Supergravitons from one loop perturbative $\mathcal{N} = 4$ SYM

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Abstract

We determine the partition function of $\frac{1}{16}$BPS operators in $\mathcal{N} = 4$ SYM at weak coupling at the one-loop level in the planar limit. This partition function is significantly different from the one computed at zero coupling. We find that it coincides precisely with the partition function of a gas of $\frac{1}{16}$BPS 'supergravitons' in $AdS_5 \times S^5$.

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1 Introduction

The AdS/CFT correspondence states an exact equivalence between $\mathcal{N} = 4$ SYM gauge theory and type IIB superstrings in an $AdS_5 \times S^5$ background [1]. It provides a fascinating new approach for studying nonperturbative properties of gauge theory. On the other hand, one can use the gauge theory knowledge to gain insight into the behaviour of (super-)gravity at the quantum level (see e.g. [2, 3, 4]). In general this is a formidable problem but progress can be made when studying configurations which preserve some fraction of supersymmetry. A dictionary between $\frac{1}{2}$BPS operators in gauge theory and dual geometries has been established in [5]. $\frac{1}{4}$- and $\frac{1}{8}$BPS states have been discussed from various points of view [6]. Of particular interest are the $\frac{1}{16}$BPS states [7] due to the existence $\frac{1}{16}$BPS black holes [8]. At low energies, the gauge theory $\frac{1}{16}$BPS states should correspond to a gas of $\frac{1}{16}$BPS supergravitons, while at high energies these states should account for the entropy of $\frac{1}{16}$BPS black holes.

In [7] $\frac{1}{16}$BPS states were counted on the gauge theory side at zero coupling. It was found that the resulting partition function overcounts both the $\frac{1}{16}$BPS supergraviton partition function (giving a different energy scaling of entropy) and the $\frac{1}{16}$BPS black hole entropy in the relevant parameter regimes. In that paper it was suggested that once gauge theory interactions are turned on, many states which were counted as $\frac{1}{16}$BPS at zero coupling would get anomalous dimensions, and that the overcounting could be cured.

The aim of this paper is to perform the enumeration of $\frac{1}{16}$BPS operators in perturbative $\mathcal{N} = 4$ SYM to one-loop order. We do the counting in the planar limit using the oscillator construction of the one-loop dilatation operator of [9]. We find *exact* agreement with the $\frac{1}{16}$BPS supergraviton partition function.

The plan of the paper is as follows. In section 2 we review the definition of $\frac{1}{16}$BPS states and fix notation. Then, in section 3, we review the counting of these states in the free theory, and in section 4 we describe what has to be done to perform the calculation at one loop. In section 5 we review the supergravity result for the $\frac{1}{16}$BPS supergraviton partition function. In section 6 we describe in some details the construction of the one loop dilatation operator and, in the following section, we determine the partition function of $\frac{1}{16}$BPS operators and perform some checks. In section 8 we compare the result with the supergravity prediction and finaly, in section 9, we discuss the possible extension to large but finite N. We close the paper with a summary.

2 $\frac{1}{16}$BPS states

In this paper we consider $\frac{1}{16}$BPS states which by definition are annihilated by the following two supercharges:

$$Q \equiv Q^{-\frac{1}{2},1}, \qquad\qquad S \equiv S^{-\frac{1}{2},1}, \tag{1}$$

where $Q^\dagger = S$ and $Q^{-\frac{1}{2},1}$, $S^{-\frac{1}{2},1}$ are as in [7]. We would like to calculate the partition function over these states. To do so it is convinient to introduce the anticommutator

$$\Delta \equiv 2\{S, Q\}. \tag{2}$$

The states annihilated by S and Q are exactly those annihilated by Δ. Moreover these states are in a 1 to 1 correspondence with the cohomology classes w.r.t. Q.

In general, states in $\mathcal{N} = 4$ SYM can be labeled by the eigenvalue of the dilatation operator H, two Lorentz spins J_1 and J_2, and three $SU(4)_R$ charges R_1, R_2 and R_3 (we use the notation of [7]). The anticommutator Δ can be evaluated in terms of these quantum numbers. We have

$$\Delta = 2\{S, Q\} = H - 2J_1 - \frac{3}{2}R_1 - R_2 - \frac{1}{2}R_3. \tag{3}$$

Hence we have to calculate the partition function

$$Z_{\frac{1}{16}BPS} = \mathrm{tr}_{\Delta=0}\, x^{2H} z^{2J_1} y^{2J_2} v^{R_2} w^{R_3}, \tag{4}$$

where we picked just one possible choice of generating parameters. R_1 is of course fixed by the condition $\Delta = 0$. The above partition function includes all single and multitrace operators. It counts *all* operators annihilated by Q and S. We thus do not restrict ourselves to operators which are $\frac{1}{16}$BPS but not $\frac{1}{8}$BPS or higher.

3 Gauge theory at zero coupling

In order to evaluate the partition function in gauge theory it is convenient to use the oscillator representation, introduced in [9], for all single trace operators. In this picture, a single trace operator $\mathrm{tr}\, O_1 O_2 O_3 \dots O_L$ is represented by L sites each occupied by the 'elementary' field O_i. Operators O_i are in

turn represented by states in a Fock space generated by 4 bosonic ($a_1^\dagger, a_2^\dagger$ and $b_1^\dagger, b_2^\dagger$) creation operators, and 4 fermionic ones ($c_1^\dagger, c_2^\dagger, c_3^\dagger, c_4^\dagger$). The Fock space is narrowed by the central charge constraint which relates the total number of oscillators of various kinds on each site:

$$n_a - n_b + n_c = 2. \tag{5}$$

For an explicit dictionary between operators and Fock space states see [9].

