UNIFORM EXISTENCE OF THE INTEGRATED DENSITY OF STATES FOR COMBINATORIAL AND METRIC GRAPHS OVER $\mathbb{Z}^d$

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Abstract. We give an overview and extension of recent results on ergodic random Schrödinger operators for models on $\mathbb{Z}^d$. The operators we consider are defined on combinatorial or metric graphs, with random potentials, random boundary conditions and random metrics taking values in a finite set. We show that normalized finite volume eigenvalue counting functions converge to a limit uniformly in the energy variable, at least locally. This limit, the integrated density of states (IDS), can be expressed by a closed Shubin-Pastur type trace formula. The set of points of increase of the IDS supports the spectrum and its points of discontinuity are characterized by existence of compactly supported eigenfunctions. This applies to several examples, including various periodic operators and percolation models.

1. Introduction

This paper deals with spectral analysis of certain random type operators on graphs with a $\mathbb{Z}^d$-structure. We consider both combinatorial graphs and quantum graphs. Randomness enters not only via potentials but, more importantly, via geometry. More precisely, we will consider certain random "perturbations" of our graph in the combinatorial setting and random boundary conditions and random lengths in the quantum graph setting. Our results are concerned with existence of the integrated density of states for such models.

The *integrated density of states* (IDS) or *spectral distribution function* is a fundamental tool in the study of such random operators. It measures the number of states (up to a given energy) per unit volume of the underlying system. Accordingly, it can be obtained as a limit of normalized eigenvalue counting functions. Existence of this limit in the sense of pointwise convergence or rather vague convergence of measures is well established for ergodic Schrödinger operators in the continuum (i.e. on $L^2(\mathbb{R}^d)$) and on the lattice (i.e. on $\ell^2(\mathbb{Z}^d)$), see for instance the early papers [Pas80, Shu82, KM82] and the recent surveys [KM07, Ves07]. Some of these approaches can be modified to give an analogous result for random Schrödinger on metric graphs [HV07, § 6].

It turns out that existence of the limit can be shown in a much stronger sense than in the pointwise one, viz in the sense of convergence in the supremum norm. Thus, the limit exists uniformly in the energy. This is particularly remarkable as

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the limiting object, the IDS, can have many points of discontinuity in the setting of geometric randomness described above. As such discontinuities are not possible for random Schrödinger operators on $\mathbb{Z}^d$ itself, they are an exclusive consequence of geometric ingredients. In fact, it turns out that they are intimately related to local features of the geometry viz to existence of compactly supported eigenfunctions. This phenomenon has attracted attention before. For periodic, abelian graphs it was observed by Kuchment in [Kuc91, Kuc05] (see [DLM$^+$03] for related material as well). For discrete models it was then studied systematically in the somewhat different context of aperiodic order in [KLS03], as well as for periodic and percolation models on quasi-transitive graphs in [Ves05b]. These two types of results were unified in the recent [LV]. Random operators on graphs based on $\mathbb{Z}^d$ allow for a particularly nice treatment of the question of uniform convergence, see [LMV, GLV07].

This paper deals with the circle of topics just discussed for models with the abovementioned $\mathbb{Z}^d$-structure. In particular, it surveys and extends the results of [LMV, GLV07]. Examples include several variants of periodic operators and percolation models. One essential tool in these works is a general ergodic type result from [LMV]. This result is not about operators, but about Banach space valued functions which are compatible with a so-called colouring. In order to apply it, we need a certain finiteness assumption to hold for the number of local geometric situations in our model. Limitations and extensions of this type of approach are discussed in the final section.

More generally, the paper is organized as follows. In Section 2 we define colourings of $\mathbb{Z}^d$ and present the mentioned general ergodic theorem from [LMV], as well as a random version. In Sections 3, 4 and 5 we apply these results to operators on combinatorial graphs, metric graphs and metric graphs with random lengths, respectively. This corresponds to three different types of behaviour for the underlying (counting) functions and allows us to cover examples ranging from periodic operator to random order, including aperiodic order and percolation. Finally, Section 6 gives an outlook on the interplay between uniform convergence and discontinuities of the integrated density of states.

The results in Section 3 are taken from [LMV]. The results in Section 4 are taken from [GLV07]. The results in Section 5 are new.

2. Colourings and Ergodic Theorems

In this section, we introduce some basic set-up and discuss the abstract ergodic theorem of [LMV] (see [Len02, LS06] for earlier results of the same type). The theorem is phrased in terms of Banach-space valued functions on patterns. It will turn out that eigenvalue counting functions of suitable (restrictions of) operators provide exactly such Banach-space valued functions.

We are concerned with the graph $\mathbb{Z}^d$. Thus, the vertex set is given by $\mathbb{Z}^d$ and vertices of Euclidean distance 1 are adjacent. There are two groups acting naturally on this graph. One is the group $\Gamma_{\text{trans}} = \mathbb{Z}^d$ acting by translations. The other group is the group Γ_{full} of all graph automorphisms. Note that Γ_{full} is generated by translations by vectors in $\mathbb{Z}^d$ and a finite set of rotations. For our results it does not matter which of the two groups we consider. Thus, we now choose Γ to be Γ_{full} or Γ_{trans}. The action will be written multiplicatively.

Let $\mathcal{A}$ be a finite set. The set of all finite subsets of $\mathbb{Z}^d$ is denoted by $\mathcal{F}$. A map $\Lambda\colon \mathbb{Z}^d \longrightarrow \mathcal{A}$ is called an $\mathcal{A}$-*colouring* of $\mathbb{Z}^d$. A map $P\colon Q(P) \longrightarrow \mathcal{A}$ with $Q(P) \in \mathcal{F}$ is called an $\mathcal{A}$-*pattern*. For $M \in \mathbb{N}$ we denote by C_M the cube at the origin with side length $M-1$, i.e.

$$C_M := \{x \in \mathbb{Z}^d : 0 \le x_j \le M-1, j = 1, \dots, d\}.$$

The set of all C_M, $M \in \mathbb{N}$, is denoted by $\mathcal{C}$, and a pattern with $Q(P) \in \mathcal{C}$ is called a *cube pattern* or a *box pattern*. The set of box patterns P with $Q(P) = C_M$ is denoted by $\mathcal{P}_0^B(M)$. For a pattern P and $Q \in \mathcal{F}$ with $Q \subset Q(P)$ we define the restriction $P \cap Q$ of P to Q in the obvious way by $P \cap Q : Q \longrightarrow \mathcal{A}$, $x \mapsto P(x)$. For a pattern P and $\gamma \in \Gamma$ we define the shifted map γP by $\gamma P : \gamma Q(P) \longrightarrow \mathcal{A}, \gamma(y) \mapsto P(y)$. On the set of all patterns we define an equivalence relation by $P \sim P'$ if and only if there exists a $\gamma \in \Gamma$ with $\gamma P = P'$. For a cube pattern P and an arbitrary pattern P' we define the number of occurrences of the pattern P in P' by

$$\sharp_P^\Gamma P' := \sharp\Big\{x \in Q(P') : P' \cap \big(x + Q(P)\big) \sim P\}.$$

Given a set $Q \subset \mathbb{Z}^d$ we denote by $V_Q^\partial \subset Q$ the inner vertex boundary of Q, i.e. the set of those vertices contained in Q which have a neighbour in the complement $\mathbb{Z}^d \setminus Q$. A sequence $(Q_l)_{l \in \mathbb{N}}$ of finite subsets of $\mathbb{Z}^d$ is called a *van Hove sequence* in $\mathbb{Z}^d$ if $\lim_{l\to\infty} \frac{|V_{Q_l}^\partial|}{|Q_l|} = 0$.

Definition 1. A map $b\colon \mathcal{F} \longrightarrow [0,\infty)$ is called a *boundary term* if $b(Q) = b(t+Q)$ for all $t \in \mathbb{Z}^d$ and $Q \in \mathcal{F}$, $\lim_{j\to\infty} |Q_j|^{-1} b(Q_j) = 0$ for any van Hove sequence (Q_j), and there exists $D > 0$ with $b(Q) \le D|Q|$ for all $Q \in \mathcal{F}$.

Definition 2. Let $(X, \|\cdot\|)$ be a Banach space and $F\colon \mathcal{F} \longrightarrow X$ be given.

(a) The function F is said to be *almost-additive* if there exists a boundary term b such that

$$\left\| F(\cup_{k=1}^m Q_k) - \sum_{k=1}^m F(Q_k) \right\| \le \sum_{k=1}^m b(Q_k)$$

for all $m \in \mathbb{N}$ and all pairwise disjoint sets $Q_k \in \mathcal{F}$, $k = 1, \dots, m$.

