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Spinning superstrings at 2-loops: strong-coupling corrections to dimensions of large-twist SYM operators

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Abstract

We consider folded (S, J) spinning strings in $AdS_5 \times S^5$ (with one spin component in AdS_5 and a one in S^5) corresponding to the $\mathrm{Tr}(D^S \Phi^J)$ operators in the $sl(2)$ sector of the $\mathcal{N} = 4$ SYM theory in the special scaling limit in which both the string mass $\sim \sqrt{\lambda} \ln S$ and J are sent to infinity with their ratio fixed. Expanding in the parameter $\ell = \frac{J}{\sqrt{\lambda} \ln S}$ we compute the 2-loop string sigma model correction to the string energy and show that it agrees with the expression proposed by Alday and Maldacena in arxiv:0708.0672. We suggest that a resummation of the logarithmic $\ell^2 \ln^n \ell$ terms is necessary in order to establish an interpolation to the weakly coupled gauge theory results. In the process, we set up a general framework for the calculation of higher loop corrections to the energy of multi-spin string configurations. In particular, we find that in addition to the direct 2-loop term in the string energy there is a contribution from lower loop order due to a finite "renormalization" of the relation between the parameters of the classical solution and the fixed spins, i.e. the charges of the $SO(2, 4) \times SO(6)$ symmetry.

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1 Introduction

The spinning folded closed string state in AdS_5 for which the difference between the energy E and the spin S scales as $\ln S$ [1] played a remarkable role in the recent progress in the quantitative understanding of the AdS/CFT duality (see, e.g., [2, 3, 4, 5, 6, 7, 8, 9, 11, 10, 12, 13, 14, 15, 16]). With spin J in S^5 added [2], this state can be thought of as being dual to $\mathrm{Tr}(D^S\Phi^J)$ operators in the $sl(2)$ sector of the $\mathcal{N}=4$ SYM theory, interpolating between the near-BMN operators for $J \gg S$ and small-twist operators for small J. The resulting quantum string energy $E(S, J, \sqrt{\lambda})$ (or the gauge theory anomalous dimension $\Delta = E - S - J$) is a non-trivial function of three arguments that can be explored in various limits, uncovering and testing important features of the underlying Bethe ansatz [8].

Our aim here will to compute the 2-loop string correction to this energy in an important J-dependent scaling limit [5, 11], extending earlier 2-loop result found in the $J = 0$ case [13, 16], and to compare it to the prediction made recently in [14] on the basis of a conjectured relation of this scaling limit to the $O(6)$ sigma model.

Let us begin by reviewing what is known about $E(S, J, \sqrt{\lambda})$ in various relevant limits. String semiclassical expansion can be organized as an expansion in the inverse tension or $\frac{1}{\sqrt{\lambda}}$ with the semiclassical parameters $\mathcal{S} = \frac{S}{\sqrt{\lambda}}$, $\mathcal{J} = \frac{J}{\sqrt{\lambda}}$ (or "frequencies") being kept fixed [2, 11]

$$E = \sqrt{\lambda}\,\mathcal{E}(\mathcal{S}, \mathcal{J}, \frac{1}{\sqrt{\lambda}}) = \sqrt{\lambda}\Big[\mathcal{E}_0(\mathcal{S}, \mathcal{J}) + \frac{1}{\sqrt{\lambda}}\mathcal{E}_1(\mathcal{S}, \mathcal{J}) + \frac{1}{(\sqrt{\lambda})^2}\mathcal{E}_2(\mathcal{S}, \mathcal{J}) + ...\Big] \ . \tag{1}$$

The semiclassical (SC) string limit is thus defined by

$$\mathrm{SC}: \qquad \mathcal{S},\ \mathcal{J} = \text{fixed} \ , \qquad \lambda \to \infty \ , \qquad \mathcal{S} \equiv \frac{S}{\sqrt{\lambda}} \ , \quad \mathcal{J} \equiv \frac{J}{\sqrt{\lambda}} \ . \tag{2}$$

The semiclassical string expansion thus explores the energy/dimension in the 3-parameter space (S, J, λ) far away from the origin along the "diagonals" with $S \sim \sqrt{\lambda}$ and $J \sim \sqrt{\lambda}$. This should be kept in mind when comparing to gauge theory, where one usually fixes the (large) values of S and J and studies the dependence on λ for fixed S, J.

Within this semiclassical limit we may consider a special "sub-limit" [5] which we shall call the "semiclassical scaling limit" or SCS (also called the "long string limit" in [11])

$$\mathrm{SCS}: \qquad \mathcal{S} \gg \mathcal{J} \gg 1 \ , \qquad \ell \equiv \frac{\mathcal{J}}{\frac{1}{\pi}\ln \mathcal{S}} = \text{fixed} \ . \tag{3}$$

Since $\ln \mathcal{S} \gg \ln \mathcal{J}$ and $S \gg \sqrt{\lambda}$ we may as well assume that this limit is defined by

$$\mathrm{SCS}: \qquad S \gg J \gg 1 \ , \qquad \ell \approx \frac{J}{\frac{\sqrt{\lambda}}{\pi}\ln S} \equiv \frac{j}{\sqrt{\lambda}} = \text{fixed} \ . \tag{4}$$

Next, we may also consider "sub-limits" in which the fixed parameter $\frac{\mathcal{J}}{\ln \mathcal{S}}$ or $\frac{\mathcal{J}}{\ln S}$ is much smaller (or much larger) than 1 so that one can expand in powers of it (or of its inverse) [11]. Here we shall focus on the first possibility, i.e. on the "semiclassical scaling small" limit or SCSS (called "slow long string limit" in [11])[1]

$$\text{SCSS :} \qquad \ell = \frac{\mathcal{J}}{\frac{1}{\pi}\ln \mathcal{S}} \ll 1 \ . \tag{5}$$

Note that the SCS and SCSS limits are just the special cases of the semiclassical limit (2) limit, so that the expansion (1) still applies, and each term $\mathcal{E}_n$ in (1) can be simplified further by taking these limits.

The (S, J) string solution, given in general in terms of elliptic functions, simplifies dramatically in the scaling limit (3) [11]: it becomes the following *homogeneous* configuration in $AdS_3 \times S^1$

$$\begin{aligned} &ds^2 = -\cosh^2\rho \ dt^2 + d\rho^2 + \sinh^2\rho \ d\theta^2 + d\phi^2 \ , \\ &t = \kappa\tau \ , \qquad \rho = \mu\sigma \ , \qquad \theta = \kappa\tau \ , \qquad \phi = \nu\tau \ , \qquad \kappa, \mu, \nu \gg 1 \ , \end{aligned} \tag{6}$$

where the conformal gauge condition requires that

$$\kappa^2 = \mu^2 + \nu^2 \ . \tag{7}$$

Here $0 \leq \sigma < 2\pi$. ν is related to the S^5 spin by $J = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma \ \partial_0\phi = \sqrt{\lambda}\ \nu$, i.e. $\nu = \mathcal{J}$. For large μ it is related to the AdS_5 spin $S = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma \ \sinh^2\rho \ \partial_0\theta = \sqrt{\lambda}\ \mathcal{S}$ by

$$\mu = \frac{1}{\pi}\ln \mathcal{S} \ , \qquad \mu \gg 1 \ , \qquad \ell = \frac{\nu}{\mu} = \text{fixed} \ . \tag{8}$$

Rescaling σ by $\mu \gg 1$ we get $\rho = \sigma$ and μ plays the role of string length $L = 2\pi\mu = 2\ln \mathcal{S} \gg 1$ that scales out of the classical action and quantum corrections. For $L \to \infty$ the closed folded string becomes effectively a combination of two infinite open strings (see also [26]). [2]

In the scaling limit (3) the classical energy $E_0 = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma \ \cosh^2\rho \ \partial_0 t = \sqrt{\lambda}\ \mathcal{E}_0$ becomes equal to $\mathcal{E} = \mathcal{S} + \kappa = \mathcal{S} + \sqrt{\mu^2 + \nu^2}$. Thus, while the classical energy $\mathcal{E}_0(\mathcal{S}, \mathcal{J})$ in (1) as a function of two general arguments cannot be written in a simple closed form (it is a solution of a system of two parametric equations involving elliptic functions [2, 3]), in the scaling limit it simplifies to [5]

$$\text{SCS :} \qquad \mathcal{E}_0 - \mathcal{S} = \ \mu\sqrt{1 + \ell^2} = \ \frac{1}{\pi}\ln \mathcal{S}\ \sqrt{1 + \frac{\pi^2 \mathcal{J}^2}{\ln^2 \mathcal{S}}} \ . \tag{9}$$

[1]The parameter ℓ is the inverse of the parameter x in [11] and is related to j in [14] by $j = \sqrt{\lambda}\ \ell$.

[2]For $J = 0$ the ends of the string that reach the S^3 boundary may be thought of as point particles following massless geodesics in AdS_5 at $\rho = \infty$ each carrying half of the infinitely large energy and spin $E = S$, while the interior of the string carries the extra energy ("anomalous dimension") given by string mass, i.e. the tension times the string length, $E - S = \frac{\sqrt{\lambda}}{2\pi} L = \frac{\sqrt{\lambda}}{\pi}\ln \mathcal{S}$.

Since in this limit $\mathcal{S} \gg \mathcal{J}$ and $\frac{S}{\sqrt{\lambda}} \gg 1$ it is not possible to distinguish between $\ln \mathcal{S}$, $\ln(\mathcal{S}/\mathcal{J}) = \ln(S/J)$ or $\ln S$. Therefore, the energy in the SCS limit can be also written as[3]

$$\text{SCS}: \qquad E_0 - S = \frac{1}{\pi} \ln S \sqrt{\lambda + \frac{\pi^2 J^2}{\ln^2 S}} + \dots \, . \tag{10}$$

Considering further the SCSS limit (5), i.e. expanding in powers of ℓ we find

$$\text{SCSS}: \qquad E_0 - S = \frac{\sqrt{\lambda}}{\pi} \mathrm{f}_0(\ell) \ \ln S + \dots \, , \tag{11}$$

$$\mathrm{f}_0(\ell) = \sqrt{1+\ell^2} = 1 + \frac{1}{2}\ell^2 - \frac{1}{8}\ell^4 + \dots \, , \tag{12}$$

or, equivalently,

$$\text{SCSS}: \qquad E_0 - S = \frac{\sqrt{\lambda}}{\pi} \ln S \ \big[1 + \frac{\pi^2 J^2}{2\lambda \ln^2 S} - \frac{\pi^4 J^4}{8\lambda^2 \ln^4 S} + \dots\big] + \dots \, . \tag{13}$$

The 1-loop string correction $\mathcal{E}_1$ in (1) was so far computed only in the SCS limit where it takes the form [11]:

$$\text{SCS}: \qquad E_1 = \frac{\sqrt{\lambda}}{\pi} \mathrm{f}_1(\ell) \ \ln S \ + \dots \, , \tag{14}$$

$$\mathrm{f}_1(\ell) = \frac{1}{\sqrt{1+\ell^2}} \Big\{ \sqrt{1+\ell^2} - 1 + 2(1+\ell^2)\ln(1+\ell^2) - \ell^2 \ln \ell^2 - \ 2(1 + \frac{1}{2}\ell^2) \ln \Big[\sqrt{2+\ell^2}(1+\sqrt{1+\ell^2}) \Big] \Big\} \, . \tag{15}$$

Expanding in small ℓ we get from (15)

$$\text{SCSS}: \qquad \mathrm{f}_1(\ell) = -3\ln 2 - 2\ell^2(\ln \ell - \frac{3}{4}) + \ell^4(\ln \ell - \frac{3}{8}\ln 2 - \frac{1}{16}) + O(\ell^6) \, . \tag{16}$$

These explicit string-theory results suggest that the strong-coupling expansion in the scaling limit may be organized, following [14], as

$$E - S = \frac{\sqrt{\lambda}}{\pi} \ \mathrm{f}(\ell, \lambda) \ \ln S + \dots \, , \tag{17}$$

$$\mathrm{f} = \mathrm{f}_0(\ell) + \frac{1}{\sqrt{\lambda}} \mathrm{f}_1(\ell) + \frac{1}{(\sqrt{\lambda})^2} \mathrm{f}_2(\ell) + \dots \, . \tag{18}$$

[3]Note that there is a similarity between this scaling limit of a folded string on $AdS_3 \times S^1$ and the giant magnon limit [17, 18] of a folded string on $R_t \times S^3$. As discussed in [19], one can understand the latter as the infinite spin limit of a folded (J_1, J_2) 2-spin solution on S^3 where one takes E and J_2 to infinity while keeping their difference finite. Then $E - J_2 = \sqrt{J_1^2 + \lambda k}$, where k is a constant which depends on the specifics of the initial solution. By starting with the (S, J) solution one can also consider the limit where E and J are sent to infinity [19]; however, a regular scaling limit appears to be the one of [5] where one sends instead S to infinity while keeping the ratio ℓ in (3) fixed. The energy then takes again the same universal square root form (10) (cf. [19]).

