

## The diagonal cosets of the Heisenberg group

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**ABSTRACT:** In this paper we study the diagonal cosets of the non-compact  $H_4$  WZW model. Generalising earlier work by Antoniadis and Obers, we provide an exact world-sheet description for several families of non-maximally symmetric gravitational plane waves with background NS fluxes. We show that the  $\sigma$ -models that correspond to the asymmetric cosets smoothly interpolate between singular and non singular plane waves. We also analyse the representations of the coset chiral algebra and derive the spectrum of all the models.

**KEYWORDS:** Conformal Field Models in String Theory.

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## Contents

|  |           |
|--|-----------|
| <b>1. Introduction</b>   | <b>2</b>  |
| <b>2. Classification of the diagonal cosets</b>  | <b>4</b>  |
| 2.1 Asymmetrically gauged Wess-Zumino-Witten models  | 4         |
| 2.2 The Heisenberg algebra and the associated group manifold   | 5         |
| 2.3 Classification of diagonal coset models  | 6         |
| 2.4 Derivation of coset data and gauge fixing  | 8         |
| <b>3. Coset geometries</b>   | <b>10</b> |
| 3.1 General plane waves  | 11        |
| 3.2 The cosets of type $(--)$  | 11        |
| 3.2.1 Symmetric gauging  | 11        |
| 3.2.2 Twisted gauging  | 13        |
| 3.3 The cosets of type $(+-)$  | 13        |
| 3.3.1 Symmetric gauging  | 14        |
| 3.3.2 Twisted gauging  | 14        |
| 3.4 The cosets of type $(++)$  | 15        |
| <b>4. Coset characters and representation theory</b>   | <b>17</b> |
| 4.1 Semi-classical analysis  | 17        |
| 4.2 Affine representation theory   | 18        |
| 4.2.1 Standard representations   | 19        |
| 4.2.2 Spectral flow representations  | 20        |
| 4.3 Tensor product decompositions for the diagonal coset   | 20        |
| 4.3.1 General strategy   | 20        |
| 4.3.2 Decomposition of the tensor products of standard representations                               | 22        |
| 4.3.3 Remarks on the decomposition of tensor products with spectral flow representations             | 25        |
| <b>5. The spectrum of the diagonal cosets</b>  | <b>25</b> |
| 5.1 The cosets of type $(--)$  | 27        |
| 5.2 The cosets of type $(+-)$  | 28        |
| 5.3 The cosets of type $(++)$  | 30        |
| <b>6. Conclusions</b>  | <b>32</b> |
| <b>A. Geometries for the fully asymmetric cosets of type <math>(--)</math> and <math>(+-)</math></b> | <b>34</b> |
| A.1 Cosets of type $(--)$  | 34        |
| A.2 Cosets of type $(+-)$  | 35        |

|   |           |
|---|-----------|
| <b>B. Tensor products of <math>H_4</math>-representations</b> | <b>37</b> |
| B.1 Adjoint times adjoint                                     | 37        |
| B.2 Continuous times adjoint                                  | 38        |

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## 1. Introduction

Curved string backgrounds provide a well-defined context to explore the properties of string theory as a theory of quantum gravity. In certain cases they can even provide an holographic description of the dynamics of a dual gauge theory [1]. For these reasons one would like to extend as much as possible the class of curved space-times for which an exact conformal field theory description is available. With this aim in mind, we started in [2] a systematic study of the non-compact coset models based on the Heisenberg group  $H_4$ . Non-compact cosets are a very interesting class of string theory backgrounds since they can be studied from two complementary points of view. On one hand their Lagrangian formulation as gauged WZW models [3, 4, 5, 6, 7, 8] provides a clear space-time interpretation, on the other hand their exact conformal field theory description [9, 10] allows to derive the spectrum of string excitations and calculate their scattering amplitudes.

Using their geometric formulation, several curved backgrounds were soon recognized as non-compact coset theories: the two-dimensional black-hole [11], the three-dimensional black string [12], the inhomogeneous Nappi-Witten cosmology [13] as well as many other examples [14, 15]. On the other hand for several years it was not possible to use conformal field theory techniques to derive the spectrum and compute the interactions, since the representation theory of the non-compact affine algebras was not properly understood and their structure constants were not known. The situation changed with the work of Tschner as well as Maldacena and Ooguri [16, 17, 18]. These authors clarified the operator content of the  $SL(2, \mathbb{R})$  WZW model and of its Euclidean analogue, the  $H_3^+$  model, and derived their structure constants. Using these results, it was finally possible to analyse in some detail the conformal field theories of abelian cosets based on  $SL(2, \mathbb{R})$  [19, 20, 21, 22].

As it is well-known, the  $SL(2, \mathbb{R})$  WZW model describes the propagation of strings in  $AdS_3$ . The Heisenberg group  $H_4$  considered in the present paper describes the propagation of strings in a four-dimensional, maximally symmetric plane wave with seven isometries [23]. The study of the representation theory of the  $H_4$  affine algebra was started in [24] and the model was exactly solved in [25, 26]. Also the exact solution of the boundary CFTs of the maximally symmetric D-branes in this background is available [27].

In our first paper [2] we considered the abelian cosets of the Heisenberg group [24, 28, 29] and showed that they provide a CFT description of several three-dimensional backgrounds such as the Melvin model [30], the conical point-particle space-times [31] and the null orbifold [32]. In the present paper we perform a detailed study of the diagonal cosets of the Heisenberg group. Both the numerator and the denominator group of these cosets are non-compact and non-abelian. To our knowledge such theories have hardly been studied from an algebraic point of view so far.

As shown by Antoniadis and Obers [33] these models have also a very interesting geometric interpretation in terms of non-maximally symmetric plane waves. In the paper mentioned above the authors discussed two special classes of diagonal cosets. The first class is given by a two-parameter family of singular geometries which are T-dual to plane gravitational waves. The second class describes a one-parameter family of gravitational waves with five isometries. Although it is well-known that  $\sigma$ -models associated with plane gravitational waves are always conformally invariant if the dilaton and the three-form flux are chosen appropriately [34, 35, 36, 37, 38], the underlying conformal field theories are not easy to identify. The work of Antoniadis and Obers provided this identification for a whole class of plane wave backgrounds.

In the present paper we generalise the analysis of Antoniadis and Obers in two respects. Firstly we extend their consideration to include further string backgrounds. In addition we go beyond the pure Lagrangian and geometric description and make a significant step towards a full conformal field theory analysis of these models. One of the main results of our paper is the classification of all possible diagonal cosets of the Heisenberg group and the explicit construction of the corresponding  $\sigma$ -models. Since  $H_4$  is non-semisimple, it admits continuous families of non-isometric automorphisms. These continuous families of automorphisms combined with the freedom of choosing different embeddings of  $H_4$  into the left and the right sectors of the original CFT lead to an extremely rich number of possible models. They can be divided into three classes that we will refer to as  $(++)$ ,  $(+-)$  and  $(--)$ . Each class depends on several continuous parameters. For certain restricted choices of the parameters of the  $(++)$  and  $(--)$  classes, one recovers the models constructed by Antoniadis and Obers.

In this paper we also present the derivation of the spectrum of the diagonal cosets using conformal field theory techniques. In order to achieve this we study the decomposition of the tensor products of affine  $H_4$  representations with respect to the embedded  $H_4$  algebra. In contrast to compact or abelian cosets the standard method of determining the branching functions fails since products of affine characters are usually divergent if considered as generating functions for the states of a representation. This problem already shows up at the level of the horizontal subalgebra, since the unitary representations of a non-compact group are infinite dimensional.

We develop a new method for the derivation of the affine coset characters which makes use both of character decompositions and of the knowledge of the tensor products of the horizontal subalgebra. The latter should be thought of as providing some analytical input which allows to deal with the mathematical difficulties of having infinite dimensional weight spaces. While we will be able to provide a complete answer for the decomposition of the tensor products of highest weight representations, we only have partial results for the tensor products of spectral flow representations. The fact that the adjoint representation of  $H_4$  is indecomposable but reducible potentially leads to further complications.

