# EIGENVALUE CORRELATIONS IN CONTINUUM ONE-DIMENSIONAL ANDERSON MODELS

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ABSTRACT. For a large class of one-dimensional, continuum random Schrö-dinger operators we prove an $N$-level Wegner estimate. Such estimates bound the probability that the corresponding finite volume Hamiltonians have $N$ eigenvalues in an energy interval $[a, b]$. Our bounds, which only employ basic Prüfer variable techniques, are proportional to the $N$-th power of the product of the size of the interval and the volume, and therefore, they demonstrate an absence of correlations between close eigenvalues at any order.

## 1. INTRODUCTION

We will consider a one-dimensional Anderson model in $L^2(\mathbb{R})$ of the form

$$
(1.1) \qquad H = H(\omega) = -\frac{d^2}{dx^2} + W(x) + V_\omega(x).
$$

The background potential $W$ is real-valued and periodic, $W(x + 1) = W(x)$. The random potential is given by

$$
(1.2) \qquad V_\omega = \sum_{n \in \mathbb{Z}} \eta_n(\omega) f_n.
$$

We will assume that the single site potentials $f_n$ are translates $f_n(x) = f(x - n)$ of a non-negative and bounded $f$ supported on $[-1, 0]$, which is strictly positive on a non-empty subinterval $I$ of $[-1, 0]$, i.e. there exist $C \geq c > 0$ such that

$$
(1.3) \qquad c\chi_I \leq f \leq C\chi_{[-1,0]}.
$$

For the random variables $\eta_n$ we assume that they are independent and identically distributed. We will also assume that the $\eta_n$ have a bounded density $\rho$ with compact support, i.e. $\|\rho\|_\infty < \infty$ and $\text{supp}(\rho) \subset [\eta_{\min}, \eta_{\max}]$. Our results and proofs allow to weaken these assumptions as we will discuss at the end of Section 4.

By $H_L = H_L(\omega)$ we denote the restriction of $H$ to $L^2(0, L)$ with Dirichlet boundary conditions.

Our main goal here is to prove the following $N$-level Wegner-type estimate for the distribution of the eigenvalues of the finite volume operators $H_L(\omega)$.

**Theorem 1.1.** *For every $N \in \mathbb{N}$ and $E_{max} \in \mathbb{R}$ there exists $C = C(N, E_{max})$ such that*

$$
(1.4) \qquad \mathbb{P}\left(H_L \text{ has at least } N \text{ eigenvalues in } [a, b]\right) \leq C|b - a|^N L^N
$$

*for all $L \in \mathbb{N}$ and intervals $[a, b]$ with $a < b \leq E_{max}$.*

Here, and throughout the rest of the paper, the number $C$ will denote a quantity that is independent of the length-scale $L$; it will depend on $f$, $\|W\|_\infty$, $\|\rho\|_\infty$, and supp$(\rho)$. When relevant, as in Theorem 1.1, we will indicate the dependence of $C$ on $N$ and $E_{\max}$.

We note that by translation invariance (1.4) holds for the restriction of $H$ to any interval $[\ell_1, \ell_2]$ with $L$ replaced by $|\ell_2 - \ell_1|$.

For $N = 1$ the bound (1.4) follows by Chebyshev from

$$(1.5) \qquad \mathbb{E}(\operatorname{tr} \chi_{[a,b]}(H_L)) \leq C|b - a|L,$$

where $\chi_{[a,b]}(H_L)$ denotes the spectral projection of $H_L$ onto the interval $[a, b]$. This is the classical Wegner estimate which has been studied in much detail due to its relevance in proofs of Anderson localization and for deriving regularity properties of the integrated density of states

$$(1.6) \qquad N(E) := \lim_{L \to \infty} \frac{1}{L} \mathbb{E}(\operatorname{tr} \chi_{(-\infty, E]}(H_L)).$$

It implies Lipshitz continuity of $N(E)$ and therefore existence and local boundedness of the density of states $N'(E)$ for almost all energies. In Section 3 we will give a short proof of (1.5) based on the methods used here, but this is known in much more generality. Combes, Hislop and Klopp [5] have recently proven a Wegner estimate in the form (1.5) in arbitrary dimension $d$, i.e. with linear dependence of the right hand side of (1.5) on $|b - a|$ as well as the box volume $L^d$ and allowing for single site potentials of small support as in (1.3).

For $N \geq 2$, Theorem 1.1 can be considered as a preliminary result on the absence of correlations (of arbitrary order) between close lying eigenvalues of $H_L$. Minami showed in [10] that results of this type are a crucial first step towards establishing Poisson statistics of the finite volume eigenvalues of the Anderson model in the localized energy regime. He considers the discrete Anderson model

$$(h_\omega u)(n) = (h_0 u)(n) + \eta_n(\omega)u(n)$$

for $u \in \ell^2(\mathbb{Z}^d)$, where $h_0$ denotes the discrete Laplacian. Minami's methods provide the bound

$$(1.7) \qquad \mathbb{P}\left(h_\omega^\Lambda \text{ has at least 2 eigenvalues in } [a, b]\right) \leq C|b - a|^2|\Lambda|^2$$

for restrictions $h_\omega^\Lambda$ of $h_\omega$ to finite boxes $\Lambda \subset \mathbb{Z}^d$ with volume $|\Lambda|$, a multi-dimensional version of (1.4) for $N = 2$. In the above form the bound (1.7) was stated in [9], where it was shown that bounds of this type can also be used to show simplicity of eigenvalues of the infinite volume Anderson model in exponentially localized intervals of the spectrum.

In attempts to prove Poisson statistics or simplicity of eigenvalues for *continuum* Anderson models, it turns out that finding an analogue of (1.7) is a central step, but also a serious obstacle. For multi-dimensional continuum Anderson models there is currently no replacement for the rank-one perturbation methods which were extensively used in [10] and also more recently in [2], where (1.7) was extended to general $N$ with a different method of proof (see also [7] for the proof of a related result using Minami's approach).