In the SCSS limit $\ell \ll 1$ we may further expand in powers of ℓ and organize the expansion as

$$\begin{aligned} \mathrm{f}(\ell, \lambda) = f(\lambda) + \ell^2 \big[q_0(\lambda) + q_1(\lambda) \ln \ell + q_2(\lambda) \ln^2 \ell + ...\big] \\ + \ell^4 \big[p_0(\lambda) + p_1(\lambda) \ln \ell + ...\big] + ... \ . \end{aligned} \tag{19}$$

Here $f(\lambda)$ corresponds on the gauge theory side to the universal scaling function f (expected [6, 8] to be the same as the cusp anomalous dimension). f and the functions $q_n, p_n, ..$ receive corrections order by order in the inverse tension expansion. Explicitly,

$$f(\lambda) = 1 - \frac{3 \ln 2}{\sqrt{\lambda}} - \frac{K}{(\sqrt{\lambda})^2} + ... \ , \tag{20}$$

$$q_0(\lambda) = \frac{1}{2} + \frac{3}{2\sqrt{\lambda}} + ... \ , \qquad q_1(\lambda) = -\frac{2}{\sqrt{\lambda}} + ... \ , \tag{21}$$

$$p_0(\lambda) = -\frac{1}{8} - \frac{1}{8\sqrt{\lambda}}(3 \ln 2 + \frac{1}{12})... \ , \qquad p_1(\lambda) = \frac{1}{\sqrt{\lambda}} + ... \ . \tag{22}$$

In $f(\lambda)$ we included the known [13, 16] 2-loop string result (K is the Catalan's constant).

The formal expansion in (19) is to be understood in the sense that higher powers of $\ln \ell$ are suppressed compared to lower ones at each given order of the strong coupling expansion in which we *first* take $\lambda \gg 1$ and *then* take ℓ to be small, i.e. in the semiclassical limit in string theory we are considering we should really to use (18) where each of the loop corrections $\mathrm{f}_n(\ell)$ is then expanded in ℓ, i.e. (11) and (16).

The presence of $\ln^n \ell$ terms appears to be an artifact of inverse tension or loop expansion on the string side: to compare to gauge theory one would need to resum the logarithmic terms (see also section 5). On the gauge side the limit is taken in a different way: by first considering $\ln S \gg J$ at small λ and then continuing to large λ. Since $\ell = \frac{\pi J}{\sqrt{\lambda} \ln S} \equiv \frac{j}{\sqrt{\lambda}}$ is naturally small also in *that* limit, the two limits may actually commute.

Let us add that the 1-loop string expression (15) was reproduced [12] from the "string" form of the Bethe ansatz (with the phase in the S matrix expanded at strong coupling [20] with the 1-loop term in it determined [21, 22] from some other 1-loop string results). For work towards the determination of the weak-coupling gauge theory predictions in the scaling limit for non-zero ℓ see [5, 23] and especially [24].

Our aim here will be to compute f_2 in (18) in the SCSS limit, i.e. to understand how the expansion (19) is modified by the *2-loop* string corrections. One important question is if the coefficient of the ℓ^2 term in (19) receives $\ln^2 \ell$ corrections as was suggested in [14] by analogy with the $O(6)$ model origin of the 1-loop $\ell^2 \ln \ell$ term in (16) coming from the light S^5 modes.

The rest of this paper is organized as follows. In section 2 we shall explain the general procedure for computing quantum corrections to the space-time energy of the string with fixed values of the spins. In section 3 we shall discuss the form of the classical solution and the $AdS_5 \times S^5$ action that we will be using for the 2-loop computation of the energy in the scaling limit. Details of the computation of the 1-loop and the 2-loop quantum contributions to the world-sheet effective action will be discussed in section 4.

In section 5 we shall present our final result for the 2-loop term in (18) and show its agreement with the prediction made in [14]. In section 6 we shall suggest that it should be possible to resum all the $\ell^2 \ln^n \ell$ terms so that they should disappear in the weak-coupling gauge theory limit. Appendix A will contain the results for some relevant 2-loop integrals. In Appendix B we will consider a model computation of the 2-loop correction to the effective action of the S^2 sigma model.

2 Structure of computation of quantum corrections to string energy

As explained above, we would like to compute the 2-loop string correction to the energy E of the spinning string as a function of the spins S and J in the scaling limit (3). The 2-loop order in the multi-spin case is the first time when we face an important conceptual subtlety that was not addressed in an earlier work. Similarly to the energy, the spins may receive quantum corrections from their classical values given by the semiclassical "frequency" parameters. In general, to be able to compare to gauge theory we need to find the energy E as a function of the exact values of the spins, i.e. E should be computed with these values fixed. In fact, we should treat the energy and the spins (i.e. all of the $SO(2,4) \times SO(6)$ charges) on an equal footing, relating them at the end via the quantum conformal gauge constraint.

It was argued in [11, 13, 16] that in the scaling limit, i.e. in the limit of large string length, the quantum string correction to the space-time energy (i.e. the global AdS_5 energy) can be computed from the string partition function. We will further justify this below; a key ingredient of the argument is that the volume of the string world sheet is large and factorizes.

In the quantum theory, the symmetry charges are found as expectation values of integrals of the corresponding Nöether currents. In the scaling limit which we are considering here (keeping only the leading power of the effective string length or $\mu = \frac{1}{\pi} \ln \mathcal{S}$) we need to take into account only corrections to the second "small" spin J. This is so because the perturbative corrections to the expectation value of any operator can depend only on the parameters of the classical solution like κ or μ which can grow with S at most as $\ln S$. Consequently, the corrections to the spin S operator (and thus their contributions to the energy) are suppressed by the inverse powers of S.

The main observation is that the space-time energy as well as the spins in AdS_5 or S^5 are conserved charges of the world-sheet theory: they correspond to the generators of the $PSU(2,2|4)$ global symmetry group of the string sigma model. Thus, they should be treated on an equal footing in a conformal-gauge world sheet calculation. The global conserved charges are related by the Virasoro constraint, i.e. by the requirement that the 2d world-sheet energy vanishes.

Let us comment on the implementation of the Virasoro constraint in a semiclassical expansion. Since one is to expand around a solution of the classical equations of motion, one should impose the classical Virasoro constraint. Both the charges and the world sheet energy receive quantum corrections order by order in sigma model perturbation theory. They are naturally expressed in terms of the parameters of the classical solution. The quantum Virasoro constraint requires the vanishing of the world sheet energy. This constraint therefore imposes an additional relation between the parameters of the classical solution which can be satisfied only if the quantum corrections to the global symmetry charges are correlated in a certain way. It is this relation that we are going to expose and exploit below.

The implementation of the fixed-charge constraint is a well-known problem in statistical mechanics leading to the notion of generalized ensemble. Below we shall first review some relevant general points and then turn to our specific string theory sigma model case.

2.1 Partition function with fixed charges

In general, one thinks of the partition function as a sum over states. It is typically hard to sum only over the states of some definite fixed charge – especially when one does not know all the states of the theory. This difficulty led to the concept of generalized statistical ensemble. The main idea is that, instead of using charges to label states, one interprets the charge as an average quantity which can then be set to any desired value.[4]

Let us review the definition and construction of the corresponding partition function. We consider a system whose states have some energy E and carry some charge Q. By definition, the partition function is simply the normalization factor of the probability that the system has energy E and charge Q. To find this probability we start with the fact that the number of states for the system in contact with a reservoir of energy and charge at fixed *total* energy $E_{\rm T}$ and *total* charge $Q_{\rm T}$ is

$$\begin{aligned}\Omega_{\rm T}(E_{\rm T}, Q_{\rm T}, \dots) &= \int \frac{dE}{\Delta} \sum_Q \Omega(E,Q)\ \Omega_R(E_{\rm T}-E, Q_{\rm T}-Q) \\ &= \int \frac{dE}{\Delta} \sum_Q\ \Omega(E,Q)\ \exp\left[S_R(E_{\rm T}-E, Q_{\rm T}-Q)\right]\ . \end{aligned} \tag{23}$$

Here Ω denotes the number of states of the system, Ω_R the number of states of the reservoir, S_R is its entropy and Δ is a coarse graining parameter which is arbitrary, apart from the fact that it should not scale with the volume of the system. It follows then that the probability for the system to have the energy E and charge Q is

$$P(E,Q) \propto \Omega(E,Q)\ \exp\left[S_R(E_{\rm T}-E, Q_{\rm T}-Q)\right]\ . \tag{24}$$

We may then expand the entropy of the reservoir $S_R(E_{\rm T}-E, Q_{\rm T}-Q)$, using the fact that its energy and charge (and therefore the total energy $E_{\rm T}$ and total charge $Q_{\rm T}$) are by definition much larger than the energy and charge of the system:

$$\begin{aligned}S_R(E_{\rm T}-E, Q_{\rm T}-Q) &= S_R(E_{\rm T}, Q_{\rm T}) + \frac{\partial S_R(E_{\rm T}, Q_{\rm T})}{\partial E_{\rm T}}(-E) + \frac{\partial S_R(E_{\rm T}, Q_{\rm T})}{\partial Q_{\rm T}}(-Q) + \dots \\ &\approx S_R(E_{\rm T}, Q_{\rm T}) - \beta E + \beta h Q\ , \end{aligned} \tag{25}$$

where β is the inverse temperature and h is the variable canonically conjugate to the charge (e.g., if Q is an electric charge then h is an electric potential). We therefore find that the probability

[4]Clearly, this standpoint is particularly appropriate if one expects that the symmetry generators receive quantum corrections.

for the system to have energy E and charge Q is given (up to overall factors independent of E and Q) by

$$P(E,Q) \;\propto\; \Omega(E,Q)\; e^{-\beta(E-h\,Q)} \quad . \tag{26}$$

Assuming that E and Q are the eigenvalues of the *commuting* operators $\widehat{H}$ and $\widehat{Q}$, the partition function is

$$Z(h) = \mathrm{Tr}[e^{-\beta(\widehat{H}-h\,\widehat{Q})}] \;, \tag{27}$$

where the trace is taken over all states.[5]

This implies, on general grounds, that the logarithm of the partition function is related to $\langle\widehat{H}\rangle$ and $\langle\widehat{Q}\rangle$ as

$$-\frac{1}{\beta}\ln Z(h) = \langle\widehat{H}\rangle - h\langle\widehat{Q}\rangle \equiv \Sigma(h) \;, \tag{28}$$

which defines the generalized thermodynamic potential $\Sigma(h)$ (which may also be loosely called free energy).

Then, the average value of the energy $\langle\widehat{H}\rangle$ over the states with fixed charge Q is the Legendre transform of Σ with respect to h. Namely, we are first to compute the partition function with the modified Hamiltonian

$$\widehat{\widetilde{H}} = \widehat{H} - h\,\widehat{Q} \;, \tag{29}$$

from which we are to find the value of the charge averaged over all states

$$\langle\widehat{Q}\rangle = \frac{1}{\beta}\frac{\partial \ln Z(h)}{\partial h} \quad . \tag{30}$$

Setting this to the desired value

$$\langle\widehat{Q}\rangle = Q \;, \tag{31}$$

we find $h = h(Q)$. Then the energy averaged over the states with fixed charge is found by evaluating

$$\langle\widehat{H}\rangle = -\frac{1}{\beta}\ln Z(h(Q)) + h(Q)\,Q \;. \tag{32}$$

In a semiclassical expansion $\langle\widehat{H}\rangle$ comes out as a series of quantum corrections to the classical energy.

This construction is completely general under the assumption that the volume of the system is large (infinite) and that the interactions are local.[6] It can of course be generalized to include two or more different charges.

[5]The presence of β in front of both $\widehat{H}$ and $h\,\widehat{Q}$ implies that $\widehat{H}-h\,\widehat{Q}$ is the evolution operator in this ensemble. We suppress the obvious argument β in Z and related quantities.

[6]Corrections to the equation (28) are suppressed by inverse powers of the volume. Moreover, the root-mean-squared fluctuations of the energy and the charge around their average values are also suppressed by inverse powers of the spatial volume. Finally, one may also argue that under these conditions $\langle\widehat{H}\rangle$ and $\langle\widehat{Q}\rangle$ are the most probable values of the energy and the charge.

In the following we will identify the "undeformed" Hamiltonian $\widehat{H}$ with the world-sheet Hamiltonian, one charge with the difference between the space-time energy E and the AdS_5 spin S and a second charge with the angular momentum J on S^5. The reason for using $E - S$ as a charge instead of introducing separately E and S is that in the present case this difference is parametrically smaller that either E or S.

2.2 Partition function of a world-sheet theory with fixed charges

The discussion of section 2.1 can be easily translated to field-theory language and applied to the semiclassical expansion of a two-dimensional string sigma model.