This paper is organised as follows. In section 2 we begin with a brief review of the construction of asymmetrically gauged WZW models. After a classification of the diagonal embeddings of the Lie algebra  $H_4$  we determine all possible diagonal cosets and derive the quantities needed for their Lagrangian description. In section 3 we present the metric,

the dilaton and the antisymmetric tensor of our three classes of models. The background data are displayed only for two particular families of parameters while the most general expressions are collected in appendix A. In section 4 we proceed to a more algebraic treatment and derive explicit formulas for the diagonal coset characters. These results are used in section 5 to compute the spectrum of the three classes of diagonal cosets. Section 6 contains our conclusions and some comments on possible extensions of our work.

## 2. Classification of the diagonal cosets

After a brief review of the Lagrangian description of the gauged WZW models, we classify all possible diagonal cosets of the Heisenberg group. We thereby generalise the analysis of [33] and provide the grounds for a thorough treatment of the cosets geometries described in section 3.

### 2.1 Asymmetrically gauged Wess-Zumino-Witten models

We begin with a short review of the Lagrangian description of general cosets  $G/H$ , where  $G$  is a Lie group and  $H$  is a Lie subgroup of  $G$ . The model is completely specified by the choice of two invariant forms  $\langle \cdot, \cdot \rangle_G$  and  $\langle \cdot, \cdot \rangle_H$  on the respective Lie algebras and the selection of two embeddings  $\epsilon, \bar{\epsilon} : H \rightarrow G$ . The coset space is then determined by the identification

$$G/H = \{g \in G \mid g \sim \epsilon(h)g\bar{\epsilon}(h^{-1}), \forall h \in H\} . \quad (2.1)$$

Provided that the consistency requirement

$$\langle \epsilon(X), \epsilon(Y) \rangle_G = \langle \bar{\epsilon}(X), \bar{\epsilon}(Y) \rangle_G = \langle X, Y \rangle_H \quad \text{for all } X, Y \in \mathfrak{h} , \quad (2.2)$$

is satisfied, these data define a conformally invariant  $\sigma$ -model on  $G/H$  via the construction of gauged Wess-Zumino-Witten models [3, 4, 5, 6, 7, 8].

The starting point of this construction is the action

$$\mathcal{S}^{G/H}(g, U, V) = \mathcal{S}^G(\epsilon(U^{-1})g\bar{\epsilon}(V)) - \mathcal{S}^H(U^{-1}V) , \quad (2.3)$$

where  $g : \Sigma \rightarrow G$  and  $U, V : \Sigma \rightarrow H$  are group valued fields. Here the symbol  $\mathcal{S}^G$  denotes the WZW Lagrangian for the group  $G$

$$\mathcal{S}^G(g) = -\frac{i}{4\pi} \int \langle g^{-1}\partial g, g^{-1}\bar{\partial}g \rangle_G dz \wedge d\bar{z} - \frac{i}{24\pi} \int \langle g^{-1}dg, [g^{-1}dg, g^{-1}dg] \rangle_G , \quad (2.4)$$

and similarly  $\mathcal{S}^H$  denotes the WZW Lagrangian for the group  $H$ . The action (2.3) is manifestly invariant under local  $H$ -transformations of the form

$$g \mapsto \epsilon(h)g\bar{\epsilon}(h^{-1}) , \quad U \mapsto hU , \quad V \mapsto hV . \quad (2.5)$$

To further simplify the action (2.3), we introduce the gauge fields  $\bar{A} = \bar{\partial}UU^{-1}$  and  $A = \partial VV^{-1}$  and use the Polyakov-Wiegmann identity [39]

$$\mathcal{S}^G(gh) = \mathcal{S}^G(g) + \mathcal{S}^G(h) - \frac{i}{2\pi} \int \langle g^{-1}\bar{\partial}g, \partial h h^{-1} \rangle_G dz \wedge d\bar{z} , \quad (2.6)$$

to write

$$\begin{aligned} \mathcal{S}^{G/H}(g, A, \bar{A}) = \mathcal{S}^G(g) + \frac{i}{2\pi} \int \left\{ -\langle \bar{A}, A \rangle_H + \langle \epsilon(\bar{A}), g\bar{\epsilon}(A)g^{-1} \rangle_G \right. \\ \left. + \langle \epsilon(\bar{A}), \partial g g^{-1} \rangle_G - \langle g^{-1} \bar{\partial} g, \bar{\epsilon}(A) \rangle_G \right\} dz \wedge d\bar{z} . \end{aligned} \quad (2.7)$$

There are no terms which depend only on  $U$  or  $V$  in the previous action because the condition (2.2) implies the relation

$$\mathcal{S}^G(\epsilon(h)) = \mathcal{S}^G(\bar{\epsilon}(h)) = \mathcal{S}^H(h) \quad \text{for all } h \in H . \quad (2.8)$$

It is convenient to introduce the following compact notation for the Lagrangian (2.7)

$$\mathcal{S}^{G/H}(g, A, \bar{A}) = \mathcal{S}^G(g) + \frac{i}{2\pi} \int \left\{ \bar{A}^T M A + \bar{b}^T A + b^T \bar{A} \right\} dz \wedge d\bar{z} , \quad (2.9)$$

where the gauge fields are expressed in coordinates with respect to some concrete basis of the Lie algebra and the matrix  $M$  and the vectors  $b$  and  $\bar{b}$  are implicitly determined by comparing the integrands of (2.7) and (2.9). The action is at most quadratic in the gauge fields and when the matrix  $M$  is non-degenerate they can be easily integrated out. The resulting action is a  $\sigma$ -model whose metric and antisymmetric tensor can be inferred from

$$\mathcal{S}^{G/H}(g) = \mathcal{S}^G(g) - \frac{i}{2\pi} \int \left\{ \bar{b}^T M^{-1} b \right\} dz \wedge d\bar{z} . \quad (2.10)$$

The background also includes a non-trivial dilaton [11] given up to constant terms by

$$\Phi = -\frac{1}{2} \ln \det M . \quad (2.11)$$

When the matrix  $M$  is degenerate, the integration over the gauge fields results in the appearance of constraints for the  $\sigma$ -model fields. As we will see below in section 2.4, the diagonal cosets in the class  $(++)$  provide an example of this type. The occurrence of this and other somewhat unusual effects is typical of gauged WZW models involving non-semisimple algebras [40].

For a more detailed discussion of asymmetric coset models in the bulk and on the boundary we refer the reader to [5, 41].

## 2.2 The Heisenberg algebra and the associated group manifold

In this paper the general construction of the previous subsection will be applied to the diagonal cosets of the Heisenberg group  $H_4$ . The underlying Lie algebra, which will be denoted by the same symbol, is a four-dimensional non-semisimple Lie algebra. Its four generators  $P_1, P_2, J$  and  $K$  satisfy the following commutation relations

$$[P_i, P_j] = \epsilon_{ij} K , \quad [J, P_i] = \epsilon_{ij} P_j , \quad (2.12)$$

with  $\epsilon_{12} = 1$ . In terms of the raising and lowering operators  $P^\pm = P_1 \pm iP_2$  the previous relations become

$$[P^+, P^-] = -2iK , \quad [J, P^\pm] = \mp iP^\pm . \quad (2.13)$$

In our conventions, the generators  $P_1$  and  $P_2$  are hermitian while  $J$  and  $K$  are anti-hermitian.

The Heisenberg algebra admits a two-parameter family of invariant bilinear forms

$$\langle P_i, P_j \rangle = \Lambda \delta_{ij} , \quad \langle J, K \rangle = \Lambda , \quad \langle J, J \rangle = 2\lambda\Lambda . \quad (2.14)$$

By a rescaling of the generators and a redefinition  $J \mapsto J - \lambda K$ , it is always possible to set  $\Lambda = 1$  and  $\lambda = 0$  so that the metric assumes the standard form

$$\langle P_i, P_j \rangle = \delta_{ij} , \quad \langle J, K \rangle = 1 , \quad (2.15)$$

without affecting the commutation relations. We also need an explicit parameterisation of the group elements. For the sake of easy comparison of our results with those of Antoniadis and Obers [33] we use

$$g = e^{xP_1} e^{uJ} e^{yP_1} e^{vK} . \quad (2.16)$$

In this coordinate system the action of a single  $H_4$  WZW model is

$$\mathcal{S}^{H_4}(g) = -\frac{i}{4\pi} \int dz \wedge d\bar{z} [\partial u \bar{\partial} v + \partial v \bar{\partial} u + \partial x \bar{\partial} x + \partial y \bar{\partial} y + 2 \cos u \partial x \bar{\partial} y] . \quad (2.17)$$

In a similar way the group elements of  $H_4 \times H_4$  will be parameterised by two sets of coordinates  $(u_i, v_i, x_i, y_i)$ ,  $i = 1, 2$ .