(b) Let $\Lambda : \mathbb{Z}^d \longrightarrow \mathcal{A}$ be a colouring. The function F is said to be Γ-Λ-*invariant* if

$$F(Q) = F(\gamma Q)$$

whenever $\gamma \in \Gamma$ and $Q \in \mathcal{F}$ obey $\gamma(\Lambda \cap Q) = \Lambda \cap (\gamma Q)$. In this case there exists a function $\widetilde{F}$ on the cubes $\mathcal{C}$ with values in X such that

$$F(\gamma Q) = \widetilde{F}\Big(\gamma^{-1}\big(\Lambda \cap (\gamma Q)\big)\Big)$$

for cubes $Q \in \mathcal{C}$ and $\gamma \in \Gamma$.

(c) The function F is said to be *bounded* if there exists a finite constant $C > 0$ such that

$$\|F(Q)\| \le C|Q|$$

for all $Q \in \mathcal{F}$.

Theorem 3. *Let $\mathcal{A}$ be a finite set, $\Lambda\colon \mathbb{Z}^d \longrightarrow \mathcal{A}$ an $\mathcal{A}$-colouring and $(X, \|\cdot\|)$ a Banach space. Let $(Q_j)_{j\in\mathbb{N}}$ be a van Hove sequence such that for every pattern*

P the frequency $\nu_P = \lim_{j\to\infty} |Q_j|^{-1} \sharp^\Gamma_P(\Lambda \cap Q_j)$ exists. Let $F : \mathcal{F} \longrightarrow X$ be a Γ-Λ-invariant, almost-additive bounded function. Then the limits

$$\overline{F} := \lim_{j\to\infty} \frac{F(Q_j)}{|Q_j|} = \lim_{M\to\infty} \sum_{P\in\mathcal{P}_0^B(M)} \nu_P \frac{\widetilde{F}(P)}{|C_M|}$$

exist in the topology of $(X, \|\cdot\|)$ and are equal.

Remark 4. (a) The theorem is proven in [LMV] for $\Gamma = \Gamma_{\text{trans}}$. The proof carries over to give the result for $\Gamma = \Gamma_{\text{full}}$ as well.

(b) We have explicit bounds on speed of convergence in terms of speed of convergence of the frequencies. For details we refer to [LMV].

The previous result does not require the context of an ergodic action. Instead existence of the frequencies is sufficient. Of course, existence of frequencies follows for ergodic actions. This is discussed next. Let $(\Omega, \mathbb{P})$ be a probability space such that Γ acts ergodically on $(\Omega, \mathbb{P})$. A random $\mathcal{A}$-colouring is a map

$$\Lambda : \Omega \longrightarrow \bigotimes_{\mathbb{Z}^d} \mathcal{A} \text{ with } \Lambda(\gamma(\omega))_{\gamma^{-1}x} = \Lambda(\omega)_x$$

for all $\gamma \in \Gamma$ and $x \in \mathbb{Z}^d$.

Lemma 5. *Let (Q_j) be an arbitrary van Hove sequence. Then, for almost every $\omega \in \Omega$ the frequency $\nu_P = \lim_{j\to\infty} |Q_j|^{-1} \sharp^\Gamma_P(\Lambda(\omega) \cap Q_j)$ exists and is independent of ω for every cube pattern P.*

Proof. For a fixed pattern P the frequency exists for almost every ω by a standard ergodic theorem. As there are only countably many P the statement follows. □

The random version of Theorem 3 then reads:

Theorem 6. *Let $\mathcal{A}$ be a finite set, $(\Lambda_\omega)_{\omega\in\Omega}$ be a random $\mathcal{A}$-colouring and $(X, \|\cdot\|)$ a Banach space. Let $(Q_j)_{j\in\mathbb{N}}$ be a van Hove sequence. Let $F_\omega : \mathcal{F} \longrightarrow X$ be a family of Γ-Λ_ω-invariant, almost-additive bounded functions which is homogeneous, i.e. $F_{\gamma(\omega)}(\gamma(Q)) = F_\omega(Q)$ for all $\gamma \in \Gamma, Q \in \mathcal{F}$. Then, for almost every $\omega \in \Omega$ the limits*

$$\overline{F}_\omega := \lim_{j\to\infty} \frac{F_\omega(Q_j)}{|Q_j|} = \lim_{M\to\infty} \sum_{P\in\mathcal{P}_0^B(M)} \nu_P \frac{\widetilde{F}(P)}{|C_M|}$$

exist in the topology of $(X, \|\cdot\|)$ and are equal. In particular, $\overline{F}_\omega$ is almost surely independent of Ω.

Proof. Almost sure existence of the limit is a direct consequence of the previous lemma and the first theorem of this section. In fact, this theorem gives the explicit formula for the limit in terms of the function $\widetilde{F}$. This shows that the limit does not depend on ω almost surely. □

Remark 7. The theorem is similar in appearance to the "usual" ergodic theorems. Let us therefore point out the differences: First of all, the result is valid for functions taking values in a Banach space. This gives quite some additional freedom. In fact, this freedom will allows us in the next sections to conclude uniform (in the energy) convergence of the integrated density of states compared to the usual pointwise (in the energy) results obtained from e.g. subadditve ergodic theorems.

Moreover, the (proof of the) theorem gives an explicit description of elements in the probability space for which the limit exists. These elements turn out to be exactly the typical elements with respect to the randomness viz the elements having frequencies.

Finally, let us mention that one can even obtain explicit error bounds on speed of convergence in terms of speed of convergence of the frequencies (see [LMV].

3. Operators on Combinatorial Graphs

In this section we introduce finite (hopping) range equivariant operators for graph-like structures over $\mathbb{Z}^d$. We then use the result of the previous section to obtain existence of the integrated density of states. Finally, we have a closer look at three instances of this situation.

We consider situations in which a fixed finite dimensional Hilbert space is attached to each vertex in $\mathbb{Z}^d$. In order to model this we need some more notation. Let $\mathcal{H}$ be a fixed Hilbert space with dimension $\dim(\mathcal{H}) < \infty$ and norm $\|\cdot\|$. Then,

$$\ell^2(\mathbb{Z}^d, \mathcal{H}) := \{u\colon \mathbb{Z}^d \longrightarrow \mathcal{H} : \sum_{x\in\mathbb{Z}^d} \|u(x)\|^2 < \infty\}$$

is a Hilbert space. The *support* of $u \in \ell^2(\mathbb{Z}^d, \mathcal{H})$ is the set of $x \in \mathbb{Z}^d$ with $u(x) \neq 0$.

For $x \in \mathbb{Z}^d$, we define the natural projection $p_x : \ell^2(\mathbb{Z}^d, \mathcal{H}) \longrightarrow \mathcal{H}$, $u \mapsto p_x(u) := u(x)$. Let $i_x\colon \mathcal{H} \longrightarrow \ell^2(\mathbb{Z}^d, \mathcal{H})$ be the adjoint of p_x. Similarly, for a subset $Q \subset \mathbb{Z}^d$ we define $\ell^2(Q, \mathcal{H})$ to be the subspace of $\ell^2(\mathbb{Z}^d, \mathcal{H})$ consisting of elements supported in Q. The projection of $\ell^2(\mathbb{Z}^d, \mathcal{H})$ on $\ell^2(Q, \mathcal{H})$ is denoted by p_Q and its adjoint by i_Q.

The operators and functions we are interested in are specified in the next two definitions.

Definition 8. Let $\mathcal{A}$ be a finite set, $\Lambda\colon \mathbb{Z}^d \longrightarrow \mathcal{A}$ a colouring and $H\colon \ell^2(\mathbb{Z}^d, \mathcal{H}) \longrightarrow \ell^2(\mathbb{Z}^d, \mathcal{H})$ a selfadjoint operator.

(a) The operator H is said to be of *finite range* if there exists a length $R_{fr} > 0$ such that $p_y H i_x = 0$, whenever $x, y \in \mathbb{Z}^d$ have distance bigger than R_{fr}.

(b) The operator H is said to be *Λ-invariant* if there exists a length $R_{inv} \in \mathbb{N}$ such that $p_y H i_x = p_{\gamma y} H i_{\gamma x}$ for all $x, y \in \mathbb{Z}^d$ and $\gamma \in \Gamma$ obeying

$$\gamma\Big(\Lambda \cap \big(C_{R_{inv}}(x) \cup C_{R_{inv}}(y)\big)\Big) = \Lambda \cap \big(C_{R_{inv}}(\gamma x) \cup C_{R_{inv}}(\gamma y)\big).$$

For a given colouring Λ, a finite-range, Λ-invariant operator H is fully determined by specifying finitely many $\dim(\mathcal{H}) \times \dim(\mathcal{H})$ matrices $p_y H i_x$. In particular, such operators H are bounded.