As explained above, we are to consider the world-sheet Hamiltonian modified by the addition of the charge operators. Anticipating the relation to the equation (6), we shall denote by $-\kappa$ the "chemical potential" h conjugate to the difference between the space-time energy E and the AdS_5 spin S and by ν – the "chemical potential" conjugate to the S^5 spin J: [7]

$$\widetilde{H}_{\rm 2d} = H_{\rm 2d} + \ \kappa\,(E - S) - \nu\, J \ . \tag{33}$$

The partition function with this modified Hamiltonian may be computed by transforming it first in a standard way into a path integral with euclidean action.[8]

To this end we note that E, S and J are momenta conjugate to world sheet fields – the global time direction t, an isometric angle θ in AdS_5 and an isometric angle ϕ on S^5 (see (6)). It is then easy to see that the Lagrangian associated to the modified Hamiltonian $\widetilde{H}_{\rm 2d}$ is obtained from the one associated to $H_{\rm 2d}$ simply by shifting the time derivatives of the relevant fields by constants κ and ν:

$$\dot{t} \mapsto \dot{t} + \kappa \ , \qquad \dot{\theta} \mapsto \dot{\theta} + \kappa \ , \qquad \dot{\phi} \mapsto \dot{\phi} + \nu \ . \tag{34}$$

An expansion around any classical solution of this modified Lagrangian is then found by including an additional background term in each of the fields t, θ and ϕ

$$t = \kappa\tau + \tilde{t}(\sigma,\tau) \ , \qquad \theta = \kappa\tau + \tilde{\theta}(\sigma,\tau) \ , \qquad \phi = \nu\tau + \tilde{\phi}(\sigma,\tau) \ . \tag{35}$$

One may want to include also the profiles of other fields (such as ρ in (6)).

Since we want to interpret the resulting Lagrangian as that of a world-sheet theory, we need to identify the corresponding Virasoro constraints (we shall use the conformal gauge). To this end we note that on a curved world sheet the momentum conjugate to a field contains a factor of the inverse metric. This implies that the replacement (34) is sufficient also in the presence of a nontrivial world sheet metric and thus the Virasoro constraint for the modified Lagrangian

[7] The sign difference between $h_{E-S} = -\kappa$ and $h_J = \nu$ has to do with the fact that κ is in a sense "time-like" – much like the entropy: the variable conjugate to the entropy is β while the one conjugate to the charges is $-\beta h$.

[8] For a discussion of a formally similar definition of free energy in two-dimensional sigma models see [27, 28, 29, 30]. The present string case will, however, be conceptually different in that we will have the Virasoro constraint.

follows from that for the original one by the same replacement (34). The Virasoro condition then relates κ and ν (cf. (7))

$$\kappa = \kappa(\nu) \ . \tag{36}$$

Consequently, the partition function is a function of only ν; therefore, it is impossible to "measure" separately $\langle E-S\rangle$ and $\langle J\rangle$. Instead of $\Sigma(\kappa,\nu)$ in (28) we get $\Sigma(\nu) \equiv \Sigma(\kappa(\nu),\nu)$ and its derivative is a combination of $\langle E-S\rangle$ and $\langle J\rangle$

$$\begin{aligned} \frac{d\Sigma(\nu)}{d\nu} &= \frac{\partial\Sigma(\kappa,\nu)}{\partial\kappa}\Big|_{\kappa=\kappa(\nu)} \frac{d\kappa(\nu)}{d\nu} + \frac{\partial\Sigma(\kappa,\nu)}{\partial\nu}\Big|_{\kappa=\kappa(\nu)} \\ &= \frac{d\kappa(\nu)}{d\nu}\langle E-S\rangle - \langle J\rangle \ . \end{aligned} \tag{37}$$

A second relation between these quantities is found by recalling that the expression for the generalized potential in terms of the average values of charges is

$$\Sigma(\nu) = \langle H_{\rm 2d}\rangle + \kappa(\nu)\langle E-S\rangle - \nu\langle J\rangle \ , \tag{38}$$

and imposing the quantum Virasoro constraint

$$\langle H_{\rm 2d}\rangle = 0 \ . \tag{39}$$

$\Sigma(\nu)$ is proportional to the world-sheet effective action $\Gamma(\nu)$

$$\Gamma(\nu) \equiv -\ln Z(\nu) = \beta\Sigma(\nu) \ . \tag{40}$$

Here we will be interested in the zero temperature limit when $\beta \equiv \mathcal{T} \to \infty$ plays the role of the length of the world-sheet time interval.

Combining (37) and (38) and setting the average values of the charges to the desired values,

$$\langle E-S\rangle = E-S \ , \qquad \langle J\rangle = J \ , \tag{41}$$

we find that

$$E-S = \frac{1}{\mathcal{T}}\left(\nu\frac{d\kappa(\nu)}{d\nu} - \kappa(\nu)\right)^{-1}\left(\Gamma(\nu) - \nu\frac{d\Gamma(\nu)}{d\nu}\right) \ , \tag{42}$$

$$J = \frac{1}{\mathcal{T}}\left(\nu\frac{d\kappa(\nu)}{d\nu} - \kappa(\nu)\right)^{-1}\left(\Gamma(\nu)\frac{d\kappa(\nu)}{d\nu} - \kappa(\nu)\frac{d\Gamma(\nu)}{d\nu}\right) \ . \tag{43}$$

Before proceeding let us point out that the discussion in this section has a trivial generalization to the multi-spin cases where instead of one parameter ν we have several of them; then the Virasoro condition implies that

$$\kappa = \kappa(\nu_1,\ldots,\nu_n) \ , \tag{44}$$

and the equation (37) generalizes to a system of equations:

$$\frac{d\Sigma(\nu_1,\ldots,\nu_n)}{d\nu_i} = \frac{\partial\kappa(\nu_1,\ldots,\nu_n)}{\partial\nu_i}\langle E-S\rangle - \langle J_i\rangle \ , \qquad i=1,\ldots,n \ . \tag{45}$$

Eq. (28) combined with the quantum Virasoro constraint becomes

$$\Sigma(\nu_1, \ldots, \nu_n) = \kappa(\nu_1, \ldots \nu_n)\langle E - S\rangle - \sum_{i=1}^{n} \nu_i \langle J_i \rangle \ . \tag{46}$$

For independent charges J_i, this system of $(n+1)$ equations for the same number of unknowns is nondegenerate.

2.3 Sigma model loop expansion

Let us now specify the discussion of the previous section to our particular solution (6) with $\rho = \mu\ \sigma,\ \ \mu \to \infty$. To make manifest the fact that it applies without modification we will rescale the world sheet coordinates τ and σ by μ. The homogeneity of the solution, reflected in the fact that the coefficients in the fluctuation Lagrangian are constant [11, 16], guarantees that μ can be rescaled out in the classical action with the parameters κ and ν replaced by

$$\hat{\kappa} = \frac{\kappa}{\mu} \ , \qquad\qquad \hat{\ell} = \frac{\nu}{\mu} \ . \tag{47}$$

Then the quantum effective action is proportional to the world-sheet volume

$$\Gamma = \frac{\sqrt{\lambda}}{2\pi}\ V_2\ \mathcal{F}(\hat{\ell}\,) \ , \tag{48}$$

$$V_2(\mu) = 2\pi\mu\ \mathcal{T} \ , \qquad \mathcal{T} = \mu\,\bar{\mathcal{T}} \ , \qquad \mu = \frac{1}{\pi}\ln\mathcal{S} \approx \frac{1}{\pi}\ln S \gg 1 \ . \tag{49}$$

$\mathcal{F}(\hat{\ell}\,)$ has a standard expansion in inverse powers of $\sqrt{\lambda}$ with a constant leading term.

Using (7) after the the rescaling of world sheet coordinates by μ, i.e.

$$\hat{\kappa}(\hat{\ell}\,) = \sqrt{1 + \hat{\ell}^{\,2}} \tag{50}$$

and evaluating (42) and (43) it is easy to find $E - S$ and J in terms of $\mathcal{F}(\hat{\ell}\,)$ and its first derivative:

$$E - S = \mathcal{M}\sqrt{1+\hat{\ell}^{\,2}}\left[\mathcal{F}(\hat{\ell}\,) - \hat{\ell}\,\frac{d\mathcal{F}(\hat{\ell}\,)}{d\hat{\ell}}\right] , \tag{51}$$

$$J = \mathcal{M}\left[\hat{\ell}\,\mathcal{F}(\hat{\ell}\,) - (1+\hat{\ell}^{\,2})\frac{d\mathcal{F}(\hat{\ell}\,)}{d\hat{\ell}}\right] , \tag{52}$$

where $\mathcal{M} = \frac{\sqrt{\lambda}}{2\pi}L$ is the "string mass" (tension times length),[9]

$$\mathcal{M} \equiv \frac{\sqrt{\lambda}}{2\pi}\frac{V_2}{\mathcal{T}} = \frac{\sqrt{\lambda}}{\pi}\ln S \ . \tag{53}$$

[9]Recall that, as discussed in section 1, the scaling limit $\sqrt{\lambda}\ln S \gg J,\ \ S \gg \sqrt{\lambda},\ \lambda \gg 1$, makes $\ln\mathcal{S}$, $\ln S$ and $\ln S/J$ indistinguishable.

Expanding in the inverse string tension we get

$$\mathcal{F}(\hat{\ell}\,) = \mathcal{F}_0 + \sum_{n=1}^{\infty} \frac{1}{(\sqrt{\lambda})^n} \mathcal{F}_n(\hat{\ell}\,) \ . \tag{54}$$

As we shall see below,

$$\mathcal{F}_0 = 1 \ . \tag{55}$$

Introducing the notation for the analog of ℓ in (3) or (4) (and using (55))

$$\ell = \frac{J}{\mathcal{F}_0 \ \mathcal{M}} = \frac{\pi J}{\sqrt{\lambda}\, \ln S} \ , \tag{56}$$

we find that the first few orders in the inverse tension expansion of the energy have the same structure as in (17):

$$E - S \ = \ \frac{\sqrt{\lambda}}{\pi} \ln S \ \left[\mathrm{f}_0 + \frac{1}{\sqrt{\lambda}} \mathrm{f}_1 + \frac{1}{(\sqrt{\lambda})^2} \mathrm{f}_2 + \ldots \right] , \tag{57}$$

$$\mathrm{f}_0 \ = \ \mathcal{F}_0 \sqrt{1 + \ell^2} \ , \tag{58}$$

$$\mathrm{f}_1 \ = \ \frac{\mathcal{F}_1(\ell)}{\sqrt{1+\ell^2}} \ , \tag{59}$$

$$\mathrm{f}_2 \ = \ \frac{1}{\sqrt{1+\ell^2}} \left[\mathcal{F}_2(\ell) + \frac{1}{2\mathcal{F}_0} \left(\frac{\ell}{\sqrt{1+\ell^2}}\, \mathcal{F}_1(\ell) - \sqrt{1+\ell^2}\, \frac{d\mathcal{F}_1(\ell)}{d\ell} \right)^2 \right] \ . \tag{60}$$

It is also straightforward to find the expression for $\hat{\ell}$ in terms of ℓ:

$$\hat{\ell} \ = \ \hat{\ell}^{\,(0)} + \frac{1}{\sqrt{\lambda}} \hat{\ell}^{\,(1)} + \frac{1}{(\sqrt{\lambda})^2} \hat{\ell}^{\,(2)} + \ldots \ , \tag{61}$$

$$\hat{\ell}^{\,(0)} \ = \ \ell \ , \tag{62}$$

$$\hat{\ell}^{\,(1)} \ = \ \frac{1}{\mathcal{F}_0} \left[(1+\ell^2) \frac{d\mathcal{F}_1(\ell)}{d\ell} - \ell\, \mathcal{F}_1(\ell) \right] \ , \tag{63}$$

$$\hat{\ell}^{\,(2)} \ = \ \frac{1}{\mathcal{F}_0} \left[(1+\ell^2) \frac{d\mathcal{F}_2(\ell)}{d\ell} - \ell\, \mathcal{F}_2(\ell) \right] + \hat{\ell}^{\,(1)}\ \frac{d\hat{\ell}^{\,(1)}}{d\ell} \ . \tag{64}$$

The fact that to the leading order we have $\hat{\ell} = \ell$ implies that the SCSS expansion (13), (19) is equivalent to the small $\hat{\ell}$ expansion of the effective action.

We are thus to compute the effective action $\Gamma(\hat{\ell}\,)$ or, equivalently, the function $\mathcal{F}(\hat{\ell}\,)$ and then expand it in small $\hat{\ell}$.

3 Classical solution and fluctuation action

The calculation of the two-loop world sheet partition function in the presence of the *"chemical potentials"* $\hat{\kappa}$ and $\hat{\ell}$ is most easily done using the T-dual version of the $AdS_5 \times S^5$ string action in the Poincaré patch of AdS_5.

Let us first discuss this map for the solution (6), following [16] and [26], and then proceed to construct the constant-coefficient action for the fluctuations around this solution.

3.1 Solution and $AdS_5 \times S^5$ superstring action in Poincaré coordinates

To transform the Minkowski-signature folded closed string solution in global coordinates into the Poincaré-patch Euclidean-signature solution let us use the embedding coordinates and perform a discrete $SO(2,4)$ transformation as well as an analytic continuation.