### 2.3 Classification of diagonal coset models

In this section we shall provide a classification of all possible diagonal cosets of the Heisenberg group  $H_4$ . We will find three inequivalent families, each depending on six real parameters. As we will explain, for special choices of the parameters the models in two of these families coincide with the models studied in [33].

We begin with a classification of all possible diagonal embeddings of the Heisenberg Lie algebra. Let us first define more precisely what we mean by a diagonal embedding. We recall that there is a canonical way to define the action of a Lie algebra  $\mathfrak{g}$  on a tensor product of two representations. The corresponding action is implemented in terms of the standard coproduct  $\Delta$  which maps a generator  $X \in \mathfrak{g}$  to  $\Delta(X) = X \otimes 1 + 1 \otimes X \in \mathfrak{g} \otimes \mathfrak{g}$ . Since the coproduct is injective and preserves the commutation relations it can be thought of as an embedding  $\Delta : \mathfrak{g} \rightarrow \mathfrak{g} \oplus \mathfrak{g}$ . It is obvious that further embeddings can be obtained if one concatenates  $\Delta$  with automorphisms of  $\mathfrak{g}$  and  $\mathfrak{g} \oplus \mathfrak{g}$ , respectively. Inner automorphisms lead to equivalent embeddings so that one can focus on the group of outer automorphisms. If  $n$  is the order of the outer automorphisms of  $\mathfrak{g}$ , one should thus consider  $2n^3$  a priori different possibilities, a factor  $n$  for each of the Lie algebras  $\mathfrak{g}$  and a factor 2 for the possibility of exchanging the two algebras in  $\mathfrak{g} \oplus \mathfrak{g}$ .

For simple Lie algebras the group of outer automorphisms is rather small, the maximum of  $n = 3$  being achieved for  $\mathfrak{g} = D_4$ . On the other hand, the non-semisimple Lie algebra  $\mathfrak{g} = H_4$  offers a great variety of automorphisms. The existence of a two-parameter family of invariant metrics (2.14) is paralleled by the existence of a continuous family of non-isometric automorphisms, which is the main reason for the significant number of diagonal

coset models that can be constructed for  $H_4$ . In fact, the automorphisms of *simple* Lie algebras are always isometric. They just correspond to symmetries of the associated Dynkin diagram.

It is possible to prove that the most general outer automorphism depends on a sign  $\eta = \pm 1$  and two continuous parameters  $\mu \in \mathbb{R}$  and  $\nu \in \mathbb{R}_+$ . Its action on the generators is the following,

$$\Omega_\eta^{(\mu, \nu)} : (P_1, P_2, J, K) \mapsto (\nu P_1, \eta \nu P_2, \eta J + \mu K, \eta \nu^2 K) . \quad (2.18)$$

These automorphisms are isometric only when  $\nu = 1$  and  $\mu = 0$  while in general one has

$$\begin{aligned} \langle \Omega_\eta^{(\mu, \nu)}(J), \Omega_\eta^{(\mu, \nu)}(K) \rangle &= \nu^2 \langle J, K \rangle , \\ \langle \Omega_\eta^{(\mu, \nu)}(J), \Omega_\eta^{(\mu, \nu)}(J) \rangle &= \langle J, J \rangle + 2\mu \langle J, K \rangle . \end{aligned} \quad (2.19)$$

Using the automorphisms  $\Omega_\eta^{(\mu, \nu)}$  we can now construct the most general diagonal embedding  $\epsilon : H_4 \rightarrow H_4 \oplus H_4$  which is given by

$$\epsilon = \Omega_{\eta_1}^{(\mu_1, \nu_1)} \times \Omega_{\eta_2}^{(\mu_2, \nu_2)} \circ \Delta \circ \Omega_\eta^{(\mu, \nu)} . \quad (2.20)$$

In fact, it is easy to see that the automorphism  $\Omega_\eta^{(\mu, \nu)}$  in (2.20) is redundant since it can be removed by a redefinition of the parameters of the other two automorphisms. Accordingly, the most general diagonal embedding is

$$\begin{aligned} \epsilon(P_1) &= \nu_1 P_1^{(1)} + \nu_2 P_1^{(2)} , & \epsilon(J) &= \eta_1 J^{(1)} + \eta_2 J^{(2)} + \mu_1 K^{(1)} + \mu_2 K^{(2)} , \\ \epsilon(P_2) &= \eta_1 \nu_1 P_2^{(1)} + \eta_2 \nu_2 P_2^{(2)} , & \epsilon(K) &= \eta_1 \nu_1^2 K^{(1)} + \eta_2 \nu_2^2 K^{(2)} , \end{aligned} \quad (2.21)$$

where the superscripts (1) and (2) refer to the two  $H_4$  factors. The embedding depends on four real parameters  $\mu_1, \mu_2, \nu_1$  and  $\nu_2$  and on the pair of signs  $(\eta_1 \eta_2)$  that will subsequently be called the class of the embedding.

We now turn to the classification of possible diagonal coset models. We first note that using the outer automorphisms just described we can always choose on  $H_4 \times H_4$  the standard invariant bilinear form with  $\Lambda = 1$  and  $\lambda = 0$  for both factors. This simply amounts to a reparameterisation of the group elements.

We can then choose two embeddings of the form (2.21) and require that they satisfy the constraint (2.2), which ensures that we are gauging an anomaly-free subgroup. In terms of the parameters  $(\eta_i, \mu_i, \nu_i)$  and  $(\bar{\eta}_i, \bar{\mu}_i, \bar{\nu}_i)$  of the two embeddings, Eq. (2.2) becomes

$$\nu_1^2 + \nu_2^2 = \bar{\nu}_1^2 + \bar{\nu}_2^2 \equiv \Lambda , \quad \eta_1 \mu_1 + \eta_2 \mu_2 = \bar{\eta}_1 \bar{\mu}_1 + \bar{\eta}_2 \bar{\mu}_2 \equiv \Lambda \lambda . \quad (2.22)$$

The left part of both equations should be read as a consistency condition while the right part defines the constants  $\Lambda$  and  $\lambda$ . It is convenient to write the general solution of the previous constraint equations in the following form

$$(\nu_1, \nu_2) = \sqrt{\Lambda} (\cos \alpha, \sin \alpha) , \quad (\bar{\nu}_1, \bar{\nu}_2) = \sqrt{\Lambda} (\cos \bar{\alpha}, \sin \bar{\alpha}) , \quad (2.23)$$

$$(\mu_1, \mu_2) = (\eta_1 \Lambda \mu, \eta_2 \Lambda (\lambda - \mu)) , \quad (\bar{\mu}_1, \bar{\mu}_2) = (\bar{\eta}_1 \Lambda \bar{\mu}, \bar{\eta}_2 \Lambda (\lambda - \bar{\mu})) , \quad (2.24)$$



with  $\alpha, \bar{\alpha} \in (0, \pi/2)$ . At this point it seems that we are left with four discrete and six continuous parameters. However two of the discrete parameters can be removed using the freedom of reparameterising the group elements. We already used this freedom to choose the standard metric on  $H_4 \oplus H_4$  but we can still act with the outer isometric automorphism  $\Omega_{-1}^{(0,1)}$ . In this way we can set for instance  $\eta_1 = \eta_2 = 1$ .

We thus arrive at the conclusion that the diagonal coset models based on the Heisenberg group  $H_4$  are specified by two discrete parameters and six continuous parameters. Since the physical properties will strongly depend on the particular choice of signs  $(\bar{\eta}_1 \bar{\eta}_2)$ , this tuple will be called the class of the coset. It labels distinct families of models. In each family there are two special choices of parameter, namely  $\bar{\alpha} = \alpha$  and  $\bar{\alpha} = \pi/2 - \alpha$ . The resulting models will be referred to as “symmetric” and “twisted” gaugings respectively. The  $(++)$  and  $(--)$  families of diagonal cosets with  $\alpha = \bar{\alpha}$  and  $\lambda = \mu = \bar{\mu} = 0$  correspond respectively to the vector gauged models and to the vector-axial gauged model studied in [33].