The Lorentz spins J_1, J_2 and the $SU(4)$ charges are simply represented by the total number of various oscillators[1]:

$$\begin{aligned} R_1 &= n_{c_2} - n_{c_1}, \\ R_2 &= n_{c_3} - n_{c_2}, \\ R_3 &= n_{c_4} - n_{c_3}, \\ \sum_{i=1}^{3} q_i &= \frac{1}{2}(n_{c_1} + n_{c_2} + n_{c_3} + n_{c_4}) - 2n_{c_1}, \\ J_1 &= \frac{1}{2}(n_{a_2} - n_{a_1}), \\ J_2 &= \frac{1}{2}(n_{b_2} - n_{b_1}). \end{aligned} \tag{6}$$

In the *free theory*, the free dilatation operator H_0 also has a similar representation

$$H_0 = n_{a_1} + n_{a_2} + \frac{1}{2}(n_{c_1} + n_{c_2} + n_{c_3} + n_{c_4}). \tag{7}$$

Consequently, in the free SYM theory, the condition $\Delta = 0$ can be evaluated to give

$$\Delta_{\lambda=0} = 2n_{a_1} + 2n_{c_1} = 0. \tag{8}$$

Therefore, $\frac{1}{16}$BPS states are exactly the operators which do not have any $a_1^\dagger$ or $c_1^\dagger$ operators in the oscillator representation. Since all the spins and charges are expressed in terms of the total number of oscillators of various kinds, it is convenient to keep track of the number of oscillators of each type

[1]The sum $\sum_{i=1}^{3} q_i$ is defined as $\frac{3}{2}R_1 + R_2 + \frac{1}{2}R_3$.

when counting $\frac{1}{16}$BPS operators. We thus consider partition functions of the form

$$Z(a_2, b_1, b_2, c_2, c_3, c_4) = \sum_{\Delta=0} a_2^{n_{a_2}} b_1^{n_{b_1}} b_2^{n_{b_2}} c_2^{n_{c_2}} c_3^{n_{c_3}} c_4^{n_{c_4}}. \tag{9}$$

A simple counting over the Fock space states, taking into consideration the central charge constraint (5), gives for the 'letter' partition function (partition function of operators at each site):

$$z_B = \frac{a_2^2 + c_2c_3 + c_2c_4 + c_3c_4}{(1 - b_1a_2)(1 - b_2a_2)}, \tag{10}$$

$$z_F = \frac{a_2(c_2 + c_3 + c_4) + (b_1 + b_2 - a_2b_1b_2)c_2c_3c_4}{(1 - b_1a_2)(1 - b_2a_2)}, \tag{11}$$

where we made a separation into bosonic and fermionic states.

The partition function of single trace operators then follows from

$$Z_{s.t.} = -\sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log\left(1 - z_B(x^n) - (-1)^{n+1} z_F(x^n)\right), \tag{12}$$

where x stands for generic arguments (e.g. $x = (a_2, b_1, b_2, c_2, c_3, c_4)$ in our case). Finally, the partition function of multitrace operators (at $N_c = \infty$) is given by

$$Z = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \left\{Z^B_{s.t.}(x^n) + (-1)^{n+1} Z^F_{s.t.}(x^n)\right\}\right). \tag{13}$$

The above formulas do not take into account finite N effects which appear, e.g. when certain long traces are equivalent to linear combinations of shorter multitrace operators (due to the Cayley-Hamilton theorem for finite matrices). For the specific case of *free* SYM, the method of character expansions of [10, 11] allows to perform an exact calculation at finite N starting directly from the letter partition function (10). The resulting fixed N partition function is given by the formula

$$Z = \int DU \exp\left\{\sum_{n=1}^{\infty} \left(z_B(x^n) + (-1)^{n+1} z_F(x^n)\right) \frac{\operatorname{tr} U^n \operatorname{tr} U^{-n}}{n}\right\}, \tag{14}$$

where the integral is over the unitary group $U(N)$. For the case at hand this has been analyzed in [7] for large N. For small values of parameters x the large N limit of (14) does not depend on N (reproducing effectively (13)), while at a finite value of x (strictly less than 1) the $\frac{1}{16}$BPS partition function exhibits a behavior $\log Z \sim N^2$.

4 Gauge theory at one loop

At one loop, $H = H_0 + \lambda \delta H$ and the anomalous part is now a nontrivial operator which acts on each two neighboring sites. The complete one loop dilatation operator was constructed in [9]. We discuss it in details in section 6 . The condition $\Delta_{1-loop} = 0$ now takes the form

$$\Delta_{1-loop} = \Delta_{\lambda=0} + \lambda \delta H = 0. \tag{15}$$

Since at one loop the eigenvalues of δH are rational/radical expressions, for generic transcendental λ this condition picks out states which satisfy *both* the free and one loop conditions separately, i.e. states with $n_{a_1} = n_{c_1} = 0$ which do not get any anomalous dimensions δH at one loop. We thus have to compute

$$Z = \sum_{\substack{n_{a_1}=n_{c_1}=0 \\ \delta H=0}} a_2^{n_{a_2}} b_1^{n_{b_1}} b_2^{n_{b_2}} c_2^{n_{c_2}} c_3^{n_{c_3}} c_4^{n_{c_4}}. \tag{16}$$

Note that now the formula (12) no longer holds and we have to identify the number of operators which do not get anomalous dimensions at one loop for each L independently.

We first determine the sum over single trace operators for fixed L by computing the above power series with some truncation on the number of oscillators. This turns out to give enough information to guess the analytical form of the generating function. Next, we test the function on various configurations which were not used in the process of obtaining the analytical form. The details of this procedure are discussed in section 7.

Summing over L gives the partition function of *all* single trace operators. Then (13) may be used to get the partition function of multitrace ones. Let us note that at one loop we do not have a counterpart of the exact formula for finite N valid for zero coupling (14). We will, nevertheless, discuss some aspects of the possible large N behavior at the end of the paper.

5 Supergraviton partition function

At strong coupling one can calculate the partition function over $\frac{1}{16}$BPS states using the supergravity/superstring side of the AdS/CFT correspondence. This has been considered in [7]. We now briefly review these results.