In the embedding coordinates, the AdS_5 part of the solution (6) is:

$$\begin{aligned} X_0 &= \cosh\mu\sigma\ \cos\kappa\tau \ , \qquad & X_5 &= \cosh\mu\sigma\ \sin\kappa\tau \ , \\ X_1 &= \sinh\mu\sigma\ \cos\kappa\tau \ , \qquad & X_2 &= \sinh\mu\sigma\ \sin\kappa\tau \ . \end{aligned} \tag{65}$$

Analytically continuing to the Euclidean world sheet time, $\tau = -i\tau'$, as well as interchanging a space-like and a time-like target space coordinates, $X_2 = iX'_5,\ X_5 = iX'_2$ (which is a discrete $SO(2,4)$ transformation) leads to

$$\begin{aligned} X_0 &= \cosh\mu\sigma\ \cosh\kappa\tau' \ , \qquad & X'_2 &= \cosh\mu\sigma\ \sinh\kappa\tau' \ , \\ X_1 &= \sinh\mu\sigma\ \cosh\kappa\tau' \ , \qquad & X'_5 &= \sinh\mu\sigma\ \sinh\kappa\tau' \ . \end{aligned} \tag{66}$$

Further discrete $SO(2,4)$ transformations in the planes $(0,5)$ and $(1,2)$ put this this solution into the form

$$\begin{aligned} X_0 &= \frac{1}{\sqrt{2}}\cosh(\mu\sigma+\kappa\tau') \ , \qquad & X_5 &= \frac{1}{\sqrt{2}}\cosh(\mu\sigma-\kappa\tau') \ , \\ X_1 &= \frac{1}{\sqrt{2}}\sinh(\mu\sigma+\kappa\tau') \ , \qquad & X_2 &= \frac{1}{\sqrt{2}}\sinh(\mu\sigma-\kappa\tau') \ . \end{aligned} \tag{67}$$

Finally, interpreting this as a background in the Poincaré patch with the metric

$$ds^2 = z^{-2}(dz^2 + dx^m dx_m)$$

yields

$$z = \sqrt{2}\, e^{-\kappa\tau'+\mu\sigma} \ , \qquad x^0 = \frac{z}{\sqrt{2}}\cosh(\kappa\tau'+\mu\sigma) \ , \qquad x^1 = \frac{z}{\sqrt{2}}\sinh(\kappa\tau'+\mu\sigma) \ . \tag{68}$$

The S^5 coordinates are affected only by the analytic continuation to the Euclidean time: the resulting profile for the S^5 field ϕ is

$$\phi = i\nu\tau' \ . \tag{69}$$

In the following we will omit the prime, denoting the Euclidean world sheet time coordinate simply by τ.

Throughout the above chain of transformations we have carefully kept track of the parameters κ and ν. While from the standpoint of the resulting solution their interpretation as chemical potentials for certain global charges is obscured, the fact that to find (68) we used only symmetries of the string action guarantees that their meaning is unchanged.

Eqs. (68), (69) define the solution we will be using from now on. To get a simple *quadratic* form of the resulting fermionic part of the fluctuation action we can view it, following [16], as corresponding to the T-dual form of the $AdS_5 \times S^5$ superstring action [25]:[10]

$$I = I_B + I_F \ , \qquad I_B = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \ \mathcal{L}_B = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \ \frac{1}{z^2}\big(dx^m dx_m + dz^s dz_s\big) \ , \tag{70}$$

$$I_F = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \ \mathcal{L}_F = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \ 4 \ \epsilon^{ab} \ \bar{\theta} \left(\partial_a x^m \Gamma_m \ + \ \partial_a z^s \Gamma_s\right) \partial_b \theta \ . \tag{71}$$

The coordinates z^s are flat coordinates on $\mathbb{R}^6$; the coset parametrization used in [25] relates them to a particular choice of coordinates on S^5 as ($i = 4, 5, 6, 8, 9$)

$$z^i \equiv \ z \ \hat{z}^i = \ z \ \frac{\hat{y}^i}{1 + \frac{1}{4}\hat{y}^2} \ , \qquad\qquad z^7 \equiv z \ \hat{z}^7 = z \ \frac{1 - \frac{1}{4}\hat{y}^2}{1 + \frac{1}{4}\hat{y}^2} \ , \tag{72}$$

in terms of which the metric takes the form

$$ds^2_{S_5} = \frac{d\hat{y}_i d\hat{y}_i}{\left(1 + \frac{1}{4}\hat{y}^2\right)^2} \ , \qquad\qquad \hat{y}^2 = (\hat{y}^4)^2 + (\hat{y}^5)^2 + (\hat{y}^6)^2 + (\hat{y}^8)^2 + (\hat{y}^9)^2 \ . \tag{73}$$

As was mentioned in [16], it is useful to introduce a new coordinate system in the Poincare patch of AdS_5

$$ds^2_{AdS_5} = dr^2 + e^{-2r} dx^m dx_m = dr^2 + (dy^m + y^m dr)(dy_m + y_m dr) \ , \tag{74}$$

where $z = e^r$ and $y^m = \frac{x^m}{z} = e^{-r} x^m$. Here $m = 0, 1, 2, 3$ and the boundary signature is $(-, +, +, +)$. Furthermore, one may define

$$y^+ = y^0 + y^1 = v \ e^w \ , \qquad y^- = y^0 - y^1 = v \ e^{-w} \ , \qquad y^k = (y^2, y^3) \ . \tag{75}$$

The choice of signs here is adapted to the special solution we are going to consider.[11] Then the AdS_5 metric (74) takes the form ($k = 2, 3$)

$$ds^2_{AdS_5} = dr^2 - (dv + v dr)^2 + v^2 dw^2 + \sum_{k=2}^{3} (dy_k + y_k dr)^2 \ . \tag{76}$$

We see that shifts of r, w are linear isometries, and that v is an apparent time-like coordinate.

---

[10] The use of this action, T-dual to the $AdS_5 \times S^5$ superstring action in a particular κ-symmetry gauge, here is a technical trick to simplify the computation of the fermionic contributions (see also [16]). The T-duality which is a formal 2d duality transformation at the level of path integral is legitimate to use in the computation of quantum corrections near a classical solution. This is true provided the solution is "covariant" under this transformation which is the case here in the scaling limit $\mu \gg 1$ when μ and κ (and τ and σ) enter on an equal footing (see also [16]).

[11] If v, w are real we are selecting the region where $y^m \geq 0$; other choices are related by analytic continuation.

To identify the isometry direction ϕ on S^5 and thus to account for the presence of the chemical potential ν we perform the following coordinate transformation ($i = 4, 5, 6, 7$)

$$\hat{z}^i = \frac{y^i}{1 + \frac{1}{4}y^2} \ , \qquad y^2 = \sum_{i=4}^{7} (y^i)^2 \ ,$$
$$\hat{z}^8 = \frac{1 - \frac{1}{4}y^2}{1 + \frac{1}{4}y^2} \cos\phi \equiv Y \cos\phi \ , \qquad \hat{z}^9 = \frac{1 - \frac{1}{4}y^2}{1 + \frac{1}{4}y^2} \sin\phi \equiv Y \sin\phi \ . \tag{77}$$

Then the S^5 metric takes the form

$$ds^2_{S^5} = \left(\frac{1 - \frac{1}{4}y^2}{1 + \frac{1}{4}y^2}\right)^2 d\phi^2 + \frac{1}{\left(1 + \frac{1}{4}y^2\right)^2} \sum_{i=4}^{7} (dy^i)^2 \ . \tag{78}$$

These coordinates are well-suited for following the computational strategy described in sections 2.1 and 2.2.

3.2 Classical value of the action

Transforming the spinning folded string solution (68) to the above coordinates and taking into account the rescaling of the world sheet coordinates by μ leads to the following form of the solution

$$\bar{r} = n_1 \cdot \boldsymbol{\sigma} + \ln\sqrt{2} \ , \qquad \bar{w} = n_2 \cdot \boldsymbol{\sigma} \ , \qquad \bar{v} = \frac{1}{\sqrt{2}} \ , \qquad \bar{\phi} = i\,\mathrm{m} \cdot \boldsymbol{\sigma} \ , \tag{79}$$

$$n_1 = (-\hat{\kappa}, 1) \ , \qquad n_2 = (\hat{\kappa}, 1) \ , \qquad \mathrm{m} = (\hat{\ell}\,, 0) \ , \qquad n_1 \cdot n_2 = -\mathrm{m} \cdot \mathrm{m} = -\hat{\ell}^{\,2} \ , \tag{80}$$

where $\boldsymbol{\sigma} = (\tau, \sigma)$ and the relation between n_1, n_2 and $\hat{\ell}$ is implied by the Virasoro constraints.

The classical string action evaluated on this solution is

$$\begin{aligned} \bar{I} &= \frac{\sqrt{\lambda}}{4\pi} \Big[n_1 \cdot n_1 - \frac{1}{2}(n_1 + n_2) \cdot (n_1 - n_2) - \mathrm{m}^2) \Big] \ V_2 \\ &= \frac{\sqrt{\lambda}}{4\pi} (n_1 \cdot n_1 - \mathrm{m}^2) V_2 = \frac{\sqrt{\lambda}}{2\pi} \ V_2 \ . \end{aligned} \tag{81}$$

Thus, as already mentioned in (55), we find that the tree-level term in the effective action (48) is

$$\mathcal{F}_0 = 1 \ . \tag{82}$$

3.3 Fluctuation Lagrangian

Since the solution (80) has linear coordinate dependence along some isometry directions of the metric (76), the straightforward expansion $\Phi = \bar{\Phi} + \tilde{\Phi}$ (Φ is a generic bosonic field with background value $\bar{\Phi}$ and $\tilde{\Phi}$ is its fluctuation) of the bosonic Lagrangian around this solution leads immediately to a fluctuation action with constant coefficients.

The propagator of the AdS_5 modes is found to be

$$
\begin{aligned}
K^{-1}_{B,AdS_5}(p) &= \begin{pmatrix}
\frac{1}{2p^2}+\frac{(n_2\cdot p)^2-(n_1\cdot p)^2}{p^2\mathcal{D}_B[p]} & \frac{in_2\cdot p}{\mathcal{D}_B[p]}+\frac{2n_1\cdot p\; n_2\cdot p}{p^2\mathcal{D}_B[p]} & \frac{in_2\cdot p}{\sqrt{2}\mathcal{D}_B[p]}-\frac{p^2}{2\sqrt{2}\mathcal{D}_B[p]} & 0_{1\times 2}\\
-\frac{in_2\cdot p}{\mathcal{D}_B[p]}+\frac{2n_1\cdot p\; n_2\cdot p}{p^2\mathcal{D}_B[p]} & \frac{1}{p^2}-\frac{2(n_2\cdot p)^2}{p^2\mathcal{D}_B[p]} & -\frac{in_2\cdot p}{\sqrt{2}\mathcal{D}_B[p]} & 0_{1\times 2}\\
-\frac{in_2\cdot p}{\sqrt{2}\mathcal{D}_B[p]}-\frac{p^2}{2\sqrt{2}\mathcal{D}_B[p]} & \frac{in_2\cdot p}{\sqrt{2}\mathcal{D}_B[p]} & -\frac{p^2}{4\mathcal{D}_B[p]} & 0_{1\times 2}\\
0_{2\times 1} & 0_{2\times 1} & 0_{2\times 1} & \frac{1_{2\times 2}}{2(p^2+n_1\cdot n_1)}
\end{pmatrix}\\
\mathcal{D}_B[p] &= (p^2)^2+2[(n_1\cdot p)^2+(n_2\cdot p)^2] \ . \qquad (83)
\end{aligned}
$$

One can show that K_B^{-1} describes four massive and one massless propagating modes. The massless mode and a similar massless mode in S^5 direction mentioned below reflect a residual conformal symmetry in the conformal gauge and decouple from the non-trivial part of the computation (their constant contribution to the 1-loop partition function cancels against that of conformal gauge ghosts).

The denominator $\mathcal{D}_B$ of this bosonic propagator is not 2d Lorentz invariant, but becomes invariant in the limit $\hat{\ell} \to 0$,

$$
(\mathcal{D}_B[p])_{\hat{\ell}\to 0} \quad \mapsto \quad p^2(p^2+4) \ . \qquad (84)
$$

There is thus a massless mode which is lifted at nonzero $\hat{\ell}$; its mass arises from an imbalance between the space-like and time-like components of the momentum, suggesting that in the quantum theory this mode will not lead to the logarithmic thresholds $\sim \ln\hat{\ell}$.

The propagator of the S^5 modes is substantially simpler,

$$
K^{-1}_{B,S^5}(p) = \begin{pmatrix} \frac{1}{2p^2} & 0_{1\times 4}\\ 0_{4\times 1} & \frac{1_{4\times 4}}{2(p^2+\hat{\ell}^{\,2})}\end{pmatrix} \ . \qquad (85)
$$

It describes one massless mode and four massive modes. The latter become massless in the $\hat{\ell} \to 0$ limit, so that one should expect that the effective action should contain $\hat{\ell}$-dependent threshold terms $\hat{\ell}^{\,n}\ln^m\hat{\ell}$ with n and m depending on loop order.