It is our ongoing bewilderment with "Minami's rank-one miracle" which has caused us to re-visit the question of eigenvalue statistics for one-dimensional continuum random Schrödinger operators. In fact, it was this setting where the first result

on Poisson statistics for random Schrödinger operators was proven by Molchanov in [12]. He studied the Markov type model

$$(1.8) \qquad -d^2/dt^2 + F(x_t(\omega)),$$

where $F$ is a smooth Morse function on a compact Riemannian manifold $K$ and $x_t$ is stationary Brownian motion on $K$. Having the detailed results from [6] and [11] on exponential localization for this model available, the hard part of the work in [12] was to control second moments of the integrated density of states. The case $N = 2$ of Theorem 1.1 can be seen as a version of this latter fact for the one-dimensional continuum Anderson model, thus substituting for Minami's miracle in a situation where rank one techniques don't apply.

As in [12], our main technical tool are Prüfer phases, which in one dimension count eigenvalues when interpreted as rotation numbers. In [12] they are studied as Markov processes, while for the Anderson model (1.1), (1.2) our proof of Theorem 1.1 will be rather elementary, using that the Prüfer phases at integer values of $x$ are Markov chains. The Prüfer phases capture the cumulative spatial effect of the coupling constants $\eta_n$ on eigenvalues. Similar mechanisms will have to be identified in higher dimension to gain physical insight into Minami's miracle and to prove an $N$-level Wegner estimate for multi-dimensional continuum models.

We mention that a combination of the ideas from [10] and [12] has also been used by Stoiciu in [14] to prove Poisson statistics for the roots of random paraorthogonal polynomials. Our proof of the $N$-level Wegner estimate is somewhat inspired by Stoiciu's proof of a corresponding 2-level result in his setting.

The $N$-level Wegner estimate (1.4) by itself does not imply Poisson statistics of eigenvalues. The latter is only expected in the localized energy regime, a fact which does not enter the proof of (1.4). In a future work we plan to show how localization bounds (in the form of fractional moment estimates for the resolvent) combine with a 2-level Wegner estimate to yield Poisson statistics for continuum Anderson models, similar to the approach in [10] and [11]. As fractional moment estimates are now available in the continuum for arbitrary dimension $d$, e.g. [1, 3], this part of the argument for Poisson statistics can be carried out for all $d$. Of course, proving a two-level Wegner estimate in the continuum for $d > 1$ remains the big open question.

## 2. Energy dependence of the Prüfer phase

Our main tools are Prüfer phases and amplitudes, which we introduce as follows: For a real potential $q \in L^1_{\mathrm{loc}}(\mathbb{R})$ and real parameters $c$, $E$ and $\theta$ let $u$ be the solution of

$$-u'' + qu = Eu$$

with $u(c) = \sin\theta$, $u'(c) = \cos\theta$. By regarding this solution and its derivative in polar coordinates, we define the Prüfer amplitude $R_c(x, E, \theta)$ and the Prüfer phase $\phi_c(x, E, \theta)$ by writing

$$(2.1) \quad u(x) = R_c(x, E, \theta)\sin\phi_c(x, E, \theta) \quad \text{and} \quad u'(x) = R_c(x, E, \theta)\cos\phi_c(x, E, \theta).$$

For fixed $E$, we declare $\phi_c(c, E, \theta) = \theta$ and require continuity of $\phi$ in $x$. In this manner we define uniquely the functions $R_c(x, E, \theta)$ and $\phi_c(x, E, \theta)$ which are jointly continuous in $x$ and $E$. We will also use the corresponding notations $u_c(x, E, \theta)$ and $u'_c(x, E, \theta)$ to denote the dependence of solutions and their $x$-derivative on the above parameters.

We will make heavy use of many well known properties of the Prüfer variables. In order to make our presentation self-contained, though, we will present proofs of a variety of facts in Section 5.

Throughout our proof we have $q = W + V_\omega$ defined through (1.1), (1.2) and we will leave the dependence of the Prüfer variables on the random parameters implicit. The Prüfer phase counts eigenvalues of $H_L$ in the sense that

$$(2.2) \qquad \phi_0\left(L, E_n(L), 0\right) = n\pi,$$

where $E_n(L)$ is the $n$-th eigenvalue of $H_L$. This is a consequence of the monotonicity of the Prüfer phase in energy, e.g. Corollary 5.5.

By (2.2) the number of eigenvalues of $H_L$ in $[a, b]$ is given by the number of integer multiples of $\pi$ in $[\phi_0(L, a, 0), \phi_0(L, b, 0)]$ and thus, up to an error of $\pm 1$, equal to $|\phi_0(L, b, 0) - \phi_0(L, a, 0)|/\pi$. To estimate the expectation of the number of eigenvalues, we will differentiate and integrate this with respect to $E$ and thus need a bound on the expectation of $\partial\phi_0/\partial E$. This is the content of Proposition 2.2. Here and later we will use

**Lemma 2.1.** *Under the above assumptions, for all $E_{max} \in \mathbb{R}$ there exists a number $C = C\left(E_{\max}\right)$ such that*

$$(2.3) \qquad \int_{\eta_{\min}}^{\eta_{\max}} R_n^{-2}(L, E, \theta)\, \rho(\eta_{n+1})\, d\,\eta_{n+1} \leq C\, \|\rho\|_\infty,$$

*for any $E \leq E_{max}$, $\theta \in \mathbb{R}$, and positive integers $n$ and $L$ with $n + 1 \leq L$.*

*Proof.* Using the product formula for the Prüfer amplitude, we may write

$$R_n^2(L, E, \theta) = R_n^2(n + 1, E, \theta)\, R_{n+1}^2\left(L, E, \phi_n(n + 1, E, \theta)\right).$$