Since the $psu(2,2|4)$ supersymmetry algebra of the gauge theory is also the symmetry group of superstrings in $AdS_5 \times S^5$, we have direct counterparts of Q and S operators and we can use them to define the $\frac{1}{16}$BPS states.

In the low energy regime the partition function should be given by supergravity fields which are annihilated by the Q and S operators. This has been done in [7] where the single particle partition function

$$Z^{single}_{gravitons} = \sum_{\Delta=0} x^{2H} z^{2J_1} y^{2J_2} v^{R_2} w^{R_3}, \tag{17}$$

was calculated with the result

$$\begin{aligned}
Z^{single}_{gravitons} &= \frac{bosons + fermions}{denominator}, && (18)\\
denominator &= (1-\frac{x^2}{w})(1-x^2 v)(1-x^2\frac{w}{v})(1-x^2\frac{z}{y})(1-x^2 zy), && (19)\\
bosons &= vx^2 + \frac{x^2}{w} + \frac{wx^2}{v} - \frac{x^4}{v} - \frac{vx^4}{w} - wx^4 + 2x^6 + \frac{x^6 z}{yv} \\
&\quad + \frac{vx^6 z}{wy} + \frac{wx^6 z}{y} - \frac{x^8 z}{y} + \frac{x^6 zy}{v} + \frac{vx^6 zy}{w} + wx^6 zy \\
&\quad - x^8 zy + x^4 z^2 + x^{10} z^2, && (20)\\
fermions &= \frac{x^3}{y} + x^3 y + \frac{x^3 z}{v} + \frac{vx^3 z}{w} + wx^3 z - 2x^5 z + vx^7 z \\
&\quad + \frac{x^7 z}{w} + \frac{wx^7 z}{v} + \frac{x^7 z^2}{y} + x^7 z^2 y. && (21)
\end{aligned}$$

The full partition function is obtained by summation over the Fock spaces of these particles using the formula

$$Z_{gravitons} = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n}\left\{Z^{single,bos}_{gravitons}(x^n,\ldots,w^n) + (-1)^{n+1} Z^{single,fer}_{gravitons}(x^n,\ldots,w^n)\right\}\right). \tag{22}$$

We note that the above formula is identical in form to the one obtained when passing from single- to multi-trace operators in gauge theory (13).

It turns out that $Z_{gravitons}$ *does not* agree with the result from free SYM theory [7]. In the case when $z = y = v = w = 1$ even the scaling of the entropy with energy is different.

Moreover, [7] obtained the partition function for $\frac{1}{8}$BPS states by taking the limit $z \to 0$. The result again was in disagreement with free SYM, but matched exactly the calculation made using properties of the chiral ring of (interacting) SYM.

In section 8 we compare this supergraviton partition function with *perturbative* computations at one loop in SYM.

When energies are large (compared with N) it is expected that the partition function for $\frac{1}{16}$BPS states will have a

$$\log Z \sim N^2, \tag{23}$$

behavior which should coincide with the one obtained from the $\frac{1}{16}$BPS black holes (see [7], section 5.3 for explicit formulas). In [7], a similar qualitative behavior was obtained at zero coupling, although the numerical details did not match. The motivation for this paper was to investigate how much of this zero coupling result survives at one loop.

6 The one loop dilatation operator

In this section we review the construction of the one loop dilatation operator δH in the oscillator picture [9].

The Fock space

Let us consider the space of operators which are traces of L adjoint fields. We represent each of those fields as a state on one of L sites. Since the trace is cyclic invariant we restrict ourselves to cyclic invariant states of the Fock space at the end of the calculation.

A generic state in Fock space is thus a linear combination of states

$$\mid s_1\rangle \otimes \ldots \otimes \mid s_L\rangle, \tag{24}$$

where on each site i the state $\mid s_i\rangle$ is obtained by acting with bosonic $a^\dagger_{1,i}, a^\dagger_{2,i}, b^\dagger_{1,i}, b^\dagger_{2,i}$ and fermionic $c^\dagger_{1,i}, c^\dagger_{2,i}, c^\dagger_{3,i}, c^\dagger_{4,i}$ creation operators on the Fock

vacuum $\mid 0\rangle_i$. An arbitrary state is labeled by the oscillator occupation numbers

$$\begin{aligned} \mid s_i\rangle &= \mid n_{a_{1,i}}, n_{a_{2,i}}, n_{b_{1,i}}, n_{b_{2,i}}, n_{c_{1,i}}, n_{c_{2,i}}, n_{c_{3,i}}, n_{c_{4,i}}\rangle \\ &= a_{1,i}^{\dagger\, n_{a_{1,i}}} a_{2,i}^{\dagger\, n_{a_{2,i}}} b_{1,i}^{\dagger\, n_{b_{1,i}}} b_{2,i}^{\dagger\, n_{b_{2,i}}} c_{1,i}^{\dagger\, n_{c_{1,i}}} c_{2,i}^{\dagger\, n_{c_{2,i}}} c_{3,i}^{\dagger\, n_{c_{3,i}}} c_{4,i}^{\dagger\, n_{c_{4,i}}} \mid 0\rangle_i. \quad (25) \end{aligned}$$

As discussed in section 3, the occupation numbers at each site are constrained by (5)

$$n_{a_{2,i}} + n_{a_{2,i}} - n_{b_{1,i}} - n_{b_{2,i}} + n_{c_{1,i}} + n_{c_{2,i}} + n_{c_{3,i}} + n_{c_{4,i}} = 2. \quad (26)$$

The one loop dilatation operator does not change the total number of oscillators of each kind and only moves them from site to site. Therefore it acts within the space with fixed total number of oscillators of any type for fixed L. It follows that we can diagonalize δH in subspaces labeled by

$$[n_{a_1}, n_{a_2}, n_{b_1}, n_{b_2}, n_{c_1}, n_{c_2}, n_{c_3}, n_{c_4}; L]. \quad (27)$$

The harmonic action

Let us now review the construction of δH [9], giving more details about the computer code implementation.

