f} = \mathfrak{p} \oplus \mathfrak{m} \oplus \mathfrak{h}$

$$\begin{gathered} \omega' = d\sigma^+(A_+ + B_+ + \ell P_+) + d\sigma^-(A_- + B_- + \ell^{-1} P_-)\,, \\ A_\pm \in \mathfrak{h}\,, \quad B_\pm \in \mathfrak{m}, \quad P_+ \in \mathfrak{a}\,, \qquad P_- \in \mathfrak{p}\,, \end{gathered} \tag{4.21}$$

one finds as above that the compatibility condition implies eqs. (4.6), i.e. $\partial_- P_+ = 0$ and $[P_+, B_-] = 0$; the latter gives again $B_- = 0$. This allows us to relate the Lax connection to that with $B_- = 0$, i.e.

$$\omega'' = d\sigma^+(A_+ + B_+ + \ell P_+) + d\sigma^-(A_- + \ell^{-1} P_-)\,, \tag{4.22}$$

whose flatness condition implies the last two equations in (4.8).[13]

As for the equations $\partial_- P_+ = 0$ and $\partial_+ P_- + [A_+ + B_+, P_-] = 0$ in (4.8), assuming they are satisfied, one can again use the conformal transformations to set $P_+ = \mu T_+$ and $\mathrm{Tr}(P_- P_-) = -2\mu^2$. As a result, the Lax connection takes the following form:

$$\omega_{\rm red} = d\sigma^+(A_+ + B_+ + \ell\mu T_+) + d\sigma^-(A_- + \ell^{-1} P_-)\,. \tag{4.23}$$

[12]More generally one, can conside asymmetrical gauge by introducing the appopriate $\mathfrak{h}$-automorphism τ. See the respective discussion in section **3.2**.

[13] Note that this reduction is local as $B_- = 0$ is an algebraic consequence of the compatibility condition, i.e. B_- is an auxiliary field.

Finally, using again the parametrisation $P_- = \mu g^{-1}T_- g$ and $A_+ + B_+ = g^{-1}\partial_+ g + g^{-1}A'_+ g$, one arrives at

$$\omega = d\sigma^+(g^{-1}\partial_+ g + g^{-1}A'_+ g + \ell\mu T_+) + d\sigma^-(A'_- + \ell^{-1}\mu g^{-1}T_- g)\,, \tag{4.24}$$

whose compatibility condition implies (4.19). It was shown in the previous subsection that by an appropriate gauge transformation one can also satisfy the on-shell relations (4.20). We thus find the relation to the Lax representation of the G/H gWZW model (cf. (3.20),(3.23)).

4.4 Reduced equations of $S^n = SO(n+1)/SO(n)$ coset model in the $A_\pm = 0$ gauge

Let us now turn to the special case of our interest: sigma model with a sphere as a target space. Using the standard $(n+1)\times(n+1)$ matrix representation for $F = SO(n+1)$ and its diagonally embedded $G = SO(n)$ subgroup we can choose $T_+ = T_-$ to have only one non-zero upper 2×2 block so that $H = SO(n-1)$ is also diagonally embedded into $G = SO(n)$ (the conditions (4.1),(4.2) are then satisfied). In this case we get for $P_\pm$ in (4.12),(4.14)

$$P_+ = \mu T_+ = \mu \begin{pmatrix} 0 & 1 & \dots & 0 \\ -1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \qquad P_- = \mu \begin{pmatrix} 0 & k_1 & \dots & k_n \\ -k_1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ -k_n & 0 & \dots & 0 \end{pmatrix}. \tag{4.25}$$

Here g in (4.14) is parametrized by k_l and $-\frac{1}{2}\mathrm{Tr}(P_+P_+) = \mu^2$. Also, $-\frac{1}{2}\mathrm{Tr}(P_-P_-) = \mu^2$ is satisfied provided

$$\sum_{s=1}^{n} k_s k_s = 1\,. \tag{4.26}$$

The subalgebras $\mathfrak{g} = so(n)$ and $\mathfrak{h} = so(n-1)$ are canonically (diagonally) embedded into $\mathfrak{f} = so(n+1)$. In addition to $B_- = 0$ from (4.6),(4.7) we have for $B_+ = (\mathcal{A}_+)_\mathfrak{m}$ (see (4.16))

$$B_+ = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & b_2 & \dots & b_n \\ 0 & -b_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & -b_n & 0 & \dots & 0 \end{pmatrix}. \tag{4.27}$$

In this case the equation $\partial_+ P_- + [A_+, P_-] = [P_-, B_+]$ in (4.8) can be solved algebraically for B_+ giving (4.27) with

$$b_l = \frac{\partial_+ k_l + [A_+, k]_l}{\sqrt{1 - \sum_{m=2}^{n} k_m k_m}}\,, \qquad l = 2, \dots, n\,. \tag{4.28}$$

Fixing the $H = SO(n-1)$ on-shell gauge as

$$A_+ = A_- = 0\,, \tag{4.29}$$

the third equation in (4.8) then gives the following reduced system of equations for the remaining $n-1$ unknown functions $k_2, ..., k_n$ (k_1 is determined from (4.26)) [19]

$$\partial_- \frac{\partial_+ k_l}{\sqrt{1-\sum_{m=2}^n k_m k_m}} = -\mu^2 k_l \,, \qquad l = 2, \ldots, n \,. \tag{4.30}$$

This is the same reduced system that follows both from the $SO(n+1)/SO(n)$ coset model [16, 18] and the $SO(n)/SO(n-1)$ gWZW model in the $A_\pm = 0$ gauge [24].

The point $g = \mathbf{1}$ is an obvious vacuum for eq. (3.19) in the $A_\pm = 0$ gauge, i.e. a trivial solution of (3.28),(3.29) with $T_+ = T_-$. According to (4.14),(4.25) it corresponds to

$$k_2 = ... = k_n = 0 \,. \tag{4.31}$$

The massive fluctuations near this vacuum in the gauge (4.29) are described by the $H = SO(n-1)$ invariant equation (4.30), i.e.

$$\partial_+\partial_- k_l + \mu^2 k_l + O(k_l^2) = 0 \,. \tag{4.32}$$

It is convenient to rewrite the equation (4.30) in terms of the new variables (φ, u_m) defined so that (4.26) is satisfied

$$k_1 = \cos 2\varphi \,, \qquad k_l = u_l \sin 2\varphi \,, \qquad u_l u_l = 1 \,, \qquad l = 2, \ldots, n \,, \tag{4.33}$$

getting [19]

$$\begin{gathered} \partial_+\partial_-\varphi - \frac{1}{2}\tan 2\varphi \, \partial_+ u_l \partial_- u_l + \frac{\mu^2}{2}\sin 2\varphi = 0 \,, \\ \partial_+\partial_- u_l + (\partial_+ u_m \partial_- u_m)\, u_l + \frac{2}{\sin 2\varphi}(\cos 2\varphi \, \partial_+\varphi \partial_- u_l + \frac{1}{\cos 2\varphi}\, \partial_-\varphi \partial_+ u_l) = 0 \,. \end{gathered} \tag{4.34}$$

Besides the obvious $SO(n-1)$ symmetry these equations are invariant under the following formal transformation

$$\varphi \;\to\; \varphi + \frac{\pi}{2} \,, \qquad\qquad \mu^2 \to -\mu^2 \,. \tag{4.35}$$

In the case of $F/G = SO(4)/SO(3)$, i.e. CSG as a reduced model, this formal transformation relates the two 2d dual reduced models with T-dual target space metrics in the corresponding reduced Lagrangians [23, 39, 29].[14]

Let us briefly describe the modifications of the above construction in the case of the $AdS_n = SO(2, n-1)/SO(1, n-1)$ coset model. The vector-space signature is diag$(-1, -1, 1, \ldots, 1)$ and the subgroup $G = SO(1, n-1)$ is diagonaly embedded. In the standard representation of $\mathfrak{f} = so(2, n-1)$ the element $T_+ = T_-$ can be choosen to have the same form as in (4.25) while the condition (4.26) takes the form $k_1 k_1 - \sum_{m=2}^n k_m k_m = 1$. Equation (4.30) is then replaced by

$$\partial_- \frac{\partial_+ k_l}{\sqrt{1+\sum_{m=2}^n k_m k_m}} = -\mu^2 k_l \,, \qquad l = 2, \ldots, n \,. \tag{4.36}$$

[14] In this case of $SO(3)/SO(2)$ gWZW model this duality is also related with the vector ($g \to h^{-1}gh$) or the axial ($g \to hgh$) gauging [45, 44].

Finally, introducing instead of (4.33) the parametrization

$$k_1 = \cosh 2\phi\,, \qquad k_l = u_l \sinh 2\phi\,, \qquad u_l u_l = 1\,, \qquad l = 2, \ldots, n\,, \tag{4.37}$$

one arrives at the system (4.34) for ϕ, u_l with the obvious replacement of $\cos\varphi, \sin\varphi, \tan\varphi$ with $\cosh\phi, \sinh\phi, \tanh\phi$. The two systems of equations are thus related by the replacement $\varphi = i\phi$, as one would expect from the standard analytic continuation argument. Remarkably, the variables u_l satisfy the same normalization condition in the S^n and the AdS_n cases and both systems are invariant under the same $H = SO(n-1)$ symmetry. Note also that in the AdS_n case the linearized equations (4.32) have exactly the same form leading to the same massive fluctuations near the vacuum $g = \mathbf{1}$.

Instead of using the parametrization of P_- in terms of k_l in (4.25) we may start with a particular choice of $g \in G$ which then determines P_- according to (4.14). Parametrising $g \in G = SO(n)$ by the generalized Euler angles and expressing P_- in terms of them one arrives at a certain multi-field generalisation of the sine-Gordon equation which is just another form of (4.34) (φ introduced in (4.33) corresponds then to the first Euler angle). In the $SO(3)/SO(2)$ case this gives the standard sine-Gordon equation

$$g = \begin{pmatrix} \cos 2\varphi & \sin 2\varphi \\ -\sin 2\varphi & \cos 2\varphi \end{pmatrix}\,, \qquad k_1 = \cos 2\varphi\,, \quad k_2 = \sin 2\varphi\,, \tag{4.38}$$

$$\partial_+\partial_-\varphi + \frac{\mu^2}{2}\sin 2\varphi = 0\,. \tag{4.39}$$

In the $SO(4)/SO(3)$ case we can parametrize $g \in SO(3)$ as

$$g = g_2 g_1 g_2\,, \qquad g_1 = \exp\left(2\varphi R_1\right)\,, \qquad g_2 = \exp\left(\chi R_2\right)\,, \tag{4.40}$$

$$R_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \qquad R_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\,. \tag{4.41}$$

The corresponding components of the unit vector k_s in (4.25),(4.33) are

$$k_1 = \cos 2\varphi\,, \qquad k_2 = \sin 2\varphi\ \cos\chi\,, \qquad k_3 = \sin 2\varphi\ \sin\chi\,. \tag{4.42}$$

The equations of motion (4.34) take the form

$$\begin{gathered} \partial_+\partial_-\varphi - \frac{1}{2}\tan 2\varphi\,\partial_+\chi\partial_-\chi + \frac{\mu^2}{2}\sin 2\varphi = 0\,, \\ \partial_+\partial_-\chi + \frac{2}{\sin 2\varphi}\left(\cos 2\varphi\,\partial_+\varphi\,\partial_-\chi + \frac{1}{\cos 2\varphi}\,\partial_-\varphi\,\partial_+\chi\right) = 0\,. \end{gathered} \tag{4.43}$$

These equations can be brought to the standard complex sine-Gordon form by a (nonlocal) change of variables (which may be interpreted as a gauge change in (3.19),(3.20)). Indeed, replacing χ by θ via

$$\partial_+\theta = \frac{\cos^2\varphi}{\cos 2\varphi}\partial_+\chi\,, \qquad \partial_-\theta = \cos^2\varphi\,\partial_-\chi\,, \tag{4.44}$$

we get [19]:

$$
\begin{aligned}
&\partial_+\partial_-\varphi - \frac{\sin\varphi}{\cos^3\varphi}\partial_+\theta\partial_-\theta + \frac{\mu^2}{2}\sin 2\varphi = 0\,, \\
&\partial_+\partial_-\theta + \frac{2}{\sin 2\varphi}\left(\partial_+\varphi\partial_-\theta + \partial_-\varphi\partial_+\theta\right) = 0\,,
\end{aligned}
\tag{4.45}
$$

which follow from the local CSG Lagrangian (2.7). If we replace eq. (4.44) by the transformation

$$
\partial_+\widetilde{\theta} = -\frac{\sin^2\varphi}{\cos 2\varphi}\partial_+\chi\,, \qquad \partial_-\widetilde{\theta} = \sin^2\varphi\,\partial_-\chi\,, \tag{4.46}
$$

we get instead of (4.45) the equations that follow from the analog of (2.7) with T-dual target space metric: $ds^2 = d\varphi^2 + \cot^2\varphi\, d\widetilde{\theta}^2$. Both the corresponding "dual" Lagrangian and its equations of motion are related, respectively, to (2.7) and (4.45) by the transformation (4.35). The fields θ in (4.44) and $\widetilde{\theta}$ in (4.46) are related of course by the 2d duality transformation.

In general, the equations (4.30) found in the $A_\pm = 0$ gauge do not follow from a local Lagrangian for the field k_m (apart from the $n = 2$, i.e. the SG case). In particular, this applies to the system (4.43): one needs a nontrivial field redefinition (4.44) (which is consistent only on the equations of motion for φ) to get a Lagrangean system (4.45).

Such a non-local field redefinition may be interpreted as corresponding to a change of the H gauge. A way to get a Lagrangean system of the reduced equations is to fix the H gauge not on $A_\pm$ (as was done in [24] and above in this section) but on g, i.e. to solve the equations for $A_\pm$ in terms of the gauge-fixed g. We shall discuss this procedure in the next section.

5 Lagrangian of reduced theory: $S^n = SO(n+1)/SO(n)$ model

As we have seen in section **4**, the reduced equations of motion of the F/G coset model are in general gauge-equivalent to the equations of motion of the G/H gWZW model with a specific integrable potential. To get a Lagrangean formulation of the reduced theory corresponding to the F/G model (or, equivalently, to the bosonic string on $R_t \times F/G$ in the conformal gauge) we may then start with the associated G/H gWZW model, fix an H-gauge on $g \in G$ and solve for the auxiliary gauge field $A_\pm$. This will produce a classically-equivalent integrable system. Here we shall concentrate on the example of the S^n sigma model.

5.1 General structure of the reduced Lagrangian

In the case of $F/G = S^n$, i.e. $G/H = SO(n)/SO(n-1)$ we will end up with an integrable theory represented by an (n–1)- dimensional sigma model with a potential[15]

$$
L = G_{mk}(x)\,\partial_+x^m\partial_-x^k - \mu^2 U(x)\,. \tag{5.1}
$$

[15]The absence of the antisymmetric B_{mn} coupling has to do with the symmetric gauging of the maximal diagonal subgroup.

The special cases are the $n = 2$ (2.6) and $n = 3$ (2.7) examples discussed above. Here x^m are the $n-1$ $(= \dim G - \dim H)$ independent components of g left over after the H gauge fixing on g.

In contrast to the metric of the usual geometric (or "right") coset $SO(n)/SO(n-1) = S^{n-1}$ the metric G_{mk} in (5.1) found from the symmetrically gauged $G/H = \frac{SO(n)}{SO(n-1)}$ gWZW model will generically have singularities and no non-abelian isometries.[16]

Following [42] we may call these geometries resulting from conformal $\frac{SO(n)}{SO(n-1)}$ gWZW models as "conformal cosets" or "conformal spheres", with the notation Σ^{n-1}. Instead of $R_{mk} = c\, G_{mk}$ for a standard sphere their metric G_{mk} satisfies $R_{mk} + 2\nabla_m \nabla_k \Phi = 0$ where Φ is the corresponding dilaton resulting from integrating out A_a. The explicit expressions for G_{mk} were worked out for a few low-dimensional cases: Σ^2 [36], Σ^3 [37, 38, 40] and Σ^4 [44].

The potential ("tachyon") term in (5.1) originates directly from the μ^2 term in (3.14). It is a relevant (and integrable) perturbation of the gWZW model and thus also of the "reduced" geometry, so that it should satisfy (see also [48])

$$\frac{1}{\sqrt{G}e^{-2\Phi}}\partial_m(\sqrt{G}e^{-2\Phi}G^{mk}\partial_k)U - M^2 U = 0 \, . \tag{5.2}$$

Below we shall comment on details of the derivation of the metric G_{mk} and write down explicitly the reduced Lagrangian (5.1) for the new non-trivial cases of $n = 4, 5$, i.e. for the string on $R_t \times S^4$ and $R_t \times S^5$, which generalize the $n = 3$ CSG model (2.7).