Definition 9. Let $\mathcal{R}$ be the Banach space of right-continuous, bounded functions equipped with the supremum norm. For a selfadjoint operator A on a finite-dimensional Hilbert space we define its *cumulative eigenvalue counting function* $n(A) \in \mathcal{R}$ by setting

$$n(A)(\lambda) := \sharp\{\text{eigenvalues of } A \text{ not exceeding } \lambda\}$$

for all $\lambda \in \mathbb{R}$, where each eigenvalue is counted according to its multiplicity.

Theorem 10. *Let $\Lambda\colon \mathbb{Z}^d \longrightarrow \mathcal{A}$ be a colouring and $(Q_j)_{j\in\mathbb{N}}$ a van Hove sequence along which the frequencies ν_P of all patterns $P \in \mathcal{P}_0^B(M)$ exist. Let $H : \ell^2(\mathbb{Z}^d, \mathcal{H}) \longrightarrow \ell^2(\mathbb{Z}^d, \mathcal{H})$ be a selfadjoint, Λ-invariant finite-range operator.*

Then, there exists a unique probability measure μ_H on $\mathbb{R}$ with distribution function N_H such that $\frac{1}{|Q_j|}n(p_{Q_j}Hi_{Q_j})$ converges to N_H with respect to the supremum norm as $j \to \infty$.

A **proof** for this theorem can be found in [LMV]). Here, we only sketch the idea. We consider the Banach space $\mathcal{R}$ of all right continuous bounded real-valued functions on the real line with the supremum norm. The operator A then gives rise to a map from $\mathcal{F}$ to $\mathcal{R}$ viz $Q \mapsto n(A_Q)$, where A_Q denotes the restriction of A to Q. By assumption on A this map satisfies the assumptions of Theorem 3. Hence, the desired averages exist by that theorem.

Linear algebra and the uniform convergence just established can be used to derive the following two corollaries.

Corollary 11. *Assume the situation of the theorem and additionally positivity of the frequency ν_P for any pattern P occurring in Λ. Then the spectrum of H is the topological support of μ_H.*

Remark 12. The assumption on positivity of frequencies is necessary in order to obtain this result. Consider e.g. id on $\ell^2(\mathbb{Z})$ and perform a rank one perturbation $B = \langle \delta_0, \cdot \rangle \delta_0$ at the origin. Then, the IDS of id and of $id + B$ coincide, but their spectra do not.

Corollary 13. *Assume the situation of the previous corollary. Then the following assertions for $\lambda \in \mathbb{R}$ are equivalent:*

(i) *λ is a point of discontinuity of N_H,*
(ii) *there exists a compactly supported eigenfunction of H corresponding to λ.*

Remark 14. For random Schrödinger operators on $\mathbb{Z}^d$ the statement of Theorem 10 can be obtained from continuity of the IDS. (More precisely, continuity of the IDS combined with the well established weak convergence of the eigenvalue counting measures gives uniform convergence of the IDS). Thus, the statement of Theorem 10 is particularly interesing for situations in which this continuity is not valid. This non-continuity arises due to local geometric structures as discussed in the previous corollary. Specific examples where discontinuites of the IDS exist are discussed in Subsection 3.3.

3.1. **Periodic Operators.** The setting just described can easily be applied to periodic operators over $\mathbb{Z}^d$. In fact, the additional Hilbert space $\mathcal{H}$ gives quite some freedom to consider situations which are only similar to $\mathbb{Z}^d$. This is discussed next.

Let G be a graph with a countable set of vertices (which we again denote by G) on which $\mathbb{Z}^d$ acts isometrically, freely and cocompactly. Let us denote by $\mathfrak{D} \subset G$ a $\mathbb{Z}^d$-fundamental domain. Thus $\mathfrak{D}$ contains exactly one element of each $\mathbb{Z}^d$-orbit in G. By the cocompactness assumption, $\mathfrak{D}$ is finite. This implies in particular that the vertex degree of G is uniformly bounded. From now on the fundamental domain $\mathfrak{D}$ will be assumed fixed.

Remark 15. A simple example of such a graph is $\mathbb{Z}^d$ with the natural action of the group $(N\mathbb{Z})^d$ for $N \in \mathbb{N}$ fixed. Another example would be the Cayley graph $G = \mathrm{Cay}(\mathcal{G}, S)$ of a direct product group $\mathcal{G} = \mathbb{Z}^d \otimes F$, where F is any finite group and S is a finite, symmetric set of generators for $\mathcal{G}$. Here the action of $\mathbb{Z}^d$ on G is induced by the (obvious) action of $\mathbb{Z}^d$ on itself. Note that even for trivial F we obtain infinitely many different graphs, namely the Cayley graphs of $\mathbb{Z}^d$.

We now turn to operators acting on $\ell^2(G)$ and $\ell^2(\mathbb{Z}^d, \mathcal{H})$. Let $A\colon \ell^2(G) \to \ell^2(G)$ be a selfadjoint linear operator satisfying the following two conditions:

- $A(x,y) = A(\gamma x, \gamma y)$ for all $x, y \in G, \gamma \in \mathbb{Z}^d$. (Covariance)
- There exists $\rho > 0$ with $A(x,y) = 0$ whenever the graph distance of x and y exceeds ρ. (Finite range).

The setting developed so far looks different from the setting discussed in the previous section. To make the connection we proceed as follows. Let $\mathcal{A}$ be a set consisting of one element and let Λ be the trivial colouring. Set $\mathcal{H} := \ell^2(\mathfrak{D})$, then $\dim(\mathcal{H}) = |\mathfrak{D}|$. We can now define a unitary operator $U\colon \ell^2(\mathbb{Z}^d, \mathcal{H}) \to \ell^2(G)$ in the following way: For a $\psi \in \ell^2(\mathbb{Z}^d, \mathcal{H})$ and $\gamma \in \mathbb{Z}^d$ write $\psi(\gamma) = \sum_{i \in \mathfrak{D}} \psi_i(\gamma)\delta_i$, where $(\delta_i)_{i\in\mathfrak{D}}$ is the standard orthonormal basis of $\ell^2(\mathfrak{D})$. Then, the coefficients $\psi_i(\gamma)$ are uniquely determined. We set $(U\psi)(x) := \psi_i(\gamma)$ where $i \in \mathfrak{D}$ and $\gamma \in \mathbb{Z}^d$ are the unique elements such that $x = \gamma i$. Then, $H = U^*AU$ can easily be seen to be a Λ-invariant operator of finite range. Moreover, the frequencies of all patterns (occurring in Λ) are positive (and in fact equal to 1). Thus, all the results of the previous section apply to H. They can then be used to infer the obvious analogues for the operator A.

3.2. **Set of Visible Points.** The set of visible points in $\mathbb{Z}^d$ is a prominent example (and counterexample) in number theory and aperiodic order [BMP00, Ple]. In particular its diffraction theory has been well studied. Still, it seems that the corresponding nearest-neighbour hopping model had not received attention until [LMV]. Here, we shortly discuss the result from there.

The set $\mathcal{VP}$ of visible points in $\mathbb{Z}^d$ consists of the origin and all $x \neq 0$ in $\mathbb{Z}^d$ with

$$\{tx : 0 < t < 1\} \cap \mathbb{Z}^d = \emptyset.$$

Thus, $x \neq 0$ belongs to $\mathcal{VP}$, if and only if the greatest common divisor of its coordinates is 1. The obvious interpretation is that such an x can be seen by an observer standing at the origin. This gives the name to this set. The characteristic function

$$\Lambda := \chi_{\mathcal{VP}}\colon \mathbb{Z}^d \longrightarrow \mathcal{A} := \{0,1\}$$

of $\mathcal{VP}$ provides a colouring. While $\mathcal{VP}$ is very regular in many respects, it has arbitrarily large holes. In particular, existence of the frequencies ν_P does not hold along arbitrary van Hove sequences. However, as was shown in [Ple] (see [BMP00] for special cases as well), the frequencies exist and can be calculated explicitly for sequences of cubes centred at the origin. Moreover, the frequencies of all patterns which occur are strictly positive.

Thus, all abstract results discussed above are valid for $\chi_{\mathcal{VP}}$-invariant operators of finite range. One relevant such operator is the adjacency operator $A_{\mathcal{VP}}$. We finish this section by defining this operator: Points $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ in $\mathbb{Z}^d$ are said to be neighbours, written as $x \sim y$, whenever

$$\sum_{j=1}^{d} |x_j - y_j| = 1.$$

Then, $A_{\mathcal{VP}}\colon \ell^2(\mathbb{Z}^d) \longrightarrow \ell^2(\mathbb{Z}^d)$ is defined by

$$(A_{\mathcal{VP}}u)(x) := \chi_{\mathcal{VP}}(x) \sum_{y\sim x : y \in \mathcal{VP}} u(y)$$

for all $x \in \mathbb{Z}^d$ and all $u \in \ell^2(\mathbb{Z}^d)$.