The action of the one-loop dilatation operator introduces an interaction between *only* the neighboring sites (the last and the first site are assumed to be neighbors). For this reason it is enough to consider a pair of such sites

$$\mid v\rangle = \mid n_{a_1}, n_{a_2}, n_{b_1}, n_{b_2}, n_{c_1}, n_{c_2}, n_{c_3}, n_{c_4}\rangle \otimes \mid m_{a_1}, m_{a_2}, m_{b_1}, m_{b_2}, m_{c_1}, m_{c_2}, m_{c_3}, m_{c_4}\rangle, \quad (28)$$

where we dropped the index i (and $i+1$) for clarity. Our object is now to calculate the hamiltonian matrix element between (28) and an arbitrary other state

$$\mid v'\rangle = \mid n'_{a_1}, n'_{a_2}, n'_{b_1}, n'_{b_2}, n'_{c_1}, n'_{c_2}, n'_{c_3}, n'_{c_4}\rangle \otimes \mid m'_{a_1}, m'_{a_2}, m'_{b_1}, m'_{b_2}, m'_{c_1}, m'_{c_2}, m'_{c_3}, m'_{c_4}\rangle. \quad (29)$$

If one is interested in calculating the element $\langle v' \mid H \mid v\rangle$ then it turns out that the harmonic action [9] can be described by the following set of rules

- consider all the possibilities of oscillator hopping from site $i \to i+1$ and from site $i+1 \to i$ such that the state (28) becomes (29)

- to each such possibility associate a number

$$c_{n,n_{12},n_{21}} = (-1)^{1+n_{12}n_{21}} \frac{\Gamma(\frac{1}{2}n_{12} + \frac{1}{2}n_{21})\Gamma(1 + \frac{1}{2}n - \frac{1}{2}n_{12} - \frac{1}{2}n_{21})}{\Gamma(1 + \frac{1}{2}n)}, \quad (30)$$

  where n_{12}, n_{21} are the numbers of oscillators hopping from $i \to i+1$, $i+1 \to i$ respectively, n is the total number of quanta at sites i and $i+1$ in the beginning

- include the -1 factors when the fermion oscillators are hopping "over" other fermions. In particular, if a fermion is hopping form $i = 1$ to $i = L$ or vice versa then all fermions in between (for $1 < i < L$) have to be considered.

- sum over all the possibilities and multiply the result by $\frac{\||v'\rangle\|}{\||v\rangle\|}$

The harmonic action can be implemented in two independent ways. One is to use the above rules as they are and compute the element $\langle v' \mid H \mid v\rangle$ indirectly by evaluating

$$H \mid v\rangle = \sum_{v'} H_{v,v'} \mid v'\rangle. \quad (31)$$

Second is to write down the formula for the matrix element $\langle v' \mid H \mid v\rangle$ and compute it explicitly. It turns out to be possible, we have

$$\langle v' \mid H \mid v\rangle = \sqrt{\prod_{t\in T} \frac{n'_t!}{n_t!}} \sum_{t'\in T} \sum_{k_{t'}=0}^{m_{t'}} (-1)^F c_{x,y,z} \prod_{t''\in T} \binom{n_{t''}}{n_{t''} - n'_{t''} + k_{t''}} \binom{m_{t''}}{k_{t''}},$$

$$T = \{a_1, a_2, b_1, b_2, c_1, c_2, c_3, c_4\},$$

$$x = \sum_{t\in T} n_t + m_t, \qquad y = \sum_{t\in T} \epsilon_t |n_t - n'_t| + k_t, \qquad z = \sum_{t\in T} \eta_t |n_t - n'_t| + k_t, \quad (32)$$

where $(-1)^F$ is the fermion number discussed in one of the rules and where the parameters ϵ_t, η_t are defined in the following way. ϵ_t is equal 1 if $n_t \geq n'_t$ and 0 otherwise, $\eta_t = 1 - \epsilon_t$.

The above formula can be justified in the following way. Let us consider n_{a_1} bosons $a_1^\dagger$. We want them to hop form $i \to i+1$ so that only n'_{a_1} of them are left (clearly we assume that $n_{a_1} \geq n'_{a_1}$). Since they commute and

are indistinguishable the number of possibilities coincides with the number of combinations $\binom{n_{a_1}}{n_{a_1}-n'_{a_1}}$. Other possibilities are when the number of such hops is $n_{a_1} - n'_{a_1} + k_{a_1}$ with $k_{a_1} > 0$. Then, we have to hop back (from $i+1 \to i$) exactly k_{a_1} oscillators $a_1^\dagger$. This can be done in $\binom{m_{a_1}}{k_{a_1}}$ ways. Therefore, the net factor for given k_{a_1} is $\binom{n_{a_1}}{n_{a_1}-n'_{a_1}+k_{a_1}}\binom{m_{a_1}}{k_{a_1}}$. To include all the possibilities we sum over all possible k_{a_1}'s, i.e from 0 to m_{a_1}.

For the other bosonic and fermionic operators $a_2^\dagger,\ b_1^\dagger,\ b_2^\dagger,\ c_1^\dagger,\ c_2^\dagger,\ c_3^\dagger,\ c_4^\dagger$ the analysis in analogous and gives the corresponding factors as in (6). To include the -1 factors coming from hopping of fermions we weight the sum (6) with the factor $(-1)^F$.

We have implemented the above construction of the one loop dilatation operator independently in two different programs and verified that the results agree. As a further check we reproduced various one loop anomalous dimensions given in [9].

In order to complete the construction we project the Hilbert space (24) to the subspace of states which are invariant under cyclic permutations, since only these states correspond to gauge theory single trace operators.

We start with the hamiltonian matrix H represented in the *non*-cyclic invariant basis B (24) constructed as above. Then, we construct a matrix representation of an operator T which translates the chain by one site. The cyclic invariant states correspond to eigenvectors $v_1, \ldots, v_n, \quad n \leq \#B$ of T with an eigenvalue equal 1. Now, we build the projection matrix $P = [v_1, \ldots, v_n]$ and perform the similarity transformation

$$H \to PHP^T,$$

on H. The result is the hamiltonian matrix represented in the cyclic invariant basis.