The expansion of the fermionic part of the superstring action is somewhat more involved. Finding a convenient (i.e. constant-coefficient) form of the fluctuation action requires a sequence of rotations as well as a rescaling. This is so because in the coordinates (71) neither r, w or ϕ correspond to linear isometries. Inspecting the action (71), it is not hard to see that the relevant transformations ($\theta\to\psi$) are of the form

$$
\begin{aligned}
&\theta = e^{-\check{r}}\mathcal{M}_{01}(\check{w})\mathcal{M}_{89}(\check{\phi})\ \psi \ ,\\
&\mathcal{M}_{89}(\check{\phi}) = \cos\frac{\check{\phi}}{2}-\sin\frac{\check{\phi}}{2}\Gamma_8\Gamma_9 \ , \qquad \mathcal{M}_{01}(\check{w}) = \cosh\frac{\check{w}}{2}+\sinh\frac{\check{w}}{2}\,\Gamma_0\Gamma_1 \ . \qquad (86)
\end{aligned}
$$

There is a certain amount of freedom in the choice of parameters for these transformations as long as they contain the background values of the fields, i.e.

$$
\check{r} = \bar{r}+\alpha\tilde{r} \ , \qquad \check{w} = \bar{w}+\beta\tilde{w} \ , \qquad \check{\phi} = \bar{\phi}+\gamma\tilde{\phi} \qquad (87)
$$

where α, β and γ are some numbers. The values of these parameters are irrelevant at the 1-loop order. Choosing $\alpha = \beta = \gamma = 1$ leads to a relatively simple Lagrangian. However, despite this simplicity the pattern of cancellations at higher loops is somewhat obscure. It turns out that $\alpha = 1$, $\beta = \gamma = 0$ makes it most transparent. The Lagrangian for $\alpha = 1$ and arbitrary β and γ is ($i = 2, 3$; $q = 4, 5, 6, 7$):

$$
\begin{aligned}
\mathcal{L}_F \;=\;& 4\,\epsilon^{ab}\,\bar\theta\Big[(v\partial_a r + \partial_a v)(\cosh(w-\check w)\Gamma_0 + \sinh(w-\check w)\Gamma_1) \\
&\quad + v dw(\sinh(w-\check w)\Gamma_0 + \cosh(w-\check w)\Gamma_1) \\
&\quad + \big(\partial_a r y^i + \partial_a y^i\big)\Gamma_i \; + \big(\partial_a r \hat z^u + \partial_a \hat z^u\big)\Gamma_u \\
&\quad + \; \big(\partial_a r Y + \partial_a Y\big)(\cos(\phi-\check\phi)\Gamma_8 + \sin(\phi-\check\phi)\Gamma_9) \\
&\quad + \; Y\partial_a\phi\;(-\sin(\phi-\check\phi)\Gamma_8 + \cos(\phi-\check\phi)\Gamma_9)\Big]\partial_b\theta \\
+\;& 2\,\epsilon^{ab}\,\bar\theta\Big[\big(\partial_a r y^i + \partial_a y^i\big)\partial_b\check w\Gamma_i\Gamma_0\Gamma_1 \; + \big(\partial_a r\hat z^q + \partial_a \hat z^q\big)\partial_b\check w\Gamma_q\Gamma_0\Gamma_1 \\
&\quad + \; \big(\partial_a r Y + \partial_a Y\big)\partial_b\check w(\cos(\phi-\check\phi)\Gamma_8 + \sin(\phi-\check\phi)\Gamma_9)\Gamma_0\Gamma_1 \\
&\quad + \; Y\partial_a\phi\partial_b\overline{w}\;(-\sin(\phi-\check\phi)\Gamma_8 + \cos(\phi-\check\phi)\Gamma_9)\Gamma_0\Gamma_1\Big]\theta \\
-\;& 2\,\epsilon^{ab}\,\bar\theta\Big[(v\partial_a r + \partial_a v)(\cosh(w-\check w)\Gamma_0 + \sinh(w-\check w)\Gamma_1)\;\partial_b\check\phi\Gamma_8\Gamma_9 \\
&\quad + v dw(\sinh(w-\check w)\Gamma_0 + \cosh(w-\check w)\Gamma_1)\;\partial_b\check\phi\;\Gamma_8\Gamma_9 \\
&\quad + \big(\partial_a r y^i + \partial_a y^i\big)\;\partial_b\check\phi\;\Gamma_i\Gamma_8\Gamma_9 \; + \big(\partial_a r\hat z^u + \partial_a\hat z^u\big)\;\partial_b\check\phi\;\Gamma_u\Gamma_8\Gamma_9\Big]\theta\;. \qquad (88)
\end{aligned}
$$

The quadratic part of this action is independent of the choice of α, β and γ. Extracting and inverting it is somewhat tedious but algorithmic; the fermion propagator turns out to be ($n \times m \equiv \epsilon^{ab} n_a m_b$)

$$
\begin{aligned}
K_F^{-1}(p) \;=\;& \frac{1}{\mathcal{D}_F[p]}\Big[2\sqrt{2}\,i\,\Big(n_1\times p\;(\Gamma_0 + \sqrt{2}\Gamma_8) + n_2\times p\;\Gamma_1\Big) - 4\,\mathrm{m}\times p\;\Gamma_8 \\
& -i\sqrt{2}\;\mathrm{m}\times n_1(\sqrt{2}\;\Gamma_{019} + \Gamma_{089} + \Gamma_{189}) + 2\,n_1\times n_2\;\Gamma_{018}\Big] \\
&\qquad\qquad \times\Big[2\,(4p^2 + 2n_1\cdot n_1 - n_1\cdot n_2)\mathbb{1} - 8\mathrm{m}\cdot p\;\Gamma_{0189}\Big]\mathcal{C}^{-1}\;, \\
\mathcal{D}_F[p] \;=\;& 8\left[16(\mathrm{m}\cdot p)^2 + (\mathrm{m}\cdot\mathrm{m} + 2n_1\cdot n_1 + 4p^2)^2\right]\;. \qquad (89)
\end{aligned}
$$

Here the matrix structure was organized to emphasize the strategy used to construct it. $\mathcal{C}$ is the charge conjugation matrix (see [13, 16]). Note that similarly to the denominator of the bosonic propagator, $\mathcal{D}_F$ becomes 2d Lorentz-invariant in the limit $\mathrm{m}\to 0$, i.e. $\hat\ell\to 0$:

$$
\mathcal{D}_F[p]_{\hat\ell\to 0} \;\mapsto\; 32(2p^2+1)^2\;. \qquad (90)
$$

4 Quantum corrections to the effective action

Let us now use the above fluctuation action to compute the 2-loop correction to the superstring partition function or the effective action, $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_2 + ...,\quad \Gamma_n = O(\frac{1}{(\sqrt{\lambda})^n})$. In the next section we will extract from it the corresponding correction to the string energy as a function of the spins in the scaling limit.

4.1 One loop

The one-loop partition function for the (S, J) solution in the scaling limit was computed in [11]. Let us review it here in the framework and notation set up in sections 2.2 and 2.3. As discussed there, we will need $\mathcal{F}_1(\hat{\ell}\,)$ in (48), (54) to find the $\mathcal{O}(\frac{1}{\sqrt{\lambda}})$ correction to the space-time energy of the folded spinning string.

The one-loop correction to the effective action is the difference between the logarithms of the determinants of the bosonic and the fermionic kinetic operators. Directly computing these determinants is quite tedious, especially for the fermions. Taking the product of the denominators that appear in the bosonic and fermionic propagators leads, however, to a useful factorization of these determinants.

Up to a constant term which simply counts the number of bosonic degrees of freedom and cancels against its fermionic counterpart, the bosonic contribution is

$$\begin{aligned}
\Gamma_{1B} &= \hat{\kappa}^2 V_2 \int_0^\infty \frac{dp}{2\pi} \Big[\sqrt{p^2 + 2 + 2\sqrt{1 + \hat{u}^2 p^2}} + \sqrt{p^2 + 2 - 2\sqrt{1 + \hat{u}^2 p^2}} \\
&\qquad\qquad\qquad + \; 2\sqrt{p^2 + 2 - \hat{u}^2} + 4\sqrt{p^2 + \hat{u}^2} \;\; \Big] \\
&= \hat{\kappa}^2 V_2 \int_0^\infty \frac{dp}{2\pi} \Big[\sqrt{4\hat{u}^2 + \big(p + \sqrt{p^2 + 4 - 4\hat{u}^2}\big)^2} + 2\sqrt{p^2 + 2 - \hat{u}^2} + 4\sqrt{p^2 + \hat{u}^2} \;\Big]
\end{aligned} \tag{91}$$

where (see (50))

$$\hat{\kappa} = \sqrt{1 + \hat{\ell}^{\,2}}\ , \qquad\qquad \hat{u} \equiv \frac{\hat{\ell}}{\hat{\kappa}} = \frac{\hat{\ell}}{\sqrt{1 + \hat{\ell}^{\,2}}} \quad . \tag{92}$$

The fermionic fluctuation spectrum is given by the solutions of the following equation:

$$16(\mathrm{m} \cdot p)^2 + (\mathrm{m} \cdot \mathrm{m} + 2n_1 \cdot n_1 + 4p^2)^2 = 0\ , \tag{93}$$

i.e.

$$\omega_\pm(p) = \sqrt{p^2 + \hat{\kappa}^2} \pm \frac{\hat{\ell}}{2} \quad . \tag{94}$$

Factorizing $\mathcal{D}_F[p]$ it is easy to find that up to a constant term which cancels against its bosonic counterpart, the fermion contribution to the 1-loop effective action is

$$\Gamma_{1F} \;=\; -8\hat{\kappa}^2 V_2 \int_0^\infty \frac{dp}{2\pi} \sqrt{p^2 + 1}\ . \tag{95}$$

Combining the bosonic and fermionic contributions to the effective action reproduces the result of [11]

$$
\begin{aligned}
\Gamma_1 \;&=\; \hat{\kappa}^2 V_2 \int_0^\infty \frac{dp}{2\pi}\Big[\sqrt{4\hat{u}^2+\big(p+\sqrt{p^2+4-4\hat{u}^2}\big)^2}+2\sqrt{p^2+2-\hat{u}^2} \\
&\qquad\qquad + 4\sqrt{p^2+\hat{u}^2}-8\sqrt{p^2+1}\;\;\Big] \\
&=\; -\frac{\hat{\kappa}^2}{2\pi}V_2\Big[1-\hat{u}^2-\sqrt{1-\hat{u}^2}+(2-\hat{u}^2)\ln[\sqrt{2-\hat{u}^2}(1+\sqrt{1-\hat{u}^2})]+2\hat{u}^2\ln\hat{u}\Big]
\end{aligned}
\tag{96}
$$

Using the expressions for $\hat{\kappa}(\hat{\ell}\,)$ and $u(\hat{\ell}\,)$ in (92) this leads to the one-loop term in $\mathcal{F}(\hat{\ell}\,)$ in (54)

$$
\begin{aligned}
\mathcal{F}_1(\hat{\ell}\,) \;&=\; -1+\sqrt{1+\hat{\ell}^{\,2}}+2(1+\hat{\ell}^{\,2})\ln(1+\hat{\ell}^{\,2})-2\,\hat{\ell}^{\,2}\ln\hat{\ell} \\
&\qquad - \;(2+\hat{\ell}^{\,2})\ln\Big(\sqrt{2+\hat{\ell}^{\,2}}\;\big(1+\sqrt{1+\hat{\ell}^{\,2}}\,\big)\Big)\;\;.
\end{aligned}
\tag{97}
$$

4.2 Two loops

As was discussed in detail in our earlier work [13, 16], the two loop order is the first order at which it is crucial to choose explicitly a consistent regularization of the world sheet superstring theory. Among the required features of such a regularization should be the preservation of the κ-symmetry of the classical action. Since this symmetry is intrinsically two-dimensional (having self-dual parameters), the standard dimensional regularization is not among the consistent choices.

A scheme advocated in [13, 16] is based on doing all algebraic manipulations in $d = 2$ and then continue the final two-dimensional momentum integrand to $d = 2 - 2\epsilon$. While such a prescription may be a source of confusion in a generic quantum field theories (e.g., it may not be completely clear what the "final integrand" actually means) in computing the partition function of a 2-dimensional theory with the conformal invariance spontaneously broken by the classical solution there should be no ambiguity in its implementation.

Indeed, it is expected that the integrands corresponding to all potential logarithmically divergent contributions cancel out before the actual integration. Only power-like divergent integrals (with no softer singularities hidden under the leading one) may remain and they can be analytically (e.g. dimensionally) regularized away. In an explicit cutoff regularization such power-like divergences would cancel against the contribution of the path integral measure and of the non-propagating κ-symmetry ghosts. An important consequence of this regularization scheme is that the BMN point-like string remains a BPS state at the two-loop order – its energy is not corrected [13]; this would not happen if one used the standard dimensional regularization.