Observe that the quantity $R_{n+1}^2\left(L, E, \phi_n(n + 1, E, \theta)\right)$ depends on the random coupling $\eta_{n+1}$ only through it's dependence on the phase $\phi_n(n + 1, E, \theta)$. We make the change of variables $t(\eta_{n+1}) = \phi_n(n + 1, E, \theta)$ and note that Lemma 5.4 implies

$$\frac{\partial t}{\partial \eta_{n+1}} = \frac{-1}{R_n^2(n + 1, E, \theta)} \int_n^{n+1} f_{n+1}(x)\, u_n^2(x, E, \theta)\, dx.$$

Basic solution estimates, e.g. Lemma 5.6 and Lemma 5.7, guarantee that there exist constants $C_1$ and $C_2$ for which

$$0 < C_1 \leq \left|\frac{\partial t}{\partial \eta_{n+1}}\right| \leq C_2 < \infty.$$

The above inequalities enable the bound

$$\int_{\eta_{\min}}^{\eta_{\max}} R_n^{-2}(L, E, \theta)\rho(\eta_{n+1})\, d\,\eta_{n+1} \leq C\, \|\rho\|_\infty \int_{t(\eta_{\max})}^{t(\eta_{\min})} R_{n+1}^{-2}(L, E, t)\, dt,$$

and they also ensure that $|t(\eta_{\max}) - t(\eta_{\min})| \leq C$. The result claimed in (2.3) now follows from the averaging formula for the Prüfer amplitude, see Corollary 5.3. $\quad\square$

**Proposition 2.2.** *For all $E_{max} \in \mathbb{R}$, there exists a number $C = C(E_{max})$ such that*

$$(2.4) \qquad \mathbb{E}\left(\frac{\partial\phi_0}{\partial E}(L, E, \theta)\right) \leq C\, L$$

*for all $E \leq E_{max}$, $L \in \mathbb{N}$, and $\theta \in \mathbb{R}$.*

One may regard Proposition 2.2 as an analogue of equation (6.15) found in [14].

*Proof.* Consider a fixed $E \leq E_{\max}$. In Corollary 5.5 of Section 5, we calculate the derivative of the Prüfer phase with respect to the energy, and the following bound readily follows:

$$(2.5) \qquad \frac{\partial \phi_0}{\partial E}(L, E, \theta) \leq \frac{1}{R_0^2(L, E, \theta)} \int_0^L R_0^2(x, E, \theta) \, dx$$

Using (2.5), it is easy to see that

$$(2.6) \qquad \frac{\partial \phi_0}{\partial E}(L, E, \theta) \leq C \sum_{n=0}^{L-1} \frac{R_0^2(n, E, \theta)}{R_0^2(L, E, \theta)} = C \sum_{n=0}^{L-1} \frac{1}{R_n^2(L, E, \phi_0(n, E, \theta))}.$$

In fact, the first inequality in (2.6) uses the basic solution estimate (5.10), and therefore, the constant $C$ depends only on $E_{\max}$. Moreover, the final equality above follows from the fact that $R_0^2(L, E, \theta) = R_0^2(n, E, \theta) R_n^2(L, E, \phi_0(n, E, \theta))$. We will now estimate the average of each term, $R_n^{-2}$, by a constant independent of $n \in \{0, \ldots, L-1\}$.

To bound the average of $R_n^{-2}(L, E, \phi_0(n, E, \theta))$, we first integrate over the random parameter $\eta_{n+1}$ and let $\hat{\mathbb{E}}(\cdot)$ denote integration with respect to the remaining variables $\eta_1, \eta_2, \ldots, \eta_n, \eta_{n+2}, \ldots, \eta_L$. Recalling that the random variable $\eta_j$ multiplies the single site potential supported on the interval $[j-1, j]$, we see that the phase $\phi_0(n, E, \theta)$ is determined by $\eta_1, \ldots, \eta_n$ and independent of $\eta_{n+1}$. Thus we infer from Lemma 2.1 that

$$\mathbb{E}\left(R_n^{-2}(L, E, \phi_0(n, E, \theta))\right) = \hat{\mathbb{E}}\left(\int_{\eta_{\min}}^{\eta_{\max}} R_n^{-2}(L, E, \phi_0(n, E, \theta)) \rho(\eta_{n+1}) d\eta_{n+1}\right)$$

$$\leq \hat{\mathbb{E}}(C\|\rho\|_\infty) = C\|\rho\|_\infty.$$

The summation in (2.6) completes the proof. $\qquad \square$

## 3. The Wegner estimate

While our main new result is the $N$-level Wegner estimate (1.4) for $N \geq 2$, we will now show how the Prüfer variable methods used here provide a proof of the classical Wegner estimate. For another proof of this result which uses one-dimensional techniques see [8].

**Theorem 3.1.** *For all $E_{max} \in \mathbb{R}$, there exists a number $C = C(E_{max})$ such that*

$$(3.1) \qquad \mathbb{P}\left(H_L \text{ has an eigenvalue in } [a, b]\right) \leq \mathbb{E}(tr \chi_{[a,b]}(H_L)) \leq C |b - a| L$$

*for all $L \in \mathbb{N}$ and intervals $[a, b]$ with $a < b \leq E_{max}$.*

The first part of (3.1) follows from Chebyshev's inequality. The main idea in the second part is that, by Proposition 2.2,

$$(3.2) \qquad \frac{1}{\pi}\mathbb{E}\left(\phi_0(L, b, 0) - \phi_0(L, a, 0)\right) \leq \frac{C}{\pi}|b - a| L$$

and that by the discussion at the beginning of Section 2 the left hand side of (3.2) is equal to $\mathbb{E}(tr \chi_{[a,b]}(H_L))$ up to an error of $\pm 1$. The error is due to the fact that $(\phi_0(L, b, 0) - \phi_0(L, a, 0))/\pi$ is insensitive to the particular boundary condition one chooses at $L$ in defining $H_L$. However, using Lemma 3.2 below, a simple observation

on averaging the number of lattice points in an interval over translations of the lattice, one can see that

$$\int_0^\pi \operatorname{tr} \chi_{[a,b]}(H_L^\gamma)\, d\gamma = \phi_0(L, b, 0) - \phi_0(L, a, 0).$$

Here $H_L^\gamma$ is the restriction of $H$ to $[0, L]$ with Dirichlet boundary condition at 0 and boundary condition $u(L)/u'(L) = \tan\gamma$ at $L$. A similar effect will be achieved in the proof of Theorem 3.1 by keeping $\gamma = 0$ and instead averaging over $\eta_L$ before exploiting Proposition 2.2.