The H gauge fixing on g and elimination of A_a from the $\frac{SO(n)}{SO(n-1)}$ gWZW Lagrangian (3.14) can be done by generalizing the discussion of the $n = 4$ case in [38]. The first step is the parametrisation of g in terms of the generalized Euler angles. Let us define the 1-parameter subgroups corresponding to the $SO(n+1)$ generators $R_{m+1,m}$ $(m = 0, 1, ..., n-1)$

$$g_m(\theta) = e^{\theta R_m}, \qquad (R_m)_i^j = (R_{m+1,m})_i^j \equiv \delta_m^j \delta_{m+1,i} - \delta_{mi}\delta_{m+1}^j \, . \tag{5.3}$$

Then $T_\pm = T$ in (3.14) is equivalent to the generator R_0 corresponding to g_0

$$T = R_0$$

and the generators of the subgroup $H = SO(n-1)$ which commutes with T contain $R_{m+1,m}$ with $m = 2, ..., n-1$. A generic element of $G = SO(n)$ can be parametrized as $g = g_{n-1}(\theta_{n-1})...g_2(\theta_2)g_1(\theta_1)h$, where h belongs to H. A convenient H gauge choice is then [38]

$$g = g_{n-1}(\theta_{n-1})...g_2(\theta_2)g_1(2\varphi)g_2(\theta_2)...g_{n-1}(\theta_{n-1}) \, , \tag{5.4}$$

so that $\varphi \equiv \frac{1}{2}\theta_1, \ \theta_2, ..., \theta_{n-2}$ are $n-1$ coordinates on the coset space Σ^{n-1}, with φ playing a distinguished role.

With this choice of the parametrisation it turns out that the potential U in (3.14),(5.1) has a universal form for *any* dimension n: it is simply proportional to $\cos 2\varphi$ as in the SG (2.6) or CSG (2.7) cases. Indeed, since $[T_\pm, g_k] = 0$ for $k \geqslant 2$, one finds

$$\mathrm{Tr}(T_+ g^{-1} T_- g) = \mathrm{Tr}(T_+ g_1^{-1} T_- g_1) = 2\cos 2\varphi \, . \tag{5.5}$$

[16]While the gauge $A_\pm = 0$ preserves the explicit $SO(n-1)$ invariance of the equations of motion, fixing the gauge on g and integrating out A_a breaks all non-abelian symmetries (the corresponding symmetries are then "hidden").

The metric and the dilaton resulting from integrating out the H gauge field A_a satisfy

$$ds^2 = G_{mk}dx^m dx^k = d\varphi^2 + \mathrm{g}_{pq}(\varphi,\theta)d\theta^p d\theta^q \ , \qquad \sqrt{G}\, e^{-2\Phi} = (\sin 2\varphi)^{n-1} \ , \tag{5.6}$$

so that the equation (5.2) is indeed solved by[17]

$$U = -\frac{1}{2}\cos 2\varphi \ , \qquad M^2 = -4n \ . \tag{5.7}$$

Let us now make few remarks.

As was already mentioned, the reduced model (5.1) has no antisymmetric tensor coupling term. The antisymmetic tensor contribution could originate either from the WZ term in the WZW action in (3.15) or in the process of solving for the gauge field A_a. It turns out that both contributions vanish if the gauge condition (5.4) is used. Details of the proof are given in the Appendix **B**

The obvious "vacuum" configurations, i.e. extrema of the potential U are $\theta_p = \mathrm{const}$ and $\varphi = \frac{\pi}{2}n,\ n = 0, 1, 2, ...$ The metric $\mathrm{g}_{pq}(\varphi,\theta)$ in (5.6) may, however, be singular near such points, i.e. they may not be reachable in a given coordinate system and more detailed analysis may be required.

One should keep in mind that the gWZW action (3.14) is the most general and universal definition of the theory, while special gauges and parametrizations may have their drawbacks and may not apply globally. For example, the elimination of the gauge fields $A_\pm$ from (3.19) or the gWZW action (3.15) requires solving the constraints in (3.20), i.e. $A_+ = (g^{-1}A_+g + g^{-1}\partial_+g)_\mathfrak{h}$ and $A_- = (gA_-g^{-1} + g\partial_-g^{-1})_\mathfrak{h}$. The corresponding operator $(\mathbf{1} - Ad_g)_\mathfrak{h}$ is singular near some points g (e.g., $g = \mathbf{1}$) implying that in their vicinity one should use a different gauge or do not directly solve for $A_\pm$.

For example, one may consider an asymmetrically gauged WZW model (see (3.26)) corresponding to a more general on-shell gauge (3.25); in this case one should use (5.4) with the left-hand-side factor $g_{n-1}(\theta_{n-1})...g_2(\theta_2)$ replaced by $\widehat{\tau}(g_{n-1}(\theta_{n-1})...g_2(\theta_2))$ where $\widehat{\tau}$ is the lift of the automorphism in (3.25). However, in the case when $\mathfrak{h}$ is simple (e.g., for the $SO(5)/SO(4)$ coset) such an automorphism can always be represented as $\tau(A) = h_\tau^{-1}Ah_\tau$ for some $h_\tau \in H$; therefore it can not be used to remove the degeneracy of the operator in the A_+A_- part of the action.[18]

Finally, let us note that both the gauge fixing and the eliminating of $A_\pm$ can be implemented at the level of the Lax connection, leading to the Lax formulation of the reduced model in terms of the generalized Euler angles, i.e. ensuring the integrability of the reduced model (5.1).

Let us now turn to specific examples.

5.2 Examples of reduced Lagrangians for S^n models

Let us first show how to get the Lagrangian (2.7) of the CSG model directly from the $\frac{SO(3)}{SO(2)}$ gWZW model (3.14). The equation for A_+ following from (3.15) reads:

$$A_+ = (g^{-1}\partial_+g + g^{-1}A_+g)_\mathfrak{h} \ . \tag{5.8}$$

[17]We fix the overall normalisation constant in the WZW action so that $\alpha' k = 1$.

[18]The nonsingular metrics known to arise in the SG and CSG cases are due to the fact that $\mathfrak{h} = 0$ in the CS case and $\mathfrak{h} = U(1)$ in the CSG case. As we will see below, the nonsingular metric in the CSG case is obtained by utilizing the automorphism $\tau(A) = -A$. This automorphism does not, however, apply to the case of a non-abelian $\mathfrak{h}$.

In the $\frac{SO(3)}{SO(2)}$ gWZW case we have from (5.4) $g = g_2(\theta) g_1(2\varphi) g_2(\theta)$ so that

$$(g^{-1}\partial_+ g)_{\mathfrak{h}} = (1+\cos 2\varphi) R_2 \partial_+\theta\,, \qquad \partial_- g g^{-1} = (1-\cos 2\varphi) R_2 \partial_-\theta\,,$$
$$A_+ = \frac{1+\cos 2\varphi}{1-\cos 2\varphi} R_2 \partial_+\theta\,. \tag{5.9}$$

One finds also

$$-\frac{1}{2}\mathrm{Tr}(g^{-1}\partial_+ g g^{-1}\partial_- g) = 2(1+\cos 2\varphi)\partial_+\theta\partial_-\theta + 4\partial_+\varphi\partial_-\varphi\,,$$
$$\mathrm{Tr}(A_+\partial_- g g^{-1}) = -2\frac{(1+\cos 2\varphi)^2}{1-\cos 2\varphi}\partial_+\theta\partial_-\theta\,. \tag{5.10}$$

Using (5.5) one finally obtains the Lagrangian

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \cot^2\varphi\,\partial_+\theta\partial_-\theta + \frac{\mu^2}{2}\cos 2\varphi\,. \tag{5.11}$$

This Lagrangian is dual to that in (2.7), i.e. the two are related by 2d duality $\theta \to \widetilde{\theta}$. As was already mentioned above, the CSG Lagrangian (2.7) is directly obtained if we start with the asymmetrically ("axially") gauged WZW model with $\tau(A_-) = -A_-$.[19] Alternatively, the two dual models are related by the formal transformation (4.35).

The explicit form of the Σ^{n-1} metric (5.6) with $n = 2, 3, 4$ as found directly from the action (3.14) with (5.4) is thus

$$ds^2_{n=2} = d\varphi^2\,, \qquad ds^2_{n=3} = d\varphi^2 + \cot^2\varphi\, d\theta^2\,, \tag{5.12}$$

$$ds^2_{n=4} = d\varphi^2 + \cot^2\varphi\,(d\theta_2 + \tan\theta_3\cot\theta_2\, d\theta_3)^2 + \tan^2\varphi\,\frac{d\theta_3^2}{\sin^2\theta_2}\,. \tag{5.13}$$

After a change of variables ($x = \cos\theta_2\ \cos\theta_3,\ y = \sin\theta_3$) we get the metric on Σ^3 [38]

$$(ds^2)_{n=4} = d\varphi^2 + \frac{\cot^2\varphi\, dx^2 + \tan^2\varphi\, dy^2}{1 - x^2 - y^2}\,. \tag{5.14}$$

Thus in the case of $n = 4$ (i.e. for the string on $R_t \times S^4$) we find from (5.14),(5.7) that the reduced theory is described by the following Lagrangian (cf. (2.7))

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \frac{\cot^2\varphi\,\partial_+ x\,\partial_- x + \tan^2\varphi\,\partial_+ y\,\partial_- y}{1 - x^2 - y^2} + \frac{\mu^2}{2}\cos 2\varphi\,. \tag{5.15}$$

An equivalent form of the metric of Σ^3 (5.14) was found in [40]

$$(ds^2)_{n=4} = \frac{db^2}{4(1-b^2)} - \frac{1+b}{4(1-b)}\frac{dv^2}{v(v-u-2)} + \frac{1-b}{4(1+b)}\frac{du^2}{u(v-u-2)}\,, \tag{5.16}$$

[19] In this case the parametrization (5.4) takes the form $g = \widehat{\tau}(g_2) g_1 g_2 = g_2(-\theta) g_1(2\varphi) g_2(\theta)$.

as one can see by setting $b = \cos 2\varphi,\ u = -2y^2,\ v = 2x^2$. The metric-dilaton background for Σ^4 (i.e. $n = 5$) case was obtained in similar coordinates (b, u, v, w) in [44]. Setting $b = \cos 2\varphi$, $w = \cos\alpha,\ v = \cos\beta$ we get

$$(ds^2)_{n=5} = d\varphi^2 + \tan^2\varphi \, \frac{du^2}{(\cos\beta - u)(u - \cos\alpha)} + \cot^2\varphi \, (\cos\beta - \cos\alpha)\left[\frac{d\alpha^2}{4(u-\cos\alpha)} + \frac{d\beta^2}{4(\cos\beta - u)}\right] . \qquad (5.17)$$

Together with the $\cos 2\varphi$ potential (5.7) this metric thus defines the reduced model for the string on $R_t \times S^5$.

5.3 Reduced model for a bosonic string in $AdS_n \times S^n$

One can similarly find the reduced Lagrangians for the $F/G = AdS_n = SO(2, n-1)/SO(1, n-1)$ coset sigma models related to the above ones by an analytic continuation. These reduced models describe strings in $AdS_n \times S^1$ spaces in the conformal gauge with the residual conformal symmetry fixed, e.g., by choosing the S^1 angle α equal to $\mu\tau$ (cf. (2.15)).

As was already discussed at the end of section **2**, the reduced model for strings on $AdS_n \times S^n$ can then be obtained by simply combining the reduced models for strings on $AdS_n \times S^1$ and on $R \times S^n$.[20]

For example, in the case of a string in $AdS_2 \times S^2$ we then find the sum of the sine-Gordon and sinh-Gordon Lagrangians (cf. (2.6),(2.13))

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \partial_+\phi\partial_-\phi + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi) \ . \qquad (5.18)$$

For a string in $AdS_3 \times S^3$ we get (cf. (2.7))

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \tan^2\varphi \, \partial_+\theta\partial_-\theta + \partial_+\phi\partial_-\phi + \tanh^2\phi \, \partial_+\chi\partial_-\chi + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi) \ . \qquad (5.19)$$

Similar bosonic actions are then found for a string in $AdS_4 \times S^4$ and in $AdS_5 \times S^5$: one is to "double" (5.15) and its analog corresponding to (5.17).[21]

Expanding (5.18) near $\varphi = \phi = 0$ we get two massive fluctuation modes. Doing similar expansion near the trivial vacuum in the case of (5.19) it may seem that only two modes (φ and ϕ) get masses μ, but, in fact, *all* 2+2 bosonic modes become massive. Indeed, as is clear from the form of kinetic terms in (5.19), the expansion near the point where all angles are zero is singular. This is like expanding

[20] Note that this is *not* the same as the reduced theory for the coset sigma model with $F/G = AdS_n \times S^n = [SO(2, n-1)/SO(1, n-1)] \times [SO(n+1)/SO(n)]$: in the latter case we would set, following [14], the components of the *total* stress tensor to be equal to a constant, while for a *string* in $AdS_n \times S^n$ the total stress tensor should vanish. The reduced theory for coset sigma model $F/G = AdS_n \times S^n$ case is of course formally equivalent to the reduced theory for a string on $AdS_n \times S^n \times S^1$.

[21] A "mnemonic" rule to get, e.g., the AdS_n counterparts of S^n Lagrangians in (2.7),(5.15) is to change $\varphi \to i\phi$ and to change the overall sign of the Lagrangian.

near $r = 0$, $\varphi = 0$ on the disc $ds^2 = dr^2 + r^2 d\varphi^2$; instead, one is first to do a transformation to "cartesian" coordinates and then expand. Since φ and ϕ play the role of the "radial" directions in the 2+2 dimensional space[22] their $\frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi)$ potential gives mass to all 4 "cartesian" fluctuations. In the CSG case this is the transformation that puts the Lagrangian (2.7) into the familiar form $\widetilde{L} = \frac{\partial^a\psi\partial_a\psi^*}{1-\psi\psi^*} - \mu^2\psi\psi^*$ where $\psi = \sin\varphi\, e^{i\theta}$.

The analogous conclusion should be true also in the general $AdS_n \times S^n$ case with $n > 3$ though there a direct demonstration of this in the gauge where $A_\pm$ are solved for is complicated by the degeneracy of the metric g_{pq} in (5.6). As we have already seen in (4.32),(4.30), in the S^n case all the $(n-1)$ fluctuation modes near the trivial vacuum get mass μ if we start with the classical equations of the reduced theory in the $A_+ = A_- = 0$ gauge. Since the mass spectrum should be gauge-invariant, the same should be true also in other gauges/parametrizations.

Thus in the $AdS_5 \times S^5$ case we should get 4+4 massive bosonic modes. Similar conclusion will be reached for the fermionic fields discussed in the next section (see (6.54)): all 8 dynamical fermionic modes will also have mass μ. The "free" spectrum will thus be the same as in the "plane-wave" limit of [7].

6 Pohlmeyer reduction of the $AdS_5 \times S^5$ superstring model

The $AdS_5 \times S^5$ superstring can be described in terms of the Green-Schwarz version of the $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ (or, equivalently, $\frac{PSU(2,2|4)}{Sp(2,2)\times Sp(4)}$) coset sigma model [2]. In the conformal gauge its bosonic part is the direct sum of the AdS_5 and S^5 sigma models. Below we shall apply the idea of the Pohlmeyer reduction to the whole action including the fermions. The important new element will be the κ-symmetry gauge fixing, reducing the number of the fermionic degrees of freedom to the same 8 (or 16 real Grassmann components) as of the bosonic ones after the solution of the conformal gauge constraint.

We shall derive the corresponding reduced Lagrangian that generalizes the bosonic Lagrangian discussed in section **5** above. We shall find that it is invariant under the 2d Lorentz symmetry.[23]

Later in section **7** we will also consider a simpler $AdS_2 \times S^2$ model which is described by a similar action for the $\frac{PSU(1,1|2)}{SO(1,1)\times SO(2)}$ coset. In this case the reduced Lagrangian happens to be invariant under the $\mathrm{N} = 2$ (i.e. (2,2)) 2d supersymmetry, and is the same as the $\mathrm{N} = 2$ supersymmetric sine-Gordon Lagrangian.

6.1 Equations of motion in terms of currents in conformal gauge

Let us start with some relevant definitions and notation. The Lie superalgebra $psl(2m|2m;\mathbb{C})$ can be identified with the quotient of $sl(2m|2m;\mathbb{C})$ by the central subalgebra of elements proportional to the unit matrix (which belongs to $sl(2m|2m;\mathbb{C})$ since its supertrace vanishes). We are interested in its real form $psu(m,m|2m)$ which is defined by the condition $M^* = -M$, where * is an appropriate antilinear anti-automorphism. This superalgebra corresponds to the Lie supergroup $\widehat{F} = PSU(m,m|2m)$.

[22]Recall also that they are related to the Lagrange multipliers for the embedding coordinates discussed in section **2** so we are then expanding near a point where the two Lagrange multipliers have constant "vacuum" values.