From the explicit construction that will be performed in the next section, it turns out that the parameters  $\alpha$  and  $\bar{\alpha}$  always label inequivalent models. On the other hand the parameter  $\Lambda$  disappears from the action. For the families  $(--)$  and  $(+-)$ , the parameters  $\mu, \bar{\mu}$  and  $\lambda$  can be removed by a simple coordinate redefinition and do not generically affect the spectrum of the model, unless the coordinates  $v_1$  or  $v_2$  are compact. Finally, the family  $(++)$  has a non trivial dependence on the difference  $\mu - \bar{\mu}$  which can however be removed by a T-duality transformation. Consequently, the number of physical parameters is smaller than indicated by our purely algebraic reasoning.

## 2.4 Derivation of coset data and gauge fixing

In this section we compute the quantities  $M, b$  and  $\bar{b}$  that were defined in Eq. (2.9). Thereby we completely specify the action of the gauged WZW model. We also describe our gauge choices for the different classes of models. The gauge field takes value in the Lie algebra  $H_4$  and can be written as

$$A = A_1 P_1 + A_2 P_2 + A_3 J + A_4 K, \quad \bar{A} = \bar{A}_1 P_1 + \bar{A}_2 P_2 + \bar{A}_3 J + \bar{A}_4 K. \quad (2.25)$$

We embed these fields in the numerator algebra using the two embeddings  $\epsilon$  and  $\bar{\epsilon}$ ,

$$\begin{aligned} \epsilon(\bar{A}) &= \nu_1 \bar{A}_1 P_1^{(1)} + \eta_1 \nu_1 \bar{A}_2 P_2^{(1)} + \eta_1 \bar{A}_3 J^{(1)} + [\eta_1 \nu_1^2 \bar{A}_4 + \mu_1 \bar{A}_3] K^{(1)} + (1 \leftrightarrow 2), \\ \bar{\epsilon}(A) &= \bar{\nu}_1 A_1 P_1^{(1)} + \bar{\eta}_1 \bar{\nu}_1 A_2 P_2^{(1)} + \bar{\eta}_1 A_3 J^{(1)} + [\bar{\eta}_1 \bar{\nu}_1^2 A_4 + \bar{\mu}_1 A_3] K^{(1)} + (1 \leftrightarrow 2). \end{aligned} \quad (2.26)$$

The expression  $(1 \leftrightarrow 2)$  indicates the presence of a similar contribution in the second factor, obtained by replacing the label 1 by the label 2 in the superscripts of the generators and

in the subscripts of the parameters. The explicit form of the matrices  $M$ ,  $b$  and  $\bar{b}$  is then

$$\begin{aligned}
M &= \begin{pmatrix} -\frac{\Delta}{2} + \nu_1 \bar{\nu}_1 c_1 & -\bar{\eta}_1 \nu_1 \bar{\nu}_1 s_1 & \bar{\eta}_1 \nu_1 y_1 s_1 & 0 \\ \nu_1 \bar{\nu}_1 s_1 & -\frac{\Delta}{2} + \bar{\eta}_1 \nu_1 \bar{\nu}_1 c_1 & -\bar{\eta}_1 \nu_1 (x_1 + y_1 c_1) & 0 \\ \bar{\nu}_1 x_1 s_1 & \bar{\eta}_1 \bar{\nu}_1 (x_1 c_1 + y_1) - \frac{\Delta \lambda}{2} + \bar{\eta}_1 \mu_1 + \bar{\mu}_1 - \frac{1}{2} \bar{\eta}_1 (x_1^2 + y_1^2 + 2x_1 y_1 c_1) - \frac{\Delta}{2} + \bar{\eta}_1 \bar{\nu}_1^2 & -\frac{\Delta}{2} + \bar{\eta}_1 \nu_1^2 & 0 \\ 0 & 0 & -\frac{\Delta}{2} + \bar{\eta}_1 \nu_1^2 & 0 \end{pmatrix} + (1 \leftrightarrow 2) \\
b &= \begin{pmatrix} \nu_1 (\partial x_1 + c_1 \partial y_1) \\ \nu_1 (s_1 \partial y_1 - x_1 \partial u_1) \\ (\partial v_1 + x_1 s_1 \partial y_1 - \frac{x_1^2}{2} \partial u_1) + \mu_1 \partial u_1 \\ \nu_1^2 \partial u_1 \end{pmatrix} + (1 \leftrightarrow 2) \\
\bar{b} &= \begin{pmatrix} -\bar{\nu}_1 (c_1 \bar{\partial} x_1 + \bar{\partial} y_1) \\ -\bar{\eta}_1 \bar{\nu}_1 (-s_1 \bar{\partial} x_1 + y_1 \bar{\partial} u_1) \\ -\bar{\mu}_1 \bar{\partial} u_1 - \bar{\eta}_1 (\bar{\partial} v_1 + y_1 s_1 \bar{\partial} x_1 - \frac{y_1^2}{2} \bar{\partial} u_1) \\ -\bar{\eta}_1 \bar{\nu}_1^2 \bar{\partial} u_1 \end{pmatrix} + (1 \leftrightarrow 2) .
\end{aligned} \tag{2.27}$$

In the previous expressions we introduced the abbreviations  $c_i = \cos(u_i)$  and  $s_i = \sin(u_i)$ . As before, the notation  $(1 \leftrightarrow 2)$  stands for an additional term identical to the first except for the relabeling of all the indices.

We now turn to the gauge fixing conditions for the local symmetry of our models

$$g \mapsto \epsilon(h) g \bar{\epsilon}(h^{-1}) , \quad A \mapsto h(A - \partial) h^{-1} , \quad \bar{A} \mapsto h(\bar{A} - \bar{\partial}) h^{-1} . \tag{2.28}$$

For the two classes  $(--)$  and  $(+-)$  we choose a gauge where the  $\sigma$ -model fields satisfy the following relations,

$$\begin{aligned}
x_1 &= x \cos \alpha , & y_1 &= y \cos \bar{\alpha} , & u_1 &= u , & v_1 &= v \\
x_2 &= x \sin \alpha , & y_2 &= -y \sin \bar{\alpha} , & u_2 &= -u , & v_2 &= -v .
\end{aligned} \tag{2.29}$$

The matrix  $M$  is non-singular and after integrating over the gauge fields one obtains a  $\sigma$ -model action of the form displayed in Eq. (2.10) with a dilaton given by Eq. (2.11). The resulting background fields are discussed in sections 3.2 and 3.3.

For the cosets of type  $(++)$  the matrix  $M$  turns out to be singular and it has the following form

$$M = \begin{pmatrix} \tilde{M} & 0 \\ 0 & 0 \end{pmatrix} , \tag{2.30}$$

where  $\tilde{M}$  is a non-singular  $3 \times 3$  matrix. We first fix the gauge freedom associated with the transformations generated by  $P_1$ ,  $P_2$  and  $J$  setting

$$x_1 = x \cos \alpha , \quad x_2 = x \sin \alpha , \quad y_1 = y_2 = 0 . \tag{2.31}$$

Since in this case the fourth row and column of the matrix  $M$  vanish, the fields  $A_4$  and  $\bar{A}_4$  appear in the Lagrangian of the gauged WZW model only in the following two linear terms

$$\mathcal{L}_{A_4} = 2(\nu_1^2 \partial u_1 + \nu_2^2 \partial u_2) \bar{A}_4 - 2(\bar{\nu}_1^2 \bar{\partial} u_1 + \bar{\nu}_2^2 \bar{\partial} u_2) A_4 . \tag{2.32}$$

Up to a total derivative the previous expression is equivalent to

$$\mathcal{L}_{A_4} = (\partial\bar{A}_4 - \bar{\partial}A_4)U + (\partial\bar{A}_4 + \bar{\partial}A_4)V \quad , \quad (2.33)$$

where

$$U \equiv (\nu_1^2 + \bar{\nu}_1^2)u_1 + (\nu_2^2 + \bar{\nu}_2^2)u_2 \quad , \quad V \equiv (\nu_1^2 - \bar{\nu}_1^2)u_1 + (\nu_2^2 - \bar{\nu}_2^2)u_2 \quad . \quad (2.34)$$