**Lemma 3.2.** *Let $[\alpha, \beta] \subset \mathbb{R}$. One has that*

$$(3.3) \qquad \int_0^\pi \#\{(\pi\mathbb{Z} + x) \cap [\alpha, \beta]\}\, dx = \beta - \alpha.$$

*Proof.* The left hand side of (3.3) is invariant under translations of $[\alpha, \beta]$. Thus we may assume $\alpha = 0$. By writing the above expression in terms of characteristic functions, the integral is straight-forward to calculate. Clearly, for each $x \in [0, \pi]$,

$$\#\{(\pi\mathbb{Z} + x) \cap [0, \beta]\} = \sum_{j \in \mathbb{Z}} \chi_{[0,\beta]}(\pi j + x).$$

Let $n$ be a non-negative integer and $c \in [0, \pi)$ such that $\beta = n\pi + c$. It is easy to see that

$$\int_0^\pi \chi_{[0, n\pi + c]}(j\pi + x)\, dx = \begin{cases} \pi, & 0 \le j \le n - 1, \\ c, & j = n, \\ 0, & \text{else.} \end{cases}$$

Summing over $j$ yields $n\pi + c = \beta$ as claimed. $\qquad\square$

*Proof.* (of Theorem 3.1) As indicated above, from Chebyshev's inequality we have that

$$\begin{aligned} \mathbb{P}(H_L \text{ has an eigenvalue in } [a, b]) &\le \mathbb{E}\left(\operatorname{Tr}\left[\chi_{[a,b]}(H_L)\right]\right) \\ &= \hat{\mathbb{E}}\left(\int \operatorname{Tr}\left[\chi_{[a,b]}(H_L)\right] \rho(\eta_L)\, d\eta_L\right), \end{aligned}$$

where $\hat{\mathbb{E}}(\cdot)$ is the expectation with respect to all variables $\eta_1, \eta_2, \ldots, \eta_{L-1}$. Moreover, regarding the Prüfer phase as a rotation number, we have already concluded that

$$(3.4) \qquad \operatorname{Tr}\left[\chi_{[a,b]}(H_L)\right] = \#\{\pi\mathbb{Z} \cap [\phi_0(L, a, 0), \phi_0(L, b, 0)]\}.$$

Now we rewrite the phases and estimate the $\eta_L$-average. Due to the monotonicity of the phase $\phi_0$ with respect to the boundary condition, see Lemma 5.2, we may express the statement that $\pi n \in [\phi_0(L, a, 0), \phi_0(L, b, 0)]$ equivalently by

$$\phi_L(L - 1, a, \pi n) \in [\phi_L(L - 1, a, \phi_0(L, a, 0)), \phi_L(L - 1, a, \phi_0(L, b, 0))].$$

The defining properties of the Prüfer phase also immediately imply both

$$\phi_L(L - 1, a, \pi n) = \phi_L(L - 1, a, 0) + \pi n,$$

and

$$(3.5) \qquad \phi_L(L - 1, a, \phi_0(L, a, 0)) = \phi_0(L - 1, a, 0).$$

With the formula derived in Corollary 5.5 and the solution estimates found in Lemma 5.6, it is easy to see that the phase evaluated over a unit interval satisfies a uniform bound, in particular we have that for any $\theta$,

$$
\begin{aligned}
|\phi_L(L-1, a, \theta) - \phi_L(L-1, b, \theta)| &\leq \int_a^b \left| \frac{\partial \phi_L}{\partial E}(L-1, E, \theta) \right| dE \\
&\leq C |b-a|,
\end{aligned}
$$

(3.6)

the constant $C$ being independent of $L$, $\theta$, and the randomness $\omega$. Putting this together, we find that the trace appearing in equation (3.4) can be rewritten as

$$
\# \left\{ (\pi \mathbb{Z} + \phi_L(L-1, a, 0)) \cap [\phi_0(L-1, a, 0), \phi_L(L-1, a, \phi_0(L, b, 0))] \right\},
$$

and, moreover, the interval contained in the expression above is a subset of

(3.7) $$ [\phi_0(L-1, a, 0), \phi_0(L-1, b, 0) + C|b-a|] =: [\alpha, \beta]. $$

The containment claimed in (3.7) follows from (3.6) and the analogue of (3.5), valid with $a$ set equal to $b$. Observe that the larger interval, which we have labeled $[\alpha, \beta]$ to ease notation, is independent of $\eta_L$.

Our arguments above demonstrate that

$$
\mathbb{P}\left( \sigma(H_L) \cap [a, b] \neq \emptyset \right) \leq \hat{\mathbb{E}} \left( \int \# \left\{ (\pi \mathbb{Z} + \phi_L(L-1, a, 0)) \cap [\alpha, \beta] \right\} \rho(\eta_L) \, d\eta_L \right).
$$

Making the change of variables $\eta_L \to \phi_L(L-1, a, 0)$, see Lemma 2.1 for similar calculations, and applying Lemma 3.2, we find that the inner integral can be estimated from above by a quantity of the form

$$
C_1 |\beta - \alpha| = C_1 \left( C|b-a| + \int_a^b \frac{\partial \phi_0}{\partial E}(L-1, E, 0) \, dE \right).
$$

