# Nonlinear Fluid Dynamics from Gravity

Sayantani Bhattacharyya$^{a*}$, Veronika E Hubeny$^{b\dagger}$,

Shiraz Minwalla$^{a\ddagger}$, Mukund Rangamani$^{b\S}$

$^a$*Department of Theoretical Physics,Tata Institute of Fundamental Research,*

*Homi Bhabha Rd, Mumbai 400005, India*

$^b$ *Centre for Particle Theory & Department of Mathematical Sciences,*

*Science Laboratories, South Road, Durham DH1 3LE, United Kingdom*

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## Abstract

Black branes in AdS$_5$ appear in a four parameter family labeled by their velocity and temperature. Promoting these parameters to Goldstone modes or collective coordinate fields – arbitrary functions of the coordinates on the boundary of AdS$_5$ – we use Einstein's equations together with regularity requirements and boundary conditions to determine their dynamics. The resultant equations turn out to be those of boundary fluid dynamics, with specific values for fluid parameters. Our analysis is perturbative in the boundary derivative expansion but is valid for arbitrary amplitudes. Our work may be regarded as a derivation of the nonlinear equations of boundary fluid dynamics from gravity. As a concrete application we find an explicit expression for the expansion of this fluid stress tensor including terms up to second order in the derivative expansion.

$^*$sayanta@theory.tifr.res.in

$^\dagger$veronika.hubeny@durham.ac.uk

$^\ddagger$minwalla@theory.tifr.res.in

$^\S$mukund.rangamani@durham.ac.uk

# Contents

# 1    Introduction

The AdS/CFT correspondence provides an important laboratory to explore both gravitational physics as well as strongly coupled dynamics in a class of quantum field theories. Using this correspondence it is possible to test general lore about quantum field theory in a non perturbative setting and so learn general lessons about strongly coupled dynamics. Conversely, it is also possible to use the AdS/CFT duality to convert strongly held convictions about the behaviour of quantum field theories into general lessons about gravitational and stringy dynamics.

In this paper we use the AdS/CFT correspondence to study the effective description of strongly coupled conformal field theories at long wavelengths. On physical grounds it is reasonable that any interacting quantum field theory equilibrates locally at high enough energy densities, and so admits an effective description in terms of fluid dynamics. The variables of such a description are the local densities of all conserved charges together with the local fluid velocities. The equations of fluid dynamics are simply the equations of local conservation of the corresponding charge currents, supplemented by constitutive relations that express these currents as functions of fluid mechanical variables. As fluid dynamics is a long wavelength effective theory, these constitutive relations are usually specified in a derivative expansion. At any given order, thermodynamics plus symmetries determine the form of this expansion up to a finite number of undetermined coefficients. These coefficients may then be obtained either from measurements or from microscopic computations.

The best understood examples of the AdS/CFT correspondence relate the strongly coupled dynamics of certain conformal field theories to the dynamics of gravitational systems in AdS spaces. In this paper we will demonstrate that Einstein's equations with a negative cosmological constant, supplemented with appropriate regularity restrictions and boundary conditions, reduce to the nonlinear equations of fluid dynamics in an appropriate regime of parameters. We provide a systematic framework to construct this universal nonlinear fluid dynamics, order by order in a boundary derivative expansion. Our work builds on earlier derivations of linearized fluid dynamics from linearized gravity by Policastro, Son and Starinets [1] and on earlier examples of the duality between nonlinear fluid dynamics and gravity by Janik, some of the current authors and collaborators [2, 3, 4, 5, 6, 7, 8, 9] (*cf,* [10, 11] for some recent work). There is a large literature in deriving linearized hydrodynamics from AdS/CFT, see [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29] for developments in this area and [30] for a review and comprehensive set of references.

Our results, together with those of earlier papers referred to above, may be interpreted from several points of view. First, one may view them as a confirmation that fluid dynamics is the correct long wavelength effective description of strongly coupled field theory dynamics. Second, one could assume the correctness of the fluid description and view our results as providing information on the allowed singularities of 'legal' solutions of gravity.

Finally, our work may be used to extract the values of all coefficients of the various terms in the expansion of the stress tensor in the fluid dynamical derivative expansion, for the fluid dual to gravity on $AdS_5$. The universal behaviour of the shear viscosity - a coefficient of a term in the expansion of the stress tensor to first order in field theory derivatives - in fluids dual to gravity [22] has already attracted attention and has impacted experimental analysis of RHIC data [31, 32, 33]. In this paper we work out the universal values of all coefficients of (nonlinear) two derivative terms stress tensor of the distinguished conformal fluid dual to gravity on $AdS_5$.

Consider any two derivative theory of five dimensional gravity interacting with other fields, that has $AdS_5$ as a solution. Examples of such theories include IIB supergravity on $AdS_5 \times M$ where $M$ is any compact five dimensional Einstein manifold with positive cosmological constant; for example $M = S^5$, $T^{1,1}$ and $Y^{p,q}$ for all $p, q$. The solution space of such systems has a universal sub-sector; the solutions of pure gravity with a negative cosmological constant.[1] We will focus on this universal sub-sector in a particular long wavelength limit. Specifically, we study all solutions that tubewise[2] approximate black branes in $AdS_5$, whose temperature and boost velocities vary as a function of boundary coordinates $x^\mu$ on a length scale that is large compared to the inverse temperature of the brane. We investigate all such solutions order by order in a perturbative expansion; the perturbation parameter is the length scale of boundary variation divided by the thermal length scale. Within the domain of validity of our perturbative procedure (and subject to a technical assumption), we establish the existence of a one to one map between these gravitational solutions and the solutions of the equations of a distinguished system of boundary conformal fluid dynamics. Implementing our perturbative procedure to second order, we explicitly construct the fluid dynamical stress tensor of this distinguished fluid to second order in the derivative expansion.

Roughly speaking, our construction may be regarded as the 'Chiral Lagrangian' for brane horizons. Recall that the isometry group of $AdS_5$ is $SO(4,2)$. The Poincare algebra plus dilatations form a distinguished subalgebra of this group; one that acts mildly on the boundary. The rotations $SO(3)$ and translations $\mathbf{R}^{3,1}$ that belong to this subalgebra annihilate the static black brane solution in $AdS_5$. However the remaining symmetry generators – dilatations and boosts – act nontrivially on this brane, generating a 4 parameter set of brane solutions. These four parameters are simply the temperature and the velocity of the brane. Our construction effectively promotes these parameters to 'Goldstone fields' (or perhaps more accurately collective coordinate fields) and determines the effective dy-

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[1]Recall that the Einstein frame Lagrangian contains no interaction terms that are linear in the non gravitational fluctuations.

[2]We will work in AdS spacetimes where the radial coordinate $r \in (0, \infty)$ and will refer to the remaining coordinates $x^\mu = (v, x_i) \in \mathbf{R}^{1,3}$ as field theory or boundary coordinates. The tubes referred to in the text cover a small patch in field theory directions, but include all values of $r$ well separated from the black brane singularity at $r = 0$; typically $r \geq r_h$ where $r_h$ is the scale set by the putative horizon.

namics of these collective coordinate fields, order by order in the derivative expansion, but making no assumption about amplitudes. Of course the collective coordinates method has a distinguished tradition in theoretic physics; see for instance the derivation of the Nambu-Goto action in [34]. Our paper, which applies these methods to black brane horizons, is strongly reminiscent of the membrane paradigm of black hole physics, and may perhaps be regarded as the precise version of this paradigm in its natural setting, *i.e.*, AdS spacetime.

Seen from inverse point of view, our construction may be regarded as a map from solutions of the relativistic fluid dynamics equations on $\mathbf{R}^{3,1}$ to the space of long wavelength, locally black brane, solutions of gravity in $AdS_5$. That is, we present a systematic procedure to explicitly construct a metric dual to any solution of the equations of the distinguished fluid dynamics alluded to above. This metric solves the Einstein's equations to a given order in the derivative expansion (one higher than the order to which the equations of fluid dynamics were formulated and solved), asymptotes to $AdS_5$ with a boundary stress tensor equal to the fluid dynamical stress tensor, and is regular away from the usual singularity of black branes (chosen by convention to be at $r = 0$).

As an important physical input into our procedure, we follow [3, 4, 7] to demand that all the solutions we study are regular away from the $r = 0$ curvature singularity of black branes, and in particular at the the location of the horizon of the black brane tubes out of which our solution is constructed. We present our construction in the analogue of Eddington-Finklestein coordinates which extend all the way to the future curvature singularity. Although we have not yet performed a careful global analysis of our solutions, it seems rather clear that they each possess a regular event horizon that shields the boundary from this curvature singularity.

This paper is organized as follows. We begin in § 2 with the basic outline of the computation expanding on the ideas presented above. In § 3 we outline in detail the logic and strategy of our perturbative procedure. We then proceed in § 4 to implement our perturbative procedure to first order in the derivative expansion. In § 5 we extend our computation to second order in the same expansion. In § 6 we demonstrate the Weyl invariance of the fluid dynamical stress tensor we obtain, and further use this stress tensor to compute corrections to the dispersion relation for sound and shear waves in this fluid. In § 7 we end with a discussion of our results and of future directions.

**Note added:** After we had completed writing this paper we learnt of related work soon to appear [35]. The authors of this paper utilize Weyl invariance to constrain the form of the second order fluid dynamical stress tensor up to 5 undetermined coefficients. They then use information from linearized gravitational quasinormal mode calculations together with an earlier computation of Janik and collaborators to determine 3 of these five coefficients. As far as we have been able to tell, their results are consistent with the full second order stress tensor (and prediction for quasinormal mode frequencies) presented herein. This is a

nontrivial check of our results. We thank the authors of [35] for sharing their results with us prior to publication.

## 2 Fluid dynamics from gravity

We begin with a description of the procedure we use to construct a map from solutions of fluid dynamics to solutions of gravity. We then summarize the results obtained by implementing this procedure to second order in the derivative expansion.

Consider a theory of pure gravity with a negative cosmological constant. With a particular choice of units ($R_{AdS} = 1$) Einstein's equations are given by[3]

$$E_{MN} = R_{MN} - \frac{1}{2}g_{MN}R - 6\,g_{MN} = 0$$
$$\Longrightarrow R_{MN} + 4\,g_{MN} = 0, \qquad R = -20. \tag{2.1}$$

Of course the equations (2.1) admit AdS$_5$ solutions. Another class of solutions to these equations is given by the 'boosted black branes'[4]

$$ds^2 = -2\,u_\mu\,dx^\mu dr - r^2\,f(b\,r)\,u_\mu\,u_\nu\,dx^\mu dx^\nu + r^2\,P_{\mu\nu}\,dx^\mu dx^\nu\;, \tag{2.2}$$

with

$$f(r) = 1 - \frac{1}{r^4}$$
$$u^v = \frac{1}{\sqrt{1 - \beta_i^2}}$$
$$u^i = \frac{\beta_i}{\sqrt{1 - \beta_i^2}}\;, \tag{2.3}$$

where the temperature $T = \frac{1}{\pi b}$ and velocities $\beta_i$ are all constants, and

$$P^{\mu\nu} = u^\mu u^\nu + \eta^{\mu\nu} \tag{2.4}$$

is the projector onto spatial directions. The metrics (2.2) describe the uniform black brane written in ingoing Eddington-Finkelstein coordinates, at temperature $T$, moving at velocity

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[3]We use upper case Latin indices $\{M, N, \cdots\}$ to denote bulk directions, while lower case Greek indices $\{\mu, \nu, \cdots\}$ refer to field theory or boundary directions. Finally, we use lower case Latin indices $\{i, j, \cdots\}$ to denote the spatial directions in the boundary.

[4]The indices in the boundary are raised and lowered with the Minkowski metric *i.e.*, $u_\mu = \eta_{\mu\nu}\,u^\nu$.

$\beta^i$.[5]

Now consider the metric (2.2) with the constant parameter $b$ and the velocities $\beta_i$ replaced by slowly varying functions $b(x^\mu), \beta_i(x^\mu)$ of the boundary coordinates.

$$ds^2 = -2\, u_\mu(x^\alpha)\, dx^\mu\, dr - r^2\, f\left(b(x^\alpha)\, r\right)\, u_\mu(x^\alpha)\, u_\nu(x^\alpha)\, dx^\mu\, dx^\nu + r^2\, P_{\mu\nu}(x^\alpha)\, dx^\mu\, dx^\nu \ . \quad (2.6)$$

Generically, such a metric (we will denote it by $g^{(0)}(b(x^\mu), \beta_i(x^\mu))$) is not a solution to Einstein's equations. Nevertheless it has two attractive features. Firstly, away from $r = 0$, this deformed metric is everywhere non-singular. This pleasant feature is tied to our use of Eddington-Finkelstein[6] coordinates.[7] Secondly, if all derivatives of the parameters $b(x^\mu)$ and $\beta_i(x^\mu)$ are small, $g^{(0)}$ is tubewise[8] well approximated by a boosted black brane. Consequently, for slowly varying functions $b(x^\mu), \beta_i(x^\mu)$, it might seem intuitively plausible that (2.6) is a good approximation to a true solution of Einstein's equations with a regular event horizon. The main result of our paper is that this intuition is correct, provided the functions $b(x^\mu)$ and $\beta_i(x^\mu)$ obey a set of equations of motion, which turn out simply to be the equations of boundary fluid dynamics.

Einstein's equations, when evaluated on the metric $g^{(0)}$, yield terms of first and second order in field theory (*i.e.*, $(x_i, v) \equiv x^\mu$) derivatives of the temperature and velocity fields.[9] By performing a scaling of coordinates to set $b$ to unity (in a local patch), it is possible to show that field theory derivatives of either $\ln b(x^\mu)$ or $\beta_i(x^\mu)$ always appear together with a factor of $b$. As a result, the contribution of $n$ derivative terms to the Einstein's equations is suppressed (relative to terms with no derivatives) by a factor of $(b/L)^n \sim 1/(T\, L)^n$. Here $L$ is the length scale of variations of the temperature and velocity fields in the neighbourhood of a particular point, and $T$ is the temperature at that point. Therefore, provided $L\, T \gg 1$, it is sensible to solve Einstein's equations perturbatively in the number of field theory derivatives.[10]

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[5] As we have explained above, the 4 parameter set of metrics (2.2) may all be obtained from

$$ds^2 = 2\, dv\, dr - r^2\, f(r)\, dv^2 + r^2\, d\mathbf{x}^2 \ , \qquad (2.5)$$

with $f = 1 - \frac{1}{r^4}$ via a coordinate transform. The coordinate transformations in question are generated by a subalgebra of the isometry group of AdS$_5$.