We can fix the gauge freedom associated with the transformations generated by  $K$  choosing the gauge  $\partial\bar{A}_4 + \bar{\partial}A_4 = 0$ . The integration over the gauge field  $A_4$  then leads to the constraint  $U = 0$ . We can modify this constraint adding to the Lagrangian the total derivative  $\mathcal{L}_\rho = -\Lambda\rho(\partial\bar{A}_4 - \bar{\partial}A_4)$  where  $\rho$  is a real constant. The constraint then becomes  $U = \Lambda\rho$  and its general solution reads

$$u_1 = (1 - \gamma)u + \rho \quad , \quad u_2 = -(1 + \gamma)u + \rho \quad , \quad (2.35)$$

where

$$\gamma = \cos(\alpha - \bar{\alpha})\cos(\alpha + \bar{\alpha}) \quad . \quad (2.36)$$

Due to the presence of the constraint  $U = \rho$ , the construction of the asymmetric models with  $\nu_i \neq \bar{\nu}_i$  might not be entirely straightforward. In particular it is not clear if it leads to non-trivial conformally invariant  $\sigma$ -models. We leave the general discussion to a future publication and in the rest of this paper we shall only consider models in the  $(++)$  class with  $\nu_i = \bar{\nu}_i$  or  $\alpha = \bar{\alpha}$ . Since the matrix  $\tilde{M}$  is non-singular, after integrating over the remaining gauge fields one obtains again a  $\sigma$ -model action of the form displayed in Eq. (2.10) with a dilaton given by Eq. (2.11). The resulting background fields are discussed in section 3.4.

### 3. Coset geometries

The coset construction for the various embeddings considered in the previous section gives rise to a number of interesting geometries. They are all particular examples of a class of string backgrounds related to plane waves by abelian T-duality transformations [42], whose properties are briefly reviewed at the beginning of this section. We then discuss the geometries associated with our three families of diagonal cosets. The general features of each class of models will be illustrated by two simple choices of parameters: the symmetric gauging  $\alpha = \bar{\alpha}$  and the twisted gauging  $\alpha = \pi/2 - \bar{\alpha}$ . As we will see the symmetric gaugings always lead to singular backgrounds while the twisted gaugings are non-singular for generic values of the parameters. Consequently the asymmetry in the parameters  $\alpha$  and  $\bar{\alpha}$  allows to interpolate between singular and non singular backgrounds. The parameters  $\lambda$ ,  $\mu$  and  $\bar{\mu}$  can generically be removed by a change of coordinates.

The background fields that correspond to the most general choices of parameters are collected in appendix A since the resulting expressions are quite lengthy.

### 3.1 General plane waves

The curved space-times that correspond to the diagonal cosets of the Heisenberg group belong to a class of four-dimensional string backgrounds introduced in [42]. These models have a covariantly constant null Killing vector and are either plane waves or are related to plane waves by a T-duality transformation with respect to an abelian non-null isometry. Following the notation in [33], the metric, dilaton and antisymmetric tensor are given by

$$\begin{aligned} ds^2 &= 2d\zeta d\bar{\zeta} - 2\left[f(u)\zeta^2 + \bar{f}(u)\bar{\zeta}^2 + F(u)\zeta\bar{\zeta}\right]du^2 - 2dudv \quad . \\ B_{\zeta\bar{\zeta}} &= ib(u) \quad , \quad \Phi = -\log g(u) \quad . \end{aligned} \quad (3.1)$$

Here  $f(u)$  is a complex function,  $g(u)$  and  $b(u)$  are real functions and the function  $F(u)$  is given by

$$F(u) = -\partial_u^2 \log g(u) + \frac{[\partial_u b(u)]^2}{4} \quad . \quad (3.2)$$

The previous equation is equivalent to the condition  $\beta^G = 0$  where  $\beta^G$  is the one-loop beta function for the metric of the  $\sigma$ -model. In fact the only non-trivial component of the Ricci tensor for the metric in (3.1) is

$$R_{uu} = 2F(u) \quad . \quad (3.3)$$

The one-loop beta functions  $\beta^\Phi$  and  $\beta^B$  for the dilaton and the antisymmetric tensor vanish identically.

The backgrounds in (3.1) were classified in [33] according to their isometries. For generic choices of the functions  $f(u)$  and  $F(u)$  they have five isometries. There is an additional isometry when  $f(u) = 0$  or when  $f(u)$  and  $F(u)$  are both constant. Finally when  $f(u) = 0$  and  $F(u)$  is constant the background (3.1) has seven isometries and coincides with the  $H_4$  WZW model [23], the maximally symmetric plane wave in four dimensions.

### 3.2 The cosets of type $(--)$

We can now follow the procedure described in the previous section and derive the background fields of the coset models that belong to the  $(--)$  class. The symmetric gauging coincides with the axial-vector gauging discussed in [33] and leads to singular plane waves. The general models with  $\alpha \neq \bar{\alpha}$  are still plane waves but now without singularities. In fact from the explicit form of the background fields of the twisted gaugings one can easily see that the corresponding models smoothly interpolate between non-singular spaces and a special singular point where the symmetric and twisted gauging coincide. For readers interested in the general asymmetric coset of type  $(--)$  we collected the corresponding background fields in appendix A. We finally note that while  $\alpha$  and  $\bar{\alpha}$  parameterise distinct backgrounds, the parameters  $\lambda$ ,  $\mu$  and  $\bar{\mu}$  can always be removed by a change of coordinates.

#### 3.2.1 Symmetric gauging

In order to make contact with the work of Antoniadis and Obers [33] it is useful to replace the angle  $\alpha = \bar{\alpha}$  by the new parameter  $\nu = \tan \alpha$ . The parameters of the model are then

specified in the following way

$$\begin{aligned}
\nu_1 &= 1 & \nu_2 &= \nu & \mu_1 &= \Lambda\mu & \mu_2 &= \Lambda(\lambda - \mu) \\
\bar{\nu}_1 &= 1 & \bar{\nu}_2 &= \nu & \bar{\mu}_1 &= -\Lambda\bar{\mu} & \bar{\mu}_2 &= -\Lambda(\lambda - \bar{\mu}) \ ,
\end{aligned} \tag{3.4}$$

with  $\Lambda = 1 + \nu^2$ . Given this choice of parameters and the gauge fixing (2.29), we obtain the following metric

$$\begin{aligned}
ds^2 &= 4dudv + \frac{r^+(u)}{2\nu^2(1 - \cos(u))} dx^2 + \frac{1 - \nu^4}{\nu^2(1 - \cos(u))} dx dy + \frac{r^-(u)}{2\nu^2(1 - \cos(u))} dy^2 \\
&\quad - \frac{(1 - \nu^2)[(1 - \nu^4)x + (4\nu^2 + r^+(u))y]}{2\nu^2(1 + \nu^2)\sin(u)} dx du \\
&\quad - \frac{(1 - \nu^4)r^-(u)x + ((1 + \nu^2)^4 - 16\nu^4 \cos(u))y}{2\nu^2(1 + \nu^2)^2 \sin(u)} dy du \\
&\quad + \frac{(1 - \nu^2)^2 r^-(u)x^2}{8\nu^2(1 + \nu^2)^2(1 + \cos(u))} du^2 + \frac{(1 - \nu^4)xy}{4\nu^2(1 + \cos(u))} du^2 \\
&\quad + \frac{(1 - \nu^2)^2(8\nu^2 + r^+(u))y^2}{8\nu^2(1 + \nu^2)^2(1 + \cos(u))} du^2 + \frac{2(1 - \nu^2)[2\lambda - (1 + \nu^2)(\mu + \bar{\mu})]}{1 + \nu^2} du^2 \ ,
\end{aligned} \tag{3.5}$$

where

$$r^\pm(u) = 1 + \nu^4 \pm 2\nu^2 \cos(u) \ . \tag{3.6}$$

The last term in the metric can be removed by the coordinate transformation  $v \rightarrow v - \xi u$  with constant  $\xi$ . Since the three-form flux  $H$  vanishes, the only other non-trivial background field is the dilaton

$$\Phi = -\frac{1}{2} \ln \sin^2(u) \ . \tag{3.7}$$

It is easy to check that the metric above coincides with the one in [33] after setting  $\nu = \sqrt{(1 - \kappa)/(1 + \kappa)}$  and rescaling the coordinate  $v$ . The resulting background is a singular plane wave belonging to the family (3.1). The only non-vanishing component of the Ricci tensor is  $R_{uu}$  and it is given by

$$F = -1/\sin^2(u) \ . \tag{3.8}$$

Following the series of coordinate transformations that have been described in [33] one can also determine the function

$$f(u) = \frac{1}{2 \sin^2(u)} [\cos(u) - i\kappa \sin(u)] e^{i\kappa u} \ , \tag{3.9}$$

which together with  $g(u) = \sin u$  and  $b(u) = 0$  completely specifies the background in (3.1).