3.3. **Percolation on Combinatorial Graphs.** In this section we add some randomness to our model. Thus we obtain random operators which are generated by a percolation process on the underlying graph. Hamilton operators on percolation subgraphs of combinatorial graphs have been considered in the literature in theoretical physics [dGLM59, KE72, CCF+86], computational physics [KB02] (and references therein), and mathematical physics [BK01, KN03, Ves05a, Ves05b, KM06, MS07, AV08, AV].

Choose $\Gamma = \mathbb{Z}^d$. We start with the deterministic part. Fix a finite range self-adjoint operator $A\colon \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ which is invariant under the trivial colouring where every element of $\mathbb{Z}^d$ has the same colour. Thus, A is $\mathbb{Z}^d$ periodic.

To define the random part, let $(\Omega, \mathbb{P})$ be a probability space and $\tau_\gamma\colon \Omega \to \Omega, \gamma \in \Gamma$, an ergodic family of measure preserving transformations. Furthermore, let $\mathcal{A}$ be an arbitrary finite subset of $\mathbb{R} \cup \{+\infty\}$ and $(\omega, x) \mapsto V(\omega, x) \in \mathcal{A}$ a random field which is invariant under the transformations $\tau_\gamma, \gamma \in \Gamma$. More precisely, for all $\gamma \in \Gamma$, $\omega \in \Omega$ and $x \in G_d$ we require $V(\tau_\gamma \omega, x) = V(\omega, \gamma x)$. Next we define random subsets of $\mathbb{Z}^d$ and $\ell^2(\mathbb{Z}^d)$ induced by the random field V. For each $\omega \in \Omega$ define the subset of vertices $G_\omega := \{x \in \mathbb{Z}^d : V(\omega, x) < \infty\}$, the natural projection operator $p_\omega\colon \ell^2(G_d) \to \ell^2(G_\omega)$ and its adjoint $i_\omega\colon \ell^2(G_\omega) \to \ell^2(G_d)$. This gives rise to the random Hamiltonian

$$H_\omega := A_\omega + V_\omega, \quad \mathcal{D}(H_\omega) := \ell^2(G_\omega).$$

Here, the hopping part is given by

$$A_\omega := p_\omega A\, i_\omega, \quad \mathcal{D}(A_\omega) := \ell^2(G_\omega).$$

and V_ω is defined by $V_\omega := p_\omega V(\omega, \cdot)\, i_\omega\colon \ell^2(G_\omega) \to \ell^2(G_\omega)$. We extend this operator to $\ell^2(\mathbb{Z}^d)$ by setting it equal to zero on the complement of G_ω. The extension will be denoted by H_ω as well. For such operators the existence of the IDS as a pointwise limit has been established in [Ves05a], and its continuity properties have been analysed in [Ves05b]. Here, we discuss uniform existence of the integrated density of states by fitting these operators into the framework presented above. For each $\omega \in \Omega$ we define a colouring by

$$\Lambda_\omega\colon \mathbb{Z}^d \to \mathcal{A}, \quad \Lambda_\omega(x) := V(\omega, x).$$

Then, the (H_ω) are a Λ_ω-invariant family of operators in the sense of Section 2. Moreover, it is not hard to see that for any pattern $P\colon Q(P) \to \mathcal{A}, Q(P) \in \mathcal{F}$, the frequency ν_P of P in Λ_ω exists almost surely along the van Hove sequence of boxes $C_j, j \in \mathbb{N}$. We can then find a set of full measure in Ω for which the frequencies of all patterns exist and the frequencies of the occurring patterns are positive. For such an ω we can then apply Theorem 10 and its two corollaries.

Let us close this section by pointing out two situations to which the results presented here can be easily extended:

(1) The operator A can be allowed to be $(N\mathbb{Z})^d$ periodic. In fact, one can do a similar analysis for models based on the graphs introduced in Section 3.1.

(2) Instead of *site*-percolation Hamiltonians one can consider Hamiltonians on *bond*-percolation graphs. For the construction of the IDS for such operators, see [KM06].

Let us mention that the asymptotics of the IDS at spectral edges for various percolation models has been analysed in [BK01, KN03, KM06, MS07, AV08, AV].

4. Operators on Metric Graphs

The typical (differential) operators of interest on metric graphs are unbounded, such that their counting functions are right-continuous, but unbounded. Subtraction of the counting function of a reference operator yields bounded functions again: spectral shift functions. Thus we can apply the results of Section 2 to the (bounded) spectral shift functions, which in turn yields results for the counting functions.

4.1. **General Results.** We define a metric graph G_d over $\mathbb{Z}^d$ in the following way. Let $e_j,\ j = 1, \ldots, d$ be the standard basis of the real d-dimensional space $\mathbb{R}^d$. Each edge $e \in E_d$ is determined by $\iota(e) = x, \tau(e) = x + e_j$ for some $x \in V_d = \mathbb{Z}^d, j \in 1, \ldots, d$. We define the metric graph G_d by identifying each edge e of the combinatorial graph with the interval $e = [x, x+e_j]$, which in turn can be identified canonically with the interval $(0, 1)$. This procedure induces an orientation on our graph. However, it turns out that all relevant quantities are independent of the choice of orientation.

We will also need to consider finite subgraphs G of G_d . By a subgraph we mean a subset of the edges of G_d together with all adjacent vertices. All functions we consider live on the topological space G_d (as a subspace of $\mathbb{R}^d$) or subgraphs of it.

The operators we are interested in will be defined on the Hilbert space

$$L^2(E_d) := \bigoplus_{e\in E_d} L^2(e)$$

and their domains of definition will be subspaces of

$$W^{2,2}(E_d) := \bigoplus_{e\in E_d} W^{2,2}(e),$$

where $W^{2,2}(e)$ is the usual Sobolev space of $L^2(e)$ functions whose (weak) derivatives up to order two are in $L^2(e)$ as well. The restriction of $f \in W^{2,2}(E_d)$ to an edge e is denoted by f_e. For an edge $e = [\iota(e), \tau(e)] = [x, x + e_j]$ and g in $W^{2,2}(e)$ the boundary values of g

$$g(\iota(e)) := \lim_{t\searrow 0} g(x + te_j),\ \ g(\tau(e)) := \lim_{t\nearrow 1} g(\iota(e) + te_j)$$

and the boundary values of g'

$$g'(\iota(e)) := \lim_{\epsilon\searrow 0} \frac{g(x + \epsilon e_j) - g(x)}{\epsilon} \quad \text{and} \quad g'(\tau(e)) := \lim_{\epsilon\searrow 0} \frac{g(\tau(e)) - g(\tau(e) - \epsilon e_j)}{-\epsilon}$$

exist by standard Sobolev type theorems. For $f \in W^{2,2}(E_d)$ and each vertex x we gather the boundary values of $f_e(x)$ over all edges e adjacent to x in a vector $f(x)$. Similarly, we gather the boundary values of $f'_e(x)$ over all edges e adjacent to x in a vector $f'(x)$.

Given the boundary values of functions, we can define boundary conditions following [KS99, Har00]. A single-vertex boundary condition at $x \in V$ is a choice of subspace L_x of $\mathbb{C}^{4d}$ with dimension $2d$ such that

$$\eta((v, v'), (w, w')) := \langle v', w\rangle - \langle v, w'\rangle$$

vanishes for all $(v, v'), (w, w') \in L_x$. An $f \in W^{2,2}(E_d)$ is said to satisfy the single-vertex boundary condition L_x at x if $(f(x), f'(x))$ belongs to L_x. A field of single-vertex boundary conditions $L := \{L_x : x \in V_d\}$ will be called boundary condition. Given such a field, we obtain a selfadjoint realization Δ_L of the Laplacian Δ on $L^2(E_d)$ by choosing the domain

$$\mathcal{D}(\Delta_L) := \{f \in W^{2,2}(E_d) : \forall x : (f(x), f'(x)) \in L_x\}$$

and by letting Δ_L act on f_e as $-f_e''$. This way all so-called graph-local boundary conditions can be realised, i.e. those which relate boundary values at the same vertex only. This includes Dirichlet boundary conditions with subspace L^D consisting of all those (v, v') with $v = 0$, Neumann conditions with subspace L^N consisting of all those (v, v') with $v' = 0$, and Kirchhoff (also known as free) boundary conditions L^K consisting of all (v, v') with v having all components equal and v' having the sum over its components equal to 0.