[23] This is similar to what happened in the expansion near the S^5 geodesic to quadratic order (i.e. plane-wave limit) in the light-cone gauge [7], but here the action contains all interaction terms, i.e. is no longer truncated at the quadratic level.

We shall consider the superalgebra $\widehat{\mathfrak{f}} = psu(m,m|2m)$ with $m = 2$ or $m = 1$ which admits a Z_4 grading [54][24]

$$\widehat{\mathfrak{f}} = \widehat{\mathfrak{f}}_0 \oplus \widehat{\mathfrak{f}}_1 \oplus \widehat{\mathfrak{f}}_2 \oplus \widehat{\mathfrak{f}}_3 \,, \qquad\qquad [\widehat{\mathfrak{f}}_i, \widehat{\mathfrak{f}}_j] \subset \widehat{\mathfrak{f}}_{i+j \,\mathrm{mod}\, 4} \,. \tag{6.1}$$

In this matrix realisation one also has $i\{\widehat{\mathfrak{f}}_l, \widehat{\mathfrak{f}}_m\} \subset \widehat{\mathfrak{f}}_{l+m+2 \,\mathrm{mod}\, 4}$, where $\{A, B\} = AB + BA$. [25] For details see Appendix **C**.

The left-invariant current $f^{-1}\partial_a f, \ f \in \widehat{F}$ can then be decomposed as

$$J_a = f^{-1}\partial_a f = \mathcal{A}_a + Q_{1a} + P_a + Q_{2a} \,, \qquad\qquad \mathcal{A} \in \widehat{\mathfrak{f}}_0, \quad Q_1 \in \widehat{\mathfrak{f}}_1, \quad P \in \widehat{\mathfrak{f}}_2, \quad Q_2 \in \widehat{\mathfrak{f}}_3 \,. \tag{6.2}$$

Here $\mathcal{A}$ corresponds to the algebra of the subgroup G defining the $\widehat{F}/G$ coset (i.e. $G = Sp(2,2) \times Sp(4)$ isomorphic to $SO(1,4) \times SO(5)$ in the $AdS_5 \times S^5$ case), P is the bosonic "coset" component, and Q_1, Q_2 are the fermionic (odd) currents.

Using this Z_4 split the Lagrangian density of the $AdS_5 \times S^5$ GS superstring [2] can be written as follows [54, 55, 3, 56][26]

$$L_{\rm GS} = \frac{1}{2}\mathrm{STr}(\gamma^{ab} P_a P_b + \varepsilon^{ab} Q_{1a} Q_{2b}) \,, \tag{6.3}$$

where $\gamma^{ab} = \sqrt{-g} g^{ab}$. Written in terms of currents this coset action has bosonic gauge symmetry with $\widehat{\mathfrak{f}}_0$-valued gauge parameter. In addition to the reparametrisations it is also invariant under the local fermionic κ-symmetry [2, 57, 58]

$$\begin{gathered} \delta_\kappa J_a = \partial_a \epsilon + [J_a, \epsilon] \,, \qquad (\delta_\kappa \gamma)^{ab} = \frac{1}{m}\mathrm{STr}\left(W([ik^a_{1(-)}, Q^b_{1(-)}] + [ik^a_{2(+)}, Q^b_{2(+)}])\right) \,, \\ \epsilon = \epsilon_1 + \epsilon_2 = \{P_{(+)a}, ik^a_{1(-)}\} + \{P_{(-)a}, ik^a_{2(+)}\} \,, \end{gathered} \tag{6.4}$$

where[27] $k_{1(-)}$ and $k_{2(+)}$ take values in the degree 1 and degree 3 subspaces of $u(m,m|2m)$ respectively (it is assumed that $k_{1(+)} = k_{2(-)} = 0$). $W = \mathrm{diag}(1,\ldots,1,-1,\ldots,-1)$ is the parity automorphism (see Appendix **C**), and the $(\pm)$ components are defined as:

$$V^a_{(\pm)} = \frac{1}{2}(\gamma^{ab} \mp \varepsilon^{ab}) V_b \,. \tag{6.5}$$

A detailed discussion of the κ-invariance can be found in the Appendix **D**.

In what follows we shall assume the *conformal gauge* condition $\gamma^{ab} = \eta^{ab}$. Then (using the standard light-cone worldsheet coordinates σ^+, σ^-) the only nonvanishing components of the metric are $\gamma^{+-} = \gamma^{-+} = 1$ while $\varepsilon^{+-} = -\varepsilon^{-+} = 1$. For any vector V_a one then has

$$V_{(+)+} = V_+ \,, \qquad V_{(+)-} = 0 \,, \qquad V_{(-)+} = 0 \,, \qquad V_{(-)-} = V_- \,. \tag{6.6}$$

In the conformal gauge the Lagrangian (6.3)

$$L_{\rm GS} = \mathrm{STr}[P_+ P_- + \frac{1}{2}(Q_{1+} Q_{2-} - Q_{1-} Q_{2+})] \tag{6.7}$$

[24] It appears that all the steps of the reduction procedure discussed below are formally valid for any value of m.

[25] Note that for A, B representing elements of $psu(m,m|2m)$ their symmetrized commutator $i\{A, B\}$ belongs to $u(m,m|2m)$ but not necessarily to $psu(m,m|2m)$.

[26] Here the overall sign is consistent with having physical signs for the bosonic AdS_5 and S^5 Lagrangians.

[27] Note that the definition of ϵ in (6.4) involves the symmetrized commutator so that the projection from $u(m,m|2m)$ to $psu(m,m|2m)$ is assumed.

leads to the following equations of motion [3]

$$\begin{aligned}
&\partial_+ P_- + [\mathcal{A}_+, P_-] + [Q_{2+}, Q_{2-}] = 0\,, \\
&\partial_- P_+ + [\mathcal{A}_-, P_+] + [Q_{1-}, Q_{1+}] = 0\,, \\
&[P_+, Q_{1-}] = 0\,, \qquad [P_-, Q_{2+}] = 0\,.
\end{aligned} \tag{6.8}$$

Formulated in terms of the current components $J_\pm = \mathcal{A}_\pm + P_\pm + Q_{1\pm} + Q_{2\pm}$, they should be supplemented by the Maurer-Cartan equation

$$\partial_- J_+ - \partial_+ J_- + [J_-, J_+] = 0\ . \tag{6.9}$$

In addition, one needs to take into account the conformal gauge (Virasoro) constraints

$$\mathrm{STr}(P_+ P_+) = 0\,, \qquad \mathrm{STr}(P_- P_-) = 0\,. \tag{6.10}$$

Our aim below is to perform the Pohlmeyer-type reduction of the above system (6.8)–(6.10). The bosonic part of the model is identical to that of the F/G sigma model where the bosonic group $F \subset \widehat{F}$ has $\widehat{\mathfrak{f}}_0 \oplus \widehat{\mathfrak{f}}_2$ as its Lie algebra and G has Lie algebra $\widehat{\mathfrak{f}}_0$. In the $psu(2,2|4)$ case of our interest $\widehat{\mathfrak{f}}_0 \oplus \widehat{\mathfrak{f}}_2$ is isomorphic to $su(2,2) \oplus su(4)$ or $so(2,4) \oplus so(6)$ while $\widehat{\mathfrak{f}}_0$ is isomorphic to $sp(2,2) \oplus sp(4)$ or $so(1,4) \oplus so(5)$ (in the $psu(1,1|2)$ case $\widehat{\mathfrak{f}}_0 \oplus \widehat{\mathfrak{f}}_2 = su(1,1) \oplus su(2)$ and $\widehat{\mathfrak{f}}_0 = sp(1,1) \oplus sp(2)$). Because of the direct sum structure of the algebras one is allowed to use the reduction gauge separately for each sector, just like in the purely bosonic case.

Performing the reduction, requires, besides partially fixing the G-gauge symmetry, to fix also the κ-symmetry gauge. As we shall discuss below, this can be achieved in two steps. First, we shall impose the partial κ-symmetry gauge condition[28]

$$Q_{1-} = 0\,, \qquad Q_{2+} = 0\,, \tag{6.11}$$

and then apply the same procedure as in the case of the Pohlmeyer reduction in the bosonic $AdS_n \times S^n$ case. The resulting reduced system will be still invariant under a residual κ-symmetry which can be fixed by an additional gauge condition. That will finally make the number of the fermionic degrees of freedom the same as the number of the physical bosonic degrees of freedom (as in the familiar examples of the light-cone gauge-fixed superstring in the flat space or in the pp-wave space).

It will turn out that the resulting system of reduced equations of motion (that originate in particular from the Maurer-Cartan equations and thus are first order in derivatives) will follow from a local Lagrangian containing only *first* derivatives of the fermionic fields. The bosonic part of the reduced Lagrangian will coincide with the gauged WZW Lagrangian with the same potential as in the bosonic model discussed in section **5**.

The possibility to make the gauge choice (6.11) can be readily justified as in the flat-space case by using an explicit coordinate parametrization of the currents, i.e. by solving first the Maurer-Cartan equations (6.9). Here we would like to use a different logic treating all equations for the currents on an equal footing. Then one way of demonstrating that the required κ-symmetry gauge choices are allowed will rely on using the consequences of the reduction gauge in the bosonic part of the model. For that technical reason below we shall discuss the reduction and the κ-symmetry gauges in parallel.

[28]This choice was suggested by R. Roiban, see also [21].

6.2 Reduction gauge and κ-symmetry gauge

As a first step we shall define a decomposition $\widehat{\mathfrak{f}}_2 = \mathfrak{a} \oplus \mathfrak{n}$ where $\mathfrak{a}$ is the subspace of elements of the form $a_1 T^1 + a_2 T^2$ such that T^1 and T^2 are represented by matrices with nonvanishing upper left and lower right blocks only (i.e. T^1 is in $su(2,2)$ and T^2 is in $su(4)$ parts of $psu(2,2|4)$). More precisely, we shall choose

$$T^1 = \frac{i}{2}\mathrm{diag}(\mathrm{t}, 0)\,, \qquad T^2 = \frac{i}{2}\mathrm{diag}(0, \mathrm{t})\,, \tag{6.12}$$

where

$$psu(2,2|4) \text{ case:} \quad \mathrm{t} = \mathrm{diag}(1,1,-1,-1)\,, \qquad psu(1,1|2) \text{ case:} \quad \mathrm{t} = \mathrm{diag}(1,-1)\,. \tag{6.13}$$

Let us also introduce the matrix

$$T = T^1 + T^2\,, \tag{6.14}$$

which will play an important role in what follows. It induces the decomposition

$$\widehat{\mathfrak{f}} = \widehat{\mathfrak{f}}^{\parallel} \oplus \widehat{\mathfrak{f}}^{\perp}\,, \qquad \zeta^{\parallel} \in \widehat{\mathfrak{f}}^{\parallel}\,, \qquad \chi^{\perp} \in \widehat{\mathfrak{f}}^{\perp}\,, \tag{6.15}$$

$$P^{\parallel}\zeta^{\parallel} = \zeta^{\parallel}\,, \qquad P^{\parallel}\chi^{\perp} = 0\,, \qquad P^{\parallel} = -[T, [T, \cdot\,]]\,. \tag{6.16}$$

This decomposition can also be written with the help of the projector to $\widehat{\mathfrak{f}}_1^{\perp} \oplus \widehat{\mathfrak{f}}_3^{\perp}$ given by

$$P^{\perp}\chi^{\perp} = \chi^{\perp}\,, \qquad P^{\perp}\zeta^{\parallel} = 0\,, \qquad P^{\perp} = -\{T, \{T, \cdot\,\}\}\,. \tag{6.17}$$

Let us note that any $\zeta \in \widehat{\mathfrak{f}}^{\parallel}$ can be written as $\zeta = [T, \lambda]$ (and $\chi \in \mathfrak{f}^{\perp}$ can be written as $\chi = \{T, \nu\}$). In particular, $[T, \{T, \zeta\}] = \{T, [T, \zeta]\} = 0$ for any $\zeta \in \widehat{\mathfrak{f}}$. Moreover, $\mathrm{STr}(\zeta^{\parallel}\chi^{\perp}) = 0$ for any $\zeta^{\parallel} \in \widehat{\mathfrak{f}}^{\parallel}$ and $\chi^{\perp} \in \widehat{\mathfrak{f}}^{\perp}$, i.e. this is an orthogonal decomposition.

The decomposition $\widehat{\mathfrak{f}} = \widehat{\mathfrak{f}}^{\parallel} \oplus \widehat{\mathfrak{f}}^{\perp}$ generalizes the bosonic decomposition (4.1) to the superalgebra case. In particular, in the bosonic sector one can easily make the following identifications:[29]

$$\mathfrak{a} = \widehat{\mathfrak{f}}_2^{\perp}\,, \qquad \mathfrak{n} = \widehat{\mathfrak{f}}_2^{\parallel}\,, \qquad \mathfrak{h} = \widehat{\mathfrak{f}}_0^{\perp}\,, \qquad \mathfrak{m} = \widehat{\mathfrak{f}}_0^{\parallel}\,, \tag{6.18}$$

while the commutation relations (4.2) follow from the Z_4-grading and the following properties:[30]

$$[\widehat{\mathfrak{f}}^{\perp}, \widehat{\mathfrak{f}}^{\perp}] \subset \widehat{\mathfrak{f}}^{\perp}\,, \qquad [\widehat{\mathfrak{f}}^{\parallel}, \widehat{\mathfrak{f}}^{\perp}] \subset \widehat{\mathfrak{f}}^{\parallel}\,, \qquad [\widehat{\mathfrak{f}}^{\parallel}, \widehat{\mathfrak{f}}^{\parallel}] \subset \widehat{\mathfrak{f}}^{\perp}\,. \tag{6.19}$$

The first two properties are obvious, while checking the last one requires using the following identities

$$\{A, [B, C]\} = \{[A, B], C\} + [A, \{B, C\}]\,, \qquad \{A, \{B, C\}\} = [[A, B], C] + \{B, \{A, C\}\}\,. \tag{6.20}$$

Let us now turn to the gauge symmetry. Because the gauge algebra $\widehat{\mathfrak{f}}_0$ is a direct sum of the subalgebras represented by upper-left and lower-right nonvanishing block matrices the gauge transformations

[29] Let us note that one can not define analogous decomposition in terms of $T_{\pm}$ for the $SO(n)/SO(n-1)$ coset in the standard representation used in Section **4** as $T_{\pm}$ in this representation do not induce the decomposition (cf. the explicit form (4.25)).

[30] These can be considered as defining an additional Z_2-grading on $\widehat{\mathfrak{f}}$ with $\widehat{\mathfrak{f}}^{\perp}$ and $\widehat{\mathfrak{f}}^{\parallel}$ being, respectively, the degree 0 and degree 1 subspaces.

are independent. It follows that by applying the polar decomposition theorem in each sector independently one can partially fix the $\widehat{\mathfrak{f}}_0$ gauge symmetry in order to put P_+ into the form

$$P_+ = p_1 T^1 + p_2 T^2 \,, \tag{6.21}$$

where p_1, p_2 are some real functions. Indeed, the components of the gauge parameter taking values in the upper-left and lower-right diagonal blocks are independent so that we can apply the same logic as in the bosonic case in section **4.1** to each block separately. The Virasoro constraint $\mathrm{STr}(P_+P_+) = 0$ in (6.10) then implies $p_1^2 - p_2^2 = 0$, so that, e.g., $p_1 = p_2 = p_+$ and thus

$$P_+ = \ p_+ \, T \,, \qquad\qquad T = T^1 + T^2 \,. \tag{6.22}$$

Applying the polar decomposition theorem to P_- and using the second Virasoro constraint in (6.10) one finds that P_- can be represented as follows

$$P_- = \ p_- \, g^{-1} T g \,, \tag{6.23}$$

where p_- is a real function and g is a G-valued function (recall that G is the Lie subgroup corresponding to the Lie subalgebra $\widehat{\mathfrak{f}}_0 \subset \widehat{\mathfrak{f}}$, i.e. $Sp(2,2) \times Sp(4)$ in the $PSU(2,2|4)$ case). In what follows we shall assume that the functions p_+ and p_- do not have zeroes.

Now we are ready to argue that using the κ-symmetry (6.4) one can choose the gauge (6.11), i.e. $Q_{1-} = Q_{2+} = 0$, provided the fermionic equations of motion as well as the Virasoro constraints are satisfied. This basically follows from the fact that in the gauge where $P_+ = p_+ T$ the equation $[P_+, Q_{1-}] = 0$ implies that Q_{1-} takes values in $\widehat{\mathfrak{f}}_1^\perp$ like the parameter $\epsilon_1 = i\{P_+, k_{1-}\}$ so that this gauge invariance can be used to put Q_{1-} to zero; an analogous argument can then be given for Q_{2+}. A complication is that the κ-transformation (6.4) does not in general preserve both the conformal gauge and the reduction gauge and that makes the precise argument more involved. A detailed proof of the possibility to fix (6.11) taking all this into account is given in Appendix **D**.