### 3.2.2 Twisted gauging

As mentioned above, all the other models in the  $(--)$  class of diagonal cosets are non-singular. Their main features are well illustrated by the following simple choice of parameters

$$\begin{aligned} \nu_1 &= 1 & \nu_2 &= \nu & \mu_1 &= \Lambda\mu & \mu_2 &= \Lambda(\lambda - \mu) \\ \bar{\nu}_1 &= \nu & \bar{\nu}_2 &= 1 & \bar{\mu}_1 &= -\Lambda\bar{\mu} & \bar{\mu}_2 &= -\Lambda(\lambda - \bar{\mu}) , \end{aligned} \quad (3.10)$$

with  $\Lambda = 1 + \nu^2$ . Note that in this case  $\nu_1 = \bar{\nu}_2$  and  $\nu_2 = \bar{\nu}_1$  which is the reason for the name of twisted gauging given to this class of models. With the previous choice of parameters and the gauge fixing (2.29) the metric reads

$$\begin{aligned} ds^2 &= 4dudv + \frac{R^+(u)}{R^-(u)} dx^2 - \frac{2(1-\nu^2)}{R^-(u)} dx dy + dy^2 - \frac{2\nu(1-\nu^2)\sin(u)}{(1+\nu^2)R^-(u)} y dx du \\ &+ \frac{2\nu(1-\nu^2)\sin(u)}{(1+\nu^2)R^+(u)} x dy du - \frac{8\nu^2(2\nu - (1+\nu^2)\cos(u))\sin(u)}{(1+\nu^2)R^+(u)R^-(u)} y dy du \\ &- \frac{(1-\nu^2)^2 R^-(u)x^2 - 2(1+5\nu^2 - 5\nu^4 - \nu^6)xy + (1-\nu^2)^2 R^+(u)y^2}{4(1+\nu^2)^2 R^+(u)} du^2 \\ &- \frac{2(1-\nu^2)^2\lambda - 2(1-\nu^4)(\mu - \bar{\mu})}{1+\nu^2} du^2 , \end{aligned} \quad (3.11)$$

where

$$R^\pm(u) = 1 + \nu^2 \pm 2\nu \cos u . \quad (3.12)$$

The dilaton is given by

$$\Phi = -\frac{1}{2} \ln(1 + \nu^4 - 2\nu^2 \cos(2u)) = -\frac{1}{2} \ln(R^+(u)R^-(u)) , \quad (3.13)$$

and in this case there is also a non-trivial two-form field  $B_{\mu\nu}$  with flux

$$H = \frac{2\nu(1-\nu^2)(1+4\nu^2+\nu^4+2\nu^2\cos(2u))\sin(u)}{(1+\nu^2)R^+(u)R^-(u)^2} dx \wedge dy \wedge du . \quad (3.14)$$

The previous background fields describe a non singular plane wave. In fact the  $R_{uu}$  component of the Ricci tensor is

$$F(u) = \frac{(1+\nu^2)^4(1-18\nu^2+\nu^4) + 8\nu^2(1+\nu^2)^2(5-2\nu^2+5\nu^4)\cos^2(u) + 16\nu^4(1-\nu^2)^2\cos^4(u)}{4(1+\nu^2)^2(R^+(u)R^-(u))^2} , \quad (3.15)$$

and it is always regular for  $\nu \neq 1$ . We then see that deforming the embedding from the symmetric to the twisted case we removed the singularity in  $F(u)$ .

### 3.3 The cosets of type $(+-)$

The second class of cosets corresponds to  $\bar{\eta}_1 = 1$  and  $\bar{\eta}_2 = -1$ . As before we shall discuss two special choices of parameters, the symmetric and the twisted gaugings. The geometric data for the general choice of parameters are displayed in appendix A. The parameters  $\lambda$ ,  $\mu$  and  $\bar{\mu}$  can be removed by a change of coordinates.

### 3.3.1 Symmetric gauging

We start our discussion with the symmetric case

$$\begin{aligned} \nu_1 &= 1 & \nu_2 &= \nu & \mu_1 &= \Lambda\mu & \mu_2 &= \Lambda(\lambda - \mu) \\ \bar{\nu}_1 &= 1 & \bar{\nu}_2 &= \nu & \bar{\mu}_1 &= \Lambda\bar{\mu} & \bar{\mu}_2 &= -\Lambda(\lambda - \bar{\mu}) \end{aligned} \quad , \quad (3.16)$$

with  $\Lambda = 1 + \nu^2$ . Following the standard procedure we obtain the metric

$$\begin{aligned} ds^2 &= 2(1 + \nu^2)/\nu^2 dudv + \frac{1 + \cos(u)}{1 - \cos(u)} \nu^2 dx^2 + \frac{2\nu^2(1 + \nu^2 - 2\cos(u))(1 + \cos(u))}{(1 + \nu^2)(1 - \cos(u))} dx dy \\ &+ \frac{\nu^2[(1 + \nu^2)^2 - (3 + 2\nu^2 - \nu^4)\cos(u) + 4(1 - \nu^2)\cos^2(u)]}{(1 + \nu^2)^2(1 - \cos(u))} dy^2 \\ &+ (1 - \nu^4)\cot(u/2)/\nu^2 x dx du \\ &+ (1 - \nu^2)[1 - \nu^4 + 2\nu^2(1 + \cos(u))]\cot(u/2)/\nu^2(1 + \nu^2) y dx du \\ &+ (1 + \nu^4 - 2\nu^2\cos(u))\cot(u/2)/\nu^2 x dy du \\ &+ (1 + 2\nu^2 + 4\nu^4 + 2\nu^6 - \nu^8 - 4\nu^4(3 - \nu^2)\cos(u))\cot(u/2)/\nu^2(1 + \nu^2)^2 y dy du \\ &- (1 - \nu^4)/4\nu^2 x^2 du^2 - (1 - \nu^2)(3 - \nu^2 + 2\cos(u))/2\nu^2 xy du^2 \\ &- \frac{(1 - \nu^2)(1 + 11\nu^2 - 5\nu^4 + \nu^6 + 4(1 + 4\nu^2 - \nu^4)\cos(u))}{4\nu^2(1 + \nu^2)^2} y^2 du^2 \\ &+ (1 + \nu^2)(-2\lambda + \mu(1 + \nu^2)^2 + \bar{\mu}(1 - \nu^4))/\nu^4 du^2 \quad . \end{aligned} \quad (3.17)$$

The last term can be removed by the coordinate transformation  $v \rightarrow v - \xi u$  with constant  $\xi$ . The B-field is pure gauge and the dilaton is given by

$$\Phi = -\frac{1}{2} \ln \sin^2 \frac{u}{2} \quad . \quad (3.18)$$

The  $R_{uu}$  component of the Ricci tensor is

$$F(u) = -\frac{1}{4} \sin^2 \frac{u}{2} \quad , \quad (3.19)$$

and therefore we obtain again a singular plane wave background.