Note that a metric graph together with a field of boundary conditions is sometimes called a "quantum graph", although this means that on a fixed quantum graph there exist only well-defined operators of fixed order (2 in this case).

Everything discussed so far including existence of limits of functions at the vertices and the notions of boundary condition extends in the obvious way to subgraphs. Moreover, for a subgraph G of G_d with edge set E, we write $W^{2,2}(E) := \oplus_{e\in E} W^{2,2}(e)$. The number of edges of a finite subgraph G of G_d is denoted by $|E|$.

In order to define random operators we need some further data including a probability space $(\Omega, \mathbb{P})$ and an action of (a subgroup of) the automorphism group of G_d on Ω and maps L, V from Ω into the space of boundary conditions and potentials, respectively. As discussed at the beginning, for us two groups will be relevant, the full automorphism group Γ_{full} and the group Γ_{trans} of translations by $\mathbb{Z}^d$. We fix one of them, denote it by Γ and assume that it acts ergodically on $(\Omega, \mathbb{P})$ via measure preserving transformations. To simplify the notation we identify $\gamma \in \Gamma$ with the associated measure preserving transformation.

Let us describe the type of random operators we consider in this section:

Assumption 1. Let $(\Omega, \mathbb{P})$ be a probability space and $\Gamma \in \{\Gamma_{\text{full}}, \Gamma_{\text{trans}}\}$ a group acting ergodically on $(\Omega, \mathbb{P})$. Let $\mathcal{B}$ be a finite subset of $L^\infty(0, 1)$ and $\mathcal{L}$ a finite set of boundary conditions. A random potential is a map

$$V : \Omega \longrightarrow \bigotimes_{e\in E_d} \mathcal{B} \text{ with } V(\gamma(\omega))_{\gamma(e)} = V(\omega)_e \tag{1}$$

for all $\gamma \in \Gamma$ and $e \in E_d$. A random boundary condition is a map

$$L : \Omega \longrightarrow \bigotimes_{v\in V_d} \mathcal{L}, \text{ with } L(\gamma(\omega))(\gamma(x)) = L(\omega)(x) \tag{2}$$

for all $\gamma \in \Gamma$ and $v \in V_d$.

A family of random operators (H_ω) on $L^2(E_d)$ can be defined with domain of definition

$$\mathcal{D}(H_\omega) := \{f \in W^{2,2}(E_d) : (f(x), f'(x)) \in L(\omega)(x) \quad \text{for all } x \in V_d\}$$

acting by

$$(H_\omega f)(e) := -f_e'' + V(\omega)_e f_e$$

for each edge e. These are selfadjoint lower bounded operators. A particularly simple random operator (H_ω) is given by the pure Laplacian $-\Delta_D$ with the domain

$$\mathcal{D}(\Delta) := \{f \in W^{2,2}(E_Q) \,|\forall x \in V : (f(x), f'(x)) \in L^D\}$$

and $-\Delta = H_\omega$ with $V(\omega) \equiv 0$.

We assume throughout the section that Assumption 1 holds, and for this reason do not repeat it in every lemma.

Remark 16. While Γ_{full} is not commutative it is a natural object to deal with. In particular, let us note that the Laplacian without potential and boundary conditions in all vertices identical to Kirchhoff conditions is invariant under Γ_{full}.

We will need to consider restrictions of our operators to finite subgraphs. These are finite subgraphs associated to finite subsets of $\mathbb{Z}^d$. The cardinality of a finite subset Q of $\mathbb{Z}^d$ is denoted by $|Q|$. We introduce the set of edges

$$E_Q := \{e \in E_d \mid \iota(e) \in Q\}$$

and the set of vertices

$$V_Q := \{v \in V_d \mid v \text{ adjacent to } e \text{ for some } e \in E_Q\}.$$

The subgraph (V_Q, E_Q) will be denoted by G_Q. Note that $V_Q \supset Q$. The set V_Q^i of inner vertices of G_Q is then given by those vertices of G_Q all of whose adjacent edges (in G_d) are contained in G_Q. The set of inner edges E_Q^i of G_Q is given by those edges whose both endpoints are inner. The vertices of G_Q which are not inner are called boundary vertices. The set of all boundary vertices is denoted by V_Q^∂. Similarly, the set of edges which are not inner is denoted by E_Q^∂.

The restriction H_ω^Q of the random operator H_ω to G_Q has domain given by

$$\begin{aligned}\mathcal{D}(H_\omega^Q) := \{f \in W^{2,2}(E_Q) \,|\forall x \in V_Q^i : (f(x), f'(x)) \in L(\omega)(x),\\ \forall x \in V_Q^\partial : (f(x), f'(x)) \in L^D\}.\end{aligned}$$

This operator is again selfadjoint, lower bounded, and has purely discrete spectrum. Let us enumerate the eigenvalues of H_ω^Q in ascending order

$$\lambda_1(H_\omega^Q) < \lambda_2(H_\omega^Q) \leq \lambda_3(H_\omega^Q) \leq \dots$$

and counting multiplicities. Then, the eigenvalue counting function n_ω^Q on $\mathbb{R}$ defined by

$$n_\omega^Q(\lambda) := \sharp\{n \in \mathbb{N} \mid \lambda_n(H_\omega^Q) \leq \lambda\}$$

is monotone increasing and right continuous, i.e. a distribution function, which is associated to a pure point measure, μ_ω^Q. Denote by

$$N_\omega^Q(\lambda) := \frac{1}{|E_Q|} n_\omega^Q(\lambda)$$

the volume-scaled version of $n_\omega^Q(\lambda)$ and note that $|E_Q| = d|Q|$ as the edge to vertex ratio in the graph (V_d, E_d) is equal to d.

For a finite subgraph H of G_d let χ_H be the multiplication operator by the characteristic function of H. Denote the trace on the operators on $L^2(E_d)$ by $\mathrm{Tr}[\cdot]$.

As discussed above, a random Schrödinger (H_ω) as well as the pure Laplacian Δ_D can be restricted to the subgraphs G_Q induced by finite sets Q of $\mathbb{Z}^d$. This yields the operator H_ω^Q and Δ_D^Q with spectral counting functions n_ω^Q and n_D^Q respectively.

Now, n_D^Q decomposes as a direct sum of operators. Thus, denoting the eigenvalue counting function of the negative Dirichlet Laplacian on $]0,1[$ by $n_D(\lambda)$ (similarly as N_ω^Q above), we have $n_D^Q = |E_Q| n_D = d|Q| n_D$. The associated spectral shift function is given as

$$\xi_\omega^Q(\lambda) := n_\omega^Q(\lambda) - d\,|Q|\,n_D(\lambda) = d\,|Q|\left(N_\omega^Q(\lambda) - n_D(\lambda)\right).$$

The crucial point is that ξ_ω falls into the framework of almost additive F introduced above. This is shown in the following lemma from [GLV07].

Lemma 17. *Let $(\mathcal{R}, \|\cdot\|_\infty)$ be the Banach space of right continuous bounded functions on $\mathbb{R}$. Then, for each $\omega \in \Omega$ the function $\xi_\omega : \mathcal{F} \longrightarrow \mathcal{R}$, $Q \mapsto \xi_\omega^Q$, is a bounded, $\Lambda(\omega)$ invariant almost additive function.*

Remark 18. The need to use a spectral shift function, i.e. the difference between n_ω and n_D, in the above lemma comes exclusively from the boundedness requirement. Note that the notion of boundedness depends on the norm of the considered Banach space, cf. also Lemma 25 below.

The key result is now the following proposition.

Proposition 19. *There is a bounded right continuous function $\Xi\colon \mathbb{R} \to \mathbb{R}$ such that for a given van Hove sequence (Q_l) for almost every $\omega \in \Omega$ the uniform convergence*

$$\lim_{l\to\infty} \left\| \frac{\xi_\omega^{Q_l}}{|E_{Q_l}|} - \Xi \right\|_\infty = 0$$

holds.