This scheme has a number of interesting and useful features that were already observed in [16] at $J = 0$. In particular, using it one finds that the two-loop terms in the partition functions of the AdS_5 and S^5 bosonic sigma models vanish when computed in our classical background. As we will see, this continues to be true also in the presence of the angular momentum on S^5 (i.e. for $\hat{\ell} \neq 0$). In particular, there will be no logarithmic UV divergences coming from bosons. A consequence of this two-loop finiteness of the bosonic sigma models is that the fermionic contribution to the two-loop partition function must also be separately finite, and, indeed, it is.

We shall illustrate our computational procedure on the example of the S^2 bosonic sigma model in Appendix B.

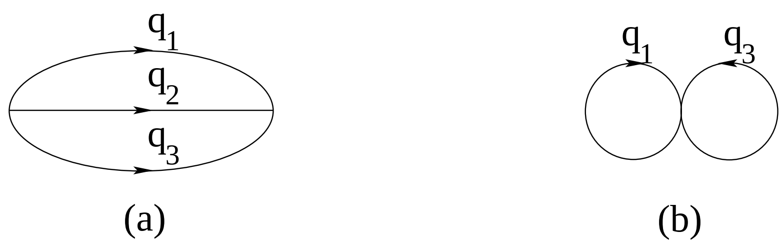

Figure 1: Topologies of possible two-loop diagrams; each line denotes either a bosonic or a fermionic propagator.

The 2-loop effective action may be written as a sum of purely bosonic and mixed boson-fermion terms:

$$\Gamma_2(\hat{\ell}\,) = \frac{1}{2\pi\sqrt{\lambda}} V_2\ \mathcal{F}_2(\hat{\ell}\,) \quad , \qquad\qquad \mathcal{F}_2(\hat{\ell}\,) = A_B(\hat{\ell}\,) + A_F(\hat{\ell}\,) \ . \tag{98}$$

Each of them receives contributions from a "sunset" topology (figure 1a, giving A_{3B}, A_{3F}) and a "double-bubble" topology (figure 1b, giving A_{4B}, A_{4F}). Including the appropriate combinatorial factors we have

$$A_B = -\frac{1}{12}A_{3B} + \frac{1}{8}A_{4B} \ , \qquad\qquad A_F = \frac{1}{16}A_{3F} + \frac{1}{8}A_{4F} \ . \tag{99}$$

It is not hard to produce integral representations for A_B and A_F using the Feynman rules corresponding to the action (71) expanded around the solution (80). The result, however, cannot be efficiently analyzed due to the Lorentz-noninvariance of the denominators of the bosonic and fermionic propagators. A way around this technical problem is to expand in $\hat{u} = \frac{\hat{\ell}}{\hat{\kappa}(\hat{\ell})}$. This is justified in light of the discussion below equation (64). Because the S^5 fluctuations become massless as $\hat{u} \to 0$ limit, we expect that the leading order correction has the structure

$$A_{B,F} = \hat{\kappa}^2\Big[A^{(0)}_{B,F} - \hat{u}^2\left(a_{B,F}\ln \hat{u}^2 + b_{B,F}\right) + \mathcal{O}(\hat{u}^4)\Big] \ . \tag{100}$$

We will first compute its derivative with respect to $\hat{u}^2$

$$A^{(1)}_{B,F} = \frac{d}{d\hat{u}^2}[\hat{u}^2\left(a_{B,F}\ln \hat{u}^2 + b_{B,F}\right)] = a_{B,F}\ln \hat{u}^2 + a_{B,F} + b_{B,F} \ , \tag{101}$$

and then integrate $A^{(1)}_{B,F}$ to determine $A_{B,F}$.

Let us define the integrals

$$I\left(\begin{smallmatrix} a_1 & a_2 & a_3 \\ m_1 & m_2 & m_3 \end{smallmatrix}\right) = \frac{1}{(2\pi)^4}\int \frac{d^2k_1 d^2k_2 d^2k_3\ \ \delta^{(2)}(\sum_{i=1}^3 k_i)}{(k_1^2+m_1^2)^{a_1}(k_2^2+m_2^2)^{a_2}(k_3^2+m_3^2)^{a_3}} \ . \tag{102}$$

They are simple generalizations of the integrals that appeared for $J = 0$ [16]. The higher powers in the denominator are related to the our expanding the 2-loop partition function in powers of u.

In terms of these integrals, the bosonic contributions are

$$A_{3B}^{(0)} = -12 I\left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & \sqrt{2} \end{smallmatrix}\right) + 6 I\left(\begin{smallmatrix} 1 & 1 \\ \sqrt{2} & \sqrt{2} \end{smallmatrix}\right) , \qquad\qquad A_{4B}^{(0)} = 4 I\left(\begin{smallmatrix} 1 & 1 \\ \sqrt{2} & \sqrt{2} \end{smallmatrix}\right) \tag{103}$$

$$\begin{aligned} A_{3B}^{(1)} = & -12 I\left(\begin{smallmatrix} 1 & 2 \\ \sqrt{2} & \sqrt{2} \end{smallmatrix}\right) + 3 I\left(\begin{smallmatrix} 1 & 1 \\ \sqrt{2} & \sqrt{2} \end{smallmatrix}\right) - 12 I\left(\begin{smallmatrix} 1 & 1 \\ \hat{u} & \hat{u} \end{smallmatrix}\right) , \\ & + 6 I\left(\begin{smallmatrix} 2 & 1 & 1 \\ 1 & 1 & \sqrt{2} \end{smallmatrix}\right) + 6 I\left(\begin{smallmatrix} 1 & 2 & 1 \\ 1 & 1 & \sqrt{2} \end{smallmatrix}\right) + 12 I\left(\begin{smallmatrix} 1 & 1 & 2 \\ 1 & 1 & \sqrt{2} \end{smallmatrix}\right) - 12 I\left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & \sqrt{2} \end{smallmatrix}\right) \end{aligned} \tag{104}$$

$$A_{4B}^{(1)} = -8 I\left(\begin{smallmatrix} 1 & 2 \\ \sqrt{2} & \sqrt{2} \end{smallmatrix}\right) + 2 I\left(\begin{smallmatrix} 1 & 1 \\ \sqrt{2} & \sqrt{2} \end{smallmatrix}\right) - 8 I\left(\begin{smallmatrix} 1 & 1 \\ \hat{u} & \hat{u} \end{smallmatrix}\right) . \tag{105}$$

Combining them as in (99) immediately leads to

$$A_B^{(0)} = I\left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & \sqrt{2} \end{smallmatrix}\right) , \tag{106}$$

$$A_B^{(1)} = -\frac{1}{2} I\left(\begin{smallmatrix} 2 & 1 & 1 \\ 1 & 1 & \sqrt{2} \end{smallmatrix}\right) - \frac{1}{2} I\left(\begin{smallmatrix} 1 & 2 & 1 \\ 1 & 1 & \sqrt{2} \end{smallmatrix}\right) - I\left(\begin{smallmatrix} 1 & 1 & 2 \\ 1 & 1 & \sqrt{2} \end{smallmatrix}\right) + I\left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & \sqrt{2} \end{smallmatrix}\right) . \tag{107}$$

It is important to note that, despite our expansion in $\hat{\ell}$, the result is still IR-convergent. This may seem surprising given the fact that some of the AdS_5 fluctuations as well as all of the S^5 fluctuations become massless as $\hat{\ell} \to 0$.[12] The absence of both UV and IR divergences can be traced to the regularization scheme we are using. Indeed, by doing the algebraic manipulations in two dimensions we find that the full bosonic two-loop S^5 contribution to the partition function is identically zero (apart from a power divergent term, see also Appendix B).

This "non-regularization" prescription is forced upon on us as the only consistent regularization scheme in the presence of the Green-Schwarz fermions. It leads to the results that are fully consistent with the expectation that the bosonic AdS_5 and S^5 sigma models are embedded in a two-dimensional conformal field theory, i.e. it can be viewed as a part of the definition of the quantum superstring theory.

The fermion contribution is somewhat more involved. Depending on the precise choice of the parameters β and γ in (87),(88), different diagrams contribute differently. In particular, if the fermionic rotation contains quantum fields then there are nontrivial cancellations between the sunset and double-bubble topologies. While we have checked that the final expression is independent of the choice of β and γ, the simplest way to state the result is by choosing $\beta = \gamma = 0$, i.e. to use a rotation involving only the classical fields. Then the double-bubble contribution vanishes identically while that of the sunset topology provides the entire fermionic contribution to the effective action.

An important subtlety is that the naive $\hat{\ell} \to 0$ expansion of the fermionic sunset graphs involving two fermionic and one bosonic propagator leads to logarithmic IR divergences which appear because the 4 "transverse" S^5 bosonic modes become massless at $\hat{\ell} = 0$.

[12]The massless AdS_5 fluctuation is not problematic because for nonzero $\hat{\ell}$ its "mass" arises from a rotation-like imbalance between the space-like and the time-like components of momenta rather than from a genuine mass term.

To isolate the relevant $\ln \hat{\ell}$ terms we shall compute the derivative of the partition function with respect to $\hat{\ell}^{\,2}$ and then set $\hat{\ell} = 0$ everywhere except in the S^5 propagators. The result is a collection of logarithms and rational functions. The rational functions are of course not trustworthy beyond $\mathcal{O}(\hat{\ell}^{\,2})$. Moreover, only the constant term in the coefficient of $\ln \hat{\ell}$ (which leads to $\hat{\ell}^{\,2} \ln \hat{\ell}$ term in the final answer) does not receive further corrections.

Following this strategy, the relevant contributions are found to be

$$A^{(0)}_{3F} = -32 I\left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & \sqrt{2} \end{smallmatrix}\right) , \qquad A^{(0)}_{4F} = 0 , \qquad A^{(1)}_{4F} = 0 , \tag{108}$$

$$\begin{aligned} A^{(1)}_{3F} = &+ 48 I\left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & \sqrt{2} \end{smallmatrix}\right) + 16 I\left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{smallmatrix}\right) + 32 I\left(\begin{smallmatrix} 1 & 1 & 2 \\ 1 & 1 & \sqrt{2} \end{smallmatrix}\right) \\ &+ 16 I\left(\begin{smallmatrix} 1 & 3 & 1 \\ 1 & 1 & \sqrt{2} \end{smallmatrix}\right) + 16 I\left(\begin{smallmatrix} 3 & 1 & 1 \\ 1 & 1 & \sqrt{2} \end{smallmatrix}\right) - 16 I\left(\begin{smallmatrix} 1 & 2 & 1 \\ 1 & 1 & \sqrt{2} \end{smallmatrix}\right) - 16 I\left(\begin{smallmatrix} 2 & 1 & 1 \\ 1 & 1 & \sqrt{2} \end{smallmatrix}\right) \\ &- 32 I\left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & \hat{u} \end{smallmatrix}\right) . \end{aligned} \tag{109}$$

Combining these terms according to (99) we are led to the following fermion contribution to the effective action:

$$A^{(0)}_{F} = -2 I\left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & \sqrt{2} \end{smallmatrix}\right) , \tag{110}$$

$$\begin{aligned} A^{(1)}_{F} = &\ 3 I\left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & \sqrt{2} \end{smallmatrix}\right) + I\left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{smallmatrix}\right) + 2 I\left(\begin{smallmatrix} 1 & 1 & 2 \\ 1 & 1 & \sqrt{2} \end{smallmatrix}\right) \\ &+ I\left(\begin{smallmatrix} 1 & 3 & 1 \\ 1 & 1 & \sqrt{2} \end{smallmatrix}\right) + I\left(\begin{smallmatrix} 3 & 1 & 1 \\ 1 & 1 & \sqrt{2} \end{smallmatrix}\right) - I\left(\begin{smallmatrix} 1 & 2 & 1 \\ 1 & 1 & \sqrt{2} \end{smallmatrix}\right) - I\left(\begin{smallmatrix} 2 & 1 & 1 \\ 1 & 1 & \sqrt{2} \end{smallmatrix}\right) \\ &- 2 I\left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & \hat{u} \end{smallmatrix}\right) . \end{aligned} \tag{111}$$

Using (106),(110),(107),(111) and the values of individual integrals listed in the appendix we find that

$$A^{(0)} = A^{(0)}_B + A^{(0)}_F \;\; = \;\; -\frac{2}{(4\pi)^2} \mathrm{K} , \tag{112}$$

$$A^{(1)} = A^{(1)}_B + A^{(1)}_F \;\; = \;\; \frac{1}{(4\pi)^2} \Big(4 \ln \hat{u} + 7\mathrm{K} + 3 \ln 2 - \frac{1}{2} \Big) , \tag{113}$$

where $\mathrm{K} = 0.915...$ is the Catalan's constant.

Reconstructing the two-loop effective action (98) implies that the first two terms in the small $\hat{\ell}$ expansion of $\mathcal{F}_2(\hat{\ell}\,)$ in (48) are

$$\mathcal{F}_2(\hat{\ell}\,) \;\; = \;\; -\hat{\kappa}^2 \left[\mathrm{K} + \frac{1}{2} \hat{u}^2 \big(4 \ln \hat{u} + 7\mathrm{K} + 3 \ln 2 - \frac{5}{2} \big) + \mathcal{O}(\hat{u}^4) \right] , \tag{114}$$

where as in (92) we have $\hat{\kappa} = \sqrt{1 + \hat{\ell}^{\,2}}$, $\quad \hat{u} = \frac{\hat{\ell}}{\sqrt{1+\hat{\ell}^{\,2}}}$. Though we are expanding in small $\hat{\ell}$, we have kept the full $\hat{u}(\hat{\ell}\,)$ instead of just its leading-order term $\hat{\ell}$ to emphasize that $\hat{u}$ is the natural world-sheet expansion parameter in the scaling limit.