### 3.3.2 Twisted gauging

Also in this case we illustrate the main features of the models that correspond to a general choice of parameters with the twisted gaugings

$$\begin{aligned} \nu_1 &= 1 & \nu_2 &= \nu & \mu_1 &= \Lambda\mu & \mu_2 &= \Lambda(\lambda - \mu) \\ \bar{\nu}_1 &= \nu & \bar{\nu}_2 &= 1 & \bar{\mu}_1 &= \Lambda\bar{\mu} & \bar{\mu}_2 &= -\Lambda(\lambda - \bar{\mu}) \end{aligned} \quad , \quad (3.20)$$

with  $\Lambda = 1 + \nu^2$ . The metric reads

$$\begin{aligned}
ds^2 = & 2(1 + \nu^2)/\nu^2 dudv + \frac{R^+(u)}{R^-(u)} dx^2 - \frac{2(1 + 4\nu^2 - \nu^4 - 4\nu^2 \cos^2(u))}{(1 + \nu^2) R^-(u)} dx dy \\
& + \frac{1 + 11\nu^2 - 5\nu^4 + \nu^6 - 2\nu(1 + \nu^2)^2 \cos(u) - 8\nu^2(1 - \nu^2) \cos^2(u)}{(1 + \nu^2)^2 R^-(u)} dy^2 \\
& + \frac{2(1 - \nu^2) \sin(u)}{\nu R^-(u)} x dx du + \frac{2(1 - \nu^2) \sin(2u)}{(1 + \nu^2) R^-(u)} y dx du \\
& - \frac{[2(1 - 3\nu^2 - \nu^4 - \nu^6) + 4\nu^3(1 + \nu^2) \cos(u)] \sin(u)}{\nu(1 + \nu^2) R^-(u)} x dy du \\
& - \frac{[2\nu(1 - 3\nu^2 - 5\nu^4 - \nu^6) + 4(1 - 3\nu^2 + 5\nu^4 + \nu^6) \cos(u)] \sin(u)}{(1 + \nu^2)^2 R^-(u)} y dy du \\
& - \frac{1 - \nu^4}{4\nu^2} x^2 du^2 + \frac{1 - 5\nu^2 + 3\nu^4 + \nu^6 - 2\nu(1 - \nu^4) \cos(u)}{2\nu^2(1 + \nu^2)} x y du^2 \\
& - \frac{(1 - \nu^2)[(1 + \nu^2)^3 - 4\nu(1 - 4\nu^2 - \nu^4) \cos(u)]}{4\nu^2(1 + \nu^2)^2} y^2 du^2 \\
& + (1 + \nu^2)[\bar{\mu}(1 - \nu^4) - \lambda(1 + 2\nu^2 - \nu^4) + \mu(1 + \nu^2)^2]/\nu^2 du^2 \ ,
\end{aligned} \tag{3.21}$$

where the functions  $R^\pm(u)$  were defined in (3.12). The background also supports a the dilaton

$$\Phi = -\frac{1}{2} \ln(R^-(u)) \ . \tag{3.22}$$

and a non-trivial three-form flux

$$H = \frac{(1 - \nu^2)(1 - \nu^2 + 2\nu \cos(u))(1 + 4\nu^2 + \nu^4 - 2\nu(1 + \nu^2) \cos(u)) \sin(u)}{\nu(1 + \nu^2) R^-(u)^2} dx \wedge dy \wedge du \ . \tag{3.23}$$

The  $R_{uv}$  component of the Ricci tensor is given by

$$\begin{aligned}
F(u) = & \frac{1}{16\nu^4 R^-(u)^2} \left[ 1 + 6\nu^2 + 3\nu^4 - 52\nu^6 + 3\nu^8 + 6\nu^{10} + \nu^{12} \right. \\
& \left. - 4\nu(1 + \nu^2)(1 + 2\nu^2 - 10\nu^4 + 2\nu^6 + \nu^8) \cos(u) + 4\nu^2(1 - \nu^4)^2 \cos^2(u) \right] \ .
\end{aligned} \tag{3.24}$$

From the previous expression we can see that also for this class of models the singular behaviour of the background fields of the symmetric gauging is regularised by an asymmetric choice of parameters.

### 3.4 The cosets of type (++)

The last family of models correspond to the choice  $\eta_1 = \eta_2 = 1$ . Only in this case, as mentioned in section 2, we restrict our analysis to the symmetric models with  $\alpha = \bar{\alpha}$ , due to potential subtleties with the constraints and the gauge fixing for the general models.



With this choice of parameters the metric, the antisymmetric tensor and the dilaton are

$$ds^2 = \frac{1}{\Delta(u)} \left[ F_+(u) dx^2 + \frac{4l_2(u)}{x} dx d\varphi + \frac{F_-(u)}{x^2} d\varphi^2 - \frac{4\Lambda^2(\mu - \bar{\mu})^2}{x^2} F_-(u) du^2 \right] - 2dudv ,$$

$$B_{u\varphi} = \frac{\Lambda(\mu - \bar{\mu})}{x^2} \frac{2F_-(u)}{\Delta(u)} , \quad \Phi = -\frac{1}{2} \ln[x^2 \Delta(u)] , \quad (3.25)$$

where we defined the auxiliary functions

$$\Delta(u) = \sin^2(2\alpha) \sin^2(u) , \quad F_{\pm}(u) = 1 + l_1^2(u) + l_2^2(u) \pm 2l_1(u) , \quad (3.26)$$

with

$$l_1(u) = \cos u \cos(\gamma u - \rho) + \gamma \sin u \sin(\gamma u - \rho) , \quad (3.27)$$

$$l_2(u) = \gamma \sin u \cos(\gamma u - \rho) - \cos u \sin(\gamma u - \rho) , \quad (3.28)$$

and  $\gamma = \cos 2\alpha$ . In the process of recovering the background data from the gauged WZW Lagrangian (2.10) we introduced the new coordinates

$$\varphi = \frac{v_1 + v_2}{2} , \quad v = -(1 - \gamma)v_1 + (1 + \gamma)v_2 , \quad (3.29)$$

and performed the following change of variables

$$\varphi \mapsto \varphi + 2\Lambda[\lambda(1 + \gamma) - (\mu + \bar{\mu})]u . \quad (3.30)$$

When  $\mu = \bar{\mu} = 0$  the curved backgrounds (3.25) coincide with the vector-gauged models discussed by Antoniadis and Obers [33]. In the general case there is an additional term in the metric and a non-trivial three-form flux, both proportional to the difference  $\mu - \bar{\mu}$ . The Ricci scalar

$$R = -\frac{4F_-(u)}{x^2 \Delta(u)} , \quad (3.31)$$

clearly exhibits the singular nature of the background. Note however that the Ricci scalar does not depend on  $\mu$  and  $\bar{\mu}$ . This is not surprising since upon performing a T-duality transformation along the  $\varphi$  direction we obtain a plane wave background which does not depend on these parameters. The explicit form of the dual background is

$$ds^2 = \frac{\Delta(u)}{F_-(u)} [dx^2 + x^2 d\theta^2] - 2dudv , \quad (3.32)$$

$$B_{\theta x} = \frac{2l_2(u)x}{F_-(u)} , \quad \Phi = -\frac{1}{2} \ln[F_-(u)] , \quad (3.33)$$

where  $\theta$  is the T-dual coordinate. The parameters  $\mu$  and  $\bar{\mu}$  were removed by the coordinate transformation  $v \rightarrow v - 2(\mu - \bar{\mu})\theta$ .

## 4. Coset characters and representation theory

The vertex operators of a coset conformal field theory  $G/H$  transform in irreducible representations of the coset chiral algebra. The goal of this section is to derive the characters of these representations in order to provide a precise description of the  $\sigma$ -model spectrum. Due to the non-compactness and non-semi-simplicity of the numerator group the standard methods of determining the branching functions fail. The reason for this failure may be attributed to unavoidable divergencies which arise in the product of characters belonging to infinite dimensional representations of the horizontal subalgebra. In this section we propose a method to circumvent this problem. Our approach makes use of both character techniques and the knowledge of tensor products of the horizontal subalgebra. Since our method only rests on the absence of singular vectors on higher energy levels it should be applicable to general non-compact coset theories. We briefly comment on subtleties which arise in connection with the decomposition of tensor products involving spectral flow representations [17] and the occurrence of representations that are not fully decomposable.

### 4.1 Semi-classical analysis

We begin this section with a discussion of the semi-classical approximation to string propagation on group manifolds and their cosets. This allows us to introduce the unitary representations of the Heisenberg algebra  $H_4$  and to illustrate in a simple context the relation between coset characters and branching functions.

When all the length scales in a problem are large compared to the string scale, there is no significant difference between the behaviour of a string and the behaviour of a point particle. In this semi-classical approximation the Hilbert space of states for a string moving on a non-compact group manifold  $G$  coincides with the space of functions  $\mathcal{F}(G)$  which are  $\delta$ -function normalizable. This space admits a left-right-regular action of  $G$  and decomposes as

$$\mathcal{F}(G) = \int d\mu V_\mu \otimes V_\mu^* , \quad (4.1)$$

where the direct integral runs over a certain set of unitary irreducible representations  $\mu$  of  $G$ . In the case of the Heisenberg group,  $\mathcal{F}(H_4)$  can be written as a direct integral over three classes of representations [43]. There are two families of so-called discrete series representations  $(\pm, p, j)$  (with  $p > 0$  and  $j \in \mathbb{R}$ ) and one family of continuous series representations  $(0, s, j)$  (with  $s \geq 0$  and  $j$  defined modulo 1). We will use the same symbols later when we talk about the associated Lie algebra.