7 The one loop $\frac{1}{16}$BPS partition function

According to the general discussion in previous sections, the tree level condition for the $\frac{1}{16}$BPS states contributing to the index is $\Delta_{\lambda=0} = 2n_{a_1} + 2n_{c_1} = 0$ hence from now on we take $n_{a_{1,i}} = n_{c_{1,i}} = 0$. Moreover, at one loop level the condition $\Delta_{1-loop} = 0$ is satisfied only for states which are eigenstates corresponding to 0 eigenvalue of the one loop dilatation operator. Let $D_{n_{a_2},n_{b_1},n_{b_2},n_{c_2},n_{c_3},n_{c_4},L}$ be the number of such states in the sector with n_{a_2}, n_{b_1},

n_{b_2}, n_{c_2}, n_{c_3}, n_{c_4} number of quanta and L sites respectively. The generating function we are looking for is

$$Z_L^{1/16th}(a_2,b_1,b_2,c_2,c_3,c_4)=\sum_{\substack{n_{a_2},n_{b_1},n_{b_2}=0,\ldots,\infty\\ n_{c_2},n_{c_3},n_{c_4}=0,\ldots,L}} D_{n_{a_2},n_{b_1},n_{b_2},n_{c_2},n_{c_3},n_{c_4},L}a_2^{n_{a_2}}b_1^{n_{b_1}}b_2^{n_{b_2}}c_2^{n_{c_2}}c_3^{n_{c_3}}c_4^{n_{c_4}},$$

(the sum over fermionic variables runs from 0 to L due to the Pauli exclusion principle). With use of computer code implementation of the rules discussed in previous section, one can determine the numbers $D_{n_{a_2},n_{b_1},n_{b_2},n_{c_2},n_{c_3},n_{c_4},L}$ exactly, but of course only for a finite number of configurations. It is by no means obvious that such data can determine the whole function $Z_L^{1/16th}(a_2,b_1,b_2,c_2,c_3,c_4)$. Nevertheless, our analysis shows that the Taylor expansion of the function $Z_L^{1/16th}(a_2,b_1,b_2,c_2,c_3,c_4)$ coincides with the expansion of certain rational function. The details of our computation are below.

For $L=2$ we analyzed the configuration with $0\leq n_{a_2},\ n_{b_1},\ n_{b_2}\leq 10$ and $0\leq n_{c_2},\ n_{c_3},\ n_{c_4}\leq 2$. There are $11^3 3^3=35937$ such possibilities however only 1494 of them satisfy the central charge constraint.

For $L=3$ we took $0\leq n_{a_2},\ n_{b_1},\ n_{b_2}\leq 5$ and $0\leq n_{c_2},\ n_{c_3},\ n_{c_4}\leq 3$. There are $6^3 4^3=13824$ such possibilities among which only 849 satisfy the central charge constraint.

For $L=4$ we analyzed the configuration with $0\leq n_{a_2},\ n_{b_1},\ n_{b_2}\leq 2$ and $0\leq n_{c_2},\ n_{c_3},\ n_{c_4}\leq 4$. There are $3^3 5^3=3375$ such possibilities and 279 which satisfy the central charge constraint.

Let us now explain how the partition function was reconstructed from the above data and consider in detail the case of $L=2$. The computer analysis gives a polynomial

$$Z_{L=2}^{1/16th,cut}(a_2,b_1,b_2,c_2,c_3,c_4)=\sum_{\substack{n_{a_2},n_{b_1},n_{b_2}=0,\ldots,10\\ n_{c_2},n_{c_3},n_{c_4}=0,1,2}} D_{n_{a_2},n_{b_1},n_{b_2},n_{c_2},n_{c_3},n_{c_4},2}a_2^{n_{a_2}}b_1^{n_{b_1}}b_2^{n_{b_2}}c_2^{n_{c_2}}c_3^{n_{c_3}}c_4^{n_{c_4}},$$

which consists of 1494 terms.

Our strategy to proceed is the following. If the full partition function $Z_{L=2}^{1/16th}(a_2,b_1,b_2,c_2,c_3,c_4)$ is a rational function then, in particular, so is $Z_{L=2}^{1/16th}(a_2,1,1,1,1,1)$. Therefore, the coefficients of $Z_{L=2}^{1/16th}(a_2,1,1,1,1,1)$ should be "easily" recognizable. Indeed, we have

$$\begin{aligned}Z_{L=2}^{1/16th,cut}(a_2,1,1,1,1,1)&=13+40a_2+72a_2^2+104a_2^3+136a_2^4+168a_2^5\\&+200a_2^6+232a_2^7+264a_2^8+296a_2^9+320a_2^{10},\end{aligned}\tag{33}$$

which (except for the last term $320a_2^{10}$) is recognized as the Taylor expansion of

$$\frac{13+14a_2+5a_2^2}{(1-a_2)^2}.$$

Next, we turn on the variable b_1, i.e. we consider $Z_{L=2}^{1/16th,cut}(a_2,b_1,1,1,1,1)$ and find the corresponding generating function. Then, we proceed analogously with $Z_{L=2}^{1/16th,cut}(a_2,b_1,b_2,1,1,1)$ and find that

$$Z_{L=2}^{1/16th}(a_2,b_1,b_2,1,1,1)=\frac{6+8a_2+3a_2^2+(3+3a_2+a_2^2)(b_1+b_2)+b_1b_2}{(1-b_1a_2)(1-b_2a_2)}.$$