Here, as before, the fact that the relevant phase, $\phi_L(L-1, a, 0)$, is only defined over an interval of size 1 guarantees that the constant $C_1$ is independent of $L$. Hence,

$$
\begin{aligned}
\mathbb{P}\left( \sigma(H_L) \cap [a, b] \neq \emptyset \right) &\leq C C_1 |b-a| + C_1 \int_a^b \hat{\mathbb{E}} \left( \frac{\partial \phi_0}{\partial E}(L-1, E, 0) \right) dE \\
&\leq \tilde{C} |b-a| L,
\end{aligned}
$$

as claimed in (3.1). For the final inequality above, we used Proposition 2.2. $\qquad \square$

## 4. THE $N$-LEVEL WEGNER ESTIMATE

*Proof.* (of Theorem 1.1) Recalling that the case of $N = 1$ was covered in Theorem 3.1, we fix an integer $N \geq 2$. Again, we express the probability of finding eigenvalues in an interval in terms of the Prüfer phase. To simplify notation, we will write $\partial_E \phi_0 = \frac{\partial \phi_0}{\partial E}(L, E, \theta)$ as the dependence on the relevant parameters is

clear. One may estimate

$\mathbb{P}\left(\,H_L \text{ has N or more eigenvalues in } [a,b]\,\right)$

$$\leq \quad \mathbb{P}(\phi_0(L,b,0) - \phi_0(L,a,0) \geq (N-1)\pi) \;=\; \mathbb{P}\left(\int_a^b \partial_E \phi_0 \, dE \;\geq\; (N-1)\pi\right)$$

$$= \quad \mathbb{P}\left(\left(\int_a^b \partial_E \phi_0 \, dE\right)^N \;\geq\; ((N-1)\pi)^N\right) \leq C_1 \, \mathbb{E}\left(\left(\int_a^b \partial_E \phi_0 \, dE\right)^N\right)$$

$$\leq \quad C_1 \, |b-a|^{N-1} \, \mathbb{E}\left(\int_a^b (\partial_E \phi_0)^N \, dE\right),$$

and we have denoted by $C_1 = C_1(N) = ((N-1)\pi)^{-N}$. Using (2.6) from Proposition 2.2, we have that

$$(\partial_E \phi_0)^N(L, E, \theta) \;\leq\; C_2 \sum_{n_1=0}^{L-1} \cdots \sum_{n_N=0}^{L-1} \prod_{j=1}^{N} R_{n_j}^{-2}\left(L, E, \phi_0(n_j)\right),$$

where here, and for the rest of this argument, we have used the notation $\phi_0(n_j) = \phi_0(n_j, E, \theta)$, and moreover, the number $C_2$ is just the $N$-th power of the constant appearing in (2.6). The goal now, much like in the proof of Proposition 2.2, is to prove that

$$(4.1) \qquad\qquad \mathbb{E}\left(\prod_{j=1}^{N} R_{n_j}^{-2}\left(L, E, \phi_0(n_j)\right)\right) \leq C_3,$$

for some $C_3$ independent of the choice of parameters $(n_1, n_2, \dots, n_N)$.

We will discuss the case $N = 2$ in detail, and then comment on how our proof generalizes to arbitrary $N > 2$. Suppose $N = 2$. As indicated by (4.1), our goal is to estimate the quantity

$$(4.2) \qquad\qquad \mathbb{E}\left(R_{n_1}^{-2}\left(L, E, \phi_0(n_1)\right) \, R_{n_2}^{-2}\left(L, E, \phi_0(n_2)\right)\right)$$

for all $n_1, n_2 \in \{0, \dots, L-1\}$. It will suffice to consider the case $n_1 + 2 \leq n_2$. We can certainly assume $n_1 \leq n_2$. In the case $n_1 \in \{n_2 - 1, n_2\}$ one argues as follows: If $n_2 \geq L - 2$ the two factors in (4.2) satisfy uniform upper bounds by the basic solution estimates in Lemma 5.6. If $n_2 < L - 2$ one may write

$$R_{n_2}^2(L, E, \phi_0(n_2)) = R_{n_2}^2(n_2 + 2, E, \phi_0(n_2)) \, R_{n_2+2}^2(L, E, \phi_{n_2}(n_2 + 2, E, \phi_0(n_2))).$$

Here the amplitude $R_{n_2}^2(n_2 + 2, E, \phi_0(n_2))$, which is only defined over an interval of length 2, is uniformly bounded by Lemma 5.6 and thus can be dropped from our considerations.

Thus we need to provide a bound on (4.2) if $n_2 \geq n_1 + 2$. The main idea here is to condition on a value of the Prüfer phase between $n_1$ and $n_2$ and see that this conditioning provides sufficient independence to factorize the integral. Let $\psi$ denote the function defined by setting $\psi(\eta_1, \dots, \eta_{n_1+1}) = \phi_0(n_1 + 1)$ and observe that there is a bijection between the variables $(\eta_1, \dots, \eta_L)$ and $(\eta_1, \dots, \eta_{n_1-1}, \psi, \eta_{n_1+1}, \dots, \eta_L)$. In particular, note that $\eta_1, \dots, \eta_{n_1-1}$ determine $\phi_0(n_1 - 1)$, $\psi$ and $\eta_{n_1+1}$ determine $\phi_0(n_1)$, and $\phi_0(n_1 - 1)$ and $\phi_0(n_1)$ determine $\eta_{n_1}$.