[6]It is perhaps better to call these generalized Gaussian null coordinates as they are constructed with the aim of having the putative horizon located at the hypersurface $r(x^\mu) = r_h$.

[7]A similar ansatz for a black branes in (for instance) Fefferman-Graham coordinates *i.e.*, Schwarzchild like coordinates respecting Poincaré symmetry, is singular at $r\, b = 1$.

[8]As explained above, any given tube consists of all values of $r$ well separated from $r = 0$, but only a small region of the boundary coordinates $x^\mu$.

[9]As $g^{(0)}$ is an exact solution to Einstein's equations when these fields are constants, terms with no derivatives are absent from this expansion.

[10]Note that the variation in the radial direction, $r$, is never slow. Although we work order by order in the field theory derivatives, we will always solve all differential equations in the $r$ direction exactly.

In § 3 we formulate the perturbation theory described in the previous paragraph, and explicitly implement this expansion to second order in $1/(LT)$. As we have mentioned above it turns out to be possible to find a gravity solution dual to a boundary velocity and temperature profile only when these fields obey the equation of motion

$$\partial_\mu T^{\mu\nu} = 0 \tag{2.7}$$

where the rescaled[11] stress tensor stress tensor $T^{\mu\nu}$ given by

$$
\begin{aligned}
T^{\mu\nu} =& (\pi\,T)^4 \left(\eta^{\mu\nu} + 4\,u^\mu u^\nu\right) - 2\,(\pi\,T)^3\,\sigma^{\mu\nu} \\
& + (\pi T)^2 \left( (\ln 2)\,T_{2a}^{\mu\nu} + 2\,T_{2b}^{\mu\nu} + (2 - \ln 2)\left[\frac{1}{3}\,T_{2c}^{\mu\nu} + T_{2d}^{\mu\nu} + T_{2e}^{\mu\nu}\right]\right)
\end{aligned}
\tag{2.8}
$$

where

$$
\begin{aligned}
\sigma^{\mu\nu} &= P^{\mu\alpha} P^{\nu\beta}\,\partial_{(\alpha} u_{\beta)} - \frac{1}{3}\,P^{\mu\nu}\,\partial_\alpha u^\alpha \\
T_{2a}^{\mu\nu} &= \epsilon^{\alpha\beta\gamma(\mu}\,\sigma^{\nu)}_{\ \gamma}\,u_\alpha\,\ell_\beta \\
T_{2b}^{\mu\nu} &= \sigma^{\mu\alpha}\sigma^\nu_{\ \alpha} - \frac{1}{3}\,P^{\mu\nu}\,\sigma^{\alpha\beta}\sigma_{\alpha\beta} \\
T_{2c}^{\mu\nu} &= \partial_\alpha u^\alpha\,\sigma^{\mu\nu} \\
T_{2d}^{\mu\nu} &= \mathcal{D}u^\mu\,\mathcal{D}u^\nu - \frac{1}{3}\,P^{\mu\nu}\,\mathcal{D}u^\alpha\,\mathcal{D}u_\alpha \\
T_{2e}^{\mu\nu} &= P^{\mu\alpha}\,P^{\nu\beta}\,\mathcal{D}\left(\partial_{(\alpha} u_{\beta)}\right) - \frac{1}{3}\,P^{\mu\nu}\,P^{\alpha\beta}\,\mathcal{D}\left(\partial_\alpha u_\beta\right) \\
\ell_\mu &= \epsilon_{\alpha\beta\gamma\mu}\,u^\alpha \partial^\beta u^\gamma.
\end{aligned}
\tag{2.9}
$$

Our conventions are $\epsilon_{0123} = -\epsilon^{0123} = 1$ and $\mathcal{D} \equiv u^\alpha \partial_\alpha$ and the brackets () around the indices to denote symmetrization, i.e., $a^{(\alpha} b^{\beta)} = (a^\alpha b^\beta + a^\beta b^\alpha)/2$.

These constraints are simply the equations of fluid dynamics expanded to second order in the derivative expansion. The first few terms in the expansion (2.8) are familiar. The derivative free terms describe a perfect fluid with pressure (i.e., negative free energy density) $\pi^4\,T^4$, and so (via thermodynamics) entropy density $s = 4\pi^4\,T^3$. The viscosity $\eta$ of this fluid may be read off from the coefficient of $\sigma^{\mu\nu}$ and is given by $\pi^3\,T^3$. Notice that $\eta/s = 1/(4\pi)$, in agreement with the famous result of Policastro, Son and Starinets [1].

Our computation of the two derivative terms in (2.8) is new; the coefficients of these terms are presumably related to the various 'relaxation times' discussed in the literature (see for instance [36]). As promised earlier, the fact that we are dealing with a particular

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[11]Throughout this paper $T^{\mu\nu} = 16\pi\,G_5\,t^{\mu\nu}$ where $G_5$ is the five dimensional Newton and $t^{\mu\nu}$ is the conventionally defined stress tensor, i.e., the charge conjugate to translations of the coordinate $v$.

conformal fluid, one that is dual to gravitational dynamics in asymptotically AdS space-times, leads to the coefficients being determined as fixed numbers. It would be interesting to check whether the stress tensor determined above fits into the framework of the so called Israel-Stewart formalism [37] (see [36, 38] for reviews). R. Loganayagam [39] is currently investigating this issue.

In § 6.1 we have checked that the minimal covariantization of the stress tensor (2.8) transforms as $T^{\mu\nu} \to e^{-6\phi} T^{\mu\nu}$ under the Weyl transformation $\eta_{\mu\nu} \to e^{2\phi} \eta_{\mu\nu}$, $T \to e^{-\phi} T$, $u^\alpha \to e^{-\phi} u^\alpha$, for an arbitrary function $\phi(x^\mu)$.[12] This transformation (together with the manifest tracelessness of $T^{\mu\nu}$) ensures Weyl invariance of the fluid dynamical equation (2.7). Note that we have computed the fluid dynamical stress tensor only in flat space. The generalization of our expression above to an arbitrary curved space could well include contributions proportional to the spacetime curvature tensor. The fact that (2.8) is Weyl invariant by itself is a bit of a (pleasant) surprise. It implies that that the sum of all curvature dependent contributions to the stress tensor must be independently Weyl invariant.

# 3   The perturbative expansion

As we have described in § 2, our goal is to set up a perturbative procedure to solve Einstein's equations in asymptotically AdS spacetimes order by order in a boundary derivative expansion. In this section we will explain the structure of this perturbative expansion, and outline our implementation of this expansion to second order, leaving the details of computation to future sections.

## 3.1   The basic set up

In order to mathematically implement our perturbation theory, it is useful to regard $b$ and $\beta_i$ described in § 2 as functions of the rescaled field theory coordinates $\varepsilon x^\mu$ where $\varepsilon$ is a formal parameter that will eventually be set to unity. Notice that every derivative of $\beta_i$ or $b$ produces a power of $\varepsilon$, consequently powers of $\varepsilon$ count the number of derivatives. We now describe a procedure to solve Einstein's equations in a power series in $\varepsilon$. Consider the metric[13]

$$g = g^{(0)}(\beta_i, b) + \varepsilon\, g^{(1)}(\beta_i, b) + \varepsilon^2\, g^{(2)}(\beta_i, b) + \mathcal{O}\left(\varepsilon^3\right), \tag{3.1}$$

where $g^{(0)}$ is the metric (2.6) and $g^{(1)}, g^{(2)}$ etc are correction metrics that are yet to be determined. As we will explain below, perturbative solutions to the gravitational equations

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[12]R. Loganayagam [39] informs us that he has succeeded in rewriting our stress tensor in a number of different compact forms, one of which makes its covariance under Weyl transformations manifest.

[13]For convenience of notation we are dropping the spacetime indices in $g^{(n)}$. We also suppress the dependence of $b$ and $\beta_i$ on $x^\mu$.

exist only when the velocity and temperature fields obey certain equations of motion. These equations are corrected order by order in the $\varepsilon$ expansion; this forces us to correct the velocity and temperature fields themselves, order by order in this expansion. Consequently we set

$$\beta_i = \beta_i^{(0)} + \varepsilon\,\beta_i^{(1)} + \mathcal{O}\left(\varepsilon^2\right), \qquad b = b^{(0)} + \varepsilon\,b^{(1)} + \mathcal{O}\left(\varepsilon^2\right), \tag{3.2}$$

where $\beta_i^{(m)}$ and $b^{(n)}$ are all functions of $\varepsilon\,x^\mu$.

In order to proceed with the calculation, it will be useful to fix a gauge. We work with the 'background field' gauge

$$g_{rr} = 0\,, \qquad g_{r\mu} \propto u_\mu\,, \qquad \mathrm{Tr}\left((g^{(0)})^{-1}g^{(n)}\right) = 0 \quad \forall\, n > 0. \tag{3.3}$$

Notice that the gauge condition at the point $x^\mu$ is given only once we know $u_\mu(v, x^i)$. In other words, the choice above amounts to choosing different gauges for different solutions, and is conceptually similar to the background field gauge routinely used in effective action computations for non abelian gauge theories.

## 3.2   General structure of perturbation theory

Let us imagine that we have solved the perturbation theory to the $(n-1)^{\text{th}}$ order, $i.e.$, we have determined $g^{(m)}$ for $m \leq n-1$, and have determined the functions $\beta_i^{(m)}$ and $b^{(m)}$ for $m \leq n-2$. Plugging the expansion (3.1) into Einstein's equations, and extracting the coefficient of $\varepsilon^n$, we obtain an equation of the form

$$H\left[g^{(0)}(\beta_i^{(0)}, b^{(0)})\right] g^{(n)}(x^\mu) = s_n. \tag{3.4}$$

Here $H$ is a linear differential operator of second order in the variable $r$ alone. As $g^{(n)}$ is already of order $\varepsilon^n$, and since every boundary derivative appears with an additional power of $\varepsilon$, $H$ is an ultralocal operator in the field theory directions. It is important to note that $H$ is a differential operator only in the variable $r$ and does not depend on the variables $x^\mu$. Moreover, the precise form of this operator at the point $x^\mu$ depends only on the values of $\beta_i^{(0)}$ and $b^{(0)}$ at $x^\mu$ but not on the derivatives of these functions at that point. Furthermore, the operator $H$ is independent of $n$; we have the same homogeneous operator at every order in perturbation theory.

The source term $s_n$ however is different at different orders in perturbation theory. It is a local expression of $n^{\text{th}}$ order in boundary derivatives of $\beta_i^{(0)}$ and $b^{(0)}$, as well as of $(n-k)^{\text{th}}$ order in $\beta_i^{(k)}$, $b^{(k)}$ for all $k \leq n-1$. Note that $\beta_i^{(n)}$ and $b^{(n)}$ do not enter the $n^{\text{th}}$ order equations as constant (derivative free) shifts of velocities and temperatures solve the Einstein's equations.

The expressions (3.4) form a set of $5 \times 6/2 = 15$ equations. It turns out that four of

these equations do not involve the unknown function $g^{(n)}$ at all; they simply constrain the velocity functions $b$ and $\beta_i$. There is one redundancy among the remaining 11 equations which leaves 10 independent 'dynamical' equations. These may be used to solve for the 10 unknown functions in our gauge fixed metric correction $g^{(n)}$, as we describe in more detail below.

### 3.2.1 Constraint equations

By abuse of nomenclature, we will refer to those of the Einstein's equations that are of first order in $r$ derivatives as constraint equations. Constraint equations are obtained by dotting the tensor $E_{MN}$ with the vector dual to the one-form $dr$. Four of the five constraint equations (*i.e.*, those whose free index is a $\mu$ index) have an especially simple boundary interpretation; they are simply the equations of boundary energy momentum conservation. In the context of our perturbative analysis, these equations simply reduce to

$$\partial_\mu T^{\mu\nu}_{(n-1)} = 0 \tag{3.5}$$

where $T^{\mu\nu}_{(n-1)}$ is the boundary stress tensor dual the solution expanded up to $\mathcal{O}\left(\varepsilon^{n-1}\right)$. Recall that each of $g^0, g^{(1)}...$ are local functions of $b, \beta_i$. It follows that the stress tensor $T^{\mu\nu}_{(n-1)}$ is also a local function (with at most $n-1$ derivatives) of these temperature and velocity fields. Of course the stress tensor $T^{\mu\nu}_{(n-1)}$ also respects 4 dimensional conformal invariance. Consequently it is a 'fluid dynamical' stress tensor with $n-1$ derivatives, the term simply being used for the most general stress tensor (with $n-1$ derivatives), written as a function of $u^\mu$ and $T$, that respects all boundary symmetries.

Consequently, in order to solve the constraint equations at $n^{th}$ order one must solve the equations of fluid dynamics to $(n-1)^{th}$ order. As we have already been handed a solution to fluid dynamics at order $n-2$, all we need to do is to correct this solution to one higher order. Though the question of how one goes about improving this solution is not the topic of our paper (we wish only to establish a map between the solutions of fluid mechanics and gravity, not to investigate how to find the set of all such solutions) a few words in this connection may be in order. The only quantity in (3.5) that is not already known from the results of perturbation theory at lower orders are $\beta_i^{n-1}$ and $b^{(n-1)}$. The four equations (3.5) are linear differential equations in these unknowns that presumably always have a solution. There is a non-uniqueness in these solutions given by the zero modes obtained by linearizing the equations of stress energy conservation at zeroth order. These zero modes may always be absorbed into a redefinition of $\beta_i^{(0)}, b^{(0)}$, and so do not correspond to a physical non-uniqueness (*i.e.*, this ambiguity goes away once you specify more clearly what your zero order solution really is).

Our discussion so far may be summarized as follows: the first step in solving Einstein's

equations at $n^{\text{th}}$ order is to solve the constraint equations – this amounts to solving the equations of fluid dynamics at $(n-1)^{th}$ order (3.5). As we explain below, while it is of course difficult in general to solve these differential equations throughout $\mathbf{R}^{3,1}$, it is easy to solve them locally in a derivative expansion about any point; this is in fact sufficient to implement our ultralocal perturbative procedure.