In the gauge $Q_{1-} = Q_{2+} = 0$ the equations of motion (6.8) become

$$\partial_+ P_- + [\mathcal{A}_+, P_-] = 0 \,, \qquad \partial_- P_+ + [\mathcal{A}_-, P_+] = 0 \,, \tag{6.24}$$

while the Maurer-Cartan equation (6.9) splits into

$$\begin{aligned} \partial_+ \mathcal{A}_- - \partial_- \mathcal{A}_+ + [\mathcal{A}_+, \mathcal{A}_-] + [P_+, P_-] + [Q_{1+}, Q_{2-}] = 0 \,, \\ \partial_- Q_{1+} + [\mathcal{A}_-, Q_{1+}] - [P_+, Q_{2-}] = 0 \,, \\ \partial_+ Q_{2-} + [\mathcal{A}_+, Q_{2-}] - [P_-, Q_{1+}] = 0 \,. \end{aligned} \tag{6.25}$$

In the reduction gauge where $P_+ = p_+ T$ and $P_- = p_- g^{-1} T g$ the second equation $\partial_- P_+ + [\mathcal{A}_-, P_+] = 0$ in (6.24) and the fact that $\mathcal{A}_-$ is block-diagonal imply that the same is true for the upper-left block projection $\partial_- P_+^1 + [A_-^1, P_+^1] = 0$. The latter implies $\partial_- \mathrm{Tr}_1(P_+ P_+) = 0$ and thus also $\partial_- \mathrm{Tr}_2(P_+ P_+) = 0$, where Tr_1 and Tr_2 are, respectively, the traces in the upper-left and the lower-right diagonal blocks (in this notation $\mathrm{STr} = \mathrm{Tr}_1 - \mathrm{Tr}_2$). Since $\mathrm{Tr}_1 T^2 \neq 0$ this leads to $\partial_- p_+ = 0$. As in the bosonic case, using an appropriate conformal transformation $\sigma^+ \to \sigma'^+(\sigma^+)$ one can then set p_+ equal to some real constant μ. Following the bosonic construction one then observes that the first equation in (6.24) leads to $\partial_+ \mathrm{Tr}_1(P_- P_-) = 0$. The conformal symmetry $\sigma^- \to \sigma'^-(\sigma^-)$ allows one to set $p_- = \mu$. Thus finally we get

$$P_+ = \mu T , \qquad P_- = \mu g^{-1} T g , \qquad \mu = \text{const} , \tag{6.26}$$

which is the direct counterpart of the reduction gauge in the bosonic case (cf. (4.12),(4.14)). Note that in terms of the notation used in the bosonic case here we have

$$T_+ = T_- = T \ . \tag{6.27}$$

Let us recall that the variable g belongs to G, i.e to the subgroup whose Lie algebra is $\widehat{\mathfrak{f}}_0$. There is a natural arbitrariness in the choice of g since P_- is invariant under $g \to hg$ if h is taking values in the subgroup of elements commuting with T. This description thus has an additional gauge symmetry which we shall use later.

By analogy with the bosonic case in addition to the decomposition $\widehat{\mathfrak{f}}_2 = \mathfrak{a} \oplus \mathfrak{n}$ we make use of the decomposition $\widehat{\mathfrak{f}}_0 = \mathfrak{m} \oplus \mathfrak{h}$ where $\mathfrak{h}$ is the centralizer of $\mathfrak{a}$ in $\widehat{\mathfrak{f}}_0$ (recall that $\mathfrak{a}$ is the subspace of elements of the form $a_1 T^1 + a_2 T^2$).[31] In the present case it is useful to identify $\mathfrak{h} = \widehat{\mathfrak{f}}_0^{\perp}$ and $\mathfrak{m} = \widehat{\mathfrak{f}}_0^{\parallel}$ so that the required decomposition of the entire superalgebra is induced by a single element T as was observed in (6.18). Accordingly, we split

$$\mathcal{A}_+ = (\mathcal{A}_+)_{\mathfrak{h}} + (\mathcal{A}_+)_{\mathfrak{m}} , \qquad \mathcal{A}_- = A_- + (\mathcal{A}_-)_{\mathfrak{m}} , \qquad A_- \equiv (\mathcal{A}_-)_{\mathfrak{h}} \in \mathfrak{h} \ . \tag{6.28}$$

The second equation in (6.24) then implies $(\mathcal{A}_-)_{\mathfrak{m}} = 0$ while the first one can be solved for $\mathcal{A}_+$ as follows

$$\mathcal{A}_+ = g^{-1} \partial_+ g + g^{-1} A_+ g , \tag{6.29}$$

where A_+ is a new field taking values in $\mathfrak{h}$.

In this way we have constructed a new parametrisation of the system in the reduction gauge: all the bosonic currents are now expressed in terms of the G-valued field g, $\mathfrak{h}$-valued field $A_\pm$, and in addition we have the fermionic currents Q_{1+}, Q_{2-}. The equations (6.25) then take the form:

$$\begin{aligned} \partial_-(g^{-1}\partial_+ g + g^{-1} A_+ g) - \partial_+ A_- + [A_-, g^{-1}\partial_+ g + g^{-1} A_+ g] \\ = - \mu^2 [g^{-1} T g, T] + [Q_{1+}, Q_{2-}] , \end{aligned} \tag{6.30}$$

$$\begin{aligned} \partial_- Q_{1+} + [A_-, Q_{1+}] &= \mu [T, Q_{2-}] , \\ \partial_+ Q_{2-} + [g^{-1}\partial_+ g + g^{-1} A_+ g, Q_{2-}] &= \mu [g^{-1} T g, Q_{1+}] . \end{aligned} \tag{6.31}$$

These equations are invariant under the following $H \times H$ gauge symmetry (H is the group whose algebra is $\mathfrak{h}$):

$$g \to h^{-1} g \bar{h} , \quad A_+ \to h^{-1} A_+ h + h^{-1} \partial_+ h , \quad A_- \to \bar{h}^{-1} A_- \bar{h} + \bar{h}^{-1} \partial_- \bar{h} , \tag{6.32}$$

$$Q_{1+} \to \bar{h}^{-1} Q_{1+} \bar{h} , \qquad Q_{2-} \to \bar{h}^{-1} Q_{2-} \bar{h} . \tag{6.33}$$

[31] In the case of our interest, i.e. $\widehat{\mathfrak{f}} = psu(2,2|4)$, the algebra $\mathfrak{h}$ is $[su(2) \oplus su(2)] \oplus [su(2) \oplus su(2)]$, i.e. is isomorphic to $so(4) \oplus so(4)$.

Let us note that this symmetry is large enough to choose the gauge $A_+ = A_- = 0$. This can be shown by a simplified version of the argument given in Appendix **E**. In particular, there is also a choice of a partial gauge in which A_+ and A_- are components of a flat connection, i.e. $F_{+-} = 0$.

The equations (6.30),(6.31) admit a Lax representation. Moreover, they can be derived from a local Lagrangian provided one uses the following parametrisation of the fermionic currents in terms of the new fermionic variables $q_1,\ q_2$ via $Q_{1+} = g^{-1}(\partial_+ q_1 + [A_+, q_1])g\,,\quad Q_{2-} = \partial_- q_2 + [A_-, q_2]$, and imposes the appropriate gauge condition on $A_\pm$. This gauge condition is analogous to the constraints (3.20) in the purely bosonic case. However, the resulting Lagrangean system is not completely satisfactory, in particular, it contains second (instead of usual first) derivatives of the fermions and thus will not be discussed below.

6.3 Gauge-fixing residual κ-symmetry

Besides the gauge symmetry (6.32),(6.33), the equations (6.30),(6.31) are also invariant under the residual κ-symmetry which can be used to eliminate some parts of the fermionic currents. To identify this symmetry let us first introduce the new fermionic variables $Q_{1+}, Q_{2-} \to \Psi_1, \Psi_2$:

$$\Psi_1 = Q_{1+}\,, \qquad\qquad \Psi_2 = gQ_{2-}g^{-1}\,. \tag{6.34}$$

The equations of motion (6.30),(6.31) then take the form

$$\begin{aligned}\partial_-(g^{-1}\partial_+ g + g^{-1}A_+ g) - \partial_+ A_- + [A_-, g^{-1}\partial_+ g + g^{-1}A_+ g]&\\ = \ -\mu^2[g^{-1}Tg, T] - [g^{-1}\Psi_2 g, \Psi_1]\,,&\end{aligned} \tag{6.35}$$

$$D_-\Psi_1 = \mu[T, g^{-1}\Psi_2 g]\,, \qquad D_+\Psi_2 = \mu[T, g\Psi_1 g^{-1}]\,, \qquad D_\pm = \partial_\pm + [A_\pm,\ ]\,. \tag{6.36}$$

Projecting the fermionic equations (6.36) to $\widehat{\mathfrak{f}}_1^\perp \oplus \widehat{\mathfrak{f}}_3^\perp$ gives

$$D_-(\Psi_1)^\perp = 0\,, \qquad D_+(\Psi_2)^\perp = 0\,. \tag{6.37}$$

Let us choose the gauge where (cf. the remark made below (6.33))

$$A_+ = A_- = 0\,. \tag{6.38}$$

Then the solution of (6.37) has the form $(\Psi_1)^\perp = \psi_1(\sigma^+)$ and $(\Psi_2)^\perp = \psi_2(\sigma^-)$.

Let us now describe the residual fermionic symmetry of the equations (6.35),(6.36). Under the infinitesimal transformation

$$\Psi_1 \to \Psi_1 + \varepsilon_1\,, \qquad \Psi_2 \to \Psi_2 + \varepsilon_2\,, \qquad g \to g + gh\,, \tag{6.39}$$

with $\varepsilon_1 \in \widehat{\mathfrak{f}}_1$, $\varepsilon_2 \in \widehat{\mathfrak{f}}_3$, and $h \in \widehat{\mathfrak{f}}_0$ these equations are invariant provided

$$\begin{aligned}\partial_-\partial_+ h + [g^{-1}\partial_+ g, h] - \mu^2[[g^{-1}Tg, h], T]&\\ + [g^{-1}\Psi_2 g, \varepsilon_1] + [g^{-1}\varepsilon_2 g, \Psi_1] + [[g^{-1}\Psi_2 g, h], \Psi_1] = 0\,,&\end{aligned} \tag{6.40}$$

$$D_-\varepsilon_1 = \mu[T, g^{-1}\varepsilon_2 g + [g^{-1}\Psi_2 g, h]]\,, \qquad D_+\varepsilon_2 = \mu[T, g\varepsilon_1 g^{-1} + g[h, \Psi_1]g^{-1}]\,. \tag{6.41}$$

Projecting the fermionic equations on $\widehat{\mathfrak{f}}^{\perp}$ one finds that $\partial_- \varepsilon_1^{\perp} = 0$ and $\partial_+ \varepsilon_2^{\perp} = 0$, implying $\varepsilon_1^{\perp}$=$\varepsilon_1^{\perp}(\sigma^+)$ and $\varepsilon_2^{\perp} = \varepsilon_2^{\perp}(\sigma^-)$. Let us consider then the projection of the fermionic equations on $\widehat{\mathfrak{f}}_1^{\|} \oplus \widehat{\mathfrak{f}}_3^{\|}$ together with the bosonic equation (6.40) as a system of equations on $\varepsilon_1^{\|}, \varepsilon_2^{\|}, h$ with $\varepsilon_1^{\perp}(\sigma^+)$ and $\varepsilon_2^{\perp}(\sigma^-)$ treated as given functions (note that their derivatives do not enter these equations). This system of partial differential equations is not overdetermined and is linear in derivatives so that it has a solution for any $\varepsilon_1^{\perp}(\sigma^+)$ and $\varepsilon_2^{\perp}(\sigma^-)$, thus giving a symmetry transformation of the equations (6.35),(6.36). The symmetry parameters $\varepsilon_1^{\perp}$ and $\varepsilon_2^{\perp}$ can, in fact, be identified as parameters of the residual κ-symmetry in (6.4) as[32]

$$\varepsilon_1^{\perp} = \partial_+\{\mu T, ik_{1-}\}\,, \qquad\qquad \varepsilon_2^{\perp} = \partial_-\{\mu T, igk_{2+}g^{-1}\}\,, \tag{6.42}$$

while the additional terms are needed to maintain the gauge conditions we have chosen. Finally, using (6.37), i.e. $\partial_- \Psi_1^{\perp} = 0$ and $\partial_+ \Psi_2^{\perp} = 0$ one concludes that $\Psi_1^{\perp}, \Psi_2^{\perp}$ can be put to zero by the residual κ-transformations. In what follows we shall thus assume the gauge where

$$\Psi_1^{\perp} = \Psi_2^{\perp} = 0\,. \tag{6.43}$$

The remaining fermionic degrees of freedom can be parametrized as follows

$$\Psi_{_R} = \frac{1}{\sqrt{\mu}}\Psi_1^{\|}\,, \qquad\qquad \Psi_{_L} = \frac{1}{\sqrt{\mu}}\Psi_2^{\|}\,, \tag{6.44}$$

taking values in $\mathfrak{h}_1^{\|}$ and $\mathfrak{h}_3^{\|}$ respectively (see (6.16),(6.17)). As we shall see below the additional factor $\mu^{-\frac{1}{2}}$ in (6.44) will simplify the structure of the 2d Lorentz invariant Lagrangian description of the resulting system (cf. (6.26)). The gauge transformations of the new fermionic variables read as follows

$$\Psi_{_R} \to \bar{h}^{-1}\Psi_{_R}\bar{h}\,, \qquad\qquad \Psi_{_L} \to h^{-1}\Psi_{_L}h\,. \tag{6.45}$$

The equations of motion (6.35),(6.36) written in the gauge (6.43) are

$$\begin{aligned} \partial_-(g^{-1}\partial_+ g + g^{-1}A_+ g) - \partial_+ A_- + [A_-, g^{-1}\partial_+ g + g^{-1}A_+ g] \\ = \; -\mu^2[g^{-1}Tg, T] - \mu[g^{-1}\Psi_{_L} g, \Psi_{_R}]\,, \end{aligned} \tag{6.46}$$

$$[T, D_-\Psi_{_R}] = -\mu(g^{-1}\Psi_{_L} g)^{\|}\,, \qquad [T, D_+\Psi_{_L}] = -\mu(g\Psi_{_R} g^{-1})^{\|}\,. \tag{6.47}$$

These equations and the gauge symmetries (6.32),(6.45) define the *reduced* system of equations of motion for the superstring on $AdS_5 \times S^5$ (or on $AdS_2 \times S^2$).

The new dynamical field variables $g, \Psi_{_L}, \Psi_{_R}$ and A_+, A_- are components of the currents, i.e. they are non-locally related to the original $AdS_5 \times S^5$ sigma model fields (coordinates on the supercoset). Note also that the bosonic equations are second-order while the fermionic equations are first-order in derivatives, as it should be for a standard 2d boson-fermion system.

Finally, let us mention that one can see explicitly that the reduced system (6.46) and (6.47) is integrable. The corresponding Lax pair encoding the equations (6.46) and (6.47) is

$$\begin{aligned} \mathcal{L}_- &= \partial_- + A_- + \ell^{-1}\sqrt{\mu}g^{-1}\Psi_{_L} g + \ell^{-2}\mu g^{-1}Tg\,, \\ \mathcal{L}_+ &= \partial_+ + g^{-1}\partial_+ g + g^{-1}A_+ g + \ell\sqrt{\mu}\Psi_{_R} + \ell^2\mu T\,. \end{aligned} \tag{6.48}$$

To show that the compatibility conditions $[\mathcal{L}_-, \mathcal{L}_+] = 0$ imply the equations of motion (6.46) and (6.47) one needs to use (6.16),(6.44), i.e. that $[T, [T, \Psi_{_{L,R}}]] = -\Psi_{_{L,R}}$.

[32] Note that in the gauge (6.11) the residual κ symmetry is determined by k_1, k_2 satisfying $\partial_- k_{1-} = 0$ and $\partial_+ k_{2+} + [g^{-1}\partial_+ g, k_{2+}] = 0$.

6.4 Reduced Lagrangian: 2d Lorentz symmetry, massive spectrum and possible 2d supersymmetry

Remarkably, it turns out that the equations of motion (6.47) and (6.46) follow from the following local Lagrangian:

$$L_{tot} = L_{\rm gWZW} + \mu^2 \, {\rm STr}(g^{-1}TgT) \\ + \tfrac{1}{2}{\rm STr}\left(\Psi_{_L}[T, D_+\Psi_{_L}] + \Psi_{_R}[T, D_-\Psi_{_R}]\right) + \ \mu \, {\rm STr}\left(g^{-1}\Psi_{_L} g \Psi_{_R}\right) \ , \tag{6.49}$$

where $L_{\rm gWZW}$ represents the G/H gWZW model (3.15) with[33]

$$\frac{G}{H} = \frac{Sp(2,2)}{SU(2)\times SU(2)} \times \frac{Sp(4)}{SU(2)\times SU(2)}$$

Note L_{tot} is explicitly H gauge-invariant under (6.32),(6.45) with $h = \bar{h}$.[34] The dimension of the bosonic target space here is the same as the dimension of the G/H coset, i.e. 4+4=8. The fermionic fields contain 8+8 independent real Grassmann components (describing 8 dynamical degrees of freedom).