The full a_2, b_1, b_2 dependence is now determined. In order to find the c_2, c_3, c_4 dependence we do the following. First, due to the Pauli exclusion principle the fermionic variables cannot be in the denominator $(1-b_1a_2)(1-b_2a_2)$. Therefore they enter only in the numerator in the form $c_2^ic_3^jc_3^k$, $i,j,k\leq L$. Second, the harmonic action is completely symmetric with respect to fermionic oscillators. This implies that the numerator of $Z_{L=2}^{1/16th}(a_2,b_1,b_2,c_2,c_3,c_4)$ is a completely symmetric function with respect to c_2, c_3 and c_4. [2]. We therefore write $Z_{L=2}^{1/16th}(a_2,b_1,b_2,c_2,c_3,c_4)$ as

$$\frac{1}{(1-b_1a_2)(1-b_2a_2)}\sum_{n=0}^{2}\sum_{m=0}^{2}\sum_{l_1,l_2,l_3=0}^{2}\tilde{D}_{n,m,l_1,l_2,l_3}A_nB_m\sigma_{l_1,l_2,l_3}(c_2,c_3,c_4),\quad(34)$$

where

$$A_0=1,\ A_1=a_2,\ A_2=a_2^2,$$
$$B_0=1,\ B_1=b_1+b_2,\ B_2=b_1b_2,$$

$\tilde{D}_{n,m,l_1,l_2,l_3}$ are some coefficients to be determined and $\sigma_{l_1,l_2,l_3}(c_2,c_3,c_4)$ are Schur polynomials [3] defined as

$$\sigma_{n_1,n_2,n_3}(x_1,x_2,x_3)=\begin{vmatrix}x_1^{n_1+2}&x_2^{n_1+2}&x_3^{n_1+2}\\x_1^{n_2+1}&x_2^{n_2+1}&x_3^{n_2+1}\\x_1^{n_3}&x_2^{n_3}&x_3^{n_3}\end{vmatrix}\Big/\begin{vmatrix}x_1^2&x_2^2&x_3^2\\x_1&x_2&x_3\\1&1&1\end{vmatrix}.\quad(35)$$

[2] By the same argument the full partition function has to be symmetric in bosonic variables b_1, b_2 hence only the combinations b_1+b_2, b_1b_2 appear in the numerator. It is not obvious why there are no other, higher order, combinations e.g. $b_1^3+b_2^3$. Clearly, this must be a property of the harmonic action.

[3] Other bases of symmetric functions, e.g. $(c_2+c_3+c_4)^i(c_2c_3+c_3c_4+c_4c_1)^j(c_2c_3c_4)^k$ are possible. However, our choice of Schur polynomials turns out to give simple expression (36).

The coefficients can be obtained by comparing the Taylor expansion of (34) with $Z_{L=2}^{1/16th,cut}(a_2, b_1, b_2, c_2, c_3, c_4)$.

The analysis for $L = 3, 4$ (with the sum over l_1, l_2, l_3 in (34) from 0 to L) is analogous. In this manner we obtain a fairly simple rational generating functions for $L = 2, 3, 4$.

Given those three functions, it was possible to guess the partition function for arbitrary L. The final result turns out to have a particularly simple expression in terms of Schur polynomials namely

$$Z_L^{1/16th}(a_2, b_1, b_2, c_2, c_3, c_4) = \frac{P}{(1 - a_2 b_1)(1 - a_2 b_2)}, \tag{36}$$

$$\begin{aligned} P = \sigma_{L,L,0} + a_2 \sigma_{L,L-1,0} + a_2^2 \sigma_{L-1,L-1,0} \\ +(b_1 + b_2) \left(\sigma_{L,L,1} + a_2 \sigma_{L,L-1,1} + a_2^2 \sigma_{L-1,L-1,1} \right) \\ +b_1 b_2 \left(\sigma_{L,L,2} + a_2 \sigma_{L,L-1,2} + a_2^2 \sigma_{L-1,L-1,2} \right), \end{aligned}$$

where $\sigma_{n_1,n_2,n_3} = \sigma_{n_1,n_2,n_3}(c_2, c_3, c_4)$ is the Schur polynomial (35).

In order to test the above result further, we performed the analysis for for $L = 5$ with $0 \leq n_{a_2},\ n_{b_1},\ n_{b_2} \leq 2$ and $0 \leq n_{c_2},\ n_{c_3},\ n_{c_4} \leq 5$. There are $3^3 6^3 = 5832$ such configurations and 414 which satisfy the central charge constraint. The corresponding generating function indeed confirms (36).

In the remaining part of this section we perform other checks of (36).

The partition function for $\frac{1}{8}$BPS states

The $\frac{1}{8}$BPS states are obtained by imposing the additional condition $J_1 = 0$ on the $\frac{1}{16}$BPS states [7]. This is equivalent to setting the corresponding constraint on the numbers of quanta, namely

$$n_{a_2} = 0. \tag{37}$$

The central charge condition in this case

$$2L + n_{b_1} + n_{b_2} = n_{c_2} + n_{c_3} + n_{c_4}, \tag{38}$$

ensures that for given L there is only a finite number of such configurations. It follows that the corresponding generating function is a polynomial in the variables b_1, b_2, c_2, c_3, c_4. Since $n_{c_2}, n_{c_3}, n_{c_4} \leq L$ the numbers n_{b_1}, n_{b_2} are also bounded, i.e. $n_{b_1}, n_{b_2} \leq L$.

These simplifications allow us to perform explicit computations of the generating function for these states for higher L (we did it for $L = 6$ and $L = 7$) and check these results with the general partition function obtained in the previous section. Indeed, we find that the resulting functions coincide with (36) after setting $a_2 = 0$, i.e. they are

$$Z_L^{1/8th}(b_1, b_2, c_2, c_3, c_4) = Z_L^{1/16th}(0, b_1, b_2, c_2, c_3, c_4)$$

$$= \sigma_{L,L,0}(c_2, c_3, c_4) + (b_1 + b_2)\sigma_{L,L,1}(c_2, c_3, c_4) + b_1 b_2 \sigma_{L,L,2}(c_2, c_3, c_4).$$