### 3.2.2 Dynamical equations

The remaining constraint $E_{rr}$ and the 'dynamical' Einstein's equations $E_{\mu\nu}$ may be used to solve for the unknown function $g^{(n)}$. Roughly speaking, it turns out to be possible to make a judicious choice of variables such that the operator $H$ is converted into a decoupled system of first order differential operators. It is then simple to solve the equation (3.4) for an arbitrary source $s_n$ by direct integration. This procedure actually yields a whole linear space of solutions. The undetermined constants of integration in this procedure are arbitrary functions of $x^\mu$ and multiply zero modes of the operator (3.4). As we will see below, for an arbitrary non-singular and appropriately normalizable source $s_n$ (of the sort that one expects to be generated in perturbation theory[14]), it is always possible to choose these constants to ensure that $g^{(n)}$ is appropriately normalizable at $r = \infty$ and non-singular at all nonzero $r$. These requirements do not yet completely specify the solution for $g^{(n)}$, as $H$ possesses a set of zero modes that satisfy both these requirements. A basis for the linear space of zero modes, denoted $g_b$ and $g_i$, is obtained by differentiating the 4 parameter class of solutions (2.2) with respect to the parameters $b$ and $\beta_i$. In other words these zero modes correspond exactly to infinitesimal shifts of $\beta_i^{(0)}$ and $b^{(0)}$ and so may be absorbed into a redefinition of these quantities. They reflect only an ambiguity of convention, and may be fixed by a 'renormalization' prescription, as we will do below.

**Summary of the perturbation analysis:** In summary, it is always possible to find a physically unique solution for the metric $g^{(n)}$, which, in turn, yields the form of the $n^{\text{th}}$ order fluid dynamical stress tensor (using the usual AdS/CFT dictionary). This process, being iterative, can be used to recover the fluid dynamics stress tensor to any desired order in the derivative expansion.

In § 3.3 and § 3.4 we will provide a few more details of our perturbative procedure, in the context of implementing this procedure to first and second order in the derivative expansion.

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[14]Provided the solution at order $n-1$ is non-singular at all nonzero $r$, it is guaranteed to produce a non-singular source at all nonzero $r$. Consequently, the non-singularlity of $s_n$ follows inductively. We think is possible to to make a similar inductive argument for for the large $r$ behaviour of the source, but have not yet formulated this argument precisely enough to call it a proof.

### 3.3 Outline of the first order computation

We now present the strategy to implement the general procedure discussed above to first order in the derivative expansion.

#### 3.3.1 Solving the constraint equations

The Einstein constraint equations at first order require that the zero order velocity and temperature fields obey the equations of perfect fluid dynamics

$$\partial_\mu T^{\mu\nu}_{(0)} = 0 \ , \tag{3.6}$$

where up to an overall constant

$$T^{\mu\nu}_{(0)} = \frac{1}{(b^{(0)})^4} \left( \eta^{\mu\nu} + 4\, u^\mu_{(0)}\, u^\nu_{(0)} \right) . \tag{3.7}$$

While it is difficult to find the general solution to these equations at all $x^\mu$, in order to carry out our ultralocal perturbative procedure at a given point $y^\mu$, we only need to solve these constraints to first order a Taylor expansion of the fields $b$ and $\beta_i$ about the point $y^\mu$. This is, of course, easily achieved. The four equations (3.6) may be used to solve for the 4 derivatives of the temperature field at $y^\mu$ in terms of first derivatives of the velocity fields at the same point. This determines the Taylor expansion of $b$ to first order about $y^\mu$ in terms of the expansion, to first order, of the field $\beta_i$ about the same point. We will only require the first order terms in the Taylor expansion of velocity and temperature fields in order to compute $g^{(1)}(y^\mu)$.

#### 3.3.2 Solving the dynamical equations

As described in the previous section, we expand Einstein's equations to first order and find the equations (3.4). Using the 'solution' of § 3.3.1, all source terms may be regarded as functions of first derivatives of velocity fields only. The equations (3.4) are then easily integrated subject to boundary conditions and we find (3.4) is given by

$$g^{(1)} = g^{(1)}_P + f_b(x_i, v)\, g_b + f_i(x_j, v)\, g_i, \tag{3.8}$$

where $g^{(1)}_P$ is a particular solution to (3.4), and $f_b$ and $f_i$ are a basis for the zero modes of $H$ that were described in the § 3.2. Plugging in this solution, the full metric $g^{(0)} + g^{(1)}$, when expanded to order first order in $\varepsilon$, is (3.1)

$$g = g^{(0)} + \varepsilon \left( g^{(1)}_P + (f_b + b^{(1)})g_b + (f_i + \beta^{(1)})g_i \right), \tag{3.9}$$

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where the four functions of $x^\mu$, $f_b + b^{(1)}$, $f_i + \beta_i^{(1)}$ are all completely unconstrained by the equations at order $\varepsilon$.

### 3.3.3 The 'Landau' Frame

Our solution (3.8) for the first order metric has a four function non-uniqueness in it. As $f_b$ and $f_i$ may be absorbed into $b^{(1)}$ and $\beta_i^{(1)}$ this non-uniqueness simply represents an ambiguity of convention, and may be fixed by a 'renormalization' choice. We describe our choice below.

Given $g^{(1)}$, it is straightforward to use the AdS/CFT correspondence to recover the stress tensor. To first order in $\varepsilon$ the boundary stress tensor dual to the metric (3.9) evaluates to

$$T^{\mu\nu} = \frac{1}{b^4}\left(\eta^{\mu\nu} + 4\,u^\mu u^\nu\right) - \frac{2}{b^3}T^{\mu\nu}_{(1)}, \tag{3.10}$$

where

$$\begin{aligned} b &= b^{(0)} + \varepsilon(b^{(1)} + f_b) \\ \beta_i &= \varepsilon(\beta_i^{(1)} + f_i) \end{aligned} \tag{3.11}$$

where $T^{\mu\nu}_{(1)}$, defined by (3.10), is an expression linear in $x^\mu$ derivatives of the velocity fields and temperature fields. Notice that our definition of $T^{\mu\nu}_{(1)}$, via (3.10), depends explicitly on the value of the coefficients $f_i$, $f_b$ of the homogeneous modes of the differential equation (3.4). These coefficients depend on the specific choice of the particular solution $g^{(1)}_P$, which is of course ambiguous up to addition of homogenous solutions. Any given solution (3.8) may be broken up in many different ways into particular and homogeneous solutions, resulting in an ambiguity of shifts of the coefficients of $f_b$, $f_i$ and thereby an ambiguity in $T^{\mu\nu}_{(1)}$. It is always possible to use the freedom provided by this ambiguity to set $u_{(0)\mu}T^{\mu\nu}_{(1)} = 0$. This choice completely fixes the particular solution $g^{(1)}_P$. We adopt this convention particular solution and then simply simply set $g^{(1)} = g^{(1)}_P$ i.e. choose $f_b = f_i = 0$. $T^{\mu\nu}_{(1)}$ is now unambiguously defined and may be evaluated by explicit computation; it turns out that

$$T^{\mu\nu}_{(1)} = \sigma^{\mu\nu}.$$

The discussion of the previous paragraph has a natural generalization to perturbation theory at any order. As the operator $H$ is the same at every order in perturbation theory, the ambiguity for the solution of $g^{(n)}$ in perturbation theory is always of the form described in (3.9). We will always fix the ambiguity in this solution by choosing $u_\mu T^{\mu\nu}_{(k)} = 0$. The convention dependence of this procedure has a well known counterpart in fluid dynamics; it is simply the ambiguity of the stress tensor under field redefinitions of the temperature and $u^\mu$. Indeed this field redefinition ambiguity is standardly fixed by precisely the 'gauge'

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choice $u_\mu T^{\mu\nu}_{(1)} = 0$. This is the so called 'Landau frame' widely used in studies of fluid dynamics.

We present the details of the first order computation in § 4 below.

## 3.4 Outline of the second order computation

Assuming that we have implemented the first order calculation described in § 3.3, it is then possible to find a solution to Einstein's equations at the next order. In this case care should be taken in implementing the constraints as we discuss below.

### 3.4.1 The constraints at second order

The general discussion of § 3.2 allows us to obtain the second order solution to Einstein's equations once we have solved the first order system as outlined in § 3.3. However, we need to confront an important issue before proceeding, owing to the way we have set up the perturbation expansion. Of course perturbation theory at second order is well defined only once the first order equations have been solved. While in principle we should solve these equations everywhere in $\mathbf{R}^{3,1}$, in the previous subsection we did not quite achieve that; we were content to solve the constraint equation (3.6) only to first order in the Taylor expansion about our special point $y^\mu$. While that was good enough to obtain $g^{(1)}$, in order to carry out the second order calculation we first need to do better; we must ensure that the first order constraint is obeyed to second order in the Taylor expansion of the fields $b^{(0)}$ and $\beta^{(0)}_i$ about $y^\mu$. That is, we require

$$\partial_\lambda \partial_\mu T^{\mu\nu}_{(0)} (y^\alpha) = 0. \tag{3.12}$$

These equations may be thought of as a set of 16 linear constraints on the coefficients of the (40+78) two derivative terms involving $b^{(0)}$ and $\beta^{(0)}_i$. We use these equations to solve for 16 coefficients, and treat the remaining coefficients as independent. This process is the conceptual analogue of our zeroth order 'solution' of fluid dynamics at the point $y^\mu$ (described in the previous subsection), obtained by solving for the first derivatives of temperature in terms of the first derivatives of velocities. Indeed it is an extension of that procedure to the next order in derivatives. See § 5 for the details of the implementation of this procedure. In summary, before we even start trying to solve for $g^{(2)}$, we need to plug a solution of (3.12) into $g^{(0)} + g^{(1)}$ expanded in a Taylor series expansion about $y^\mu$. Otherwise we would be expanding the second order equations about a background that does not solve the first order fluid dynamics.

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### 3.4.2 Nature of source terms

As we have explained above, the Einstein's equations, to second order, take the schematic form described in (3.4)

$$H\left[g^{(0)}(\beta_i^{(0)}, b^{(0)})\right] g^{(2)} = s_a + s_b \tag{3.13}$$

We have broken up the source term above into two pieces, $s_a$ and $s_b$, for conceptual convenience. $s_a$ is a local functional of $\beta_i^{(0)}$ and $b^{(0)}$ of up to second order in field theory derivatives. Terms contributing to $s_a$ have their origin both in two field theory derivatives acting on the metric $g^{(0)}$ and exactly one field theory derivative acting on $g^{(1)}$ (recall that $g^{(1)}$ itself is a local function of $\beta_i^{(0)}$ and $b^{(0)}$ of first order in derivatives). The source term $s_b$ is new: it arises from first order derivatives of the velocity and temperature corrections $\beta_i^{(1)}$ and $b^{(1)}$. This has no analogue in the first order computation.

As we have explained above, $\beta_i^{(0)}, b^{(0)}$ are absolutely any functions that obey the equations (3.6). In particular, if it turns of that the functions $\beta_i^{(0)} + \varepsilon\,\beta_i^{(1)}$ and $b^{(0)} + \varepsilon\,b^{(1)}$ obey that equation (to first order in $\varepsilon$) then $\beta_i^{(1)}$ and $b^{(1)}$ may each simply be set to zero by an appropriate redefinition of $\beta_i^{(0)}$ and $b_i^{(0)}$. This results in a 'gauge' ambiguity of the functions $\beta_i^{(1)}, b^{(1)}$. In our ultralocal perturbative procedure, we choose to fix this ambiguity by setting $\beta_i^{(1)}$ to zero (at our distinguished point $y^\mu$) while leaving $b^{(1)}$ arbitrary.[15]

### 3.4.3 Solution of the constraint equations

With the source terms in place, the procedure to solve for $g^{(2)}$ proceeds in direct imitation of the first order calculation. The constraint equations reduce to the expansion to order $\varepsilon$ of the equation of conservation of the stress tensor

$$T^{\mu\nu} = \frac{1}{b^4}\,(4\,u^\mu u^\nu + \eta^{\mu\nu}) - 2\,\varepsilon\,\frac{1}{b^3}\,\sigma^{\mu\nu} \tag{3.14}$$

with $\beta^i = \beta^{(0)}$, $b = b^{(0)} + \varepsilon\,b^{(1)}$. These four equations may be used to solve for the four derivatives $\partial_\mu b^{(1)}$ at $x^\mu$. Consequently the constraint equations plus our choice of gauge, uniquely determined the first order correction of the temperature field $\beta^{(1)}$ and velocity field $\beta_i^{(1)}$ as a function of the zeroth order solution.

Note that the gauge $\beta_i^{(1)}(y^\mu) = 0$ may be consistently chosen at any one point $y^\mu$, but not at all $x^\mu$. Nonetheless the results for $g^{(2)}$ that we obtain using this gauge will, when appropriately covariantized be simultaneously applicable to every spacetime point $x^\mu$. The reason for this is that all source terms depend on $b^{(1)}$ and $\beta_i^{(1)}$ only through the expansion to order $\varepsilon$ of $\partial_\mu T^{\mu\nu} = 0$ with $T^{\mu\nu}$ given by (3.14). Note that this source term is

---

[15] The functions $b_i^{(1)}, \beta_i^{(1)}$ have sixteen independent first derivatives, all but four of which may be fixed by the gauge freedom. We choose use this freedom to set all velocity derivatives to zero.

'gauge invariant' (recall that 'gauge' transformations are simply shifts of $b^{(1)}$ and $\beta_i^{(1)}$ by zero modes of this equation). It follows that $g^{(2)}$ determined via this procedure does not depend on our choice of gauge, which was made purely for convenience.

### 3.4.4 Solving for $g^{(2)}$ and the second order stress tensor

Now plugging this solution for $b^{(1)}$ into the source terms it is straightforward to integrate (3.13) to obtain $g^{(2)}$. We fix the ambiguity in the choice of homogeneous mode in this solution as before, by requiring $T_{(2)}^{\mu\nu} u_{(0)\nu} = 0$. This condition yields a unique solution for $g^{(2)}$ as well as for the second order correction to the fluid dynamical stress tensor $T_{(2)}^{\mu\nu}$, giving rise to the result (2.9). We present the details of the second order computation in § 5.

In the rest of this paper we will present our implementation of our perturbative procedure described above, to first and second order in the derivative expansion.