The variations over g and $\Psi_{_L}, \Psi_{_R}$ indeed lead to (6.46),(6.47). Thus in order to show that the reduced model (6.46),(6.47) is described by (6.49) one is to demonstrate that the constraint equations that arise from varying this action with respect to $A_\pm$ represent an admissible gauge condition for the equations of motion.[35] These constraints read as

$$A_+ = (\widehat{A}_+)_{\mathfrak{h}} \, , \qquad \widehat{A}_+ \equiv g^{-1}\partial_+ g + g^{-1}A_+ g - \frac{1}{2}[[T, \Psi_{_R}], \Psi_{_R}] \, , \tag{6.50}$$

$$A_- = (\widehat{A}_-)_{\mathfrak{h}} \, , \qquad \widehat{A}_- \equiv g\partial_- g^{-1} + gA_- g^{-1} - \frac{1}{2}[[T, \Psi_{_L}], \Psi_{_L}] \, . \tag{6.51}$$

In the Appendix **E** we show that they can be satisfied by an appropriate on-shell gauge transformation. Note that once these constraints are satisfied the original $H \times H$ "on-shell" gauge symmetry (6.32),(6.45) of the equations of motion having independent h and $\bar{h}$ parameters reduces to the H gauge symmetry with $h = \bar{h}$ which is the "off-shell" gauge symmetry of the Lagrangian (6.49).[36]

Let us now discuss several properties of this reduced action.

The Lagrangian (6.49) is formulated in terms of the left-invariant $\widehat{F}$ current variables (cf. (6.26), (6.44)) that are "blind" to the original $\widehat{F} = PSU(2,2|4)$ symmetry. Note that since the original coset

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[33] Here $L_{\rm gWZW}$ is given by (3.15) with Tr replaced by the $-$STr. The minus sign is needed to compensate for the definition of the supertrace which includes the S^m sector with a minus sign (the use of supertrace in the first two bosonic terms means of course just the sum of the reduced models for the AdS_5 and the S^5 parts). The corresponding reduced action $S_{tot} = \int \frac{d^2\sigma}{2\pi} \, L_{tot}$ is real (as can be seen by applying the conjugation $*$ defined in Appendix C to the expression under the trace).

[34] As was already mentioned above, our reduction procedure formally applies and leads to the Lagrangian (6.49) if one starts with any $psu(m, m|2m)$; in particular, the $m = 1$ case corresponds to $AdS_2 \times S^2$ superstring model.

[35] Note that in the $AdS_2 \times S^2$ case the subalgebra $\mathfrak{h}$ is empty and so this step is trivial.

[36] More generally, similarly to the purely bosonic case, one can consider an asymmetric gauge determined by an automorphism τ of $\mathfrak{h}$ preserving the supertrace. In this case the residual gauge transformations are $g \to h^{-1}g\widehat{\tau}(h)$, $\Psi_{_R} \to \widehat{\tau}(h^{-1})\Psi_{_R}\widehat{\tau}(h)$ with transformations of the remaining variables unchanged. The Lagrangian of the asymmetrically gauged model is given by (6.49) with A_- in $A_- \, g^{-1}\partial_+ g - g^{-1}A_+ g A_-$ terms in (3.15) replaced with $\tau(A_-)$.

$\widehat{F}/G = PSU(2,2|4)/[Sp(2,2) \times Sp(4)]$ has the purely *bosonic* factor G, the reduced action (6.49) has only the *bosonic* global and gauge symmetries, i.e. it has no target-space supersymmetry (but may have 2d supersymmetry, see below).

It is interesting to notice that the Lagrangian (6.49) can be rewritten as

$$L_{tot} = \widehat{L}_{gWZW} + L_{add}\,, \qquad\qquad L_{add} = \mathrm{STr}\left[P_+P_- + \frac{1}{2}(Q_{1+}Q_{2-} - Q_{1-}Q_{2+})\right]. \qquad (6.52)$$

Here $\widehat{L}_{gWZW}$ is the G/H bosonic gWZW Lagrangian supplemented with the "free" fermionic terms $\frac{1}{2}\mathrm{STr}\,(\Psi_{_L}[T, D_+\Psi_{_L}] + \Psi_{_R}[T, D_-\Psi_{_R}])$ while L_{add} stands for the sum of the remaining μ dependent terms in (6.49). Here we restored the original notations for the current components, i.e. used that $P_+ = \mu T,\ P_- = \mu g^{-1}Tg$ (see (6.26)), that $Q_{1+} = Q_{2-} = 0$ due to the κ-symmetry gauge condition (6.11), and that $Q_{1+} = \Psi_{_R}, Q_{2-} = g^{-1}\Psi_{_L}g$ in (6.34). Remarkably, $L_{add} = \mu^2\,\mathrm{STr}(g^{-1}TgT) + \mu\,\mathrm{STr}\,(g^{-1}\Psi_{_L}g\Psi_{_R})$ is thus nothing but the original superstring Lagrangian (6.7) rewritten in terms of the new variables $g, \Psi_{_R}, \Psi_{_L}$. At the same time, the equations following from $\widehat{L}_{gWZW}$ encode the Maurer-Cartan equations (6.25) for the $\widehat{F}$ currents. It is then clear that once the conformal gauge (Virasoro) constraints are imposed, L_{tot} describes, at least at the level of the corresponding equations of motion and up to the various gauge transformations and fixing the values of the conserved quantities in terms of μ, the same field configurations as the original superstring sigma-model Lagrangian (6.3),(6.7). An interesting question is whether one can implement a similar argument 'off-shell" or even at the quantum level in terms of path-integral transformations.[37]

Despite the fact that the 2d Lorentz invariance may appear to be broken by various gauge choices made above and that $\Psi_{_L}$ and $\Psi_{_R}$ originated from the 2d vector components of the fermionic currents (cf. (6.34),(6.44)) it is remarkable that it is still possible to assign the fermions the $SO(1,1)$ Lorentz transformation rules of the components of the left and right 2d Majorana-Weyl spinors. Then the Lagrangian (6.49) becomes invariant under the standard 2d Lorentz symmetry

$$\sigma^+ \to \Lambda\sigma^+\,, \quad \sigma^- \to \Lambda^{-1}\sigma^-\,, \qquad \Psi_{_L} \to \Lambda^{1/2}\Psi_{_L}\,, \qquad \Psi_{_R} \to \Lambda^{-1/2}\Psi_{_R}\,, \qquad (6.53)$$

with g and $A_\pm$ having the usual scalar and vector transformation laws. Choosing a parametrisation for the matrix variables $\Psi_{_L}$ and $\Psi_{_R}$ which satisfy the "parallel" constraint in (6.44),(6.16)[38] one can put the fermion kinetic terms in (6.49) into the familiar form $\psi_{_L}\partial_+\psi_{_L} + \psi_{_R}\partial_-\psi_{_R} + ...$

As in the case of the bosonic reduced theory the classical conformal invariance of the original superstring sigma model in the conformal gauge is broken by the μ-dependent interaction terms in (6.49): the residual conformal diffeomorphism symmetry was used (cf. (6.26)) to perform the reduction procedure. This breaking is "spontaneous" being due to the presence of the "background field" $T = T_+ = T_-$. This is similar to what happened in the light-cone gauge in the plane-wave model [7]

[37] A natural idea is to start with the original superstring sigma model path integral in the conformal gauge (i.e. with the delta-function insertions $\delta(T_{++})\delta(T_{--})$), fix the κ-symmetry gauge and change variables from coset coordinates to $PSU(2,2|4)$ currents. The $\widehat{L}_{gWZW}$ term in the path integral action may then appear due to this change of variables. This procedure can work only if the original path integral represents a 2d conformal theory: in the reduction procedure we used the residual conformal symmetry.

[38] The "parallel" subspace is formed by anti-diagonal matrices with fermionic 2×2 blocks.

where the mass terms (proportional to the light-cone momentum, i.e. appearing from the ∂x^+ terms) were spontaneously breaking the classical conformal invariance of the original sigma model action.

Again as in the bosonic case discussed in section **5**, the form of the reduced Lagrangian expressed in terms of only "physical" bosonic and fermionic fields may be found by imposing an H gauge fixing condition on g and then integrating out the H gauge field components $A_\pm$. This leads to a sigma-model with 4+4 dimensional bosonic part (5.1) supplemented by the fermionic terms, with the following general structure (cf. (5.1))

$$\widetilde{L} = G(x)\partial_+ x\partial_- x - \mu^2 U(x) + \psi_{_L}\mathcal{D}_+\psi_{_L} + \psi_{_R}\mathcal{D}_-\psi_{_R} + F(x)\psi_{_L}\psi_{_L}\psi_{_R}\psi_{_R} + 2\mu H(x)\psi_{_L}\psi_{_R} \ . \quad (6.54)$$

Here x stands for 8 real bosonic fields in (5.1) (i.e. for the independent variables in gauge-fixed g which parametrize G/H) and $\psi_{_L}, \psi_{_R}$ – for 8+8 independent real Grassmann fields which are the components of the matrices $\Psi_{_L}, \Psi_{_R}$. The quartic fermionic term originates from the $D_\pm$ terms in (6.49) upon integrating out $A_\pm$ ($\mathcal{D}_\pm$ in (6.54) are the standard x-dependent covariant derivatives). As discussed below, the structure of (6.49) looks very similar to that of the supersymmetric gWZW model modified by the bosonic potential and the fermionic "Yukawa" terms, and so the presence of the quartic fermionic terms in (6.54) may be interpreted as reflecting the curvature of the target space.

Let us now discuss the vacuum structure and the corresponding mass spectrum of the reduced model (6.49). Since $[T, H] = 0$ the obvious vacuum solution of the equations of motion (6.46),(6.47) for (6.49) corresponds to g being any constant element h_0 of H, i.e.

$$g_{\rm vac} = h_0 = {\rm const} \ , \qquad (A_+)_{\rm vac} = (A_-)_{\rm vac} = 0 \ , \qquad (\Psi_{_L})_{\rm vac} = (\Psi_{_R})_{\rm vac} = 0 \ , \qquad (6.55)$$

i.e. the space of vacua is equivalent to $H = [SU(2)]^4$. By a global H transformation we can always set $h_0 = 1$, i.e. the mass spectrum should not depend on h_0. Expanding the equations of motion (6.46),(6.47) near $g = \mathbf{1}$, i.e. $g = \mathbf{1} + v + ...$, and projecting to the algebra of H and its complement in $\mathfrak{g}$ we find a massive equation for $v \in \mathfrak{m} \equiv \mathfrak{f}_0^{||}$ (i.e. $v = [[T, v], T]$, see (6.16)) as well as $F_{+-} = 0$.[39] That all bosonic coset directions get mass μ was mentioned already in section 5.3 and follows also directly from the equations of motion in the $A_+ = A_- = 0$ on-shell gauge in the parametrization used in (4.30),(4.32). The linearized bosonic and fermionic equations are thus

$$\partial_+\partial_- v + \mu^2 v \ = \ 0 \, , \qquad (6.56)$$

$$[T, \partial_-\Psi_{_R}] + \mu\Psi_{_L} = 0 \, , \quad [T, \partial_+\Psi_{_L}] + \mu\Psi_{_R} = 0 \quad \to \quad \partial_+\partial_-\Psi_{_{L,R}} + \mu^2\Psi_{_{L,R}} = 0 \ , \quad (6.57)$$

where we used that $[T, [T, \Psi_{_{L,R}}]] = -\Psi_{_{L,R}}$ (see (6.16),(6.44)). The 8+8 independent real Grassmann components of the fermionic matrix fields thus represent 8 massive 2d Majorana fermions having the same mass μ as the bosonic modes. The corresponding fermionic Lagrangian is then

$$\psi_{_L}\partial_+\psi_{_L} + \psi_{_R}\partial_-\psi_{_R} - 2\mu\psi_{_L}\psi_{_R} + ... \ ,$$

[39] Equivalently, expanding the action (6.49) to quadratic order in fluctuations the A_+A_- term will cancel while the term linear in A_+, A_- will project v to the coset part $\mathfrak{m}$ of the algebra $\mathfrak{g}$.

where the mass term originates from the last "Yukawa" term in (6.49),(6.54).[40]

The small-fluctuation spectrum we get is thus formally the same as in the plane-wave limit [7]. In contrast to the case of the original $AdS_5 \times S^5$ superstring expanded near the S^5 geodesic in the light-cone gauge where one scatters "magnons" which are small fluctuations of the superstring coordinates and the remaining symmetry is $[PSU(2|2)]^2$ [11, 10], here we scatter the fluctuations of the current components which are invariants of the original supergroup $PSU(2,2|4)$. The manifest global symmetry of the S-matrix corresponding to (6.49) in the vacuum (6.55) appears to be just the bosonic $H = [SU(2)]^4$ one.[41]

Indeed, while the Lagrangian (6.54) obtained by integrating out the H gauge fields does not have manifest non-abelian global symmetry, it is natural to expect that the tree-level S-matrix for scattering of the massive excitations near the vacuum (6.55) can be extracted directly from the classical equations of motion (6.46),(6.47). The latter admit larger on-shell $H \times H$ gauge symmetry allowing us to choose the $A_+ = A_- = 0$ gauge in which the global H-symmetry of the remaining non-linear equations and thus of the resulting (gauge-independent) S-matrix becomes manifest. The same H symmetry is expected also to be present in the full quantum S-matrix.[42]

Let us now comment on the meaning of the parameter μ which plays a crucial role in our reduction procedure and sets the mass scale.[43] μ entered first through the conditions $P_+ = \mu T, \;\; P_- = \mu g^{-1} T g$ (4.12),(6.26) on the $\pm$ components of the coset-space part of the current that solve the conformal gauge constraints. In the vacuum (6.55) we thus have (cf. (6.12),(6.13))

$$(P_+)_{\rm vac} = (P_-)_{\rm vac} = \ \mu\, T \ , \qquad\qquad T = \frac{i}{2} {\rm diag}(1,1,-1,-1;1,1,-1,-1) \ . \qquad (6.58)$$

Thus μ determines the scale while T – the structure of the background values of the coset currents. The corresponding charges (defined assuming the world sheet is a cylinder) thus have both the AdS_5 and S^5 non-zero components. Though $P_\pm$ are invariants of $PSU(2,2|4)$ their non-zero vacuum values appear to translate, in particular, into the non-zero values of the quadratic Casimirs for $SO(2,4)$ and $SO(6)$ group. This suggests again a close relation to the BMN limit.[44]

In general, to relate the reduced or "current" formulation of the theory to the original $AdS_5 \times S^5$ superstring model (6.3) (and thus to gauge theory within the AdS/CFT duality) one would need to supplement the quantum theory based on (6.49) by a list of "observables" which are intrinsic to the $AdS_5 \times S^5$ string in its original coordinate-space formulation. This list should include, in particular, the components of the $PSU(2,2|4)$ charges. They cannot be computed directly without

[40]This and other points discussed in this section can be illustrated on the $AdS_2 \times S^2$ example discussed in the next section (see, e.g., (7.16) below where one is to expand near $\varphi = \phi = 0$).

[41]If we start with the closed string picture with the sigma model defined on a cylinder $R \times S^1$ we need to take the $\mu \to \infty$ limit (which "decompactifies" the spatial world sheet direction) to define the scattering matrix. An interesting question then is how to generalize the *relativisic* (cf. [11]) S-matrix for the CSG model [52] to the full reduced model for $AdS_5 \times S^5$.

[42]The S-matrix should also have higher hidden symmetries presumably related to those of the S-matrix in [11]; we thank R.Roiban for a discussion of this point.

[43]We thank S. Frolov for asking this question and useful discussions.

[44]In a certain sense, our reduction procedure may then be interpreted as an "invariant version" of the expansion near the BMN vacuum.

supplementing the reduced action with a linear problem for the associated Lax pair, but according to the above remarks about the vacuum values of currents in (6.58) we are guaranteed to have at least some components of the AdS_5 and S^5 charges to be non-zero in the vacuum (6.55) of the reduced theory.

Finally, let us discuss possible 2d supersymmetry of the action corresponding to (6.49). As was already mentioned above, the number (8) of independent bosonic degrees of freedom in the reduced Lagrangian (6.54) matches that of the fermionic ones (8+8), exactly as in a 2d supersymmetric model. Moreover, we saw that the spectrum of small fluctuations near the vacuum state (6.56),(6.57) is also supersymmetric.