5 Quantum corrections to the string energy

Having computed the effective action as a function of $\hat{\ell}$ we are now in position to reconstruct the difference $E - S$ to two-loop order using (47),(57)–(60).

At zero and one loops we find f_0 and f_1 that reproduce the tree-level [2] and one-loop [11] terms in the energy of the (S,J) spinning string in the scaling limit as given in (11) and (15). Using the expressions for $\mathcal{F}_0$ and $\mathcal{F}_1$ in (82) and (97) and $\mathcal{F}_2$ from (114) we can then use the general relation (60) to get the final expression for the 2-loop coefficient in $E-S$ in (57). Let us write (60) as

$$\mathrm{f}_2 = \bar{\mathrm{f}}_2 + \Delta\mathrm{f}_2 \ , \qquad \bar{\mathrm{f}}_2 = \frac{1}{\sqrt{1+\ell^2}}\,\mathcal{F}_2(\ell) \ , \qquad \Delta\mathrm{f}_2 = \frac{1}{2}(1+\ell^2)^{3/2}\left(\frac{d\mathrm{f}_1(\ell)}{d\ell}\right)^2 \ , \tag{115}$$

where $\bar{\mathrm{f}}_2$ is the "genuine" 2-loop contribution coming directly from the effective action (i.e. from the 2-loop graphs) while $\Delta\mathrm{f}_2$ is the "one-loop" correction due to the shift of ℓ in (63) which arises because of the finite renormalization (61) of the relation between the $PSU(2,2|4)$ charges and the parameters of the classical solution.

Using that to the leading order $\hat{u}$ in (92) can be replaced by (see (62))

$$u = \frac{\ell}{\sqrt{1+\ell^2}} \ , \tag{116}$$

but keeping as above in (114) the full dependence on u in $\bar{\mathrm{f}}_2$ we get

$$\bar{\mathrm{f}}_2 = -\sqrt{1+\ell^2}\,\Big[\,\mathrm{K} + \frac{1}{2}\,u^2\big(4\ln u + 7\mathrm{K} + 3\ln 2 - \frac{5}{2}\big) + \mathcal{O}(u^4)\Big] \ , \tag{117}$$

$$\begin{aligned}\Delta\mathrm{f}_2 = \frac{1}{2}\,\frac{\ell^2\,(1+\ell^2)^{-3/2}}{(1+\sqrt{1+\ell^2})^2}\Big[&-(2+\ell^2) + (1+\sqrt{1+\ell^2})\Big(2(\,2+\ell^2)\ln\ell \\ &+ \ell^2\ln\big[\sqrt{2+\ell^2}(1+\sqrt{1+\ell^2})\big] - 2(1+\ell^2)\ln(1+\ell^2)\Big)\Big]^2\end{aligned} \tag{118}$$

Expanding these two contributions to f_2 to order ℓ^2 we find

$$\bar{\mathrm{f}}_2 \simeq -\mathrm{K} - \ \ell^2\Big(2\ln\ell + \ 4\mathrm{K} + \frac{3}{2}\ln 2 - \frac{5}{4}\Big) + \mathcal{O}(\ell^4) \ , \tag{119}$$

$$\Delta\mathrm{f}_2 \simeq 2\ell^2\Big(2\ln\ell - \frac{1}{2}\Big)^2 + \mathcal{O}(\ell^4) \ , \tag{120}$$

so that finally (cf. (19))

$$\mathrm{f}_2 = -\mathrm{K} + \ell^2\Big(\,8\ln^2\ell - 6\ln\ell + q_{02}\Big) + \mathcal{O}(\ell^4) \ , \tag{121}$$

$$q_{02} = -4\mathrm{K} - \frac{3}{2}\ln 2 + \frac{7}{4} \ . \tag{122}$$

We observe that the $\ell^2\ln^2\ell$ and the $\ell^2\ln\ell$ terms are precisely the same as was proposed in [14].

The leading $\ell^2\ln^2\ell$ term originates solely from the contribution of the one-loop "charge renormalization" while the subleading $\ell^2\ln\ell$ term receives contributions from both the genuine two-loop term (the fermion graph with one bosonic propagator from S^5, i.e. with the mass proportional to ℓ) and the one-loop "charge renormalization".

It was argued in [14] that all the fermionic terms can be ignored in the computation of the coefficients of the leading n-loop terms $\ell^2 \ln^n \ell$. Our direct computation confirms this. At the same time, the coefficient of the subleading 2-loop term $\ell^2 \ln \ell$ is sensitive to the fermionic contributions. In [14] the value of this term was predicted by using the coefficient contained in the ℓ^2 part of the 1-loop superstring correction (15),(16).[13]

Let us now comment on higher-order terms. An information on the structure of the logarithmic terms at higher loops can be inferred from the equations (51) and (52) and the form of the fluctuation action. It is plausible that the S^5 contribution to the partition function will be trivial to all orders in perturbation theory in our regularization scheme. Logarithmic terms can arise, however, only from diagrams containing light S^5 fields and thus the leading one should come from diagrams with the maximal number of such fields. Since there is no direct coupling between the AdS_5 and S^5 fluctuations in the conformal gauge, such diagrams must necessarily include fermionic fields. At each loop order n it is not hard to identify diagrams containing $(n-1)$ light S^5 fields (for example, such is an n-loop sunset graph with two fermionic propagators). Barring miraculous cancellations, it is then guaranteed that the coefficient of the $\ln^{n-1} \ell$ term in the energy should receive genuine n-loop corrections.[14]

At 3-loop order one can see directly that diagrams with more that two S^5 fluctuations have, in fact, two fields of mass $\hat{\ell}$ and all other fields of mass zero and thus cannot produce higher than $\ln^2 \ell$ contribution. A similar analysis can be carried out at the 4-loop order. It is therefore natural to expect that the leading n-loop logarithms $b_n(\ell) \ln^n \ell$ will be "induced" by the "charge renormalization" procedure from the 1-loop partition function, while the genuine n-loop corrections are relevant only for the first subleading term $b_{n-1}(\ell) \ln^{n-1} \ell$.

6 Resuming the logarithms: towards interpolation to gauge-theory results

To summarize, combining the small ℓ expansions of the tree-level (11), one-loop (16) and two-loop (121) terms in (17),(18) we get

$$E - S = \frac{\sqrt{\lambda}}{\pi} \Big\{ f(\lambda) \quad + \quad \big[\, q_{00} \, \ell^2 + \mathcal{O}(\ell^4) \big]$$

[13]The relation between our result and the arguments of in [14] can be understood as follows. If we assume that we first integrate all "heavy" modes out getting an effective action for S^5 modes and keeping only the 2-derivative term in it (which is the one relevant for the ℓ^2 corrections) then the coefficient of the $O(6)$ model term will be shifted, e.g., by the fermionic loop contribution. The subsequent computation of the partition function amounts to closing the loop for S^5 modes. This is then effectively the same contribution as coming from the 2-loop graph with two fermionic and one S^5 propagator which in our case contributed to the $\ell^2 \ln \ell$ coefficient.

[14]The leading IR singularity in the relevant sunset graph is $\lim_{u\to 0} \int d^2x \, (K_0(ux))^{n-1} \, (K_0(x))^2 \propto \ln^{n-1} u$ (cf. (117)).

$$+ \quad \frac{1}{\sqrt{\lambda}}\Big[\ \ell^2\ (q_{11}\ln\ell + q_{01}) + \mathcal{O}(\ell^4)\Big] \tag{123}$$

$$+ \quad \frac{1}{(\sqrt{\lambda})^2}\Big[\ \ell^2\ \big(2q_{11}^2\ln^2\ell + 2q_{11}q_{01}\ln\ell + q_{02}\big) + \mathcal{O}(\ell^4)\Big] + \mathcal{O}(\frac{1}{(\sqrt{\lambda})^3})\Big\}$$

where $f(\lambda)$ is given by (20) and the coefficients q_{rn} are defined by (see (21))[15]

$$q_0(\lambda) \;=\; \sum_{n=0}^{\infty}\frac{q_{0n}}{(\sqrt{\lambda})^n} = \frac{1}{2} + \frac{3}{2\sqrt{\lambda}} + \frac{1}{(\sqrt{\lambda})^2}(-4\mathrm{K} - \frac{3}{2}\ln 2 + \frac{7}{4}) \;+\; \mathcal{O}(\frac{1}{(\sqrt{\lambda})^3})\ , \tag{124}$$

$$q_1(\lambda) \;=\; \sum_{n=1}^{\infty}\frac{q_{1n}}{(\sqrt{\lambda})^n} = -\frac{2}{\sqrt{\lambda}} + \mathcal{O}(\frac{1}{(\sqrt{\lambda})^2})\ . \tag{125}$$

Remarkably, all of the explicitly written leading terms in (123) can be reproduced by expanding in $\frac{1}{\sqrt{\lambda}}$ the following expression (see (20), (21))

$$E - S = \frac{\sqrt{\lambda}}{\pi}\Big[\ f(\lambda) + \frac{q_0(\lambda)\ \ell^2}{1 - 2q_1(\lambda)\ln\ell} + ... +\ \mathcal{O}(\ell^4)\ \Big]\ , \tag{126}$$

The coefficient of $\ln\ell$ in the denominator $1 - 2q_{11}\ln\ell = 1 + \frac{4}{\sqrt{\lambda}}\ln\ell$ has its origin in the value (4=6-2) of the 1-loop β function of the S^5 sigma-model (cf. (16)). The above expression obviously resembles the RG running coupling, in agreement with the discussion in [14].[16] In fact, as was pointed out in [14], the closed expression that reproduces the expected coefficients of the first two leading $\ell^2\ln^n\ell$ and $\ell^2\ln^{n-1}\ell$ terms at n-th loop order in $\frac{1}{\sqrt{\lambda}}$ expansion is

$$E - S = \frac{\sqrt{\lambda}}{\pi}\Big\{\ f(\lambda) + \frac{\ell^2}{1 - \frac{2q_{11}}{\sqrt{\lambda}}\ln\ell}\Big(q_{00} + \frac{q_{01}}{\sqrt{\lambda}}\frac{1 + \ln(1 - \frac{2q_{11}}{\sqrt{\lambda}}\ln\ell)}{1 - \frac{2q_{11}}{\sqrt{\lambda}}\ln\ell}\Big) + ... +\ \mathcal{O}(\ell^4)\ \Big\}\ . \tag{127}$$

This expression does not include the information about the 2-loop coefficient q_{02} in (121) we have computed here since, in contrast to its 1-loop counterpart q_{01}, it does not influence the coefficients of the two leading powers of $\ln\ell$. One may try to guess a further generalization of (127) which also incorporates the features of (126). As an illustration, an example of possible expression which encodes the complete information about the function $q_0(\lambda)$ is[17]

$$E - S = \frac{\sqrt{\lambda}}{\pi}\Big\{\ f(\lambda)\ +\ \frac{\ell^2}{1 - 2q_1(\lambda)\ln\ell}\sum_{n=0}^{\infty}\frac{q_{0n}}{(\sqrt{\lambda})^n}\Big(\frac{1 + \ln[1 - 2q_1(\lambda)\ln\ell]}{1 - 2q_1(\lambda)\ln\ell}\Big)^n + \mathcal{O}(\ell^4)\ \Big\}\ . \tag{128}$$

A resummation of the logarithms of ℓ appears to be necessary in order to interpolate between the string perturbative expansion ($\lambda \gg 1$) and the gauge theory perturbative expansion $\lambda \ll 1$) in a similar limit.

[15]Note that, unlike the universal scaling function $f(\lambda)$, the coefficients thus defined do not have definite transcendentality properties. It is possible that this is an artifact is the small ℓ expansion at large λ.

[16]Notice also that the subleading 2-loop $\ln\ell$ term in (123) originates from the product of the 1-loop logarithm in the denominator and the 1-loop "shift" (3/2) in $q_0(\lambda)$.

[17]Generalizations of (127) and (126) involving coefficients of definite transcendentality and recovering q_0 in a large λ expansion are also easy to construct.

The gauge-theory expansion corresponds to fixing the value of $j = \sqrt{\lambda}\ell = \frac{\pi J}{\ln S}$ for any λ and expanding first in λ and then in j. At the 1-loop order in the $sl(2)$ sector it was found in [5] that the anomalous dimension scales as $\Delta \equiv E - S - J = \lambda(a_{10} j + a_{30} j^3 + ...) \ln S + ...$ Surprisingly, the absence of the j^2 term extends to all orders in the weak coupling expansion. As was very recently shown by a perturbative solution of $J \neq 0$ generalization of the BES equation in the $S \to \infty$, small j limit, one has [24]

$$\Delta = \big[a_1(\lambda)\ j +\ a_3(\lambda)\ j^3 +\ a_4(\lambda)\ j^4\ +\ a_5(\lambda)\ j^5\ + ...\big]\ \ln S\ , \qquad j \equiv \frac{\pi J}{\ln S}\ , \tag{129}$$

where the functions $a_k(\lambda)$ are given by convergent serii in λ [24].