Checks for operators with many derivatives

One puzzling feature of the generating function (36) is that the denominators contain only *two* factors: $(1 - a_2 b_1)(1 - a_2 b_2)$. This suggests that the derivatives in $\frac{1}{16}$BPS states are essentially commutative (we will discuss this point further in section 9) which has a crucial impact on the singularity structure of the $\frac{1}{16}$BPS partition function. The test of $\frac{1}{8}$BPS states checks the numerator and does not involve any derivative terms. In order to check for derivatives we looked at the following configurations for $L = 5$ and $n_{c_3} = n_{c_4} = 5$: i) 7 and 10 $a_2 b_1$ derivatives, i.e. [0,7,7,0,0,5,5,5], [0,10,10,0,0,5,5,5]; ii) 6 derivatives of both types, i.e. [0,6,6,0,0,5,5,5], [0,6,5,1,0,5,5,5], ... [0,6,3,3,0,5,5,5], iii) states with derivatives and an additional c_2 oscillator, i.e. [0,5,5,0,1,4,5,5], [0,5,4,1,1,4,5,5]. In all cases, despite the large number of derivatives we found only a single $\frac{1}{16}$BPS state in those sectors, which is consistent with (36).

Letter partition function

Finally, let us note that, although we guessed the partition function for $\frac{1}{16}$BPS states starting from $L = 2$ and proceeding to $L > 2$, substituting $L = 1$ in (36), we recover the letter partition function (10) which correspond to $\frac{1}{16}$BPS operators in a $U(N)$ gauge theory. This is another consistency check of our analytical formula (36).

8 Comparision with supergravity

The one loop $\frac{1}{16}$BPS partition function can be calculated from the length L partition functions obtained in the previous section in two steps. First, the

single trace partition function is obtained through

$$Z_{s.t.} = \sum_{L=1}^{\infty} Z_L^{1/16th}, \tag{39}$$

where we sum from $L = 1$ since we are considering the partition function in a $U(N)$ gauge theory[4]. Then, the full $\frac{1}{16}$BPS partition function is obtained by passing to multitrace operators through

$$Z = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n}\left\{Z^B_{s.t.}(x^n,\ldots) + (-1)^{n+1} Z^F_{s.t.}(x^n,\ldots)\right\}\right). \tag{40}$$

In this step we are using the fact that only the planar one loop dilatation operator is considered. So we are in the strict $N \to \infty$ limit.

The first sum (39) can be carried out analytically, and the result is

$$Z_{s.t.} = \frac{bosons + fermions}{denominator}, \tag{41}$$

where

$$\begin{aligned}
denominator &= (1-a_2b_1)(1-a_2b_2)(1-c_2c_3)(1-c_2c_4)(1-c_3c_4),\\
bosons &= a_2^2 + c_2c_3 + (c_2+c_3)(1+(a_2(b_1+b_2)-1)c_2c_3)c_4\\
&\quad + c_2c_3(a_2b_1 + a_2b_2 - 1 + (1 + b_1b_2 + a_2^2b_1b_2 - a_2(b_1+b_2))c_2c_3)c_4^2,\\
fermions &= (b_1+b_2)c_2c_3c_4 + a_2^2(b_1+b_2)c_2c_3c_4\\
&\quad + a_2(c_3+c_4+b_1b_2c_2^2c_3c_4(c_3+c_4) + c_2(c_3c_4-1)(b_1b_2c_3c_4-1)).
\end{aligned}$$

Let us now compare this result with the single particle supergraviton $\frac{1}{16}$BPS partition function (18). In order to make the comparison possible we express the variables x, v, w, z in terms of a_2, c_2, c_3, c_4 and b_1, b_2 in terms of y. Using the definitions (6) and (7) we find that $b_1 = 1/y$, $b_2 = y$ and we obtain the dictionary

$$x = c_2^{\frac{1}{3}} c_3^{\frac{1}{3}} c_4^{\frac{1}{3}}, \tag{42}$$

$$v = c_2^{-\frac{2}{3}} c_3^{\frac{1}{3}} c_4^{\frac{1}{3}}, \tag{43}$$

$$w = c_2^{-\frac{1}{3}} c_3^{-\frac{1}{3}} c_4^{\frac{2}{3}}, \tag{44}$$

$$z = a_2 c_2^{-\frac{2}{3}} c_3^{-\frac{2}{3}} c_4^{-\frac{2}{3}}. \tag{45}$$

[4]This will turn out to be crucial for comparison with the supergraviton partition functions.

Remarkably enough, with these substitutions the supergraviton $\frac{1}{16}$BPS partition function (18) coincides *exactly* with the one loop single trace $\frac{1}{16}$BPS partition function (41). Thus, the full $\frac{1}{16}$BPS partition functions coincide also, since the summation over multitrace operators is mathematically equivalent to summation over the supergraviton Fock spaces (c.f. (22) and (40)).

As a byproduct we note that the resulting $\frac{1}{8}$BPS partition function obtained from the one loop perturbative dilatation operator exactly coincides with the gauge theory result obtained from the chiral ring reasoning as in [7].

9 Discussion of large N asymptotics

Ultimately we are interested in the behavior of the partition function of $\frac{1}{16}$BPS states which scales like $\log Z \propto N^2$ and which therefore can account for the entropy of $\frac{1}{16}$BPS charged black holes in $AdS_5 \times S^5$. In [7] a calculation in the free theory showed that, for values of the chemical potentials below a certain value, the partition function has a $N \to \infty$ limit, while above that value one obtains $\log Z \propto N^2$ scaling. This analysis follows from formula (14) which is exact for any N.

At one loop, we do not have a similar exact formula since the $\frac{1}{16}$BPS states are very specific and form just a tiny fraction of all operators made from the 'letters' (10). It is thus interesting to understand whether staying within the $N = \infty$ phase one can see the transition to the 'black hole' phase. Firstly, at finite N, the number of states is diminished due to trace identities following from the Cayley-Hamilton theorem. Thus, there is a chance of observing $\log Z \propto N^2$ at a certain value of the parameters only when the corresponding $N = \infty$ partition function has a singularity there or is divergent.