Using Lemma 5.4 again, we see that

$$\frac{\partial \psi}{\partial \eta_{n_1}} = \frac{-1}{R_0^2(n_1+1, E, \theta)} \int_{n_1-1}^{n_1} f_{n_1}(x) u_0^2(x, E, \theta) \, dx$$

The estimates provided in Lemmas 5.6 and 5.7 guarantee that there exist constants $C_1$ and $C_2$ for which

$$(4.3) \qquad 0 < C_1 \leq \left| \frac{\partial \psi}{\partial \eta_{n_1}} \right| \leq C_2 < \infty.$$

Now given $\eta_1, \ldots, \eta_{n_1-1}$, the quantity $\psi$ is completely determined by the variables $\eta_{n_1}$ and $\eta_{n_1+1}$. Using monotonicity of the phase as a function of the couplings, we find that for all $\eta_{n_1}$ and $\eta_{n_1+1}$

$$(4.4) \qquad \text{Range}\,(\psi) \subset [a(\eta_1, \ldots, \eta_{n_1-1}),\, b(\eta_1, \ldots, \eta_{n_1-1})],$$

with

$$(4.5) \qquad |b(\eta_1, \ldots, \eta_{n_1-1}) - a(\eta_1, \ldots, \eta_{n_1-1})| \leq C,$$

where $C$ is a constant uniform in the parameters $\eta_1, \ldots, \eta_{n_1-1}$. This follows from the formula for the derivative of the phase with respect to the couplings, i.e. Lemma 5.4, and the fact that the solution estimates in Lemma 5.6 are uniform over intervals of fixed size (here size 2). Thus by setting the couplings $(\eta_{n_1}, \eta_{n_1+1})$ equal to either $(\eta_{\min}, \eta_{\min})$ or $(\eta_{\max}, \eta_{\max})$ we may, respectively, maximize or minimize the phase over an interval of size 2 and be assured that the relative difference is independent of the initial phase..

To focus on the relevant dependencies, let $\eta_{\text{rest}}$ denote all random couplings except $\eta_{n_1}$ and $\eta_{n_1+1}$ and $\hat{\mathbb{E}}$ integration with respect to $\eta_{\text{rest}}$. For fixed $\eta_{\text{rest}}$ write

$$g(\eta_{n_1}, \eta_{n_1+1}) = R_{n_1}^{-2}\left(L, E, \phi_0(n_1)\right) R_{n_2}^{-2}\left(L, E, \phi_0(n_2)\right).$$

We make the change of variables $\eta_{n_1} \mapsto \psi$ to find that

$$\mathbb{E}\left(g(\eta_{n_1}, \eta_{n_1+1})\right)$$
$$= \hat{\mathbb{E}}\left(\int_{\eta_{\min}}^{\eta_{\max}} \int_{\eta_{\min}}^{\eta_{\max}} g(\eta_{n_1}, \eta_{n_1+1})\, \rho(\eta_{n_1})\, \rho(\eta_{n_1+1})\, d\eta_{n_1}\, d\eta_{n_1+1}\right)$$
$$\leq C\|\rho\|_\infty \hat{\mathbb{E}}\Big(\int_{\eta_{\min}}^{\eta_{\max}} \int_{\psi(\eta_{\max})}^{\psi(\eta_{\min})} R_{n_1}^{-2}(L, E, \phi_{n_1+1}(n_1, E, \psi))$$
$$R_{n_2}^{-2}(L, E, \phi_{n_1+1}(n_2, E, \psi)) \rho(\eta_{n_1+1}) d\eta_{n_1+1} d\psi\Big),$$

where (4.3) was used. As $[\psi(\eta_{\max}), \psi(\eta_{\min})] \subset [a(\eta_1, \ldots, \eta_{n_1-1}), b(\eta_1, \ldots, \eta_{n_1-1})]$ by (4.4) and the latter is independent of $\eta_{n_1+1}$ we can estimate this further by

$$\leq C\, \|\rho\|_\infty \hat{\mathbb{E}}\Big( \int_{a(\eta_1, \ldots, \eta_{n_1-1})}^{b(\eta_1, \ldots, \eta_{n_1-1})} R_{n_2}^{-2}(L, E, \phi_{n_1+1}(n_2, E, \psi))$$
$$\int_{\eta_{\min}}^{\eta_{\max}} R_{n_1}^{-2}(L, E, \phi_{n_1+1}(n_1, E, \psi))\, \rho(\eta_{n_1+1})\, d\eta_{n_1+1}\, d\psi \Big).$$

Here we have also used that $R_{n_2}^{-2}(L, E, \phi_{n_1+1}(n_2, E, \psi))$ does not depend on $\eta_{n_1+1}$. The inner integral above can be bounded using the substitution

$$\eta_{n_1+1} \mapsto \phi_{n_1+1}(n_1, E, \psi)$$

and the argument in the proof of Lemma 2.1. We have proven that
(4.6)
$$\mathbb{E}\left(g(\eta_{n_1}, \eta_{n_1+1})\right) \leq C \, \|\rho\|_\infty^2 \, \hat{\mathbb{E}}\left(\int_{a(\eta_1,\ldots,\eta_{n_1-1})}^{b(\eta_1,\ldots,\eta_{n_1-1})} R_{n_2}^{-2}(L, E, \phi_{n_1+1}(n_2, E, \psi)) \, d\psi\right).$$

With $\psi \in [a(\eta_1,\ldots,\eta_{n_1-1}), b(\eta_1,\ldots,\eta_{n_1-1})]$ fixed, the phase $\phi_{n_1+1}(n_2, E, \psi)$ is independent of the random coupling $\eta_{n_2+1}$. For this reason, we may apply Lemma 2.1 again to carry out the $\eta_{n_2+1}$-average (which is a part of $\hat{\mathbb{E}}$). The claimed result now follows using (4.5).

We finally comment on the extension of our arguments to general $N > 2$. We only need to prove (4.1) for $L \geq 2N$, as for $L < 2N$ a deterministic $N$-dependent (but $L$-independent) bound for the product on the left hand side of (4.1) follows directly from Lemma 5.6. We may also assume without loss that

(4.7)
$$n_{j+1} > n_j \text{ for all } j = 0, \ldots, N-1.$$

This is seen by an argument as in the discussion after (4.2), showing that more general sequences $n_j$ can be reduced to this case by shifting each $n_j$ by no more than $2N$, giving an additional factor in the bound (4.1) which only depends on $N$.