Proof. Given the lemma above, the proposition is a direct consequence of Theorem 6. □

By identifying the limit above and adding it to the IDS of the Dirichlet operator (see [GLV07]) we obtain a Shubin-Pastur type formula:

Theorem 20. *Let Q be a finite subset of $\mathbb{Z}^d$. Then, the function $N = N_H$ defined by*

$$N(\lambda) := \frac{1}{|E_Q|} \int_\Omega \operatorname{Tr}\left[\chi_{G_Q} \chi_{]-\infty,\lambda]}(H_\omega)\right] d\mathbb{P}(\omega) \tag{3}$$

does not depend on the choice of Q, is the distribution function of a measure $\mu = \mu_H$, and for any van Hove sequence (Q_l) in $\mathbb{Z}^d$

$$\lim_{l\to\infty} \|N_\omega^{Q_l} - N\|_\infty = 0$$

for almost every $\omega \in \Omega$. In particular, for almost every $\omega \in \Omega$, $N_\omega^{Q_l}(\lambda)$ converges as $l \to \infty$ pointwise to $N(\lambda)$ for every $\lambda \in \mathbb{R}$.

While the definition of the IDS involves an ergodic theorem, there are other spectral features of H_ω whose almost sure independence of ω uses only the ergodicity of the group action. Prominent examples are the spectrum $\sigma(H_\omega)$ and its subsets $\sigma_{pp}(H_\omega)$, $\sigma_{sc}(H_\omega)$, $\sigma_{ac}(H_\omega)$, $\sigma_{disc}(H_\omega)$, $\sigma_{ess}(H_\omega)$ according to the spectral type. In fact, by applying the general framework of [LPV07] we immediately infer the following theorem.

Theorem 21. *There exist subsets of the real line Σ, Σ_{pp}, Σ_{sc}, Σ_{ac}, Σ_{disc}, Σ_{ess} and an $\Omega' \subset \Omega$ of full measure such that $\sigma(H_\omega) = \Sigma$ and $\sigma_\bullet(H_\omega) = \Sigma_\bullet$ for all these spectral types $\bullet \in \{pp, sc, ac, disc, ess\}$ and all $\omega \in \Omega'$.*

In the following we discuss three types of percolation models. These models are based solely on random boundary conditions. The potential of the operators is identically equal to zero. Unlike in the percolation models on combinatorial graphs "deleted" edges are not removed completely from the graph but only cut off by Dirichlet boundary conditions. The reason is that removing edges would mean removing infinite dimensional subspaces from our Hilbert space. This would result in a spectral distribution function which is not comparable to the one of the Laplacian with the concerned edge included.

Before giving details we would like to emphasize the following: The examples below include cases in which the graphs contain infinitely many finite components giving rise to compactly supported eigenfunctions. In particular, the integrated density of states has a dense set of discontinuities. In fact, in the subcritical phase the IDS is a step function, albeit with dense jumps. However, despite all these jumps our result on uniform convergence does hold!

4.2. **Site Percolation on Metric Graphs.** The percolation process is defined by the following procedure: toss a (possibly biased) coin at each vertex and – according to the outcome – put either a Dirichlet or a Kirchhoff boundary condition on this vertex. Do this at every vertex independently of all the others. To be more precise, let $p \in (0,1)$ and $q = 1-p$ be given. Let $\mathcal{A} := \{L^D, L^K\}$ and the probability measure $\nu := p\delta_{L^K} + (1-p)\delta_{L^D}$ on $\mathcal{A}$ be given. Define Ω as the cartesian product space $\times_{x\in V_d}\mathcal{A}$ with product measure $\mathbb{P} := \otimes_{x\in V_d}\nu$. Let L be the stochastic process with coordinate maps $L(\omega)(x) := \omega(x)$. These data yield a family of random operators $-\Delta_\omega := H_\omega$ acting like the free Laplacian with domain given by

$$D(\Delta_\omega) = \{f \in W^{2,2}(E) : (f(x), f'(x)) \in \omega(x)\ \forall x \in V_d\}.$$

Intuitively, placing a Dirichlet boundary condition at a vertex means "removing" it from the metric graph. The $2d$ formerly adjacent edges do not "communicate" any longer through the vertex. A fundamental result of percolation theory tells us that for sufficiently small values of p the percolation graph consists entirely of finite components almost surely. For these values of p our Laplace operators decouple completely into sums of operators of the form $-\Delta^G$ for finite connected subgraphs G of G_d. Here, Δ^G acts like the free Laplacian and has Dirichlet boundary conditions on its deleted vertices (boundary vertices) and Kirchhoff boundary conditions in its vertices which have not been deleted by the percolation process (interior vertices). We introduce an equivalence relation on the set of connected subgraphs of G_d with a finite number of edges by setting $G^1 \sim G^2$ iff there exists a $\gamma \in \Gamma$ such that $\gamma G^1 = G^2$. For such an equivalence class $\mathcal{G}$ we define $n^{\mathcal{G}}$ as the eigenvalue counting function of $-\Delta^G$ for some $G \in \mathcal{G}$, and set $N^{\mathcal{G}} = \frac{n^{\mathcal{G}}}{|E_G|}$. Defining the density $\nu_{\mathcal{G}}$ of an equivalence class of finite subgraphs of G_d within the configuration ω in the obvious way, we obtain as integrated density of states for the family H_ω

$$N = \sum_{\mathcal{G}} \nu_{\mathcal{G}} N^{\mathcal{G}},$$

where the sum runs over all equivalence classes $\mathcal{G}$ of finite connected subgraphs of G_d. Thus, the integrated density of states is a pure point measure in this case with many jumps. More interestingly, all these jumps remain present (even if their height is diminished) when we start increasing p. This yields models in which the operators are not given as a direct sum of finite graph operators but still have lots of jumps in their integrated density of states. Related phenomena for combinatorial Laplacians have been studied e.g. in [CCF+86, Ves05b].

4.3. **Edge Percolation on Metric Graphs.** The basic idea is to decide for each edge independently whether Dirichlet boundary conditions are put on both ends or not. All other boundary conditions are Kirchhoff type. The problem when defining this edge percolation model is that our stochastic processes are indexed by vertices rather than edges. We thus have to relate edges to vertices. This is done by going to each vertex and then tossing a (biased) coin for each $j = 1, \dots, d$ to decide how to deal with the edge $[x, x + e_j]$.

More precisely: Let $p_0 \in (0, 1)$ and $p_1 = 1-p_0$ be given. Let $\mathcal{A}$ consist of all maps S from $\{1, \dots, d\}$ to $\{0, 1\}$. Put a probability measure ν on $\mathcal{A}$ by associating the value $\prod_{j=1}^{d} p_{S(j)}$ to the element S. Now, Ω is the cartesian product space $\times_{x\in V_d}\mathcal{A}$ with product measure $\mathbb{P} := \otimes_{x\in V_d}\nu$. To each $\omega \in \Omega$ we associate the operator $-\Delta_\omega = H_\omega$ which acts like the free Laplacian and has boundary conditions as follows: The edge $e = [x, x + e_j]$ has Dirichlet boundary conditions on both ends if the random variable associated to the vertex x has the j-th component equal to 1. Otherwise the boundary condition is chosen to be Kirchhoff. Here, again the operator decouples completely into operators on finite clusters for small enough values of p_0.

4.4. **Site-Edge Percolation on Metric Graphs.** Similar to the previous two models one can consider a percolation process indexed by pairs (x, e) of adjacent vertices and edges. As in the last model consider a colouring $\mathcal{A}$ consisting of all maps S from $\{1, -1, 2, -2 \dots, d, -d\}$ to $\{0, 1\}$. The probability space and measure are defined similarly as before. Each ω gives rise to a Laplace operator with the following boundary conditions: if the $-j$-th component of the random variable associated to the vertex x has the value one, then the edge $[x - e_j, x]$ is decoupled from x by a Dirichlet boundary condition. If the j-th component of the same random variable has value one then the edge $[x, x + e_j]$ is decoupled from x by a Dirichlet boundary condition. Conversely, those components of the random variable which are zero correspond to Kirchhoff boundary conditions.

4.5. **Operators with Magnetic Fields.** Our set-up is general enough to include magnetic fields as well. To this end, let G be a metric graph and L a choice of boundary conditions as in Section 4. The most general symmetric first order perturbation of $-\frac{\mathrm{d}^2}{\mathrm{d}t^2}$ on an edge $e \in E$ is, up to zeroth order terms, given by

$$H(a)_e := -\left(\frac{\mathrm{d}}{\mathrm{d}t} - \imath a_e\right)^2$$

for arbitrary real valued $a_e \in C^1(\bar{e})$, where $\bar{e}$ is the closure of the edge e, i.e. identified with the closed interval $[0, 1]$. The selfadjoint realization of $H(a)$ corresponding to L is then given by the domain

$$\mathcal{D}(H_L(a)) = \{f \in W^{2,2}(E) : \forall x \in V : (f(x), f'(x) - \imath(af)(x)) \in L_x\}$$

as the usual partial integration argument shows; i.e. one has to specify mixed Dirichlet and (magnetic) Neumann boundary conditions as expected.