The structure of (6.49) is essentially that of a supersymmetric gWZW model [60, 61],

$$L_{\rm SgWZW} = L_{\rm gWZW} + \psi_{_L} D_+ \psi_{_L} + \psi_{_R} D_- \psi_{_R} \ , \tag{6.59}$$

modified by the μ-dependent interaction terms. If we first set $\mu = 0$, i.e. ignore the potential and Yukawa interaction terms in (6.49), then we should expect to find the same (1,1) supersymmetry as found in the component description of supersymmetric gWZW model [60, 61], i.e.

$$\delta g \sim \epsilon_{_L} \psi_{_R} g + \epsilon_{_R} g \psi_{_L} , \quad \delta \psi_{_R} \sim \epsilon_{_L} (g^{-1} D_+ g)_{_{G/H}} , \quad \delta \psi_{_L} \sim \epsilon_{_R} (g D_- g^{-1})_{_{G/H}} , \quad \delta A_\pm = 0 \ . \tag{6.60}$$

Here $\epsilon_{_L}$ and $\epsilon_{_R}$ are parameters of the (1,0) and (0,1) supersymmetries.

For this to work the fermions should transform under the H gauge transformation as elements of the coset part of $\mathfrak{g}$, i.e. $\mathfrak{m} = \widehat{\mathfrak{f}}_0^{\|}$, considered as a representation of the gauge algebra $\mathfrak{h} = \widehat{\mathfrak{f}}_0^{\perp}$. It appears, however, that for the case of $psu(2,2|4)$ the fermions $\Psi_{_R}, \Psi_{_L}$ take values in $\widehat{\mathfrak{f}}_{1,2}^{\|}$ which is, in general, a different representation of the gauge algebra $\mathfrak{h}$. More precisely, $\widehat{\mathfrak{f}}_1^{\|}$ and $\widehat{\mathfrak{f}}_0^{\|}$ considered as representations of $\mathfrak{h}$ are inequivalent representations related by an appropriate automorphism τ of the gauge algebra $\mathfrak{h}$.[45] In the absence of μ-dependent terms in (6.49) one can of course modify the gauge transformation law of the fermions by replacing, e.g., A_- with its image under that automorphism $\tau(A_-)$ in the kinetic term for $\Psi_{_R}$. This does not, however, directly apply for $\mu \neq 0$; for example, the gauge invariance of the fermionic interaction term $\mu {\rm STr}(g^{-1} \Psi_{_L} g \Psi_{_R})$ in (6.49) determines the gauge transformation law of the fermions in terms of that of the field g.

We leave the question whether the full (6.49) in the $psu(2,2|4)$ case does have a 2d supersymmetry, i.e. if it can be identified with a supersymmetric extension of the corresponding bosonic non-abelian Toda theory, for a future investigation.[46] Our conjecture is that the answer is yes and the supersymmetry should be the extended (2,2) one.[47]

[45] One can see that $\widehat{\mathfrak{f}}_1^{\|}$ and $\widehat{\mathfrak{f}}_0^{\|}$ are inequivalent by, e.g., observing that for a subalgebra $\mathfrak{h}_1$ represented by the upper-left block matrices there are no invariant vectors in $\widehat{\mathfrak{f}}_{1,2}^{\|}$ but all the elements from $\widehat{\mathfrak{f}}_0^{\|}$ represented by lower-right block matrices are invariant. The automorphism τ simply interchanges $su(2)$ factor in the upper left block with the $su(2)$ factor in the lower-right block in the matrix representation of $\mathfrak{h}$.

[46] Supersymmetric extensions of generic non-abelian Toda theories were not previously discussed in the literature (apart from the complex sine-Gordon case [49, 50]). For some references on supersymmetric extensions of sigma models with potentials and, in particular, of abelian Toda models see [62, 63].

[47] The conditions for existence of the (2,2) supersymmetry in the (1,1) supersymmetric G/H gWZW model (i.e. in our $\mu = 0$ case) were discussed in [61] (see also [64, 65]).

As we shall show in the next section in a similar but simpler case of the $AdS_2 \times S^2$ superstring model where $psu(2,2|4)$ is replaced by the $psu(1,1|2)$ superalgebra (with trivial $\mathfrak{h}$ so that the complication of extending the supersymmetry from the "free" to $\mu \neq 0$ level is absent) the corresponding reduced Lagragian (6.49) is indeed invariant under the (2,2) supersymmetry.

An interesting question related to the existence of (2,2) supersymmetry is about finiteness property of the quantum theory defined by (6.49). A (supersymmetric) gWZW model corresponds to a (super)conformal theory, but including potential terms may in general introduce UV divergences. These divergences should cancel out if this model has (2,2) supersymmetry. We conjecture that this is indeed the case; then this reduced model has a chance to be useful for a quantum description of the $AdS_5 \times S^5$ superstring.

7 Example: reduced model for superstring in $AdS_2 \times S^2$ as $\mathrm{N} = 2$ super sine-Gordon model

Let us now specialise the construction of the previous section to the simplest case of $AdS_2 \times S^2$ superstring model [66, 54] where $\widehat{\mathfrak{f}} = psu(1,1|2)$. As we shall see below, here the reduced Lagrangian (6.49),(6.54) is equivalent to that of the $\mathrm{N} = 2$ supersymmetric sine-Gordon theory. This demonstrates the existence of the (2,2) world-sheet supersymmetry in the reduced version of this GS superstring model. Assuming one may consider the reduced theory as a legitimate starting point for the quantisation, this also implies the UV finiteness of the $AdS_2 \times S^2$ superstring and its quantum integrability.

7.1 Explicit parametrisation of $psu(1,1|2)$

The bosonic subspaces $\widehat{\mathfrak{f}}_0$ and $\widehat{\mathfrak{f}}_2$ in (6.1) here are represented by block-diagonal matrices of the form

$$f = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \qquad \Sigma A^\dagger \Sigma = -A\,, \qquad B^\dagger = -B\,, \tag{7.1}$$

with A, B being traceless 2×2 matrices and Σ given by (C.16), i.e. $A \in su(1,1)$ and $B \in su(2)$. The subspace $\widehat{\mathfrak{f}}_0$ is formed by matrices satisfying also

$$-KA_0^t K = A_0\,, \qquad -KB_0^t K = B_0\,, \tag{7.2}$$

with $K = \Sigma$ in (C.16). It is usefull to parametrise these matrices as

$$A_0 = \begin{pmatrix} 0 & \phi \\ \phi & 0 \end{pmatrix}, \qquad B_0 = \begin{pmatrix} 0 & i\varphi \\ i\varphi & 0 \end{pmatrix}, \tag{7.3}$$

where ϕ, φ are real. The elements of the subspace $\widehat{\mathfrak{f}}_2$ are determined by the additional conditions

$$KA_2^t K = A_2^t\,, \qquad KB_2^t K = B_2^t\,, \tag{7.4}$$

$$A_2 = \begin{pmatrix} ib & ic \\ -ic & -ib \end{pmatrix}, \qquad B_2 = \begin{pmatrix} iq & r \\ -r & -iq \end{pmatrix}, \tag{7.5}$$

where b, c, q, r are real. For the fermionic subspace $\widehat{\mathfrak{f}}_1$ the reality condition together with $M^\Omega = iM$ (see Appendix C) imply

$$M = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}, \qquad KY^tK = iX\,, \qquad i\Sigma Y^\dagger = X\,. \tag{7.6}$$

Since $\Sigma = K$ gives $Y^+ = -Y^tK$, $\widehat{\mathfrak{f}}_1$ can be parametrized as

$$Y_1 = \begin{pmatrix} i\alpha & i\beta \\ \gamma & \delta \end{pmatrix}, \qquad X_1 = \begin{pmatrix} \alpha & i\gamma \\ -\beta & -i\delta \end{pmatrix}. \tag{7.7}$$

For $\widehat{\mathfrak{f}}_3$ we have $KY^tK = -iX$ and $i\Sigma Y^\dagger = X$ giving $Y^\dagger = Y^tK$ and

$$Y_3 = \begin{pmatrix} \lambda & \nu \\ i\rho & i\sigma \end{pmatrix}, \qquad X_3 = \begin{pmatrix} i\lambda & \rho \\ -i\nu & -\sigma \end{pmatrix}. \tag{7.8}$$

The fixed element $T = T^1 + T^2$ in (6.14),(6.27) can be chosen in the form:

$$T = \tfrac{1}{2}\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}. \tag{7.9}$$

The subspaces $\widehat{\mathfrak{f}}_1^{\|}$ and $\widehat{\mathfrak{f}}_3^{\|}$ defined in (6.16) are then represented by (7.7) and (7.8) with

$$\alpha = \delta = 0\,, \qquad \lambda = \sigma = 0\,. \tag{7.10}$$

The field $g \in G$ introduced in (6.23) takes values in the direct product of two one-dimensional subgroups of $SU(1,1) \times SU(2)$ isomorphic to $SO(1,1)$ and $SO(2)$; it can be parametrized as

$$g = \exp\begin{pmatrix} A_0 & 0 \\ 0 & B_0 \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & \cos\varphi & i\sin\varphi \\ 0 & 0 & i\sin\varphi & \cos\varphi \end{pmatrix}. \tag{7.11}$$

7.2 Reduced Lagrangian

Let us write down the explicit form of the reduced Lagrangian (6.49) using the parametrisation introduced above. Here the subgroup H is trivial so that $A_+ = A_- = 0$. The "kinetic" WZW term is simply

$$\frac{1}{2}\mathrm{STr}(g^{-1}\partial_+ g g^{-1}\partial_- g) = \partial_+\phi\partial_-\phi + \partial_+\varphi\partial_-\varphi\,. \tag{7.12}$$

The potential term in (6.49) is

$$\mu^2\mathrm{STr}(g^{-1}TgT) = -\frac{\mu^2}{2}(\cosh 2\phi - \cos 2\varphi)\,. \tag{7.13}$$

The fermionic terms in (6.49) are

$$\begin{aligned} \frac{1}{2}\mathrm{STr}(\Psi_{_R}[T,\partial_-\Psi_R]) = \mathrm{Tr}(\partial_- Y_1[T^1,X_1]) &= -\mathrm{Tr}(\partial_- X_1[T^2,Y_1]) = \beta\partial_-\beta + \gamma\partial_-\gamma\,, \\ \frac{1}{2}\mathrm{STr}(\Psi_{_L}[T,\partial_+\Psi_{_L}]) = \mathrm{Tr}(\partial_+ Y_3[T^2,X_3]) &= -\mathrm{Tr}(\partial_+ X_3[T^2,Y_3]) = \nu\partial_+\nu + \rho\partial_+\rho\,, \end{aligned} \tag{7.14}$$

$$\begin{aligned} \mu\mathrm{STr}(g\Psi_{_R}g^{-1}\Psi_{_L}) &= \mu\mathrm{Tr}(g_1X_1g_2^{-1}Y_3) - \mathrm{Tr}(g_2Y_1g_1^{-1}X_3) \\ &= -2\mu[\cosh\phi\cos\varphi\,(\beta\nu+\gamma\rho) + \sinh\phi\sin\varphi\,(\beta\rho-\gamma\nu)]\,, \end{aligned} \tag{7.15}$$

where we have used the explicit form of the diagonal blocks $T^1 = T^2 = \frac{i}{2}\,\mathrm{diag}(1,-1) = \frac{i}{2}\Sigma$ in (7.9).

Thus the final expression of the corresponding reduced Lagrangian (6.49) in terms of the two bosonic ϕ,φ and the four fermionic β,γ,ν,ρ field variables is given by (cf. (5.18))[48]

$$\begin{aligned} L_{tot} = \partial_+\varphi\partial_-\varphi + \partial_+\phi\partial_-\phi + \frac{\mu^2}{2}&(\cos 2\varphi - \cosh 2\phi) \\ &+ \beta\partial_-\beta + \gamma\partial_-\gamma + \nu\partial_+\nu + \rho\partial_+\rho \\ &- 2\mu\,[\cosh\phi\;\cos\varphi\,(\beta\nu+\gamma\rho) + \sinh\phi\;\sin\varphi\,(\beta\rho-\gamma\nu)]\;. \end{aligned} \tag{7.16}$$

7.3 Equivalence to $\mathrm{N}=2$ supersymmetric sine-Gordon model

The bosonic part of the $AdS_2\times S^2$ reduced Lagrangian in (5.18),(7.16) happens to be exactly the same as the bosonic part of the $\mathrm{N}=2$ supersymmetric sine-Gordon Lagrangian [53]. Furthermore, the number of the fermionic fields in (7.16) is the same as in the $\mathrm{N}=2$ SG theory. This suggests that the $AdS_2\times S^2$ reduced model (7.16) may have a hidden $\mathrm{N}=2$ world-sheet supersymmetry.

Indeed, (7.16) is equivalent to the $\mathrm{N}=2$ SG theory. A generic $\mathrm{N}=2$ (i.e. (2,2)) superfield Lagrangian is

$$\begin{aligned} L &= \int d^4\vartheta\;\widehat{\Phi}^*\widehat{\Phi} + [\int d^2\vartheta\;W(\widehat{\Phi}) + h.c.]\,, \\ \widehat{\Phi} &= \Phi + \vartheta_1\psi_{_L} + \vartheta_2\psi_{_R} + \vartheta_1\vartheta_2\mathcal{D}\,, \end{aligned} \tag{7.17}$$

where $\widehat{\Phi}$ is a chiral $\mathrm{N}=2$ superfield, $\Phi=\varphi+i\phi$ is a complex scalar and $\psi_{_L},\psi_{_R}$ are complex fermions. In components

$$L = \partial_+\Phi\partial_-\Phi^* - |W'(\Phi)|^2 + \psi^*_{_L}\partial_+\psi_{_L} + \psi^*_{_R}\partial_-\psi_{_R} + \left[W''(\Phi)\psi_{_L}\psi_{_R} + W^{*\prime\prime}(\Phi^*)\psi^*_{_L}\psi^*_{_R}\right]\,. \tag{7.18}$$

The sine-Gordon choice is

$$W(\Phi) = \mu\cos\Phi\,, \qquad |W'(\Phi)|^2 = \frac{\mu^2}{2}(\cosh 2\phi - \cos 2\varphi)\,. \tag{7.19}$$

Splitting $\psi_{_L}$, $\psi_{_R}$ into the real and imaginary parts

$$\psi_{_L} = \nu + i\rho\,, \qquad \psi_{_R} = -\beta + i\gamma\,, \tag{7.20}$$

[48] As expected, the Lagrangian is real (the fermionic fields are real).

we indeed find the agreement between (7.18) and (7.16).

Let us note that it is possible to write down the $\mathrm{N}=2$ supersymmetry transformations of the fields in (7.16) in terms of the original matrix parametrisation used in (6.49). Let us consider separately the (2,0) and (0,2) supersymmetries. To describe the (2,0) transformation let us introduce a matrix fermionic parameter $\epsilon_{_L}$ taking values in $\widehat{\mathfrak{f}}_1$ in (6.1) and satisfying in addition $[T,\epsilon_{_L}]=0$. This ensures that $\epsilon_{_L}$ contains two independent fermionic parameters (α and δ in the parametrisation (7.7)). The (2,0) supersymmetry transformation of the matrix fields in (6.49) then reads as

$$\delta_{\epsilon_L} g = g[T,[\Psi_{_L},\epsilon_{_L}]]\,, \qquad \delta_{\epsilon_L}\Psi_{_L} = [g^{-1}\partial_+ g,\epsilon_{_L}]\,, \qquad \delta_{\epsilon_L}\Psi_{_R} = \mu[T, g\epsilon_{_L} g^{-1}]\,. \tag{7.21}$$

In checking the invariance of the action we have to use (besides the Z_4 grading and definition of $\epsilon_{_L}$) that $[T,[T,\Psi_{_L}]]=-\Psi_{_L}\,,\ \ [[T,[\Psi_{_L},\epsilon_{_L}]],\Psi_{_L}]=0\,$, etc. The (0,2) transformation with parameter $\epsilon_{_R}$ looks similarly.