## 4 The metric and stress tensor at first order

In this section we will determine the solution, to first order in the derivative expansion. As we have described in § 3, the equations that determine $g^{(1)}$ at $x^\mu$ are ultralocal; consequently we are able to solve the problem point by point. It is always possible to choose coordinates to set $u^\mu = (1, 0, 0, 0)$ and $b^{(0)} = 1$ at any given point $x^\mu$. Making that choice, the metric (2.6) expanded to first order in derivatives in the neighbourhood of $x^\mu$ (chosen to be the origin of $\mathbf{R}^{3,1}$ for notational simplicity) is given by

$$
\begin{aligned}
ds_{(0)}^2 = {}& 2\, dv\, dr - r^2\, f(r)\, dv^2 + r^2\, dx_i\, dx^i \\
& - 2\, x^\mu\, \partial_\mu \beta_i^{(0)}\, dx^i\, dr - 2\, x^\mu \partial_\mu \beta_i^{(0)}\, r^2\, (1 - f(r))\, dx^i\, dv - 4\, \frac{x^\mu\, \partial_\mu b^{(0)}}{r^2}\, dv^2 \ .
\end{aligned}
\tag{4.1}
$$

In order to implement the perturbation programme described in the previous section, we need to find the first order metric $g^{(1)}$ which, when added to (4.1), gives a solution to Einstein's equations to first order in derivatives.

The metric (4.1) together with $g^{(1)}$ has a background piece (the first line in (4.1)) which is simply the metric of a uniform black brane. In addition it has small first derivative corrections, some of which are known (the second line of (4.1)), and the remainder of which ($g^{(1)}$) we have to determine. Now note that the background black brane metric preserves a spatial $SO(3)$ rotational symmetry. This symmetry allows us to solve separately for the $SO(3)$ scalars, the $SO(3)$ vector and $SO(3)$ symmetric traceless two tensor ($\mathbf{5}$) components of $g^{(1)}$ and lies at the heart of the separability of the matrix valued linear operator $H$ into a set of ordinary linear operators.

In the following we will discuss each of these sectors separately and determine $g^{(1)}$. Subsequently, in § 4.4 we present the full solution to order $\varepsilon$ and proceed to calculate the stress tensor in § 4.5.

## 4.1   Scalars of $SO(3)$

The scalar components of $g^{(1)}$ are parameterized by the functions $h_1(r)$ and $k_1(r)$ according to[16]

$$g_{ii}^{(1)}(r) = 3\,r^2\,h_1(r)$$
$$g_{vv}^{(1)}(r) = \frac{k_1(r)}{r^2} \qquad\qquad (4.2)$$
$$g_{vr}^{(1)}(r) = -\frac{3}{2}\,h_1(r).$$

Here $g_{ii}^{(1)}$ and $g_{vr}^{(1)}$ are related to each other by our gauge choice $Tr((g^{(0)})^{-1}g^{(1)}) = 0$.

The scalar Einstein's equations (*i.e.*, those equations that transform as a scalar of $SO(3)$) may be divided up into constraints and dynamical equations. The constraint equations are obtained by contracting Einstein's equations (the first line of (2.1)) with the vector dual to the one form $dr$. The first scalar constraint is

$$r^2\,f(r)\,E_{vr} + E_{vv} = 0 \ , \qquad\qquad (4.3)$$

which evaluates to

$$\partial_v b^{(0)} = \frac{\partial_i \beta_i^{(0)}}{3} \ . \qquad\qquad (4.4)$$

Below, we will interpret (4.4) as the expansion of the fluid dynamical stress energy conservation, expanded to first order. The second constraint equation,

$$r^2\,f(r)\,E_{rr} + E_{vr} = 0 \ , \qquad\qquad (4.5)$$

leads to

$$12\,r^3\,h_1(r) + (3r^4 - 1)\,h_1'(r) - k_1'(r) = -6\,r^2\,\frac{\partial_i \beta_i^{(0)}}{3} \ . \qquad\qquad (4.6)$$

To this set of constraints we need add only one dynamical scalar equation,[17] the simplest of which turns out to be

$$5\,h_1'(r) + r\,h_1''(r) = 0 \ . \qquad\qquad (4.7)$$

---

[16]In the spatial $\mathbf{R}^3 \subset \mathbf{R}^{3,1}$ we will often for ease of notation, avoid the use of covariant and contravariant indices and adopt a summation convention for repeated indices *i.e.*, $g_{ii}^{(1)} = \sum_{i=1}^{3} g_{ii}^{(1)}$.

[17]We have explicitly checked that the equations listed here imply that the second dynamical equation is automatically satisfied.

The LHS of (4.7) and (4.6) are the restriction of the operator $H$ of (3.4) to the scalar sector. The RHS of the same equations are the scalar parts of the source terms $s_1$. Notice that $H$ is a first order operator in the variables $h_1'(r)$ and $k_1(r)$. Consequently the equation (4.7) may be integrated for an arbitrary source term. The resulting solution is regular at all nonzero $r$ provided that the source shares this property, and the growth $h_1(r)$ at infinity is slower than a constant – the behaviour of a non normalizable operator deformation – provided the source in (4.7) grows slower than $1/r$ at large $r$. Once $h_1(r)$ has been obtained $k_1(r)$ may be determined from (4.6) by integration, for an arbitrary source term. Once again, the solution will be regular and grows no faster than $r^3$ at large $r$, provided the source in that equation is regular and normalizable. The two source terms of this subsection satisfy these regularity and growth requirements, and it seems clear that this result will extend to arbitrary order in perturbation theory (see the next section).

The general solution to the system (4.6) and (4.7), obtained by the integration described above, is

$$h_1(r) = s + \frac{t}{r^4}, \qquad k_1(r) = \frac{2\, r^3\, \partial_i \beta_i^{(0)}}{3} + 3\, r^4\, s - \frac{t}{r^4} + u\ , \tag{4.8}$$

where $s, t$ and $u$ are arbitrary constants (in the variable $r$). In the solution above, the parameter $s$ multiplies a non normalizable mode (which represents a deformation of the field theory metric) and so is forced to zero by our boundary conditions. A linear combination of the pieces multiplied by $t$ and $u$ is generated by the action of the coordinate transformation $r' = r\, (1 + a/r^4)$ and so is pure gauge, and may be set to zero without loss of generality. The remaining coefficient $u$ corresponds to an infinitesimal temperature variation, and is forced to be zero by our renormalization condition on the stress tensor $u_{(0)}^{\mu}\, T_{\mu\nu} = 0$ (see the subsection on the stress tensor below). In summary, each of $s, t, u$ may be set to zero and the scalar part of the metric $g^{(1)}$, denoted $g_S^{(1)}$, is

$$\left( g_S^{(1)} \right)_{\alpha\beta} dx^\alpha dx^\beta = \frac{2}{3}\, r\, \partial_i \beta_i^{(0)}\, dv^2. \tag{4.9}$$

Two comments about this solution are in order. First note that $k_1(r)$ is manifestly regular at the unperturbed 'horizon' $r = 1$, as we require. Second, it grows at large $r$ like $r^3$. This is intermediate between the $r^0$ growth of finite energy fluctuations and the $r^4$ growth of a field theory metric deformation. As $g^{(0)} + g^{(1)}$ obeys the Einstein's equations to leading order in derivatives, the usual Fefferman-Graham expansion assures us that the sum of first order fluctuations in $g^{(0)} + g^{(1)}$ must (in the appropriate coordinate system) die off like $1/r^4$ compared to terms that appear in the zeroth order metric (this would correspond to $k_1(r)$ constant at infinity). Consequently the unusually slow fall off at infinity of our metric $g^{(1)}$ must be compensated for by an equal but opposite effect from a first order fluctuation piece in the second line of (4.1). This indeed turns out to be the

case. While an explicit computation of the boundary stress tensor dual to (4.1) yields a result that diverges like $r^3$, this divergence is precisely cancelled when we add $g^{(1)}$ above to the metric, and the correct value of the stress dual to $g^{(0)} + g^{(1)}$ is in fact zero in the scalar sector, in agreement with our renormalization condition $u_{(0)\mu} T^{\mu\nu} = 0$.

## 4.2   Vectors of $SO(3)$

In the vector channel the relevant Einstein's equations are the constraint $r^2 f(r) E_{ri} + E_{vi} = 0$ and a dynamical equation which can be chosen to be any linear combination of the Einstein's equations $E_{ri} = 0$ and $E_{vi} = 0$. The constraint evaluates to

$$\partial_i b^{(0)} = \partial_v \beta_i^{(0)} \ , \tag{4.10}$$

which we will later interpret as a consequence of the conservation of boundary momentum. In order to explore the content of the dynamical equation (we choose $E_{ri} = 0$), it is convenient to parameterize the vector part of the fluctuation metric by the functions $j_i^{(1)}$, as

$$\left(g_V^{(1)}\right)_{\alpha\beta} dx^\alpha dx^\beta = 2\, r^2\, (1 - f(r))\, j_i^{(1)}(r)\, dv\, dx^i. \tag{4.11}$$

The dynamical equation for $j_i(r)$ turns out to be

$$\frac{d}{dr}\left(\frac{1}{r^3}\frac{d}{dr}j_i^{(1)}(r)\right) = -\frac{3}{r^2}\,\partial_v \beta_i^{(0)}. \tag{4.12}$$

The LHS of (4.12) is the restriction of the operator $H$ of (3.4) to the vector sector, and the RHS of this equation is the projection of $s_1$ to the vector sector. $H$ is of first order in the variable $j^{(1)'}(r)$ and so may be integrated for an arbitrary source term. The resulting solution is regular and normalizable provided the source is regular and decays at infinity faster than $1/r$. This condition is obeyed in (4.12); it seems rather clear that it will continue to be obeyed at arbitrary order in perturbation theory (see the next section).

Returning to (4.12), the general solution of this equation is

$$j_i^{(1)}(r) = \partial_v \beta_i^{(0)}\, r^3 + a_i\, r^4 + c_i \tag{4.13}$$

for arbitrary constants $a_i, c_i$. The coefficient $a_i$ multiplies a non normalizable metric deformation, and so is forced to zero by our choice of boundary conditions. The other integration constant $c_i$ multiplies an infinitesimal shift in the velocity of the brane. It turns out (see below) that a nonzero value for $c_i$ leads to a nonzero value for $T_{0i}$ which violates our

'renormalization' condition, consequently $c_i$ must be set to zero. In summary,

$$\left(g_V^{(1)}\right)_{\alpha\beta} dx^\alpha dx^\beta = 2\,r\,\partial_v\beta_i\,dv\,dx^i. \tag{4.14}$$

As in the scalar sector above, this solution grows by a factor of $r^3$ faster at the boundary than the shear zero mode. This slow fall off leads to a divergent contribution to the stress tensor which precisely cancels an equal and opposite divergence from terms in the expansion of $g^{(0)}$ to first order in derivatives. As we will see below, the full contribution of $g^{(0)} + g^{(1)}$ to the vector part of the boundary stress tensor is just zero, again in agreement with our renormalization conditions.

## 4.3  The symmetric tensors of $SO(3)$

We now turn to $g_T^{(1)}$, the part of $g^{(1)}$ that transforms in the **5**, the symmetric traceless two tensor representation, of $SO(3)$. Let us parameterize our metric fluctuation by

$$\left(g_T^{(1)}\right)_{\alpha\beta} dx^\alpha\,dx^\beta = r^2\,\alpha_{ij}^{(1)}(r)\,dx^i\,dx^j, \tag{4.15}$$

where $\alpha_{ij}$ is traceless and symmetric. The Einstein's equation $E_{ij} = 0$ yield

$$\frac{d}{dr}\left(r^5\,f(r)\,\frac{d}{dr}\alpha_{ij}^{(1)}\right) = -6\,r^2\,\sigma_{ij}^{(0)}\ , \tag{4.16}$$

where we have defined a symmetric traceless matrix

$$\sigma_{ij}^{(0)} = \partial_{(i}\beta_{j)}^{(0)} - \frac{1}{3}\,\delta_{ij}\,\partial_m\beta_m^{(0)}\ . \tag{4.17}$$

The LHS of (4.16) is the restriction of the operator $H$ of (3.4) to the tensor sector, and the RHS of this equation is the tensor part of the source term $s_1$. Note that $H$ is a first order operator in the variable $\alpha_{ij}^{(1)\prime}(r)$ and so may be integrated for an arbitrary source term. The solution to this equation with arbitrary source term $s(r)$ is given by (dropping the tensor indices):

$$\alpha^{(1)} = -\int_r^\infty \frac{dx}{f(x)\,x^5} \int_1^x s(y)\,dy\ . \tag{4.18}$$

Note that the lower limit of the inner integral in (4.18) has been chosen to be unity. Provided that $s(x)$ is regular at $x = 1$ (this is true of (4.16) and will be true at every order in perturbation theory), $\int_1^x s(x)$ has a zero at $x = 1$. It follows that the outer integrand in (4.18) is regular at nonzero $x$ (and in particular at $x = 1$) despite the explicit zero in the factor $f(x)$ in the denominator. The solution for $\alpha^{(1)}$ is also normalizable provided the

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source is regular and grows at infinity slower than $r^3$. This condition is obeyed in (4.16) and is expected to continue to be obeyed at arbitrary order in perturbation theory (see the next section).

Applying (4.18) to the source term in (4.16) we find that the solution for $\alpha_{ij}^{(1)}$ is given by

$$(g_T^{(1)})_{\alpha\beta}\, dx^\alpha dx^\beta = 2\, r^2\, F(r)\, \sigma_{ij}^{(0)}\, dx^i dx^j. \tag{4.19}$$

with

$$F(r) = \int_r^\infty dx\, \frac{x^2 + x + 1}{x(x+1)\,(x^2+1)} = \frac{1}{4}\left[\ln\left(\frac{(1+r)^2(1+r^2)}{r^4}\right) - 2\,\arctan(r) + \pi\right] \tag{4.20}$$

At large $r$ it evaluates to

$$(g_T^{(1)})_{\alpha\beta}\, dx^\alpha dx^\beta = 2\left(r - \frac{1}{4\,r^2}\right)\sigma_{ij}^{(0)}\, dx^i dx^j. \tag{4.21}$$

As in the previous subsections, the first term in (4.21) yields a contribution to the stress tensor that diverges like $r^3$, but precisely cancels the corresponding divergence from first derivative terms in the expansion of $g^{(0)}$. However the second term in this expansion yields an important finite contribution to the stress tensor, as we will see below.