Now, simple calculations show that this operator is unitarily equivalent to a nonmagnetic operator with boundary conditions $\tilde{L} = uL$, where $(uL)_x := u_x L_x$ for $x \in V$, and u_x is defined as follows: Set $\varphi_e(t) = \int_0^t a_e(s)\,\mathrm{d}s$ for each edge e. Then

u_x is a diagonal matrix in $\mathbb{C}^{2\deg x}$, and the entry belonging to an edge e incident to x is $e^{-\imath\varphi(\iota(e))} = 1$ if $x = \iota(e)$ and $e^{-\imath\varphi(\tau(e))}$ if $x = \tau(e)$.

Finally, let us note that the above implies that our results for Schrödinger operators with random (or fixed) boundary conditions lead to the same results for magnetic Schrödinger operators with random (or fixed) magnetic fields, specified through the phases $\varphi_e(\tau(e))$ at the endpoints.

4.6. **Decorated Graphs over $\mathbb{Z}^d$.** Just as in Sections 3.1 and 3.3, we may consider more general graphs: Let G be a metric graph such that the group $\Gamma = \mathbb{Z}^d$ acts isometrically, freely and cocompactly on G. This metric space can be viewed as the standard graph over $\mathbb{Z}^d$, but decorated with compact graphs corresponding to a fundamental domain $\mathfrak{D}$ for the Γ-action. Assume that there is an ergodic action $\tau_\gamma, \gamma \in \Gamma$ of the group by measure preserving transformations on the probability space $(\Omega, \mathbb{P})$. Let a random potential V and a random set of boundary conditions L be given which take on only finitely many values and satisfy the compatibility conditions (1) and (2), where now the edge set E_d is replaced by the edge set $E(G)$ of the graph G. This gives rise to a random Schrödinger operator (H_ω) defined on the domain $\mathcal{D}(H_\omega) \subset W^{2,2}(E(G))$ which is determined by boundary conditions L_ω. For a finite cube $Q \subset \mathbb{Z}^d$ set $\mathfrak{D}(Q) := \bigcup_{\gamma \in Q} \gamma\mathfrak{D}$ and denote by (H_ω^Q) the restriction of (H_ω) to $L^2(\mathfrak{D}(Q))$. Now we can again define the functions $n_\omega^Q(\lambda), N_\omega^Q(\lambda)$ and $\xi_\omega^Q(\lambda)$ as in Section 4.1. For these objects the results formulated in Section 4.1 hold true.

5. Random Graph Metrics

In this section we discuss Laplace operators on metric graphs which have random edge lengths. In this case, even the shift functions will be unbounded. Nevertheless, there are two ways in which we can apply our results from Section 2: Restrict our attention to a bounded energy interval on which the shift functions are uniformly bounded; or use a different global norm, adjusted to the common growth rate of these functions. In fact, the IDS will have the same growth as the shift function in this case, which is why there is no loss in working with the IDS directly.

Let us provide, resp. recall, the notation used in this context.

Let $0 < l_- \leq l_+ < \infty$ and $\tilde{\mathcal{A}}$ a finite subset of $[l_-, l_+]$. Let $l_e \colon \Omega \to \tilde{\mathcal{A}}$ be a collection of random variables indexed by the edges $e \in E$. Similarly as in Section 4.1, for each $\omega \in \Omega$ a metric graph $G_d = G_d(\omega)$ is given by the combinatorial graph (V_d, E_d) and by identifying each edge $e \in E_d$ with the interval $\big(0, l_e(\omega)\big)$. Recall that there is a group Γ acting by isometries on the set of vertices as well as on the set of edges. The same group acts ergodically by measure preserving transformations on the probability space Ω. We assume that the random family of edge lengths obeys the transformation rule

$$l_{\gamma(e)}(\gamma\omega) = l_e(\omega).$$

In particular, the distribution of the random variable l_e is independent of e.

Now we define a colouring of the vertex set. Let $\mathcal{A} = \tilde{\mathcal{A}}^d$. For each $\omega \in \Omega$ define a colouring map by $\tilde{\Lambda}(\omega) \colon \mathbb{Z} \to \mathcal{A}$ by $\tilde{\Lambda}(\omega)(v) = \big(l_{e_1}(\omega), \dots, l_{e_d}(\omega)\big)$. Here e_j is the edge with $\iota(e_j) = v = (v_1, \dots, v_d)$ and $\tau(e_j) = (v_1, \dots, v_{j-1}, v_j + 1, v_{j+1}, \dots, v_d) \in \mathbb{Z}^d$.

The value $l_e(\omega)$ denotes the length of the edge e of the metric graph $G_d(\omega)$ in the configuration ω. For each configuration ω we thus obtain a Laplace operator

H_ω on $G_d(\omega)$. If $Q \subset \mathbb{Z}^d$ is a cube in the vertex set of $G_d(\omega)$ we introduce the set of edges $E_Q = \{e \in E_d \mid \iota(e) \in Q\}$ and the set of vertices $V_Q = \{v \in V_d \mid v \text{ adjacent to } e \text{ for some } e \in E_Q\}$ as above. Note that the sets V_Q and E_Q are independent of ω. The subgraph (V_Q, E_Q) with length function $l(\omega)|_{E_Q}$ will be denoted by $G_Q := G_Q(\omega)$. In this situation we consider again the restriction (with Dirichlet b.c.) of H_ω to $G_Q(\omega)$ and denote it by H^Q_ω, while its eigenvalue counting function is denoted by $n^Q_\omega \colon \mathbb{R} \to \mathbb{R}$.

Next we show an analogue of Lemma 21 in [GLV07].

Lemma 22. *Let $a < b \in \mathbb{R}$ be given. Let $(\mathcal{R}, \|\cdot\|_\infty)$ be the Banach space of right continuous bounded functions on $[a, b]$ equipped with the supremum norm. Then, for each $\omega \in \Omega$ the function $n_\omega : \mathcal{F} \to \mathcal{R}$, $Q \mapsto n^Q_\omega$, is a $\tilde{\Lambda}(\omega)$-invariant, almost additive, bounded function.*

Proof. Almost additivity and invariance are shown exactly as in Lemma 21 in [GLV07]. We show now that n_ω is bounded in the sense of Definition 1. For the Dirichlet Laplace operator on $[0, l]$ the eigenvalue counting function $n_d(l, \lambda) \colon \mathbb{R} \to \mathbb{R}$ is given by

$$n_d(l, \lambda) = \left\lfloor \frac{l}{\pi}\sqrt{\lambda} \right\rfloor .$$

It follows that

$$n_d(l, \lambda) \le \frac{\sqrt{\lambda}}{\pi}\, l + 1 \tag{4}$$

Now we can argue again as in Lemma 21 in [GLV07]. First we change all boundary conditions of H^Q_ω to Dirichlet ones. This corresponds to a perturbation operator of rank at most $2|E_Q|$. Consequently, it contributes an error term of at most $2|E_Q| = 2d|Q|$. Subsequently we use for each decoupled edge the estimate (4). It follows that for $\lambda \in [a, b]$

$$\begin{aligned} |n^Q_\omega(\lambda)| &\le 2|E_Q| + \sum_{e \in E_Q} N(l_e, \lambda) \\ &\le 2|E_Q| + \sum_{e \in E_Q} \left(\frac{\sqrt{\lambda}}{\pi} l_e + 1 \right) \\ &= 3|E_Q| + \frac{\sqrt{\lambda}}{\pi} \sum_{e \in E_Q} l_e \\ &\le 3|E_Q| + \frac{\sqrt{b}}{\pi} |E_Q| l_+ \\ &= |Q| d \left(3 + \frac{\sqrt{b}}{\pi} l_+ \right) \end{aligned} \tag{5}$$

□

Remark 23. The reason why we obtain uniform convergence only on bounded energy intervals and not on the whole axis is the last step in the proof of Lemma 22. Using the SSF instead of the IDS we would merely obtain a smaller constant, but the same qualitative behaviour.

The previous Lemma allow us to prove

Proposition 24. *There is a right continuous function $N\colon \mathbb{R} \to \mathbb{R}$ such that for a given van Hove sequence (Q_l) and for any bounded interval $I \subset \mathbb{R}$ the uniform convergence*

$$\lim_{l\to\infty} \sup_{\lambda\in I} \Big| \frac{n_\omega^{Q_l}}{|E_{Q_l}|}(\lambda) - N(\lambda) \Big| = 0$$

holds for almost every $\omega \in \Omega$.