The (2,0) supersymmetry transformation law (7.21) can be formally generalized to the algebraically analogous models described by (6.49) *provided* $\widehat{\mathfrak{f}}_1^{\perp}$ contains a nontrivial element commuting with the entire gauge algebra $\mathfrak{h}$. Indeed, suppose $\epsilon_{_L}$ belongs to $\widehat{\mathfrak{f}}_1^{\perp}$ and is satisfying in addition $[\epsilon,h]=0$ for any $h\in\mathfrak{h}=\widehat{\mathfrak{f}}_0^{\perp}$ (in other words, $\epsilon_{_L}$ should belong to the centraliser of $\mathfrak{h}$ in $\mathfrak{f}_1^{\perp}$). Then the supersymmetry transformation reads

$$\begin{gathered}\delta_{\epsilon_L} g = g[T,[\Psi_{_R},\epsilon_{_L}]]\,, \qquad \delta_{\epsilon_L}\Psi_{_R} = [(g^{-1}D_+ g)^{\parallel},\epsilon_{_L}]\,, \qquad \delta_{\epsilon_L}\Psi_{_L} = \mu[T,g\epsilon_{_L}g^{-1}]\,,\\ \delta_{\epsilon_L}A_+ = 0\,, \qquad \delta_{\epsilon_L}A_- = \mu[(g^{-1}\Psi_{_L} g)^{\perp},\epsilon_{_L}]\,,\end{gathered} \tag{7.22}$$

where the superscript $\parallel$ or $\perp$ denotes the projection to $\widehat{\mathfrak{f}}^{\parallel}$ or $\widehat{\mathfrak{f}}^{\perp}$ respectively. Note that for $\mu\neq 0$ the field A_- starts transforming under the supersymmetry.[49] Since the action is invariant under the exchange $+\leftrightarrows -$, $L\leftrightarrows R$, and $g\leftrightarrows g^{-1}$ one finds also the "right" counterpart of the "left" supersymmetry (7.22) with $\epsilon_{_L}\to\epsilon_{_R}$ where $\epsilon_{_R}$ is taking values in $\widehat{\mathfrak{f}}_3^{\perp}$ and is annihilated by $\mathfrak{h}$.

In the case of $psu(1,1|2)$ the subalgebra $\mathfrak{h}$ is empty and $\epsilon_{_L}$ is an arbitrary element of the two-dimensional space $\widehat{\mathfrak{f}}_1^{\perp}$ (and similarly $\epsilon_{_R}\in\widehat{\mathfrak{f}}_3^{\perp}$) so that (7.22) defines a consistent (2,0) (and also (0,2)) supersymmetry transformation. However, in the case of $psu(2,2|4)$, none of the elements in $\widehat{\mathfrak{f}}_{1,2}$ commute with the entire $\mathfrak{h}$ so that (7.22) does not directly apply (cf. the discussion at the end of section (6.4)). The existence of 2d supersymmetry of (6.49) in the $AdS_5\times S^5$ case thus remains an interesting open question.[50]

Let us finally mention that the complex sine-Gordon model (2.7) also admits an $\mathrm{N}=2$ supersymmetric version [49, 50]. The same applies to its "double" in (5.19) which has 2+2 dimensional target space which is a direct sum of the two Kähler spaces. We expect that the corresponding $\mathrm{N}=2$ model should be equivalent to the reduced model for the superstring on $AdS_3\times S^3$ [67] with (5.19) as its bosonic part.

[49] In checking the invariance of the action one is to use the algebraic properties $[[T,\Psi_{_R}],\Psi_{_R}]\in\widehat{\mathfrak{f}}_0^{\perp}$, $[[\epsilon_{_L},\Psi_{_R}],\Psi_{_R}]\in\widehat{\mathfrak{f}}_0^{\perp}$, which follow upon the application of the projectors to $\widehat{\mathfrak{f}}^{\parallel,\perp}$ and the use of the identities (6.20).

[50] Among other interesting questions let us mention also the construction of reduced models for non-critical AdS_n superstrings [56, 68] and their possible world-sheet supersymmetry.

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While this paper was in preparation we were informed by A. Mikhailov and S. Schäfer-Nameki about their closely related forthcoming paper [71] in which an equivalent reduced action for $AdS_5 \times S^5$ superstring is found.

Appendix A: Proof of gauge equivalence in section 3.2

Here we provide some details of the argument in section **3.2**. Let us introduce the following combinations

$$\widehat{A}_+ = g^{-1}\partial_+ g + g^{-1}A_+ g\,, \qquad \widehat{A}_- = g\partial_- g^{-1} + gA_- g^{-1} \tag{A.1}$$

Under the gauge transformations (3.24) $\widehat{A}_\pm$ transform as follows:

$$\widehat{A}_+ \to \bar{h}^{-1}\widehat{A}_+\bar{h} + \bar{h}^{-1}\partial_+\bar{h}\,, \qquad \widehat{A}_- \to h^{-1}\widehat{A}_+ h + h^{-1}\partial_- h\,. \tag{A.2}$$

It follows from the commutation relations $[\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$ that their $\mathfrak{h}$ projections also transform in the same way. Then the constraints (3.20) take the form

$$A_+ = (\widehat{A}_+)_\mathfrak{h}\,, \qquad A_- = (\widehat{A}_-)_\mathfrak{h}\,. \tag{A.3}$$

They are not invariant under the transformations (A.2) unless $h = \bar{h}$. Using (3.24) one can then set

$$(\widehat{A}_+)_\mathfrak{h} = A_+ = (g^{-1}\partial_+ g + g^{-1}A_+ g)_\mathfrak{h}\,. \tag{A.4}$$

This condition can be satisfied by applying the transformation (3.24) with $h = \mathbf{1}$. Under this transformation A_+ is unchanged while $(\widehat{A}_+)_\mathfrak{h} = (g^{-1}\partial_+ g + g^{-1}A_+ g)_\mathfrak{h}$ transforms as an H connection, so it is possible to find $\bar{h}$ so that transformed value of $(\widehat{A}_+)_\mathfrak{h}$ is equal to A_+.

Next, once $(\widehat{A}_+)_{\mathfrak{h}} = A_+$, eq. (3.19) implies that A_+, A_- are components of a flat 2d connection, i.e. satisfy (3.21).[51] This, together with the equation on g contained in (3.19) and the remaining part of gauge invariance (3.24) allows one to show that the second relation in (3.20) can also be satisfied.

Indeed, let us show that one can find such h_0 that the transformation (3.24) with $h = h_0$ and $\bar{h} = \mathbf{1}$ preserves $A_+ = (\widehat{A}_+)_{\mathfrak{h}}$ and transforms A_- and g so that $A_- = (\widehat{A}_-)_{\mathfrak{h}}$ (note that $\widehat{A}_-$ is unchanged under such transformation). It is enough to find h_0 in any admissible gauge that can be reached by the gauge transformation with $h = \bar{h}$ (both conditions $(\widehat{A}_+)_{\mathfrak{h}} = A_+$ and $(\widehat{A}_-)_{\mathfrak{h}} = A_-$ are invariant under such gauge transformations). Without loss of generality we can choose this gauge to be $A_+ = A_- = 0$ (this gauge can always be reached by a gauge transformation with $h = \bar{h}$). In this gauge the equation (3.19) and the constraint $(\widehat{A}_+)_{\mathfrak{h}} = A_+$ take the form (3.28) and the first equation in (3.29) respectively. Equation (3.28) can be written equivalently as

$$\partial_+(g\partial_- g^{-1}) = \mu^2 [T_-, gT_+g^{-1}]\,, \tag{A.5}$$

implying $\partial_+(g\partial_- g^{-1})_{\mathfrak{h}} = 0$. This means that $(g\partial_- g^{-1})_{\mathfrak{h}}$ is a function of σ^- only and therefore can be represented as $(g\partial_- g^{-1})_{\mathfrak{h}} = h_0 \partial_- h_0^{-1}$ for some H-valued function $h_0(\sigma^-)$. By performing the gauge transformation with $\bar{h} = \mathbf{1}$ and $h = h_0$ one then arrives at $(\widehat{A}_-)_{\mathfrak{h}} = (g\partial_- g^{-1})_{\mathfrak{h}} = 0$ while still satisfying $A_\pm = 0$ and $(\widehat{A}_+)_{\mathfrak{h}} = 0$.

Appendix B: Vanishing of the antisymmetric tensor coupling in the reduced Lagrangian in section 5.1

Here we provide details of the argument mentioned at the end of section **5.1** that the reduced Lagrangian (5.1) does not contain a WZ-type term. Indeed, all possible antisymmetric tensor contributions that may result from integrating out the gauge field of the gWZW model vanish.

Let us consider the following automorphism of the orthogonal matrix group and its Lie algebra:

$$\widetilde{M^i_j} = M^i_j (-1)^{i+j}\,, \qquad \widetilde{MN} = \widetilde{M}\widetilde{N}\,. \tag{B.1}$$

It is easy to check that

$$\mathrm{Tr}\widetilde{M} = \mathrm{Tr}M\,, \qquad \det\widetilde{M} = \det M\,, \quad \widetilde{M}^{-1} = \widetilde{M^{-1}}\,, \quad \widetilde{M}^T = \widetilde{M^T}\,. \tag{B.2}$$

If g has the gauge-fixed form (5.4) then $\widetilde{g} = g^{-1}$: this is obviously correct for any $g_k = e^{\theta_k R_k}$ because $\widetilde{R}_k = -R_k$ while g^{-1} has the same form with all g_k replaced with g_k^{-1}.

The integrand of the WZ term in (3.14),(3.15) then satisfies

$$\begin{aligned}\mathrm{Tr}(g^{-1}dgg^{-1}dgg^{-1}dg) &= \mathrm{Tr}((g^{-1}dg\widetilde{g^{-1}dg}g^{-1}dg)) \\ &= \mathrm{Tr}(gdg^{-1}gdg^{-1}gdg^{-1}) = -\mathrm{Tr}(g^{-1}dgg^{-1}dgg^{-1}dg)\,,\end{aligned} \tag{B.3}$$

and thus should vanish.

[51]Note that contrary to the discussion before (3.21) now we do not assume that both constraints (3.20) are satisfied.

Another possible contribution may originate from the gauge field dependent term in the gWZW Lagrangian (3.15)

$$L_A = \mathrm{Tr}\big(A_+\partial_- g g^{-1} - A_-\, g^{-1}\partial_+ g - g^{-1}A_+ g A_- + A_+ A_-\big)\,, \tag{B.4}$$

where $A_\pm$ should be replaced by the solutions of their equations of motion

$$A_+ = (g^{-1}\partial_+ g + g^{-1}A_+ g)_{\mathfrak{h}}\,, \qquad A_- = (g\partial_- g^{-1} + gA_- g^{-1})_{\mathfrak{h}}\,. \tag{B.5}$$

This gives

$$L_A = \mathrm{Tr}\big(A_+\partial_- g g^{-1}\big) = -\mathrm{Tr}(A_-\, g^{-1}\partial_+ g)\,. \tag{B.6}$$

It follows from the explicit form of Eqs. (B.5) that there exists a function $\mathbf{A}(g, \partial g)$ such that

$$A_+(g, \partial_+ g) = \mathbf{A}(g, \partial_+ g)\,, \qquad A_-(g, \partial_- g) = \mathbf{A}(g^{-1}, \partial_- g^{-1})\,. \tag{B.7}$$

Moreover, assuming the analyticity in g one finds

$$\mathbf{A}\widetilde{(g, \partial_\pm g)} = \mathbf{A}(g^{-1}, \partial_\pm g^{-1})\,, \tag{B.8}$$

provided $\widetilde{g} = g^{-1}$. In particular, this holds in the gauge (5.4)).

Since $A_\pm$ are linear in $\partial_\pm g$ the vanishing of the antisymmetric part of the metric is equivalent to $L_A(g, \partial_+ g, \partial_- g) = L_A(g, \partial_- g, \partial_+ g)$. Assuming $\widetilde{g} = g^{-1}$ one gets

$$\begin{aligned} L_A(g, \partial_- g, \partial_+ g) &= \mathrm{Tr}(\mathbf{A}(g, \partial_- g)\partial_+ g g^{-1}) = \mathrm{Tr}(\mathbf{A}(g, \widetilde{\partial_- g^{-1}})\partial_+ g g^{-1}) \\ &= \mathrm{Tr}(\mathbf{A}(g^{-1}, \partial_- g^{-1})\partial_+ g^{-1} g) = -\mathrm{Tr}(A_- g^{-1}\partial_+ g) = L_A(g, \partial_+ g, \partial_- g)\,. \end{aligned} \tag{B.9}$$

This shows that the antisymmetric tensor contribution to the reduced Lagrangian indeed vanishes in the gauge (5.4).

Appendix C: Matrix superalgebras: definitions and notations

Here we summarize some basic definitions and notation used in sections **6** and **7**.

Let Λ be a Grassmann algebra. The algebra $Mat(n, l; \Lambda)$ is that of $(n+l)\times(n+l)$ matrices over Λ whose diagonal block entries are even elements of Λ while off-diagonal block entries are odd.[52] The super-transposition st is defined as follows:

$$\begin{pmatrix} A & X \\ Y & B \end{pmatrix}^{st} = \begin{pmatrix} A^t & -Y^t \\ X^t & B^t \end{pmatrix}\,, \qquad (MN)^{st} = N^{st}M^{st}\,. \tag{C.1}$$

Note that in general $(M^{st})^{st} \neq M$. More precisely, $(M^{st})^{st} = WMW$ where W is the parity automorphism given by

$$W = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)\,. \tag{C.2}$$

[52] This corresponds to considering even matrices. In general one can also allow for both even and odd ones; this would lead to additional sign factors in the equations below.

A real form of a complex matrix Lie (super)algebra can be described in terms of an antilinear anti-automorphism $*$ satisfying

$$(MN)^* = M^* N^* \,, \qquad (M^*)^* = M \,, \qquad (aM)^* = \bar{a} M^* \,, \;\; a \in \mathbb{C} \,. \tag{C.3}$$

The real subspace of elements satisfying $M^* = -M$ is then a real Lie superalgebra.

We are interested in the case of $n = l$, i.e. $Mat(n|n, \Lambda)$. Suppose first that the corresponding $*$ operation is defined on Λ so that $(a^*)^* = a$ and $(ab)^* = a^* b^* = (-1)^{|a||b|} b^* a^*$ where $|a|$ denotes the Grassmann parity of a. Let us extend $*$ to arbitrary supermatrices according to

$$\left(\begin{array}{cc} A & X \\ Y & B \end{array} \right)^* = \left(\begin{array}{cc} \Sigma^{-1} A^\dagger \Sigma & -i \Sigma^{-1} Y^\dagger \\ -i X^\dagger \Sigma & B^\dagger \end{array} \right) , \tag{C.4}$$

where $\dagger$ applied to the block denotes standard hermitian conjugation, i.e. transposition combined with the $*$-conjugation of entries. It is useful to represent it as

$$M^* = \mathbf{\Sigma}^{-1} M^\dagger \mathbf{\Sigma} \,, \qquad \mathbf{\Sigma} = \left(\begin{array}{cc} \Sigma & 0 \\ 0 & \mathbf{1} \end{array} \right) , \qquad \left(\begin{array}{cc} A & X \\ Y & B \end{array} \right)^\dagger = \left(\begin{array}{cc} A^\dagger & -i Y^\dagger \\ -i X^\dagger & B^\dagger \end{array} \right) . \tag{C.5}$$

It is easy to see that * is involutive provided $\Sigma^2 = \mathbf{1}$ and $\Sigma^\dagger = \Sigma$. Note that $(MN)^\dagger = N^\dagger M^\dagger$ and $(M^\dagger)^\dagger = M$. Note also that $(M^\dagger)^{st} = W (M^{st})^\dagger W$ where W is the parity automorphism introduced above. Let us also note that the $*$ conjugation induces the real form of the respective Lie group. Namely, the condition $g^* = g^{-1}$ selects the real subgroup of the complex group. It is obviously compatible with the conjugation for the Lie algebra due to the representation $g = e^M$ and $M^* = -M$.