**Summary of the first order calculation:** In summary, our final answer for $g^{(0)} + g^{(1)}$, expanded to first order in boundary derivatives about $y^\mu = 0$, is given explicitly as

$$\begin{aligned}
ds^2 &= 2\, dv\, dr - r^2 f(r)\, dv^2 + r^2\, dx_i\, dx^i \\
&\quad - 2\, x^\mu\, \partial_\mu \beta_i^{(0)}\, dr\, dx^i - 2\, x^\mu\, \partial_\mu \beta_i^{(0)} r^2(1 - f(r))\, dv\, dx^i - 4\frac{x^\mu \partial_\mu b^{(0)}}{r^2}\, dv^2 \\
&\quad + 2\, r^2\, F(r)\, \sigma_{ij}^{(0)}\, dx^i\, dx^j + \frac{2}{3}\, r\, \partial_i \beta_i^{(0)}\, dv^2 + 2\, r\, \partial_v \beta_i^{(0)}\, dv\, dx^i.
\end{aligned} \tag{4.22}$$

This metric solves Einstein's equations to first order in the neighbourhood of $x^\mu = 0$ provided the functions $b^{(0)}$ and $\beta_i^{(0)}$ satisfy

$$\begin{aligned}
\partial_v b^{(0)} &= \frac{\partial_i \beta_i^{(0)}}{3} \\
\partial_i b^{(0)} &= \partial_v \beta_i^{(0)}.
\end{aligned} \tag{4.23}$$

## 4.4  Global solution to first order in derivatives

In the previous subsection we have computed the metric $g^{(1)}$ about $x^\mu$ assuming that $b^{(0)} = 1$ and $\beta_i^{(0)} = 0$ at the origin. Since it is possible to choose coordinates to set an

arbitrary velocity to zero and an arbitrary $b^{(0)}$ to unity at any given point (and since our perturbation procedure is ultralocal), the results of the previous subsection contain enough information to write down the metric $g^{(1)}$ about any point. A simple way to do this is to construct a covariant metric[18], as a function of $u_\mu$ and $b$, which reduces to (4.22) when $b^{(0)} = 1$ and $\beta_i^{(0)} = 0$. It is easy to check that

$$
\begin{aligned}
ds^2 &= -2\, u_\mu\, dx^\mu dr - r^2\, f(b\,r)\, u_\mu u_\nu\, dx^\mu dx^\nu + r^2\, P_{\mu\nu}\, dx^\mu dx^\nu \\
&\quad + 2\, r^2\, b\, F(b\,r)\, \sigma_{\mu\nu}\, dx^\mu dx^\nu + \frac{2}{3}\, r\, u_\mu u_\nu\, \partial_\lambda u^\lambda\, dx^\mu dx^\nu - r\, u^\lambda \partial_\lambda\left(u_\nu u_\mu\right)\, dx^\mu dx^\nu,
\end{aligned}
\tag{4.24}
$$

does the job, up to terms of second or higher order in derivatives. Here we have written the metric in terms of $\sigma_{\mu\nu}$ defined in (2.9) and the function $F(r)$ introduced in (4.20). Furthermore, it is easy to check that the metric above is the unique choice respecting the symmetries (again up to terms of second or higher order in derivatives). It follows that (4.24) is the metric $g^{(0)} + g^{(1)}$. It is also easily verified that the covariant version of (4.23) is (3.6). We will interpret this as an equation of stress energy conservation in the next subsection.

## 4.5 Stress tensor to first order

Given the solution to the first order equations, we can utilize the AdS/CFT dictionary to construct the boundary stress tensor using the prescription of [40] (see also [41]). For the metric (4.24) it is not difficult to compute the stress tensor; all we need to do is compute the extrinsic curvature tensor $K_{\mu\nu}$ to the surface at fixed $r$. By convention, we choose the unit normal to this surface to be outward pointing, *i.e.* pointing towards the boundary, in the definition of $K_{\mu\nu}$. Using then the definition

$$
T^\mu_\nu = -2 \lim_{r \to \infty} r^4 \left(K^\mu_\nu - \delta^\mu_\nu\right),
\tag{4.25}
$$

on our solution (4.24), we find the result is given simply as

$$
T^{\mu\nu} = \frac{1}{b^4}\left(4\, u^\mu u^\nu + \eta^{\mu\nu}\right) - \frac{2}{b^3}\, \sigma^{\mu\nu}.
\tag{4.26}
$$

where $\sigma^{\mu\nu}$ was defined in (2.9) and all field theory indices are raised and lowered with the boundary metric $\eta_{\mu\nu}$. As explained in the introduction, this stress tensor implies that the ratio of viscosity to entropy density of our fluid is $1/(4\pi)$. Note that as mentioned previously, the expression (4.26) is only correct up to first derivative terms in the temperature

---

[18]By abuse of notation, we will refer to expressions transformation covariantly in the boundary metric (chosen here to be $\eta_{\mu\nu}$) as covariant. In particular, we are not interested in full bulk covariance as we will continue to restrict attention to a specific coordinatization of the fifth direction.

$(T = 1/b)$ and velocities.

## 5  The metric and stress tensor at second order

In order to obtain the metric and stress tensor at second order in the derivative expansion, we follow the method outlined in § 3 and implemented in detail in § 4 to leading order. Concretely, we choose coordinates such that $\beta_i^{(0)} = 0$ and $b^{(0)} = 1$ at the point $x^\mu = 0$. The metric $g^{(0)} + g^{(1)}$ given in (4.24) may be expanded to second order in derivatives. This involves Taylor expanding $g^{(0)}$ to second order and $g^{(1)}$ to first order, the second order analogue of (4.1). As we have explained in § 3.4, at this stage we also make the substitution $b^{(0)} \to b^{(0)} + b^{(1)}$, and treat $b^{(1)}$ as an order $\varepsilon$ term, and so retain only those expressions that are of first derivative order in $b^{(1)}$ (and contain no other derivatives). This process is straightforward and we will not record the (rather lengthy) resultant expression here. To this expression we add the as yet undetermined metric fluctuation

$$g_{\alpha\beta}^{(2)} \, dx^\alpha dx^\beta = -3 \, h_2(r) \, dv \, dr + r^2 \, h_2(r) \, dx_i \, dx^i + \frac{k_2(r)}{r^2} \, dv^2 + 2 \frac{j_i^{(2)}(r)}{r^2} \, dv \, dx^i + r^2 \alpha_{ij}^{(2)} \, dx^i \, dx^j. \tag{5.1}$$

We plug this metric into Einstein's equations and obtain a set of linear second order differential equations that determine $h_2, k_2, j_i^{(2)}, \alpha_{ij}^{(2)}$. As in the previous section, $SO(3)$ symmetry ensures that the equations for the scalars $h_2, k_2$, the vectors $j_i^{(2)}$, and the tensor $\alpha_{ij}^{(2)}$ do not mix. Moreover, as we have explained in § 3, the equations that determine these unknown functions are identical to their first order counterparts in the homogeneous terms, but differ from those equations in the sources. As a result, the only new calculation we have to perform in order to obtain the metric at second order is the computation of the source terms. Once these terms are available, the corresponding equations may easily be integrated, as in the previous section.

Recall that the input metric into Einstein's equations includes terms that arise out of the Taylor expansion of $g^{(0)} + g^{(1)}$ that have explicit factors of the coordinates $x^\mu$. Nonetheless, a very simple argument assures us that the source terms in the equations that determine $g^{(2)}$ must all be independent of $x^\mu$. The argument runs as follows: We have explicitly constructed $g^{(1)}$ in the previous section so that $E_{MN}\left(g^{(0)} + g^{(1)}\right) = O_{MN}$ where $O_{MN}$ is a local expression constructed out of second order or higher $x^\mu$ derivatives of velocity and temperature fields. It follows that $x^\mu$ dependence of sources, which may be obtained by Taylor expanding $O_{MN}$ about $x^\mu = 0$, occurs only at the three derivative level or higher. It follows that source terms at the two derivative level have no $x^\mu$ dependence. Clearly, this argument has a direct analogue at arbitrary order in perturbation theory.

A crucial input into the argument of the last paragraph was the fact that $g^{(0)} + g^{(1)}$

satisfies Einstein's equations in a neighbourhood of $x^\mu = 0$ (and not just at that point). As we have seen in the previous section, the fact that the energy conservation equation is obeyed at $x^\mu = 0$ allows us to express all first derivatives of temperature in terms of first derivatives of velocities (see (4.10) and (4.4)). In addition, $\beta_i^{(0)}$ and $b^{(0)}$ must be chosen so that (3.12) is satisfied. The sixteen equations (3.12) can be grouped into sets that transform under $SO(3)$ as two scalars, three vectors and one tensor (i.e., 5). We will now explain how these constraints may be used to solve for 16 of the independent expressions of second order in derivatives of velocity and temperature fields.

In order to do this, let us first list all two derivative 'source' terms that can be built out of second derivatives of $b^{(0)}$ or $\beta_i^{(0)}$, or out of squares of first derivatives of $\beta_i^{(0)}$. These expressions may be separated according to their transformation properties under $SO(3)$ as scalars, vectors and tensors and higher order terms. The higher order pieces will not be of interest to us. An exhaustive list of these expressions that transform in the **1**, **3** or **5** is given in Table 1.[19] We define the vector $\ell_i$ as the curl of the velocity i.e.,

$$\ell_i = \epsilon_{ijk}\, \partial^j \beta^k \ , \tag{5.2}$$

and the symmetric traceless tensor $\sigma_{ij}$ has been previously defined in (4.17).

As a simple check on the completeness of expressions in Table 1, notice that the number of degrees of freedom in those of the tabulated expressions that are formed from a product of two single derivatives is 5 (in the scalar sector), $5 \times \mathbf{3}$ (in the vector sector), and $7 \times \mathbf{5}$ in the tensor sector, leading to a total of 55 real parameters. Together with degrees of freedom from the two **7**s and one **9** that can also be formed from the product of two derivatives (but will play no role in our analysis) this gives 78 degrees of freedom. This is in agreement with the expected $\frac{1}{2} \times 12 \times 13 = 78$ ways of getting a symmetric object from twelve parameters (the first derivatives of the velocity fields). On the other hand, the genuinely two derivative terms in Table 1 have $3 \times \mathbf{1} + 5 \times \mathbf{3} + 3 \times \mathbf{5} = 33$ degrees of freedom which together with a two derivative term that transforms in the **7** (which however plays no role in our analysis) is the expected number $40 = 10 \times 4$ of two derivative terms arising from temperature and velocity fields.

Assuming that we have already employed the first order conservation equation (3.6) to eliminate the first derivatives of $b$, we have to deal with the constraint equation (3.12) at the second order. Using the list of second order quantities given in Table 1, it is possible to show that (3.12) take the form of the following linear relations between these two derivative

---

[19]Note that the tensors are symmetric in their indices. The symmetrization as usual is indicated by parentheses.

| **1** of $SO(3)$ | **3** of $SO(3)$ | **5** of $SO(3)$ |
|---|---|---|
| $\mathbf{s}_1 = \frac{1}{b}\,\partial_v^2 b$ | $\mathbf{v}_{1i} = \frac{1}{b}\,\partial_i\partial_v b$ | $\mathbf{t}_{1ij} = \frac{1}{b}\,\partial_i\partial_j b - \frac{1}{3}\,\mathbf{s}_3\,\delta_{ij}$ |
| $\mathbf{s}_2 = \partial_v\partial_i\beta_i$ | $\mathbf{v}_{2i} = \partial_v^2\beta_i$ | $\mathbf{t}_{2ij} = \partial_{(i}\ell_{j)}$ |
| $\mathbf{s}_3 = \frac{1}{b}\,\partial^2 b$ | $\mathbf{v}_{3i} = \partial_v\ell_i$ | $\mathbf{t}_{3ij} = \partial_v\sigma_{ij}$ |
| $\mathfrak{S}_1 = \partial_v\beta_i\,\partial_v\beta_i$ | $\mathbf{v}_{4i} = \frac{9}{5}\partial_j\sigma_{ji} - \partial^2\beta_i$ | $\mathfrak{T}_{1ij} = \partial_v\beta_i\,\partial_v\beta_j - \frac{1}{3}\,\mathfrak{S}_1\,\delta_{ij}$ |
| $\mathfrak{S}_2 = \ell_i\,\partial_v\beta_i$ | $\mathbf{v}_{5i} = \partial^2\beta_i$ | $\mathfrak{T}_{2ij} = \ell_{(i}\,\partial_v\beta_{j)} - \frac{1}{3}\,\mathfrak{S}_2\,\delta_{ij}$ |
| $\mathfrak{S}_3 = (\partial_i\beta_i)^2$ | $\mathfrak{V}_{1i} = \frac{1}{3}(\partial_v\beta_i)(\partial_j\beta^j)$ | $\mathfrak{T}_{3ij} = 2\,\epsilon_{kl(i}\,\partial_v\beta^k\,\partial_{j)}\beta^l + \frac{2}{3}\,\mathfrak{S}_2\,\delta_{ij}$ |
| $\mathfrak{S}_4 = \ell_i\,\ell^i$ | $\mathfrak{V}_{2i} = -\epsilon_{ijk}\,\ell^j\,\partial_v\beta^k$ | $\mathfrak{T}_{4ij} = \partial_k\beta^k\,\sigma_{ij}$ |
| $\mathfrak{S}_5 = \sigma_{ij}\,\sigma^{ij}$ | $\mathfrak{V}_{3i} = \sigma_{ij}\,\partial_v\beta^j$ | $\mathfrak{T}_{5ij} = \ell_i\,\ell_j - \frac{1}{3}\,\mathfrak{S}_4\,\delta_{ij}$ |
|  | $\mathfrak{V}_{4i} = \ell_i\,\partial_j\beta^j$ | $\mathfrak{T}_{6ij} = \sigma_{ik}\,\sigma^k_j - \frac{1}{3}\,\mathfrak{S}_5\,\delta_{ij}$ |
|  | $\mathfrak{V}_{5i} = \sigma_{ij}\,\ell^j$ | $\mathfrak{T}_{7ij} = 2\,\epsilon_{mn(i}\,l^m\,\sigma^n_{j)}$ |

Table 1: An exhaustive list of two derivative terms in made up from the temperature and velocity fields. In order to present the results economically, we have dropped the superscript on the velocities $\beta_i$ and the inverse temperature $b$, leaving it implicit that these expressions are only valid at second order in the derivative expansion.

terms:

$$\mathbf{s}_1 = \frac{1}{3}\,\mathbf{s}_3 - \mathfrak{S}_1 + \frac{1}{9}\,\mathfrak{S}_3 + \frac{1}{6}\,\mathfrak{S}_4 - \frac{1}{3}\,\mathfrak{S}_5$$

$$\mathbf{s}_2 = \mathbf{s}_3 - \mathfrak{S}_1 + \frac{1}{2}\,\mathfrak{S}_4 - \mathfrak{S}_5$$

$$\mathbf{v}_{1i} = \frac{10}{9}\,\mathbf{v}_{4i} + \frac{1}{9}\,\mathbf{v}_{5i} + \frac{1}{3}\,\mathfrak{V}_{1i} - \frac{1}{3}\,\mathfrak{V}_{2i} - \frac{2}{3}\,\mathfrak{V}_{3i}$$

$$\mathbf{v}_{2i} = \frac{10}{9}\,\mathbf{v}_{4i} + \frac{1}{9}\,\mathbf{v}_{5i} - \frac{2}{3}\,\mathfrak{V}_{1i} + \frac{1}{6}\,\mathfrak{V}_{2i} - \frac{5}{3}\,\mathfrak{V}_{3i}$$

$$\mathbf{v}_{3i} = -\frac{1}{3}\,\mathfrak{V}_{4i} + \mathfrak{V}_{5i}$$

$$\mathbf{t}_{1ij} = \mathbf{t}_{3ij} + \mathfrak{T}_{1ij} + \frac{1}{3}\,\mathfrak{T}_{4ij} + \frac{1}{4}\,\mathfrak{T}_{5ij} + \mathfrak{T}_{6ij}\ .$$

(5.3)

Given these relations we now proceed to analyze the potential source terms arising from the metric (4.24) at $\mathcal{O}\left(\varepsilon^2\right)$. The analysis, as before, can be done sector by sector – the computations for the scalar, vector and tensor sectors are given in § 5.1, § 5.2 and § 5.3, respectively.