Proof. By Lemma 22, each n_ω is invariant, almost additive and bounded. The family $(n_\omega)_{\omega\in\Omega}$ is homogeneous by construction, so that Theorem 6 applies. □

Now we present a variant of the above result. We consider a different Banach space, namely

$$\tilde{\mathcal{R}} := \{ f\colon \mathbb{R} \to \mathbb{R} \mid f \text{ is right continuous and } \sup_{\mathbb{R}} \big| f(x)/\sqrt{|x|+1} \big| < \infty \}$$

with the norm $\sup_{\mathbb{R}} \big| f(x)/\sqrt{|x|+1} \big|$. Again we have:

Lemma 25. *For each $\omega \in \Omega$ the function $n_\omega\colon \mathcal{F} \to \tilde{\mathcal{R}}$, $Q \mapsto n_\omega^Q$, is $\tilde{\Lambda}(\omega)$-invariant, almost additive and bounded.*

Proof. Almost additivity and invariance are shown as before. To show boundedness we note that from (5) it follows that

$$\sup_{\lambda\in\mathbb{R}} |n_\omega^Q(\lambda)/\sqrt{|\lambda|+1}| \le \frac{3|E_Q|}{\sqrt{|\lambda|+1}} + \frac{\sqrt{\lambda}}{\sqrt{|\lambda|+1}} \frac{|E_Q|}{\pi} l_+ \le const|E_Q|$$

Here $const$ is some constant independent of Q, ω and λ. Recall that $|E_Q| = d|Q|$. Thus we have proven boundedness. □

Thus the statement of Proposition 24 can be strengthened to

$$\lim_{l\to\infty} \sup_{\lambda\in\mathbb{R}} \Big| \Big(\frac{n_\omega^{Q_l}}{|E_{Q_l}|}(\lambda) - N(\lambda) \Big) \frac{1}{\sqrt{|\lambda|+1}} \Big| = 0$$

for almost every $\omega \in \Omega$.

6. Outlook: Uniform Convergence and Jumps of the IDS

In this section we provide an outlook beyond the theorems stated in the main text. This includes a discussion of the relation to other (recent) papers, as well as some open questions which we plan to address in the future.

Uniform Convergence and Discontinuities of the IDS. Our goal here is to elaborate on the relationship between the almost sure spectrum of H_ω and its IDS.

We start by noting two corollaries of Theorem 10 for the setting of combinatorial graphs and of Theorem 20 for quantum graphs respectively. Note that in contrast to Corollaries 11 and 13 we do not require positivity of frequencies, since we are in the random setting now. We recall the notions of topological support $\operatorname{supp}\mu$ of $\mu = \mu_H$ and the set $S_p(\mu) := \{\lambda \in \mathbb{R} : \mu(\{\lambda\}) > 0\}$ of atoms of μ to the spectrum of H_ω. Note that the set of discontinuities of the IDS is precisely $S_p(\mu)$.

Corollary 26. Σ *equals the topological support* $\operatorname{supp}\mu$ *of* μ.

As usual an $f \in \ell^2(\mathbb{Z}^d, \mathcal{H})$ is said to be compactly supported if $f(v) = 0$ for all but finitely many $v \in V$; $f \in W^{2,2}(E_d)$ is said to be compactly supported if $f_e \equiv 0$ for all but finitely many edges e.

Corollary 27. *Denote by Σ_{cmp} the set of energies $\lambda \in \mathbb{R}$ such that there exists almost surely a compactly supported square integrable eigenfunction f_ω with $H_\omega f_\omega = \lambda f_\omega$. Then*

$$S_p(\mu) = \Sigma_{cmp}. \tag{6}$$

Remark 28. (a) Note that there are many examples where the IDS has discontinuities: the free Laplacian (i.e. the Schrödinger operator with identically vanishing potential) with Dirichlet, Neumann, or Kirchhoff boundary conditions; percolation models such as those in Sections 3.3, 4.2, 4.3 and 4.4; for other percolation and tiling Hamiltonians, and quantum graphs, see also [CCF⁺86, KLS03, KS04, Ves05b].

(b) If the randomness entering the potential of the operator is sufficiently strong it is natural to expect a smoothing effect on the IDS. In fact, in [HV07] for a class of alloy-type random potentials the Lipschitz-continuity of the IDS was established. In [GV08, GHV] we show for a different class of random potentials how one can estimate the modulus of continuity of the IDS. Such estimates are relevant in the context of spectral localisation for random Schrödinger operators, see the disussion in [GHV] in this volume.

These corollaries provide a criterion for deducing the existence of discontinuities of the IDS from uniform convergence (w.r.t. the energy parameter) of the IDS. More precisely, they prove that (A) implies (B'), where

(A) $\lim_{l\to\infty} \|N_\omega^{Q_l} - N_H\|_\infty = 0$,
i.e the IDS is uniformly approximated by its finite volume analoga

and

(B') The IDS has discontinuities precisely at those energies, which are eigenvalues of H_ω with compactly supported eigenfunctions almost surely.

This line of argument stems originally from [KLS03]. See [LMV, GLV07] for the present context. It is possible to turn the argument around. More precisely, property

(B) The positions and the sizes of the jumps of the IDS are approximated by the analogous data of the finite volume approximands $N_\omega^{Q_l}$.

implies already (A).

Ideas of this type have been used in [Eck99, Ele03, MSY03]. More recently, in [LV] it was proven that (B), and hence (A), holds for general ergodic, equivariant, selfadjoint, finite hopping range operators on discrete structures. The equivariance of the operator is supposed to hold w.r.t. an amenable group.

Thus for many models it is possible to pursue two different routes to obtain the same results:

(i) Either one first establishes a Banach-space valued ergodic theorem, implying (A), and then deduces (B). This was the choice made in the present work.
(ii) Or one first proves the statement (B) about jumps, and then concludes (A).

The latter approach has three advantages:

- It does not require any finite local complexity property, and thus works even for random operators where a single matrix element may assume infinitely many values. (This is e.g. the case for Anderson-percolation Hamiltonians, cf. [Ves05b].)
- It works for models which are equivariant w.r.t. a non-abelian group, as long as it is amenable.
- Furthermore, the equivariance group may be discrete (like $\mathbb{Z}^d$) or connected (like $\mathbb{R}^d$).

On the other hand the former approach (i) has the advantage of providing certain information beyond that obtained from approach (ii). In particular, it allows one to control

- the set of measure zero where the convergence fails as well as
- the convergence speed in terms of an error estimate.

In certain situations, such as the setting of minimal, uniquely ergodic dynamical systems, the control of the exceptional set of measure zero implies actually that it is empty (see [LS06])! Thus convergence holds for all configurations of the randomness rather than for almost all only.

Absence of Discontinuities. Above we discussed characterisations of the positions and sizes of jumps of the IDS. This allows in particular to show that several models have an IDS with (many) discontinuities. Intuitively, the discontinuities are related to two facts, namely that the models in question

- are not *too random*, for instance satisfy a finite local complexity condition, and
- do not satisfy the (appropriate) unique continuation property.

On the other hand, for certain random Hamiltonians where even single matrix elements have continuous distribution it can be shown that the IDS is Lipschitz-continuous. These results go under the name of Wegner estimates, cf. [Weg81], and are well established for operators on $\ell^2(\mathbb{Z}^d)$ and $L^2(\mathbb{R}^d)$, see e.g. [KM07, Ves07] for recent surveys. Meanwhile such bounds have been also established for quantum graphs in [HV07, GV08, GHV, LPPV].

Furthermore, if a model satisfies an appropriate version of the unique continuation property the IDS has no jumps, regardless of whether the finite local complexity condition is fulfilled or not (cf. e.g. Proposition 5.2. in [Ves05b]). An example of such a random operator is the Anderson model for which the continuity of the IDS was established in [CS83, DS84].

This can be compared nicely to the properties of mixed Anderson-percolation Hamiltonians. There, due to the dilution of the lattice, the operator does not have the unique continuation property. For this model it turns out that the IDS has jumps if and only if the distribution of the matrix elements has atoms (apart form the point mass at ∞ which corresponds to the deletion of a vertex).

Questions. Let us formulate two questions which concern Banach-space valued ergodic theorems more general than those formulated in the present review:

- Is it possible to show, using a similar line of argument as in [Len02, LMV], that a Banach-space valued ergodic theorem holds if the lattice $\mathbb{Z}^d$ is replaced by a finitely generated, discrete group of polynomial growth?

We plan to address this question in the future. The hope is that — since such groups are both amenable and residually finite — one can use a similar covering argument as in [LMV].

- Is there a version of a Banach-space valued ergodic theorem which is applicable both to models with $\mathbb{Z}^d$-equivariance as well as models with $\mathbb{R}^d$-equivariance, and which provides a unified treatment of the results in [LS06] and [LMV]? This would mean that one does not need to distinguish between a discrete and a continuous group action.

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