In order to show the bound (4.1) under the assumption (4.7) we proceed as above, using the change of variables $(\eta_1, \ldots, \eta_L) \mapsto (\eta_1, \ldots, \eta_{n_1-1}, \psi, \eta_{n_1+1}, \ldots, \eta_L)$ and carrying the extra factors for $j = 3, \ldots, N$ throughout. In place of (4.6) one arrives at the bound

$$C \, \|\rho\|_\infty^2 \, \mathbb{E}_1 \int_{a(\eta_1,\ldots,\eta_{n_1-1})}^{b(\eta_1,\ldots,\eta_{n_1-1})} \mathbb{E}_2 \left(\prod_{j=2}^N R_{n_j}^{-2}(L, E, \phi_{n_1+1}(n_j, E, \psi))\right) d\psi,$$

where we have split $\hat{\mathbb{E}}$ into integrations $\mathbb{E}_1$ over the variables $(\eta_1, \ldots, \eta_{n_1-1})$ and $\mathbb{E}_2$ over $(\eta_{n_1+2}, \ldots, \eta_L)$. From here we proceed by induction as $\mathbb{E}_2(\ldots)$ is of the form (4.1) with $N$ replaced by $N-1$ and $\psi$-boundary condition at $n_1 + 1$ rather than $\theta$-boundary condition at 0. For the $\psi$-integration we use (4.5) again and the $(\eta_1, \ldots, \eta_{n_1-1})$ averages become trivial.

$\square$

All the results established in the previous three sections, including our main result Theorem 1.1, can be shown under considerably weaker assumptions with only minor changes in the proofs. The only effect of these generalizations is that $H_\omega$ is not ergodic any longer and thus the integrated density of states (1.6) may not exist. First of all, the background potential $W$ merely needs to be bounded rather than periodic and one could also include singularities as long as they are locally uniformly distributed. The single site potentials $f_n$ do not have to be translates of a fixed $f$, but merely need to satisfy an assumption of the type (1.3) on each interval $[n-1, n]$ with uniform constants $c$ and $C$ and intervals $I_n \subset [n-1, n]$ of uniform length. Finally, the coupling constants $\eta_n$ do not need to be identically distributed, as long as they are absolutely continuous with densities $\rho_n$ of uniformly bounded support and uniform $L^\infty$ bound.

## 5. Appendix: Basic facts

In this section, we will collect some basic facts about Prüfer variables and two basic a-priori solution estimates which we use repeatedly throughout the main text.

Lemmas 5.2 and 5.4 as well as their Corollaries 5.3 and 5.5 have been frequently used in connection with spectral averaging techniques, e.g. [4]. Solution estimates like Lemma 5.6 and 5.7 are standard tools in the theory of Sturm-Liouville operators. We provide their proofs merely to make the paper self-contained.

For given potential $q$ we define the Prüfer phase and amplitude as in Section 2 by (2.1). Observe that $\phi_c(x, E, \theta + \pi) = \phi_c(x, E, \theta) + \pi$.

**Lemma 5.1.** *For fixed $c$, $E$, and $\theta$, one has that*

$$(5.1) \qquad \frac{\partial}{\partial x} \ln \left[ R_c^2(x, E, \theta) \right] = (1 + q(x) - E) \sin \left( 2\phi_c(x, E, \theta) \right),$$

*and*

$$(5.2) \qquad \frac{\partial}{\partial x} \phi_c(x, E, \theta) = 1 - (1 + q(x) - E) \sin^2 \left( \phi_c(x, E, \theta) \right).$$

*Proof.* It is clear that $R_c^2(x, E, \theta) = u^2(x, E, \theta) + (u')^2(x, E, \theta)$, and (5.1) follows from a simple calculation. To see (5.2), observe the following two equations: $u' = R_c' \sin(\phi_c) + R_c \cos(\phi_c)\phi_c'$ and $(q - E)u = u'' = R_c' \cos(\phi_c) - R_c \sin(\phi_c)\phi_c'$. Solving for $\phi_c'$ yields (5.2). $\qquad\square$

**Lemma 5.2.** *For any $c$, $x$, $E$, and $\theta$, one has that*

$$(5.3) \qquad \frac{\partial}{\partial \theta} \phi_c(x, E, \theta) = R_c^{-2}(x, E, \theta).$$

*Proof.* Differentiating (5.2) with respect to $\theta$, one finds that

$$(5.4) \qquad \frac{\partial^2}{\partial \theta \partial x} \phi_c(x, E, \theta) = - \frac{\partial}{\partial x} \ln \left[ R_c^2(x, E, \theta) \right] \frac{\partial}{\partial \theta} \phi_c(x, E, \theta).$$

Using this, we conclude that

$$(5.5) \qquad \frac{\partial}{\partial x} \ln \left[ R_c^2(x, E, \theta) \frac{\partial}{\partial \theta} \phi_c(x, E, \theta) \right] = 0,$$

for almost every pair $(x, \theta)$. As $R_c^2(c, E, \theta)\frac{\partial}{\partial \theta}\phi_c(c, E, \theta) = 1$, one has that $\ln \left[ R_c^2(x, E, \theta)\frac{\partial}{\partial \theta}\phi_c(x, E, \theta) \right] = 0$ for almost every $\theta$; hence, for every $\theta$ by continuity. The formula (5.3) readily follows. $\qquad\square$

A simple consequence of $\phi_c(x, E, \theta + \pi) = \phi_c(x, E, \theta) + \pi$ and Lemma 5.2 is the following, well-known, averaging result for the Prüfer amplitudes.

**Corollary 5.3.** *For any $c$, $x$, $E$, and $\theta$, one has that*

$$(5.6) \qquad \frac{1}{\pi} \int_\theta^{\theta + \pi} R_c^{-2}(x, E, \theta') \, d\theta' = 1.$$

Similar to Lemma 5.2 one can give the following formula for the derivative of the Prüfer phase with respect to a coupling constant at a potential.