## 5.1   Solution in the scalar sector

Given the general second order fluctuation (5.1), we parameterize scalar components of $g^{(2)}$ in terms of the functions $h_2(r)$ and $k_2(r)$ according to

$$
\begin{aligned}
g_{ii}^{(2)}(r) &= 3\, r^2\, h_2(r) \\
g_{vv}^{(2)}(r) &= \frac{k_2(r)}{r^2} \\
g_{vr}^{(2)}(r) &= -\frac{3}{2} h_2(r) \ .
\end{aligned}
\tag{5.4}
$$

As we have explained in the § 4.1, the constraint Einstein's equations in this sector are given by the $r$ and $v$ component of the one-form formed by contracting the Einstein tensor with the vector dual to the one-form $dr$. The $v$ component of this constraint, *i.e.* the second order expansion of (4.3), evaluates to

$$
\frac{1}{b^{(0)}} \partial_v b^{(1)} = \frac{1}{b^{(1)}}\, \mathfrak{S}_5 \ .
\tag{5.5}
$$

This equation enables us to solve for the first $v$ derivative of $b^{(1)}$ in terms of two derivative terms made up of $\beta_i^{(0)}$, but imposes no further constraints on $b^{(0)}, \beta_i^{(0)}$. (5.5) has a simple physical interpretation; it is simply the time component of the conservation equation for the stress tensor (4.26), expanded to second order in derivatives. Consequently (5.5) is the Navier Stokes equation!

The $r$ component of the constraint, *i.e.* (4.5), gives us one relation between the functions $h_2(r)$ and $k_2(r)$ and their derivatives. As in § 4.1, to this constraint we must add one dynamical equation. We obtain the following equations

$$
\begin{aligned}
5\, h_2'(r) + r\, h_2''(r) &= S_h(r) \\
k_2'(r) &= S_k(r) \\
&= 12\, r^3\, h_2(r) + (3\, r^4 - 1)\, h_2'(r) + \widehat{S}_k(r) \ ,
\end{aligned}
\tag{5.6}
$$

26

where

$$S_h(r) \equiv \frac{1}{3\,r^3}\,\mathfrak{S}4 + \frac{1}{2}\,W_h(r)\,\mathfrak{S}5$$

$$\widehat{S}_k(r) \equiv -\frac{4\,r}{3}\,\mathbf{s}3 + 2\,r\,\mathfrak{S}1 - \frac{2\,r}{9}\,\mathfrak{S}3 + \frac{1 + 2\,r^4}{6\,r^3}\,\mathfrak{S}4 + \frac{1}{2}\,W_k(r)\,\mathfrak{S}5\;. \tag{5.7}$$

The functions $W_h(r)$ and $W_k(r)$ are given by

$$W_h(r) = \frac{4}{3}\,\frac{\left(r^2 + r + 1\right)^2 - 2\left(3\,r^2 + 2r + 1\right)\,F(r)}{r\,(r+1)^2\,(r^2+1)^2}\;,$$

$$W_k(r) = \frac{2}{3}\,\frac{4\left(r^2 + r + 1\right)\left(3\,r^4 - 1\right)F(r) - \left(2r^5 + 2r^4 + 2r^3 - r - 1\right)}{r\,(r+1)\,(r^2+1)}\;.$$

As advertised, it is clear that the differential operator acting on the functions $h_2(r)$ and $k_2(r)$ is identical to the one encountered in the first order computation in § 4.1. The equation (5.6) can be explicitly integrated; to do so it is useful to record the leading large $r$ behaviour of the source term $S_h(r)$:

$$S_h(r) \to \frac{1}{r^3}\,S_h^\infty \equiv \frac{1}{r^3}\left(\frac{1}{3}\,\mathfrak{S}4 + \frac{2}{3}\,\mathfrak{S}5\right)\;. \tag{5.8}$$

The first equation in (5.6) can be integrated given this asymptotic value to obtain the leading behaviour of the function $h_2(r)$. One finds

$$h_2(r) = -\frac{1}{r^2}\,S_h^\infty + \int_r^\infty \frac{dx}{x^5} \int_x^\infty dy\; y^4 \left(S_h(y) - \frac{1}{y^3}\,S_h^\infty\right). \tag{5.9}$$

The integral expression above can be shown to be of $\mathcal{O}\left(r^{-5}\right)$ and hence the asymptotic behaviour of $h_2(r)$ is controlled by $s_h$. Given $h_2(r)$, one can integrate up the second equation of (5.6) for $k_2(r)$. The leading large $r$ behaviour of the source term $S_k(r)$ is given by

$$S_k(r) \to r\,S_k^\infty \equiv r\left(-\frac{4}{3}\,\mathbf{s}3 + 2\,\mathfrak{S}1 - \frac{2}{9}\,\mathfrak{S}3 - \frac{1}{6}\,\mathfrak{S}4 - 14\,\mathfrak{S}5\right)\;, \tag{5.10}$$

and hence we have

$$k_2(r) = \frac{r^2}{2}\,S_k^\infty - \int_r^\infty dx\;\left(S_k(x) - x\,S_k^\infty\right)\;. \tag{5.11}$$

In this case the integral makes a subleading contribution starting at $\mathcal{O}\left(r^{-1}\right)$. As in § 4.1, we have chosen the coefficients of homogeneous solutions to this differential equation so as to ensure normalizability and vanishing scalar contribution to the stress tensor (according to our renormalization conditions).

## 5.2 Solution in the vector sector

The analysis of the vector fluctuations at second order mimics the computation described in § 4.2. The vector fluctuation in $g^{(2)}$ is chosen as described in (5.1) to be

$$g_{vi}^{(2)} = \frac{j_i^{(2)}}{r^2} \ . \tag{5.12}$$

Once again, the analysis is easily done by looking at the constraint equations which are obtained by contracting the tensor $E_{MN}$ with the vector dual to $dr$. The $i^{\text{th}}$ constraint equation evaluates to

$$18 \, \partial_i b^{(1)} = 5 \, \mathbf{v}_{4i} + 5 \, \mathbf{v}_{5i} + 15 \, \mathfrak{V}_{1i} - \frac{15}{4} \, \mathfrak{V}_{2i} - \frac{33}{2} \, \mathfrak{V}_{3i} \ . \tag{5.13}$$

This equation allows us to solve for the spatial derivatives of $b^{(1)}$ in terms of derivatives of $\beta_i^{(0)}$ and $b^{(0)}$. (5.13) is simply the expansion to second order in derivatives of the conservation of momentum of the stress tensor (4.26).

To complete the solution in the vector channel, we need to solve for $j^{(2)}(r)$, which can be shown to satisfy a dynamical equation

$$\frac{d}{dr} \left( \frac{1}{r^3} \frac{d}{dr} j_i^{(2)}(r) \right) = \mathbf{B}_i(r). \tag{5.14}$$

Note that the LHS of this expression has the vector part of the operator $H$ acting on $j^{(2)}$. Here $\mathbf{B}_i(r)$ is the source term which is built out of the second derivative terms transforming in the $\mathbf{3}$ of $SO(3)$ given in Table 1.

$$\mathbf{B}(r) = \frac{p(r) \, \mathbf{B}^\infty + \mathbf{B}^{\text{fin}}}{18 \, r^3 \, (r+1) \, (r^2+1)} \tag{5.15}$$

with

$$\begin{aligned}
\mathbf{B}^\infty &= 4 \, (10 \, \mathbf{v}_4 + \mathbf{v}_5 + 3 \, \mathfrak{V}_1 - 3 \, \mathfrak{V}_2 - 6 \, \mathfrak{V}_3) \\
\mathbf{B}^{\text{fin}} &= 9 \, (20 \, \mathbf{v}_4 - 5 \, \mathfrak{V}_2 + 6 \, \mathfrak{V}_3) \,,
\end{aligned} \tag{5.16}$$

and we have introduced the polynomial:

$$p(r) = 2 \, r^3 + 2 r^2 + 2 \, r - 3 \ . \tag{5.17}$$

Clearly $p(r)$ determines the large $r$ behaviour of the vector perturbation; asymptotically

$\mathbf{B}(r) \to \frac{1}{9\,r^3}\,\mathbf{B}^\infty$. Hence, integrating (5.14) we find that $j^{(2)}(r)$ is given as

$$j_i^{(2)}(r) = -\frac{r^2}{36}\,\mathbf{B}_i^\infty + \int_r^\infty dx\, x^3 \int_x^\infty dy\, \left(\mathbf{B}_i(y) - \frac{1}{9\,y^3}\,\mathbf{B}_i^\infty\right)\,, \qquad (5.18)$$

where once again we have chosen the coefficients of homogeneous modes in order to maintain normalizability and our renormalization condition. As with the first order computation described in § 4.2, the solution (5.18) makes no contribution to the stress tensor of the field theory.

### 5.3   Solution in the tensor sector

Finally, we turn to the tensor modes at second order where we shall recover the explicit form of the second order contributions to the stress tensor. Our task is now to determine the functions $\alpha_{ij}^{(2)}(r)$ in (5.1). As in § 4.3, in the symmetric traceless sector of $SO(3)$ one has only the dynamical equation given by

$$\frac{1}{2\,r}\frac{d}{dr}\left[r^5\left(1-\frac{1}{r^4}\right)\frac{d}{dr}\alpha_{ij}^{(2)}(r)\right] = \mathbf{A}_{ij}(r) \qquad (5.19)$$

where

$$\mathbf{A}_{ij}(r) = \mathbf{a}_1(r)\left(\mathfrak{T}1_{ij} + \frac{1}{3}\mathfrak{T}4_{ij} + \mathbf{t}3_{ij}\right) + \mathbf{a}_5(r)\,\mathfrak{T}5_{ij} + \mathbf{a}_6(r)\,\mathfrak{T}6_{ij} - \frac{1}{4}\,\mathbf{a}_7(r)\,\mathfrak{T}7_{ij}$$

with the coefficient functions

$$\begin{aligned}
\mathbf{a}_1(r) &= \frac{3\,p(r)+11}{p(r)+5} - 3\,r\,F(r) \\
\mathbf{a}_5(r) &= \frac{1}{2}\left(1+\frac{1}{r^4}\right) \\
\mathbf{a}_6(r) &= \frac{4}{r^2}\,\frac{r^2\,p(r)+3\,r^2-r-1}{p(r)+5} - 6\,r\,F(r) \\
\mathbf{a}_7(r) &= 2\,\frac{p(r)+1}{p(r)+5} - 6\,r\,F(r)\,.
\end{aligned} \qquad (5.20)$$

The functions $F(r)$ and $p(r)$ are defined in (4.20) and (5.17), respectively.

The desired solution to (5.19) can be found by intergrating the right hand side of the equation twice and choosing the solution to the homogenous solution such that we retain regularity[20] at $r=1$ and appropriate normalizability at infinity. The solution with these

---

[20]We have imposed the requirement that all metric functions are well behaved in its neighbourhood of

properties is

$$\alpha_{ij}^2(r) = -\int_r^\infty \frac{dx}{x\,(x^4-1)} \int_1^x dy\, 2\,y\,\mathbf{A}_{ij}(y)$$

The quantity of prime interest to us is the leading large $r$ behaviour of $\alpha_{ij}^{(2)}$. This can be inferred from the expressions for the coefficient functions given in (5.20) and evaluates to

$$
\begin{aligned}
\alpha_{ij}^{(2)}(r) &= \frac{1}{r^2}\left(-\frac{1}{2}\,\mathfrak{T}7_{ij} + \mathfrak{T}6_{ij} - \frac{1}{4}\,\mathfrak{T}5_{ij}\right)\\
&\quad + \frac{1}{2r^4}\left[\left(1-\frac{\ln 2}{2}\right)\left(\frac{1}{3}\,\mathfrak{T}4_{ij} + \mathfrak{T}1_{ij} + \mathbf{t}3_{ij}\right) + \frac{\ln 2}{4}\,\mathfrak{T}7_{ij} + \mathfrak{T}6_{ij}\right]
\end{aligned}
\tag{5.21}
$$

The leading term here will give a divergent contribution to the stress tensor, which is necessary to cancel the divergence arising from the expansion of $g^{(0)} + g^{(1)}$ to second order. The subleading piece in (5.21) is the term that will provide us with the second order stress tensor. Before proceeding to evaluate the stress tensor we present the full solution to second order, appropriately covariantized.