Quite remarkably, our single trace partition function is finite for all arguments less than 1. This is in stark contrast with the free $\frac{1}{16}$BPS partition function which blows up much earlier (see section 5 in [7]). Let us note that there, this conclusion was reached from the exact formula (14). However one can see this behavior studying directly the single trace partition function (12). We checked that calculating (12) as a power series even for the simplest case of *two* noncommutative letters, and studying its radius of convergence recovers exactly the leading singularity (strictly below 1) which coincides with the transition point obtained from (14). This experiment gives us confidence that the knowledge of $N = \infty$ partition function can be a reliable guide

to the singularity structure and hence to the position of the phase transition.

In order to obtain some rough idea about the structure of the one loop $\frac{1}{16}$BPS states we tried to see whether one can introduce some 'effective letters' which would then reproduce the one loop single trace partition function (41). We define effective letter functions $z_B^{eff.}(x)$ and $z_F^{eff.}(x)$ for bosons and fermions respectively using the formula

$$Z_{s.t.} = -\sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log\left(1 - z_B^{eff.}(x^n) - (-1)^{n+1} z_F^{eff.}(x^n)\right). \qquad (46)$$

where on the left hand side we put the generating function of single trace $\frac{1}{16}$BPS operators obtained in the present paper. Here, the most natural choice of chemical potentials is $a_2 = x^3$, $c_2 = c_3 = c_4 = x$ and $b_1 = b_2 = 1$ (see [7, 12]). Expanding the l.h.s and r.h.s of (46) it is possible to solve nonlinear equations for the coefficients of the Taylor expansion of $z_B^{eff.}(x)$ and $z_F^{eff.}(x)$. We have found unique solutions up to the 28th order of x. All the coefficients are integers however they become very large and negative which indicates that any 'noncommutative' letters drastically overcount the $\frac{1}{16}$BPS states.

The above experiment suggests that the building blocks of $\frac{1}{16}$BPS states are predominantly commutative similarly to the building blocks of $\frac{1}{8}$BPS states as discussed in section 6.1 of [7]. This is further supported by the structure of the denominator in (41) which essentially means that only the total number of blocks like $a_2 b_1$ etc. matters – and not their ordering.

Using the above guiding principle of commutativity we tried to apply the 'plethystic' formalism of [13]. In this formalism, the partition function at finite N can be reconstructed from the $N = \infty$ one in the following manner. Suppose that the single trace bosonic and fermionic partition functions at $N = \infty$ are given by

$$Z_{s.t.}^B = \sum_{n=0}^{\infty} a_n x^n, \qquad Z_{s.t.}^F = \sum_{n=0}^{\infty} b_n x^n, \qquad (47)$$

then the finite N partition function $Z_N(x)$ is obtained from the infinite product expansion:

$$\frac{\prod_{n=1}^{\infty}(1 + g x^n)^{b_n}}{\prod_{n=1}^{\infty}(1 - g x^n)^{a_n}} = \sum_N Z_N(x) g^N, \qquad (48)$$

which in fact exactly reproduces the finite N structure of $\frac{1}{8}$BPS states obtained from chiral ring arguments taking as input only the $N = \infty$ result.

However we do not see any chance of a $\log Z \propto N^2$ behavior when we apply this formalism to our partition function.

It has been suggested in the literature [14] that the dual states contributing mainly to the black hole entropy would be of a determinant type. For fixed N, states which are determinants of some matrices can be expressed as combinations of multitrace operators. So at least formally, these states are within the space of states that we consider (which consists of all multitrace operators).

Our conclusion is that in order to see the $\log Z \propto N^2$ scaling, one has to use the whole nonplanar one loop dilatation operator the properties of which probably have a huge impact on the counting of $\frac{1}{16}$BPS states with very many traces.

10 Summary

In this paper we determined the partition function of $\frac{1}{16}$BPS operators in planar perturbative $\mathcal{N} = 4$ SYM at one loop. We used the oscillator representation of gauge theory operators and of the planar dilatation operator. In order to obtain the partition function we determined the number of $\frac{1}{16}$BPS operators for a certain set of restrictions on the number of oscillators and for operators of lengths less than 5. Then, we reconstructed a generating function (assuming that it is a rational function) which reproduced all these results. Subsequently we made numerous further checks by evaluating the dilatation operator for higher length and larger number of (some) oscillators and checking the result with the proposed generating function.

The main result that we found is an *exact* agreement with the partition function of $\frac{1}{16}$BPS supergravitons in $AdS_5 \times S^5$. Consequently we also reproduce exactly, using the one loop perturbative dilatation operator, the counting of $\frac{1}{8}$BPS states which was previously done on the gauge theory side using chiral ring reasonings [7].

Using the identification of single particle supergraviton states in terms of short representations of $psu(2,2|4)$ (see [7]) and the equality with the gauge theory partition function extracted in the present work we may identify all $\frac{1}{16}$BPS states as descendents of $\mathrm{tr}\, Z^L$ operators. Thus these states will also persist to be $\frac{1}{16}$BPS at higher loop orders. In the process of extracting the partition function from the 1-loop hamiltonian data we did not use in any way any information about the representation theory of $psu(2,2|4)$. The

fact that we recover all states in these multiplets is a further check of this procedure.

However it is perhaps a bit surprising, in view of the applications to black hole entropy, that we do not observe any other new primary states (or their descendants). As a word of caution we note that these might in principle appear for higher lengths and oscillator occupancy numbers than we could check. However, given the various checks and consistency with $\frac{1}{8}$BPS and extrapolation to $L = 1$ letters we do not think that this is very probable.

The huge reduction of the number of $\frac{1}{16}$BPS states with respect to the free theory reinstated agreement with supergraviton partition function. However, the transition to a phase with black hole like scaling which was seen at zero coupling seems to disappear. The form of our partition function suggests that the constituents generating the $\frac{1}{16}$BPS states behave much more like commutative objects than 'noncommutative letters'. We speculate that in order to see the black hole phase explicitly from gauge theory, one has to use the complete nonplanar dilatation operator.

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