**Lemma 5.4.** *Let $V$ and $q$ be real valued functions in $L_{\mathrm{loc}}^1(\mathbb{R})$. For real parameters $E$ and $\lambda$, let $u$ be the real valued solution of*

$$-u'' + Vu + \lambda q u = Eu$$

*normalized so that $u(c) = \sin(\theta)$ and $u'(c) = \cos(\theta)$. One has that*

$$(5.7) \qquad \frac{\partial}{\partial \lambda} \phi_c(x, \lambda) = -R_c^{-2}(x, \lambda) \int_c^x q(t) \, u^2(t) \, dt.$$

Note that we only keep track of the dependence of the Prüfer variables on the relevant parameters $x$ and $\lambda$ here.

*Proof.* Using both (5.1) and (5.2) from Lemma 5.1 above, one finds that

$$\frac{\partial^2}{\partial\lambda\partial x}\phi_c(x,\lambda) = -q(x)\sin^2(\phi_c(x,\lambda)) - \frac{\partial}{\partial x}\ln\left[R_c^2(x,\lambda)\right]\frac{\partial}{\partial\lambda}\phi_c(x,\lambda),$$

This implies that
(5.8)
$$\frac{\partial}{\partial x}\left(R_c^2(x,\lambda)\frac{\partial}{\partial\lambda}\phi_c(x,\lambda)\right) = -q(x)R_c^2(x,\lambda)\sin^2(\phi_c(x,\lambda)) = = -q(x)u^2(x),$$

for almost every pair $(x,\lambda)$. Since $\frac{\partial}{\partial\lambda}\phi_c(c,E,\lambda) = 0$, (5.7) follows immediately from (5.8). □

As a special case one finds the energy derivative of the Prüfer phase.

**Corollary 5.5.** *Let $u$ be the real valued solution of $-u'' + Vu = Eu$ normalized so that $u(c) = \sin(\theta)$ and $u'(c) = \cos(\theta)$. Then*

$$(5.9) \qquad \frac{\partial}{\partial E}\phi_c(x,E,\theta) = R_c^{-2}(x,E,\theta)\int_c^x u^2(t)\,dt.$$

*Proof.* This follows from Lemma 5.4 by setting $q$ constant to $-1$. □

We also provide two basic solution estimates which we use often in the main text. They are as follows.

**Lemma 5.6.** *Let $E$ be a real number, $q$ be a function in $L^1_{loc}(\mathbb{R})$, and $u$ be a solution of*

$$-u'' + qu = Eu$$

*on an interval $[c,d]$. One has that*

$$(5.10) \qquad |u(d)|^2 + |u'(d)|^2 \leq \left(|u(c)|^2 + |u'(c)|^2\right)\exp\left(\int_c^d |1 + q(x) - E|\,dx\right).$$

*Proof.* Setting $R(t) := |u(t)|^2 + |u'(t)|^2$, one easily calculates that

$$(5.11) \qquad R'(t) = 2\mathrm{Re}\left[(1 + q(t) - E)\,u(t)\overline{u'(t)}\right],$$

and hence

$$(5.12) \qquad |R'(t)| \leq |1 + q(t) - E|\,R(t).$$

Since (5.12) bounds the derivative of the logarithm of $R(t)$, the lemma is proven. □

**Lemma 5.7.** *Let $E_{\min} \leq E \leq E_{\max}$ and consider $q \in L^1_{loc}(\mathbb{R})$. For any non-trivial interval $I = [c,d] \subset \mathbb{R}$, there exists $C > 0$ only depending on the length of $I$ and the local $L^1$-norm of $q$ such that, given any real valued solution of $-u'' + qu = Eu$,*

$$(5.13) \qquad \int_c^d |u(t)|^2 dt \geq C\left(|u(c)|^2 + |u'(c)|^2\right).$$

*Proof.* First, we observe that, by rescaling, it is sufficient to prove (5.13) for real valued solutions with $|u(c)|^2 + |u'(c)|^2 = 1$. By Lemma 5.6, there are constants $0 < C_1, C_2 < \infty$, depending only on $E_{\min}$, $E_{\max}$, the local $L^1$-norm of $q$, and the length of the interval $I$ for which any real-valued solution of $-u'' + qu = Eu$ satisfies

$$(5.14) \qquad C_1 \leq |u(x)|^2 + |u'(x)|^2 \leq C_2,$$

for all $x \in I$; given the above mentioned normalization. With $C_3 := (C_1/2)^{1/2}$ and $C_4 := (2C_2)^{1/2}$, we also have that

$$(5.15) \qquad C_3 \leq |u(x)| + |u'(x)| \leq C_4.$$

We now claim that for every $0 < \alpha < |I|(2 + |I|)^{-1}$ exists an $x_0(\alpha) = x_0 \in I$ for which

$$(5.16) \qquad |u(x_0)| \geq \alpha C_3.$$

If, for such a fixed value of $\alpha$, this is not the case, then for all $x \in I$,

$$(5.17) \qquad |u(x)| < \alpha C_3,$$

and from (5.15) we may also conclude that

$$(5.18) \qquad |u'(x)| \geq C_3 - |u(x)| > (1 - \alpha)C_3 > 0.$$

Hence the derivative, $u'$, is strictly signed. With this we may estimate,

$$
\begin{aligned}
2\alpha C_3 > |u(d) - u(c)| &= \left| \int_c^d u'(x)\,dx \right| \\
&= \int_c^d |u'(x)|\,dx \\
(5.19) \qquad\qquad &> (1 - \alpha)C_3|I|.
\end{aligned}
$$

This contradicts the initial assumption on the range of $\alpha$, and we have proven (5.16).

The bound (5.13) now follows as

$$|u(x) - u(x_0)| \leq \int_{x_0}^x |u'(t)|\,dt \leq C_4|x - x_0|,$$

implies that, in particular, $|u(x)| \geq \alpha C_3/2$ for all $x \in I$ for which $|x - x_0| \leq \alpha C_3/(2C_4)$. $\qquad\square$

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