## 5.4   Global solution to second order in derivatives

Consider the following metric

$$ds^2 = -2\,u_\mu\,dx^\mu dr - r^2\,f(b\,r)\,u_\mu u_\nu\,dx^\mu dx^\nu + r^2\,P_{\mu\nu}\,dx^\mu dx^\nu + 3\,b^2\,h_2(b\,r)\,u_\mu\,dx^\mu dr + \mathcal{G}_{\mu\nu}\,dx^\mu dx^\nu ,\tag{5.22}$$

where we have defined a symmetric tensor $\mathcal{G}_{\mu\nu}$ by combing the contributions in the field theory directions from the first and second order metrics $g^{(0)} + g^{(1)}$

$$
\begin{aligned}
\mathcal{G}_{\mu\nu} &= r^2\left(2\,b\,F(b\,r)\,\sigma_{\mu\nu} + b^2\,\alpha_{\mu\nu}^{(2)}(b\,r)\right) + \frac{1}{r^2}\left(\frac{2}{3}\,r^3\,\partial_\lambda u^\lambda\,u_\mu u_\nu + \frac{k_2(b\,r)}{b^2}\right)\\
&\quad + r^2\,b^2\,h_2(b\,r)\,P_{\mu\nu} + \frac{1}{r^2}\left(-2\,r^3\,\mathcal{D}u_\alpha + \frac{1}{b^2}\,j_\alpha^{(2)}(b\,r)\right)P_\nu^\alpha\,u_\mu .
\end{aligned}
\tag{5.23}
$$

The covariant expression for $\alpha_{\mu\nu}^{(2)}$ is given by (5.19) with the replacements

$$
\begin{aligned}
\mathfrak{T}1_{ij} &\rightarrow (T_{2d})_{\mu\nu}, & \mathfrak{T}4_{ij} &\rightarrow (T_{2c})_{\mu\nu}, & \mathfrak{T}5_{ij} &\rightarrow \ell_\mu\,\ell_\nu - \frac{1}{3}P_{\mu\nu}\,\ell^\alpha\ell_\alpha\\
\mathfrak{T}6_{ij} &\rightarrow (T_{2b})_{\mu\nu}, & \mathfrak{T}7_{ij} &\rightarrow 2\,(T_{2a})_{\mu\nu}, & \mathbf{t}3_{ij} &\rightarrow (T_{2e})_{\mu\nu} .
\end{aligned}
\tag{5.24}
$$

---

$r = 1$, a regular point in the spacetime manifold. Note that $r = 1$ will not represent the horizon of our perturbed solution, but may well lie very near this horizon manifold.

Further, $j_\mu^{(2)}$ given by (5.18) with $\mathbf{B}_i(r) \to B_\nu(r)$, where $\mathbf{B}_i(r)$ is given by (5.15) and we make the following replacements

$$
\mathbf{v}_{4i} \quad \to \quad \frac{9}{5} \left[ P_\nu^\alpha \, P^{\beta\gamma} \, \partial_\gamma \, \partial_{(\beta} u_{\alpha)} - \frac{1}{3} \, P^{\alpha\beta} \, P_\nu^\gamma \, \partial_\gamma \partial_\alpha u_\beta \right] - P^{\alpha\beta} \, \partial_\alpha \partial_\beta u_\nu \, , \qquad \mathbf{v}_{5i} \quad \to \quad P^{\alpha\beta} \partial_\alpha \partial_\beta u_\nu
$$

$$
\mathfrak{V}_{1i} \quad \to \quad \partial_\alpha u^\alpha \, \mathcal{D} u_\nu \, , \qquad \mathfrak{V}_{2i} \quad \to \quad \epsilon_{\alpha\beta\gamma\nu} \, u^\alpha \, \mathcal{D} u^\beta \, \ell^\gamma \, , \qquad \mathfrak{V}_{3i} \quad \to \quad \sigma_{\alpha\nu} \, \mathcal{D} u^\alpha \, .
$$

$$
(5.25)
$$

Finally, $h_2(r)$ and $k_2(r)$ are given by (5.9) and (5.11) respectively, and in the functions $S_h(r)$, $S_k(r)$, $S_h^\infty$ and $S_k^\infty$ defined in (5.8) and (5.10) we are required to make the replacements

$$
\begin{aligned}
\mathbf{s}_3 \quad &\to \quad \frac{1}{b^{(0)}} P^{\alpha\beta} \, \partial_\alpha \partial_\beta b^{(0)} \qquad \mathfrak{S}_1 \quad \to \quad \mathcal{D} u^\alpha \, \mathcal{D} u_\alpha \, , \qquad \mathfrak{S}_2 \quad \to \quad \ell_\mu \, \mathcal{D} u^\mu \\
\mathfrak{S}_3 \quad &\to \quad (\partial_\mu u^\mu)^2 \, , \qquad \mathfrak{S}_4 \quad \to \quad \ell_\mu \, \ell^\mu \, , \qquad \mathfrak{S}_5 \quad \to \quad \sigma_{\mu\nu} \, \sigma^{\mu\nu} \, .
\end{aligned}
$$

$$
(5.26)
$$

It may be checked that this metric is the unique (up to terms that differ at third or higher order in derivatives) covariant expression that reduces to two derivative solution determined in the previous subsections, in the neighbourhood of any point $y^\mu$ after making the coordinate change that sets $b^{(0)} = 1$ and $\beta_i^{(0)} = 0$ at that point. It follows that (5.22) is the desired metric $g^{(0)} + g^{(1)} + g^{(2)}$.

## 5.5 Stress tensor to second order

The stress tensor dual to the solution to second order described in § 5.4 can be obtained by using the standard formula (4.25). To determine the extrinsic curvature at large $r$, it suffices to know the asymptotic form of the metric since we are interested in terms that have a finite limit as we take $r \to \infty$. Consequently, in order to compute the stress tensor it is sufficient to replace the various functions of $r$ that have appeared in the computation in § 5.1, § 5.2 and § 5.3 by their large $r$ asymptotics. The stress tensor may the be computed in a straightforward fashion, yielding

$$
\begin{aligned}
(T_2)_{vv} &= (T_2)_{vi} = 0 \, , \\
(T_2)_{ij} &= -\frac{\ln 2}{4} \, \mathfrak{T}_{7ij} - \mathfrak{T}_{6ij} + \left( -1 + \frac{\ln 2}{2} \right) \left( \mathbf{t}_{3ij} + \mathfrak{T}_{1ij} + \frac{1}{3} \, \mathfrak{T}_{4ij} \right) \, .
\end{aligned}
$$

$$
(5.27)
$$

The vanishing of $(T_2)_{v\mu}$ is actually guaranteed by our renormalization condition. It is easy to check that the covariant form of the expression (5.27) is indeed the stress tensor quoted in (2.8). This result is the main prediction of our fluctuation analysis to second order in the derivative expansion.

# 6   Second order fluid dynamics

In the previous section we have derived the precise form of the fluid dynamical stress tensor dual to gravity on AdS$_5$ including all terms with no more than two derivatives. In this section we initiate a study of the physics of this stress tensor. In § 6.1 below we will demonstrate that our stress tensor transforms homogeneously under Weyl transformations. In § 6.2 we compute the dispersion relation for low frequency sound and shear waves that follows from our stress tensor.

## 6.1   Weyl transformation of the stress tensor

Thus far we have extracted the stress tensor for a conformal fluid in flat space $\mathbf{R}^{3,1}$. We would like to ensure that the second order stress tensor we have derived transforms homogeneously under Weyl rescaling. In order to check this we perform the obvious minimal covariantization of our stress tensor to generalize it to a fluid stress tensor about an arbitrary boundary metric $g_{\mu\nu}$.[21] and study its Weyl transformation properties.

Consider the Weyl transformation of the boundary metric

$$
\begin{aligned}
g_{\mu\nu} = e^{2\phi}\,\widetilde{g}_{\mu\nu} \;\;&\Rightarrow\;\; g^{\mu\nu} = e^{-2\phi}\widetilde{g}^{\mu\nu} \\
\&\;\; u^{\mu} = e^{-\phi}\,\widetilde{u}^{\mu}\,, \qquad & T = e^{-\phi}\widetilde{T}.
\end{aligned}
\tag{6.1}
$$

It is well known that the first order truncation of the stress tensor (2.8) transforms as $T^{\mu\nu} = e^{-6\phi}\,\widetilde{T}^{\mu\nu}$ under this transformation (see for instance Appendix D of [42]). We proceed to show that this transformation rule holds for the two derivative stress tensor as well. This transformation property, together with the tracelessness of the stress tensor, ensures Weyl invariance of the fluid dynamical equations $\nabla_{\mu}T^{\mu\nu}$, appropriate for a conformal fluid.

It follows from (6.1) that $P^{\mu\nu} = g^{\mu\nu} + u^{\mu}u^{\nu} = e^{-2\phi}\,\widetilde{P}^{\mu\nu}$. The Christoffel symbols transform as [42]

$$
\Gamma^{\nu}_{\lambda\mu} = \widetilde{\Gamma}^{\nu}_{\lambda\mu} + \delta^{\nu}_{\lambda}\,\partial_{\mu}\phi + \delta^{\nu}_{\mu}\,\partial_{\lambda}\phi - \widetilde{g}_{\lambda\mu}\,\widetilde{g}^{\nu\sigma}\,\partial_{\sigma}\phi\,.
$$

The transformation of the covariant derivative of $u^{\mu}$ is given by

$$
\nabla_{\mu}u^{\nu} = \partial_{\mu}u^{\nu} + \Gamma^{\nu}_{\mu\lambda}\,u^{\lambda} = e^{-\phi}\left[\widetilde{\nabla}_{\mu}\,\widetilde{u}^{\nu} + \delta^{\nu}_{\mu}\,\widetilde{u}^{\sigma}\,\partial_{\sigma}\phi - \widetilde{g}_{\mu\lambda}\,\widetilde{u}^{\lambda}\,\widetilde{g}^{\nu\sigma}\,\partial_{\sigma}\phi\right].
\tag{6.2}
$$

This equation can be used to derive the transformation of various quantities of interest in

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[21] All metrics in this subsection refer to the metric on the boundary, *i.e..*, the background spacetime on which the fluid is propagating.

fluid dynamics, such as the acceleration $a^\mu$, shear $\sigma^{\mu\nu}$, *etc.*.

$$\theta = \nabla_\mu u^\mu = e^{-\phi} \left( \widetilde{\nabla}_\mu \widetilde{u}^\mu + 3\, \widetilde{u}^\sigma\, \partial_\sigma \phi \right) = e^{-\phi} \left( \widetilde{\theta} + 3\, \widetilde{\mathcal{D}} \phi \right) \,,$$

$$a^\nu = \mathcal{D} u^\nu = u^\mu \nabla_\mu u^\nu = e^{-2\phi} \left( \widetilde{a}^\nu + \widetilde{P}^{\nu\sigma}\, \partial_\sigma \phi \right) ,$$

$$\sigma^{\mu\nu} = P^{\lambda(\mu} \nabla_\lambda u^{\nu)} - \frac{1}{3}\, P^{\mu\nu}\, \nabla_\lambda u^\lambda = e^{-3\phi}\, \widetilde{\sigma}^{\mu\nu}, \tag{6.3}$$

$$\ell^\mu = u_\alpha\, \epsilon^{\alpha\beta\gamma\mu} \nabla_\beta u_\gamma = e^{-2\phi}\, \widetilde{\ell}^\mu$$

where in the last equation we have accounted for the fact that all epsilon symbols in (2.9) should be generalized in curved space to their covariant counterparts. The objects with correct tensor transformation properties scale as metric determinants *i.e.*, $\epsilon_{\alpha\beta\gamma\delta} \propto \sqrt{g}$, and $\epsilon^{\alpha\beta\gamma\delta} \propto \frac{1}{\sqrt{g}}$, from which it is easy to infer their scaling behaviour under conformal transformations; in particular, $\epsilon_{\alpha\beta\gamma\delta} = e^{4\phi}\, \widetilde{\epsilon}_{\alpha\beta\gamma\delta}$ and $\epsilon^{\alpha\beta\gamma\delta} = e^{-4\phi}\, \widetilde{\epsilon}^{\alpha\beta\gamma\delta}$.

The Weyl transformation of the two derivative terms that occur in the stress tensor (2.9) is given by

$$\begin{aligned} T_A^{\mu\nu} &= e^{-4\phi}\, \widetilde{T}_A^{\mu\nu} \,, & \text{for } A = \{2a, 2b\} \\ T_B^{\mu\nu} &= e^{-4\phi} \left( \widetilde{T}_B^{\mu\nu} + \widetilde{\delta T}_B^{\mu\nu} \right) \,, & \text{for } B = \{2c, 2d, 2e\} \end{aligned} \tag{6.4}$$

where the inhomogeneous terms arising in the Weyl transformation are:

$$\delta T_{2c}^{\mu\nu} = 3\, \mathcal{D}\phi \left( \nabla^{(\mu} u^{\nu)} + u^{(\mu}\, a^{\nu)} - \frac{1}{3}\, \theta\, P^{\mu\nu} \right)$$

$$\delta T_{2d}^{\mu\nu} = 2\, a^{(\mu}\, \nabla^{\nu)} \phi + 2\, u^{(\mu}\, a^{\nu)}\, \mathcal{D}\phi - \frac{2}{3}\, a^\alpha\, \nabla_\alpha \phi$$

$$+ 2\, u^{(\mu}\, \nabla^{\nu)}\phi\, \mathcal{D}\phi + u^\mu\, u^\nu\, (\mathcal{D}\phi)^2 - \frac{1}{3}\, P^{\mu\nu}\, (\mathcal{D}\phi)^2 + \nabla^\mu \phi\, \nabla^\nu \phi - \frac{1}{3}\, P^{\mu\nu}\, \nabla^\alpha \phi\, \nabla_\alpha \phi$$

$$\delta T_{2e}^{\mu\nu} = -\nabla^{(\mu} u^{\nu)}\, \mathcal{D}\phi - 3\, u^{(\mu} a^{\nu)}\, \mathcal{D}\phi + \frac{1}{3}\, P^{\mu\nu}\, \theta\, \mathcal{D}\phi - 2\, a^{(\mu}\, \nabla^{\nu)}\phi + \frac{2}{3}\, P^{\mu\nu}\, a^\alpha\, \nabla_\alpha \phi$$

$$- u^\mu\, u^\nu (\mathcal{D}\phi)^2 + \frac{1}{3}\, P^{\mu\nu}\, (\mathcal{D}\phi)^2 - 2\, u^{(\mu}\, \nabla^{\nu)}\phi\, \mathcal{D}\phi - \nabla^\mu \phi\, \nabla^\nu \phi + \frac{1}{3}\, P^{\mu\nu}\, \nabla_\alpha \phi\, \nabla^\alpha \phi \tag{6.5}$$

While the conformal transformation involves the inhomogeneous terms presented in (6.5) we need to ensure that the full stress tensor is Weyl covariant. Satisfyingly, these inhomogeneous terms cancel among themselves in the precise combination that occurs in (2.8); consequently the linear combination of terms that occurs in the stress tensor transforms covariantly. Note that the cancelation of inhomogeneous terms depends sensitively on the ratio of coefficients of $T_{2c}$, $T_{2d}$ and $T_{2e}$; and so provides a check of our results. Note

however that $T_{2a}$ and $T_{2b}$ are separately Weyl covariant. In summary, our result for the two derivative stress tensor is a linear combination (with precisely determined coefficients) of three independently Weyl covariant forms, with scaling weight $-4$ (for upper indices).