The corresponding weak-coupling, small $j = \sqrt{\lambda}\ \ell$ continuation of the above string theory expression (123) (or the one like in (128)) should be expected to reproduce the absence of all terms of the type $j^k \ln^n j$ as well as the complete j^2 term in the energy. The appearance of the logarithmic $\ell^k \ln^n \ell$ terms in (123) should therefore be an artifact of the string perturbative expansion.

The string expression (123) was obtained by assuming that $\lambda \gg 1$ for fixed $\ell = \frac{j}{\sqrt{\lambda}}$ while in the gauge theory expansion one assumes that $j = \sqrt{\lambda}\ell$ is fixed and expands at small λ. Thus, to compare the gauge and the string theory results we are supposed to start with the exact form of the string (strong-coupling) expansion, fix j and start decreasing λ. While the expansion in (123) is singular in this limit, the resummed expression (126) or (127) leads to an apparent suppression of the terms proportional to j^2 present in (123): instead of ℓ^2 we get $\frac{\ell^2 \ln^n \ell}{\ln^{n+1}\lambda}$ terms which vanish for small 't Hooft coupling.[18]

This gives an indication that a resummation of the string perturbative expansion may allow one to smoothly connect it with the gauge-theory expansion (129). For example, the expression like (128) may produce a function $j^2\ k(\lambda, \ln j)$ which vanishes in the small λ limit. To systematically address this issue one needs, in fact, to find the exact form of the string result to all orders in ℓ.

In particular, it would be important to find the exact ℓ-dependence of the 2-loop contribution in (18). It is easy to see that the $\frac{1}{(\sqrt{\lambda})^2}\ell^2 \ln^2 \ell$ term in (18), (121) is part of a more general term

$$\mathrm{f}_2 =\ b_2(\ell)\ \ln^2 \ell + ...\ , \qquad\qquad b_2(\ell^2) = \frac{2\ell^2 (2+\ell^2)^2}{(1+\ell^2)^{3/2}}\ . \tag{130}$$

Here $b_2(\ell)$ is completely determined by the 1-loop calculation, i.e. by (15). While for small ℓ the function b_2 scales as ℓ^2, its small λ, fixed $j = \sqrt{\lambda}\ell$ asymptotics (relevant for a discussion of an interpolation to gauge theory) is $\sim j^3$.

It would be interesting to find a similar exact form of the subleading ℓ-dependent terms in f_2 in (18) to determine how its ℓ^2 behavior discussed above may change in the gauge theory limit. Note that the exact tree-level (12) and the 1-loop (15) terms contain ℓ^2 terms in their expansion at small ℓ but their expansion in the gauge theory regime starts with j for f_0 and j^{-2} for f_1. We expect similar negative-power asymptotics for f_2 at small λ for fixed $j = \sqrt{\lambda}\ell$.

[18] The unnatural 't Hooft coupling dependence also points in a direction of a required resummation.

Finally, let us recall that a consequence of the strong coupling perturbative solution of the BES equation in [15] is that all the coefficients in the inverse string tension expansion are negative and grow factorially. Notice, however, that in the equation (60) the inclusion of the ℓ dependence corrects the $\mathcal{F}_2$ term (which contains the complete 2-loop contribution to the universal scaling function) by a positive (and potentially large) contribution. It is then tempting to speculate that a natural definition of the non-Borel-summable [15] series expansion for the scaling function (cusp anomalous dimension) at strong coupling should be to start with a more general $\hat{\ell} \neq 0$ case, resum the series and then consider the $\hat{\ell} \to 0$ limit.

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Appendix A: Some 2-loop integrals

Here we list the integrals relevant for deriving the equations (112) and (113):

$$I\left(\begin{smallmatrix}1&1&1\\1&1&2\end{smallmatrix}\right) = \frac{2\ln 2}{(4\pi)^2} \tag{A.1}$$

$$I\left(\begin{smallmatrix}1&1&1\\1&1&\sqrt{2}\end{smallmatrix}\right) = \frac{2\mathrm{K}}{(4\pi)^2} \tag{A.2}$$

$$I\left(\begin{smallmatrix}1&1&2\\1&1&\sqrt{2}\end{smallmatrix}\right) = \frac{1}{(4\pi)^2}\frac{\ln 2}{2} \tag{A.3}$$

$$I\left(\begin{smallmatrix}1&2&1\\1&1&\sqrt{2}\end{smallmatrix}\right) = \frac{1}{(4\pi)^2}\left(\mathrm{K} - \frac{1}{2}\ln 2\right) \tag{A.4}$$

$$I\left(\begin{smallmatrix}1&3&1\\1&1&\sqrt{2}\end{smallmatrix}\right) = \frac{1}{(4\pi)^2}\left(\mathrm{K} - \frac{1}{4} - \frac{1}{2}\ln 2\right) \tag{A.5}$$

$$\begin{aligned} I\left(\begin{smallmatrix}1&1&1\\1&1&\sqrt{\alpha}\end{smallmatrix}\right) &= \frac{1}{(4\pi)^2}\frac{1}{\sqrt{\alpha(\alpha-4)}}\Bigg[\ln\frac{2-\sqrt{\alpha(\alpha-4)}-\alpha}{2+\sqrt{\alpha(\alpha-4)}-\alpha}\ln\alpha \qquad\qquad (\mathrm{A.6}) \\ &\qquad + 2\mathrm{Li}_2\left(\frac{2\sqrt{\alpha}}{\sqrt{\alpha}-\sqrt{\alpha-4}}\right) - 2\mathrm{Li}_2\left(\frac{2\sqrt{\alpha}}{\sqrt{\alpha}+\sqrt{\alpha-4}}\right)\Bigg] \\ &= -\frac{\ln\alpha}{(4\pi)^2} + \ldots \qquad\qquad (\mathrm{A.7}) \end{aligned}$$

where the ellipsis stands for terms that are not relevant for us here. K is the Catalan's constant.

In recovering (A.1) and (A.2) from (A.6) one should be careful in identifying the correct branch for the dilogarithms (e.g. by requiring that the result is real).

Appendix B: 2-loop effective action of S^2 sigma model

To illustrate the discussion of our regularisation prescription in section 4.2 here we shall discuss a simple example: the computation of the 2-loop correction to the partition function of the S^2 sigma model.

Let us start with the Lagrangian (in our string-theory context one may set $g^{-2} = \frac{\sqrt{\lambda}}{2\pi}$)

$$\mathcal{L} = \frac{1}{2g^2}\left[\,(\partial\theta)^2 + \cos^2\theta(\partial\phi)^2\right] \tag{B.1}$$

We shall choose the Euclidean signature and expand in the standard way around a solution with a linear profile in the isometric direction ϕ:

$$\bar{\theta} = \tilde{\theta}\ , \qquad\qquad \bar{\phi} = i\nu\tau + \tilde{\phi}\ \ . \tag{B.2}$$

Ignoring the total derivative term gives

$$\begin{aligned}\mathcal{L} &= \frac{1}{2g^2}\left[\,(\partial\tilde{\theta})^2 + (1-\tilde{\theta}^2+\frac{1}{3}\tilde{\theta}^4)(\,-\nu^2+2i\nu\partial_0\tilde{\phi}+(\partial\tilde{\phi})^2)\right]\\ &= \frac{1}{2g^2}\left[\,(\,\partial\tilde{\theta})^2+\nu^2\tilde{\theta}^2+(\partial\tilde{\phi})^2-2i\nu\partial_0\tilde{\phi}\,\tilde{\theta}^2-\tilde{\theta}^2(\partial\tilde{\phi})^2-\frac{1}{3}\nu^2\tilde{\theta}^4\right]\ \ .\end{aligned} \tag{B.3}$$

The propagators then are

$$K_\theta^{-1} = \frac{1}{p^2+\nu^2}\ , \qquad\qquad K_\phi^{-1} = \frac{1}{p^2}\ \ . \tag{B.4}$$

The 1-loop effective action is given by (Λ is an UV cutoff)

$$\Gamma_1 = \frac{1}{2}\int\frac{d^2p}{(2\pi)^2}\left[\ \ln(p^2+\nu^2)+\ln p^2\ \right] = \frac{\nu^2}{8\pi}\Big(1-\ln\frac{\nu^2}{\Lambda^2}\Big)\ \ . \tag{B.5}$$

As in section 4.2, the 2-loop effective action receives contributions from diagrams with topologies shown in figure 1

$$\Gamma_2 = g^2\int d^2pd^2q\ \left[-\frac{1}{12}A_3+\frac{1}{8}(A_4+A_4')\right]\ \ . \tag{B.6}$$

The sunset diagram (figure 1a) contributes:

$$A_3 = -12\nu^2\frac{(p+q)_0\,(p+q)_0}{(p^2+\nu^2)(q^2+\nu^2)(p+q)^2}\ , \tag{B.7}$$

while the double bubble diagram (figure 2b) with a θ^4 vertex gives:

$$A_4 = -\frac{4\nu^2}{(p^2+\nu^2)(q^2+\nu^2)} \ . \tag{B.8}$$

Finally, the double bubble diagram (figure 2b) with a $(\partial\phi)^2\theta^2$ vertex yields a quadratically-divergent contribution

$$A_4' = 2\frac{p^2}{p^2(q^2+\nu^2)} = \frac{2}{q^2+\nu^2} \ , \tag{B.9}$$

which we shall ignore (it is cancelled by the measure contribution, cf. [31]).

Let us now compare the evaluation of the integrals of (B.7) and (B.8) in dimensional regularization and in the regularization used in our computation in section 4.

In dimensional regularization we continue the momentum integrals to $d = 2-2\epsilon$ dimensions from the outset. On Lorentz-invariance grounds we can therefore write

$$\int \frac{d^dp\, d^dq}{(2\pi)^{2d}} \frac{(p+q)_a\,(p+q)_b}{(p^2+\nu^2)(q^2+\nu^2)(p+q)^2} = \frac{1}{d}\int \frac{d^dp\, d^dq}{(2\pi)^{2d}} \frac{\delta_{ab}}{(p^2+\nu^2)(q^2+\nu^2)} \ . \tag{B.10}$$

Taking the 00-component of this tensor relation one finds that the 2-loop effective action is given by

$$\Gamma_2 = g^2\frac{d-2}{2d}\int \frac{d^dp\, d^dq}{(2\pi)^{2d}\mu^{4\epsilon}} \frac{1}{(p^2+\nu^2)(q^2+\nu^2)} \ , \tag{B.11}$$

which upon evaluation leads to a nonvanishing and divergent answer: the $d-2$ factor cancels one of the two $\frac{1}{d-2}$ poles from the standard tadpole integral (this is what happens in the computation of the 2-loop sigma model beta-function [32]).

Within our regularization prescription we stay in two dimensions. We can use the same Lorentz-invariance argument as above to write

$$\int \frac{d^2p\, d^2q}{(2\pi)^4} \frac{(p+q)_a\,(p+q)_b}{(p^2+\nu^2)(q^2+\nu^2)(p+q)^2} = \frac{1}{2}\int \frac{d^2p\, d^2q}{(2\pi)^4} \frac{\delta_{ab}}{(p^2+\nu^2)(q^2+\nu^2)} \tag{B.12}$$

and then project onto the 00-component. Alternatively, we may simply notice that the denominator $(p^2+\nu^2)(q^2+\nu^2)(p+q)^2$ is invariant under the simultaneous transformation

$$p_0 \leftrightarrow p_1 \ , \qquad\qquad q_0 \leftrightarrow q_1 \ . \tag{B.13}$$

Then symmetrizing A_3 under this transformation we get

$$\begin{aligned} A_3 &= -12\nu^2 \int \frac{d^2p\, d^2q}{(2\pi)^4} \frac{(p+q)_0\,(p+q)_0}{(p^2+\nu^2)(q^2+\nu^2)(p+q)^2} \\ &= -6\nu^2 \int \frac{d^2p\, d^2q}{(2\pi)^4} \frac{(p+q)_0\,(p+q)_0 + (p+q)_1\,(p+q)_1}{(p^2+\nu^2)(q^2+\nu^2)(p+q)^2} \end{aligned}$$

$$= \quad -6\nu^2 \int \frac{d^2p\, d^2q}{(2\pi)^4} \frac{1}{(p^2+\nu^2)(q^2+\nu^2)} \tag{B.14}$$

Combining this with B.8) leads to the trivial result for the 2-loop effective action

$$\Gamma = g^2 \left(\frac{1}{2} - \frac{1}{2} \right) \int \frac{d^2p\, d^2q}{(2\pi)^4} \frac{1}{(p^2+\nu^2)(q^2+\nu^2)} = 0 \quad . \tag{B.15}$$

Thus, in our regularization prescription the 2-loop term in the free energy of the S^2 sigma model is identically zero.

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