To define Z_4 anti-automorphism let us first consider the following automorphism

$$\left(\begin{array}{cc} A & X \\ Y & B \end{array} \right)^\Omega = - \left(\begin{array}{cc} K^{-1} A^t K & -K^{-1} Y^t K \\ K^{-1} X^t K & K^{-1} B^t K \end{array} \right) , \tag{C.6}$$

where K is some matrix required to satisfy $K^2 = \pm \mathbf{1}$ and $K^t = \pm K^{-1}$. It is useful to represent $^\Omega$ as follows

$$M^\Omega = -\mathbf{K}^{-1} M^{st} \mathbf{K} \,, \qquad \mathbf{K} = \left(\begin{array}{cc} K & 0 \\ 0 & K \end{array} \right) , \tag{C.7}$$

so that we have the property

$$(MN)^\Omega = -N^\Omega M^\Omega \,. \tag{C.8}$$

A Lie superalgebra $\mathfrak{f}^{\mathbb{C}}$ admits a Z_4 automorphism if it can be decomposed into a direct sum of eigenspaces of Ω-anti-automorphism

$$\mathfrak{f}^{\mathbb{C}} = \mathfrak{f}^{\mathbb{C}}_0 \oplus \mathfrak{f}^{\mathbb{C}}_1 \oplus \mathfrak{f}^{\mathbb{C}}_2 \oplus \mathfrak{f}^{\mathbb{C}}_3 \,, \tag{C.9}$$

where $\mathfrak{f}^{\mathbb{C}}_l$ denotes the eigenspace with eigenvalue i^l, i.e.

$$M^\Omega = i^m M \,, \quad ([M,N])^\Omega = i^{m+n} [M,N] \,, \qquad M \in \mathfrak{f}^{\mathbb{C}}_m, \quad N \in \mathfrak{f}^{\mathbb{C}}_n \,. \tag{C.10}$$

To see under which conditions Ω is compatible with the reality condition we note that

$$\begin{aligned} -\mathbf{K}^{-1} (\mathbf{\Sigma}^{-1} M^\dagger \mathbf{\Sigma})^{st} \mathbf{K} &= -((\mathbf{K}^{-1} \mathbf{\Sigma}^{-1} M \mathbf{\Sigma} \mathbf{K})^\dagger)^{st} \\ &= -W (\mathbf{\Sigma}^{-1} \mathbf{K}^{-1} M^{st} \mathbf{K} \mathbf{\Sigma})^\dagger W = (-i)^m W \mathbf{\Sigma}^{-1} M^\dagger \mathbf{\Sigma} W \,, \end{aligned} \tag{C.11}$$

where we used

$$\mathbf{K}^{st} = \pm\mathbf{K}^{-1}\,, \qquad \mathbf{\Sigma}^\dagger = \mathbf{\Sigma}^{-1} = \mathbf{\Sigma}\,, \tag{C.12}$$

and also assumed that

$$[\Sigma, K] = 0\,, \qquad \mathbf{K}^\dagger = \pm\mathbf{K}^{-1}\,, \qquad \mathbf{\Sigma}^{st} = \mathbf{\Sigma}\,. \tag{C.13}$$

If in addition the eigenvectors with odd m belong to the off-diagonal blocks (which is the case for $psl(2m|2m)$ superalgebra) one finds

$$(-i)^m W\mathbf{\Sigma^{-1}}M^\dagger\mathbf{\Sigma}W = i^m\mathbf{\Sigma^{-1}}M^\dagger\mathbf{\Sigma}\,, \tag{C.14}$$

so that $(M^*)^\Omega = i^m M^*$ provided $M^\Omega = i^m M$. This proves that Z_4 grading restricts to the real form implying its decomposition (6.1).

The explicit form of Σ and K in the case of $psu(2,2|4)$ is[53]

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{C.15}$$

In the case of $psu(1,1|2)$ we take[54]

$$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{C.16}$$

which satisfy all the conditions above.

Appendix D: κ-symmetry transformations and gauge fixing in section 6

To prove that the gauge condition $Q_{1-} = Q_{2+} = 0$ (6.11) is reachable it is useful to introduce the tangent frame field e^a_α so that the 2d metric is expressed as $g^{ab} = e^a_\alpha e^b_\beta \eta^{\alpha\beta}$ where $\eta^{\alpha\beta}$ is the tangent-space metric. We shall use the standard local frame where in the $\pm$ basis $\eta^{+-} = \eta^{-+} = 1$ and $\eta^{++} = \eta^{--} = 0$. The frame components of the currents are defined in the standard way as $J_\alpha = e^a_\alpha J_a$.

In terms of this parametrization the Lagrangian density for the superstring sigma-model can be written as (cf. (6.3))

$$L_{\rm GS} = \mathrm{STr}\big[P_+P_- + \frac{1}{2}(Q_{1+}Q_{2-} - Q_{1-}Q_{2+})\big]\; e^+ \wedge e^-\,. \tag{D.1}$$

Recall that the $\pm$ components of the currents are defined as $J_\pm = f^{-1}e^a_\pm \partial_a f$. [55] The WZ term can be written also as $Q_1 \wedge Q_2$ and does not of course depend on the frame field. Using e^α_a instead of γ^{ab} introduces a local 2d Lorentz invariance (with the corresponding the new gauge degree of freedom

[53] Here we follow the notation of [57, 58].

[54] This choice is different from the one used in [54].

[55] Note that here we use $\pm$ for the light-cone frame components contrary to the genuine light-cone components in the conformal gauge in the main text. They of course coincide if one chooses the adapted frame and $\sigma^\pm$ coordinates.

entering through e^α_a). The analog of the Virasoro constraints in this formulation are the equations of motion obtained by varying the action with respect to the frame field. Note the following useful relations:

$$\frac{\partial}{\partial e^a_-} L_{\rm GS} = e^+_a {\rm STr}(P_+P_+)\, e^+\wedge e^- \,, \qquad \frac{\partial}{\partial e^a_-} L_{\rm GS} = e^-_a {\rm STr}(P_-P_-)\, e^+\wedge e^- \,, \tag{D.2}$$

where $e^+\wedge e^- = d\sigma^1\wedge d\sigma^2(\det e^a_\alpha)^{-1}$.

The variation of the Lagrangian under the κ-transformation of the currents $\delta_\kappa J_a = \partial_a\epsilon + [J_a,\epsilon]$ with $\epsilon = \epsilon_1+\epsilon_2 = \{P_+, ik_{1-}\} + \{P_-, ik_{2+}\}$ is given by:

$$\begin{aligned}\delta^J_\kappa L_{\rm GS} &= 2\,{\rm STr}\big([P_+,Q_{1-}]\{P_+,ik_{1-}\} + [P_-,Q_{2+}]\{P_-,ik_{2+}\}\big)\; e^+\wedge e^- \\ &= 2{\rm STr}\big(P_+P_+[Q_{1-},ik_{1-}] + P_-P_-[Q_{2+},ik_{2+}]\big)\; e^+\wedge e^- \,. \end{aligned}\tag{D.3}$$

The last expression can be rewritten as

$$\delta^J_\kappa L_{\rm GS} = \frac{1}{2m}\Big({\rm STr}(P_+P_+){\rm STr}(W[Q_{1-},ik_{1-}]) + {\rm STr}(P_-P_-){\rm STr}(W[Q_{2+},ik_{2+}])\Big)\; e^+\wedge e^- \,, \tag{D.4}$$

where m is the integer in the definition of $psu(m,m)|2m)$.

To show thus (e.g. for the first term) it is convenient to use the gauge (6.21) where $P_+ = p_1T^1 + p_2T^2$. The matrices $T^1, T^2 \in \widehat{\mathfrak{f}}_2$ are defined in (6.12),(6.13) for $m=1,2$ (and can be obviously generalized to other m). In this gauge $P_+P_+ = -\frac{1}{4}(p_1^2\mathbf{1}_1 + p_2^2\mathbf{1}_2)$ where $\mathbf{1}_1$ and $\mathbf{1}_2$ are matrices with unit upper-left and lower-right blocks respectively so that one finds

$${\rm STr}\left(P_+P_+[Q_{1-},ik_{1-}]\right) = \frac{1}{4m}{\rm STr}(P_+P_+){\rm STr}(W[Q_{1-},ik_{1-}]) \tag{D.5}$$

where W is the parity automorphism (C.2) and we used that ${\rm STr}([Q_{1-},ik_{1-}]) = 0$ and $p_1^2 - p_2^2 = -\frac{2}{m}{\rm STr}(P_+P_+)$.

The variation $\delta^J_\kappa L_{\rm GS}$ can be compensated by the following variation of the frame field

$$\delta_\kappa e^a_- = -\frac{1}{2m} e^a_+ {\rm STr}(W[Q_{1-},ik_{1-}])\,, \qquad \delta_\kappa e^a_+ = -\frac{1}{2m} e^a_- {\rm STr}(W[Q_{2+},ik_{2+}])\,. \tag{D.6}$$

In particular, for the variation of the metric $g^{ab} = e^a_\alpha e^b_\beta \eta^{\alpha\beta} = e^a_+e^b_- + e^a_-e^b_+$ one finds

$$\delta_\kappa g^{ab} = \frac{1}{m}\left[e^a_+e^b_+{\rm STr}(W[ik_{1-},Q_{1-}]) + e^a_-e^b_-{\rm STr}(W[ik_{2+},Q_{2+}])\right]. \tag{D.7}$$

This can be rewritten in terms of the tangent components as

$$\delta_\kappa g^{ab} = \frac{1}{m\sqrt{-g}}\left[{\rm STr}(W[ik^b_{1(-)},Q^a_{1(-)}]) + {\rm STr}(W[ik^b_{2(+)},Q^a_{2(+)}])\right], \tag{D.8}$$

where we have used that (cf. (6.5)) $V^a_{(\pm)} = \sqrt{-g}e^a_\mp V_\pm = (\det e)^{-1}e^a_\mp V_\pm$. Taking into account the fact that $\delta_\kappa\sqrt{-g} = 0$ one indeed finds that this variation determines the variation of $\gamma^{ab} = \sqrt{-g}g^{ab}$ given in (6.4).

Let us now turn to the question of κ-symmetry gauge fixing in terms of the current components. The κ-variation of the frame components of the current is

$$\delta J_\alpha = (\delta_\kappa e^a_\alpha) J_a + e^a_\alpha(\partial_a \epsilon + [J_a, \epsilon]) = (\delta_\kappa e)^a_\alpha e^\beta_a J_\beta + e^a_\alpha \partial_a \epsilon + [J_\alpha, \epsilon] \,. \tag{D.9}$$

The fermionic equations of motion written in terms of the frame components $\pm$ of the currents take exactly the same form as in the usual "light-cone" coordinates (cf. last line in (6.8))

$$[P_+, Q_{1-}] = 0 \,, \qquad [P_-, Q_{2+}] = 0 \,. \tag{D.10}$$

As we have seen above the same applies to the Virasoro constraints expressed in terms of the frame components:

$$\mathrm{STr}(P_+P_+) = 0 \,, \qquad \mathrm{STr}(P_-P_-) = 0 \,. \tag{D.11}$$

Under the gauge transformation with G-valued gauge parameter the components $P_\pm$ transform as $P_\pm \to g_0^{-1} P_\pm g_0$. Using the Virasoro constraints and applying exactly the same argument as in the discussion of the reduction gauge in terms of the original light-cone components in section 6.2 one can assume that $P_+ = p_+ T$ and $P_- = p_- g^{-1} T g$ where $p_\pm$ are some real functions and g is a G-valued function.

In this gauge the κ-transformation of the component Q_{1-} becomes

$$\delta_\kappa Q_{1-} = (\delta_\kappa e)^a_- e^\alpha_a Q_{1\alpha} + e^a_- \partial_a \epsilon + [\mathcal{A}_-, \epsilon_1] + [P_-, \epsilon_2] + [Q_{1-}, h] \,, \tag{D.12}$$

where $h = h(J, \epsilon_1, \epsilon_2)$ is the $\widehat{\mathfrak{f}}_0$-valued parameter of the compensating gauge transformation needed to maintain the gauge condition $P_+ = p_+ T$. In fact, in this gauge $[P_-, \epsilon_2] = 0$ because $\epsilon_2 = i\{P_-, k_{2+}\}$ and $[T, \{T, M\}] = 0$ vanishes for any matrix M. The term with the κ-symmetry transformation of the frame field is given explicitly by

$$(\delta_\kappa e)^a_- e^\alpha_a Q_{1\alpha} = f^+_- Q_{1+} \,, \qquad f^+_- = \frac{1}{2m} \mathrm{STr}(W[ik_{1-}, Q_{1-}]) \,. \tag{D.13}$$

The transformation (D.12) then takes the form (cf. (6.4))

$$\delta Q_{1-} = e^a_- \partial_a \epsilon_1 + [\mathcal{A}_-, \epsilon_1] + Q_{1+} f^+_- + [Q_{1-}, h] \,. \tag{D.14}$$

Applying the decomposition $\widehat{\mathfrak{f}} = \widehat{\mathfrak{f}}^\perp \oplus \widehat{\mathfrak{f}}^\parallel$ to the κ-symmetry transformation of Q_{1-} in the reduction gauge where $P_+ = p_+ T$ one observes that ϵ_1 takes values in $\widehat{\mathfrak{f}}^\perp_1$ (cf. (6.4)) and at the same time the equation $[P_+, Q_{1-}] = 0$ implies that Q_{1-} is also $\widehat{\mathfrak{f}}^\perp_1$-valued. Because (D.14) is the symmetry of the equation $[P_+, Q_{1-}] = 0$ preserving the structure of P_+, the variation δQ_{1-} also belongs to $\widehat{\mathfrak{f}}^\perp_1$. One then concludes that Q_{1-} can be put to zero by an appropriate choice of $\widehat{\mathfrak{f}}^\perp_1$-valued ϵ_1. This in turn implies that such ϵ_1 can be represented as $i\{P_+, k_{1-}\}$.

Note that once Q_{1-} is set to zero, any transformation with an arbitrary $\epsilon_2 = i\{P_-, k_{1+}\}$ and $\epsilon_1 = i\{P_+, k_{1-}\}$ satisfying $e^a_- \partial_a \epsilon_1 + [\mathcal{A}_-, \epsilon_1] = 0$ preserves $Q_{1-} = 0$ because f^+_- in (D.13) also vanishes when $Q_{1-} = 0$. This statement is invariant under the $\widehat{\mathfrak{f}}_0$-gauge transformations and therefore holds in any $\widehat{\mathfrak{f}}_0$-gauge. Analogous considerations for Q_{2+} in the gauge where $P_- = p_- T$ show that one can also set $Q_{2+} = 0$. Finally, using a local Lorentz transformation and choosing the appropriate coordinates $\sigma^\pm$ one can bring e^α_a to the standard form where the only nonvanishing components are $e^+_+ = e^-_- = 1$. We then arriving at the gauge choice (6.11) for the two components of the fermionic currents.

Appendix E: Details of gauge fixing in section 6.4

In order to show that the reduced model of section **6.2** is indeed described by (6.49) one is to demonstrate that the constraint equations that arise from varying this action with respect to $A_\pm$ represent an admissible gauge condition for the equations of motion (6.35),(6.36). To see this let us introduce the following quantities (cf. (A.1))

$$\widehat{A}_+ = g^{-1}\partial_+ g + g^{-1}A_+ g - \frac{\mu}{2}[[T,\Psi_{_R}],\Psi_{_R}]\,, \tag{E.1}$$

$$\widehat{A}_- = g\partial_- g^{-1} + gA_- g^{-1} - \frac{\mu}{2}[[T,\Psi_{_L}],\Psi_{_L}]\,. \tag{E.2}$$

Under the gauge transformation (6.32), (6.45) they transform as follows

$$\widehat{A}_+ \to \bar{h}^{-1}\widehat{A}_+\bar{h} + \bar{h}^{-1}\partial_+\bar{h}\,, \qquad \widehat{A}_- \to h^{-1}\widehat{A}_+ h + h^{-1}\partial_- h\,. \tag{E.3}$$

Their $\mathfrak{h}$ projections $(\widehat{A}_\pm)_{\mathfrak{h}}$ obviously have the same transformations properties. The variation of the action (6.49) with respect to $A_\pm$ gives

$$A_+ = (\widehat{A}_+)_{\mathfrak{h}}\,, \qquad A_- = (\widehat{A}_-)_{\mathfrak{h}}\,. \tag{E.4}$$

The first equation in (6.35) can be written (upon using the other two equations) as

$$\partial_-\widehat{A}_+ - \partial_+ A_- + [A_-,\widehat{A}_+] + \mu^2[g^{-1}Tg,T] - \frac{\mu}{2}[T,[D_-\Psi_{_R},\Psi_{_R}]] = 0\,, \tag{E.5}$$

or, equivalently, as

$$\partial_+\widehat{A}_- - \partial_- A_+ + [A_+,\widehat{A}_-] + \mu^2[gTg^{-1},T] - \frac{\mu}{2}[T,[D_-\Psi_{_L},\Psi_{_L}]] = 0\,. \tag{E.6}$$

Since $([T,u])_{\mathfrak{h}} = 0$ (note that $[T,u] \in \widehat{\mathfrak{f}}^{\parallel}$ while $\mathfrak{h} = \widehat{\mathfrak{f}}_0^{\perp}$) and projecting this equation on $\mathfrak{h}$ one finds that A_- and $(\widehat{A}_+)_{\mathfrak{h}}$ are the two components of a flat connection. Repeating the argument used in the bosonic case one then concludes that one can set $A_+ = (\widehat{A}_+)_{\mathfrak{h}}$ by an appropriate gauge transformation with $h = \mathbf{1}$. In this gauge A_- and A_+ are then components of a flat connection and can be put to zero by a gauge transformation with $h = \bar{h}$.

In the gauge $A_+ = A_- = 0$ the equation (E.6) implies:

$$\partial_+(\widehat{A}_-)_{\mathfrak{h}} = 0\,, \tag{E.7}$$

where we again made use of the fact that $([T,u])_{\mathfrak{h}} = 0$ for any $u \in \widehat{\mathfrak{f}}_0 \oplus \widehat{\mathfrak{f}}_2$. Then $(\widehat{A}_-)_{\mathfrak{h}}$ is a function of σ^- only and therefore can be set to zero by a gauge transformation with $\bar{h} = \mathbf{1}$ and $h = h(\sigma^-)$. As in the bosonic case such a gauge transformation does not spoil the conditions $A_+ = A_- = (\widehat{A}_+)_{\mathfrak{h}} = 0$.

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