Using the transformation of the temperature (6.1) it follow that the full stress tensor transforms under Weyl transformation as

$$T^{\mu\nu} = e^{-6\phi} \, \widetilde{T}^{\mu\nu} \, . \tag{6.6}$$

R. Loganayagam [39] informs us that he has found a compact way of rewriting our stress tensor $T^{\mu\nu}$ (2.8) that makes the Weyl invariance of each of its three pieces manifest.

## 6.2 Spectrum of small fluctuations

Consider a static bath of homogeneous fluid at temperature $T$. Given the two derivative stress tensor derived above (2.9), it is trivial to solve for the spectrum of small oscillations of fluid dynamical modes about this background. As the background is translationally invariant, these fluctuations can be taken to have the form

$$\begin{aligned}
\beta_i(v, x^j) &= \delta\beta_i \, e^{i\,\omega\,v + i\,k_j x^j} \\
T(v, x^j) &= 1 + \delta T \, e^{i\,\omega\,v + i\,k_j x^j}
\end{aligned} \tag{6.7}$$

Plugging (6.7) into the equations of fluid dynamics (2.7), and working to first order in $\delta\beta_i$ and $\delta T$, these equations reduce to a set of four homogeneous linear equations in the amplitudes $\delta\beta_i$ and $\delta T$. The coefficients of these equations are functions of $\omega$ and $k_i$. These equations have nontrivial solutions if and only if the matrix formed out of these coefficient functions has zero determinant. Setting the determinant of the matrix of coefficients to zero one can find the following two dispersion relations:

$$\text{Sound mode}: \quad \boldsymbol{\omega}(\mathbf{k}) = \pm \frac{\mathbf{k}}{\sqrt{3}} + \frac{i\,\mathbf{k}^2}{6} \pm \frac{(3 - \ln 4)}{24\sqrt{3}} \, \mathbf{k}^3 + \mathcal{O}\left(\mathbf{k}^4\right), \tag{6.8}$$

$$\text{Shear mode}: \quad \boldsymbol{\omega}(\mathbf{k}) = \frac{i\,\mathbf{k}^2}{4} + \frac{i}{32}\,(2 - \ln 2)\,\mathbf{k}^4 + \mathcal{O}\left(\mathbf{k}^6\right), \tag{6.9}$$

where we have defined the rescaled energy and momenta

$$\boldsymbol{\omega} = \frac{\omega}{\pi\,T} \, , \qquad \mathbf{k} = \frac{k}{\pi\,T} \, . \tag{6.10}$$

It would be interesting to check our prediction against the quasinormal mode analysis of [23].

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# 7   Discussion

We have demonstrated how to start from a general, stationary black brane solution describing perfect fluid dynamics and promote the parameters in the gravitational solution to physical fluctuation modes. This procedure allows us to set up a fluctuation analysis which can be used to extract the boundary stress tensor of fluids dual to gravity in asymptotically AdS$_5$ spacetimes, in a derivative expansion. Our procedure is ultralocal: we obtain our solution by patching together local tubes of the black brane solution into a global solution of Einstein's equations. The fact that our solutions tubewise approximate black branes (see [3, 4, 7] for related observations) is the gravitational analogue of the fact that the fluid dynamics approximation only works when the fluid is in local equilibrium. We find this structure of our solutions quite fascinating and feel that it might have the potential to teach us important lessons about black brane dynamics.

Equation (2.9) is a prediction for the stress tensor of all four dimensional conformal fluids that admit a dual gravitational description. As we have described in the introduction, there exists an infinite number of examples of conformal field theories with a gravitational dual that differ substantially in their field content, spectrum of operators, etc. Nonetheless, up to an overall normalization, each of these theories has the same fluid dynamical expansion! Consequently, the fluid dynamics described in this paper has a degree of universality associated with it. At the one derivative level, the fluid stress tensor has a single undetermined parameter - the shear viscosity. The value of $\eta$ that we find is in agreement with earlier work, $\eta/s = 1/(4\pi)$. This relationship has been shown to have a larger degree of universality than is apparent from our work; it applies to all field theories, whether conformal or not, that have a gravitational dual. This relationship has also been conjectured to act as a lower bound on the viscosity of a relativistic field theory. It would be interesting to investigate whether any of the new two derivative coefficients we have found in this paper display extended universality features and also whether they are sensitive to higher derivative terms as discussed recently for the shear viscosity to entropy ratio in [43, 44].

As we have remarked in § 2, it would be interesting to investigate whether our result for the stress tensor is consistent with the so called Israel-Stewart formulation of fluid dynamics [39], a framework that has been employed in several practical investigations of fluid flows.

Relatedly, we note that recent claims [45] that the RHIC plasma violate the viscosity to entropy bound referred to above are based on the analysis of RHIC plasma flows using first order fluid dynamics. However, a satisfactory analysis of these flows should include contributions from higher order terms in the fluid dynamical expansion. It is possible that the stress tensor derived in this paper will be useful in this regard.[22]

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[22]We thank O. Aharony for this suggestion.

It may be possible to use the formalism presented in this paper to obtain a better understanding of the formal structure of the fluid dynamical expansion of quantum field theories. In this context it is useful to recall that the spectrum of regular small oscillations about a uniform black brane hosts an infinite spectrum of quasinormal modes. In this paper we have effectively constructed the 'chiral Lagrangian' corresponding to those of the quasinormal fluctuations that are Goldstone modes (and so have zero frequency when at zero $k$). The remaining quasinormal modes played no role in our analysis, as they are nonperturbatively massive in the inverse temperature ($\omega \sim T = 1/b$). The existence of these non perturbative modes probably implies that the fluid dynamical expansion is asymptotic rather than convergent, and might allow us to predict the location of the first singularity in the Borel transform of this perturbation series.

Recall that metric fluctuation, for any asymptotically AdS$_5$ solution to Einstein's equations, decays at large $r$ like $1/r^4$ relative to the background. The coefficients of this $1/r^4$ decay are functions of the four field theory coordinates $x^\mu$; in a particular gauge these functions may be identified with the 9 components of the traceless boundary stress tensor. This stress tensor is constrained to obey the equations of energy momentum conservation, but is otherwise unconstrained by local analysis. The Fefferman-Graham [46] method (or equivalently the formalism of holographic renormalization, see [47, 48] for reviews) demonstrates that any such conserved stress tensor, regarded as a boundary condition to Einstein's equations, leads to a unique and well defined power series expansion (in $1/r$) of an asymptotically AdS metric. Local analysis near the boundary thus appears to indicate that the space of solutions to Einstein's equations in AdS space is parameterized by the set of all conserved energy momentum tensors in four dimensions. This would be very surprising from the dual field theory viewpoint, as a set of four equations does not define a well posed initial value problem for nine functions.

The results of our paper suggest a (perhaps not unanticipated) resolution to this puzzle. In the derivative expansion in which we work, all except a four function set of this naive nine function class of metrics are unacceptably singular and so do not constitute a legal solution to Einstein's equations. Generic data result in singularities that develop at a finite value of $r$ ($r = 1/b$ in our set up) and so are not easily visible in the Fefferman-Graham expansion, which is guaranteed to work only in an open neighbourhood of the boundary. The class of boundary stress tensors that generate acceptable metrics are parameterized by four functions ($\beta_i(x^\mu)$ and $b(x^\mu)$) rather than nine. These four functions are further constrained to obey the four equations of stress energy conservation. As four equations constitute a well defined[23] initial value problem for a set of four functions, the set of

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[23]Note that our notion of a well posed PDE system is simply that we do not have an under-constrained system of equations. We are not making any claim regarding the well posedness of generic initial data; only initial data in the regime of our perturbation analysis together with the boundary conditions is guaranteed to lead to regular solutions. The general question of global regularity of Navier-Stokes equation is of course

legal solutions to Einstein's equations are parameterized by data that consists of functions of 3 spatial rather than 4 spacetime boundary variables, in agreement with field theory expectations. It would of course be of very great interest to understand how these results of the previous paragraph generalize beyond the boundary derivative expansion.

In this context it is also relevant to note that the equations of fluid dynamics themselves develop singularities under certain situations. It would be interesting to investigate the gravitational dual of this process of singularity formation.[24] More generally, the map from solutions of fluid dynamics to solutions of gravity could allow one to use the insight gained from the hundred year long study of the equations of fluid dynamics to understand qualitatively new gravitational solutions. For example, one might hope that the inhomogeneous brane solutions discovered in the study of the Gregory-Laflamme transition could admit a description in terms of an appropriate fluid dynamical system.

As we have described, in this paper we have derived explicit formulae for the metric dual to any solution of the Navier-Stokes equations. We have not yet investigated the global structure of the resulting spacetime. It seems very plausible that (under suitable physical conditions) the spacetimes we have constructed have regular event horizons. The event horizon is a null surface; we expect it to closely approximate[25] the surface $r\, b(x^\mu) = 1$. If this is the case we should be able to compute an explicit expression for this surface order by order in perturbation theory. It may then be possible to use our understanding of the horizon to define a locally positive divergence entropy current (a 'pullback' of the natural vector field of tangents to null generators of the event horizon back onto the boundary might play a role in such a construction). In the most optimistic scenario such an exercise could relate classic results about the positivity of null congruence expansions (resulting from the Raychaudhuri's equation with the usual proviso of energy conditions) to the local positivity of entropy production in fluid dynamics; a result that would be of obvious interest. The language of dynamical horizons[26] [49] may well prove to be the appropriate framework for such a discussion. We hope to return to this intriguing issue in the future.

Recall that the construction presented in this paper yields the gravitational dual of every solution of the equations of fluid dynamics. Standard field theory lore asserts that generic field theory evolutions are well described by solutions to the equations of fluid dynamics in the regime of interest to this paper. Consequently the AdS/CFT correspondence seems to imply that the construction described here yields the generic legal solution of gravity in $\mathrm{AdS}_5$, within its domain of applicability. If our guess of the previous paragraph is correct – *i.e.*, if all our solutions possess a regular event horizon that shield the

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an interesting open problem.

[24]We thank D. Berenstein for discussions on this question.

[25]We would like to thank H. Reall for many useful discussions on this point.

[26]Note that the homogeneous spacetime background being simply the uniform black brane, has a trapped surface; under the fluctuations we generically expect the trapped surface to be generated earlier in the radial evolution *i.e.*, at a larger radial coordinate (assuming of course appropriate energy conditions).

boundary from the singularity – then our results appear to be of relevance to the cosmic censorship conjecture [50, 51]. However we emphasize that our analysis applies only in a long wavelength expansion, and so presumably does not apply to several classes of scenarios that putatively violate this conjecture. More physically, fluid dynamics applies only under the assumption of local thermal equilibrium. Presumably naked singularities (if they exist) are dual to 'far from local equilibrium' boundary physics.

Several other natural generalizations of the work presented in this paper immediately suggest themselves. First, the results of our paper are likely to have an analogue for $d$ dimensional gravitational theories with a negative cosmological constant for every $d \geq 4$. Second, it should be possible to extend the results of our paper to spaces that asymptote to $\text{AdS}_{d+1}$ in global coordinates (and whose dual description is, therefore, fluid dynamics on $S^{d-1} \times \mathbf{R}_t$). More ambitiously, it may be possible to extend the results of our work to field theories whose spacetime metric asymptotes to

$$ds^2 = \frac{dr^2}{r^2} + r^2 \, ds^2_{bdy}$$

for a more general class of metrics $ds^2_{bdy}$. In particular, the generalization to time dependent metrics would permit the study of the gravitational dual of forced fluids, a subject of interest to the study of turbulence. Finally, it should not be difficult to generalize our study to two derivative theories of gravity interacting with gauge fields. We expect that the dual description of such a system will be the fluid dynamics of a system with a number of additional conserved charges (equal to the number of commuting vector fields). Note however that unlike the uncharged system, this charged fluid dynamics will not be universal at nonlinear order, as gauged supergravities do not in general admit a consistent truncation to the Einstein-Maxwell sector. For instance couplings of the form $f(\phi)F_{\mu\nu}F^{\mu\nu}$, for an arbitrary scalar field $\phi$, constitute a source for $\phi$; this is an effect that plays an important role in studies of the attractor mechanism.

Despite this non universality, IIB supergravity on $\text{AdS}_5 \times S^5$ (for instance) should be dual to a completely well defined charged fluid dynamics. It would be of interest to use the methods of this paper determine the form of this fluid dynamical stress tensor. Among other things, this exercise would allow us to zero in on the origin of the worrying apparent discrepancy between the formulas of charged black hole thermodynamics and the formulas of fluid dynamics, as reported in [9].

It should prove relatively straightforward to generalize the study of this paper to the fluid dynamics of non conformal backgrounds of gravity. For example, Scherk-Schwarz compactifications of AdS spaces yield a particularly simple set of gravitational backgrounds dual to confining gauge theories. Indeed, a moment's thought is enough to convince oneself that the deconfined phase fluid dynamical stress tensor of the 2+1 dimensional confining

gauge theory (dual to Scherk-Schwarz compactified $\mathcal{N} = 4$ Yang Mills) is simply the dimensional reduction of the stress tensor of $d = 4$, $\mathcal{N} = 4$ Yang Mills (*i.e.*, the stress tensor presented in this paper, (2.8)) plus a constant additive piece. This seemingly trival additive piece is physically very important; it leads to qualitatively new phenomena. For instance, [52, 53] have plasmaballs and plasmarings; static finite lumps of fluid with a boundary. Such configurations have qualitatively new classes of excitations; localized collective coordinates associated with fluctuations of the boundary. These new collective coordinates will interact with those studied in this paper. If it proves to be technically possible, it would be fascinating to formulate and study the resulting dynamics.

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