All these representations are infinite dimensional, thereby reflecting the non-compactness of the group manifold  $H_4$ . The  $J$  and  $K$  eigenvalues of the states in a given representation  $\mu$  are encoded by the characters

$$\rho_\mu(z, w) = \text{tr}_\mu \left[ z^{-iJ} w^{-iK} \right] , \quad (4.2)$$

whose explicit expressions are

$$\rho_{(+|p,j)}(z, w) = \frac{z^j w^p}{1-z} , \quad \rho_{(-|p,j)}(z, w) = \frac{z^j w^{-p}}{1-z^{-1}} , \quad \rho_{(0|s,j)}(z, w) = \sum_{n \in \mathbb{Z}} z^{n+j} . \quad (4.3)$$

The characters should be thought of as formal sums counting the multiplicity of states with given quantum numbers.

In a similar way the Hilbert space of a coset theory is given by  $\mathcal{F}(G/H)$ . This space coincides with the subspace of  $\mathcal{F}(G)$  consisting of the  $H$ -invariant functions

$$\text{Inv}_H \mathcal{F}(G) := \{f \in \mathcal{F}(G) \mid f(g) = f(\epsilon(h)g\bar{\epsilon}(h^{-1})) , \forall g \in G, \forall h \in H\} , \quad (4.4)$$

where  $\epsilon$  and  $\bar{\epsilon}$  denote the two embeddings of  $H$  in  $G$  used to define the coset. Since  $\mathcal{F}(G)$  can be decomposed according to Eq. (4.1), we can obtain an explicit description of  $\mathcal{F}(G/H)$  by first restricting all  $G$ -representations  $V_\mu$  to  $H$ -representations  $V_\mu|_H = \oplus_a b_\mu^a V_a$  and then taking the  $H$ -invariant part by coupling the tensor product of left and right factors to the trivial representation.

In the case of diagonal embeddings the branching coefficients are just the tensor product coefficients. In order to deduce the branching functions for the diagonal cosets of the  $H_4$  WZW model we will thus need the following tensor products of representations of the Heisenberg group [43]

$$\begin{aligned} (\pm|p_1, j_1) \otimes (\pm|p_2, j_2) &= \bigoplus_{n=0}^{\infty} (\pm|p_1+p_2, j_1+j_2 \pm n) \\ (\pm|p_1, j_1) \otimes (\mp|p_2, j_2) &= \begin{cases} \bigoplus_{n=0}^{\infty} (\tau| |p_1 - p_2|, j_1+j_2 - \tau n) , & \tau = \pm \text{sign}(p_1 - p_2) \quad , p_1 \neq p_2 \\ \int_0^\infty ds s (0|s, j_1+j_2) & , p_1 = p_2 \end{cases} \\ (\pm|p_1, j_1) \otimes (0|\sigma, j_2) &= \bigoplus_{n \in \mathbb{Z}} (\pm|p_1, j_1+j_2+n) \end{aligned} \quad (4.5)$$

$$(0|s_1, j_1) \otimes (0|s_2, j_2) = \int_0^{2\pi} \frac{d\psi}{2\pi} (0|s(\psi), j_1+j_2) , \quad s^2(\psi) = s_1^2 + s_2^2 + 2s_1 s_2 \cos \psi .$$

Writing these tensor products in terms of characters one can derive some formal rules to interpret the following a priori ill-defined products

$$\begin{aligned} \rho_{(+|p, j_1)} \rho_{(-|p, j_2)} &= \frac{1}{1-z} \frac{1}{1-z^{-1}} := \int_0^\infty ds s \sum_{n \in \mathbb{Z}} z^{n+j_1+j_2} , \\ \rho_{(0|s_1, j_1)} \rho_{(0|s_2, j_2)} &= \sum_{n, m \in \mathbb{Z}} z^{n+m+j_1+j_2} := \int_0^{2\pi} \frac{d\psi}{2\pi} \sum_{n \in \mathbb{Z}} z^{n+j_1+j_2} . \end{aligned} \quad (4.6)$$

These rules will be a valuable aid below when it comes to decomposing certain products of affine characters.

## 4.2 Affine representation theory

The symmetry algebra of the WZW model based on the Heisenberg group is generated by an affine  $H_4$  algebra. In this section we define the  $H_4$  algebra giving the commutation relations of the modes of the affine currents. We then discuss two classes of representations, the standard representations and the spectral flow representations [17]. We emphasise that the spectrum of the Virasoro generator  $L_0$  is not bounded from below in the spectral flow representations.

### 4.2.1 Standard representations

The affine  $\hat{H}_4$  algebra is defined by the following commutation relations

$$[P_n^+, P_m^-] = 2n \delta_{n+m,0} - 2iK_{n+m} , \quad [J_n, P_m^\pm] = \mp i P_{n+m}^\pm , \quad [J_n, K_m] = n \delta_{n+m,0} , \quad (4.7)$$

with  $n, m \in \mathbb{Z}$ . The simplest class of irreducible representations of  $\hat{H}_4$  are the highest-weight representations, generated by acting with all the negative modes of the currents on an irreducible unitary representation  $\mu$  of the horizontal subalgebra. Generalising the definition given for the horizontal subalgebra, we introduce the following characters

$$\chi_\mu^{\hat{H}_4}(q, z, w) = \text{tr}_\mu \left[ q^{L_0 - \frac{c}{24}} z^{-iJ_0} w^{-iK_0} \right] . \quad (4.8)$$

Here  $J_0$  and  $K_0$  are the zero modes of the corresponding affine currents and  $L_0$  is the zero mode of the energy-momentum tensor

$$T = \frac{1}{2} (P_1^2 + P_2^2 + JK + K^2) . \quad (4.9)$$

Explicit expressions for the characters are easily computed

$$\begin{aligned} \chi_{(+|p,j)}^{\hat{H}_4}(q, z, w) &= \frac{q^{h_{(+,p,j)} - \frac{1}{12}} z^j w^p}{(1-z)\eta(q)^2 \prod_{n=1}^{\infty} (1-zq^n)(1-z^{-1}q^n)} \quad (|q| < |z| < 1) \\ \chi_{(-|p,j)}^{\hat{H}_4}(q, z, w) &= \frac{q^{h_{(-,p,j)} - \frac{1}{12}} z^j w^{-p}}{(1-z^{-1})\eta(q)^2 \prod_{n=1}^{\infty} (1-zq^n)(1-z^{-1}q^n)} \quad (|q|^{-1} > |z| > 1) \\ \chi_{(0|s,j)}^{\hat{H}_4}(q, z, w) &= \frac{q^{h_{(0,s,j)} - \frac{1}{12}} \sum_{n \in \mathbb{Z}} z^{n+j}}{\eta(q)^2 \prod_{n=1}^{\infty} (1-zq^n)(1-z^{-1}q^n)} = \frac{q^{h_{(0,s,j)}}}{\eta(q)^4} \sum_{n \in \mathbb{Z}} z^{n+j} . \end{aligned} \quad (4.10)$$

The conformal weights of the ground states of these representations coincide with the eigenvalues of the modified Casimir operator associated with the energy-momentum tensor in Eq. (4.9). They are given by

$$h_{(\pm|p,j)} = \frac{p}{2}(1-p) \mp pj \quad , \quad h_{(0|s,j)} = \frac{s^2}{2} . \quad (4.11)$$

For future reference we also show how these expressions are modified when the invariant metric on  $H_4$  is not the standard one but has the general form (2.14). The conformal dimensions then become

$$h_{(\pm|p,j)} = \frac{1}{\Lambda} \left( \frac{p}{2} \mp pj \right) + \lambda p^2 - \frac{p^2}{2\Lambda^2} , \quad h_{(0|s,j)} = \frac{s^2}{2\Lambda} . \quad (4.12)$$

As it is the case for the  $SL(2, \mathbb{R})$  WZW model [17], only a subset of the highest-weight representations is part of the spectrum of the theory. For  $\hat{H}_4$  the allowed highest-weight representations are  $(0|s, j)$  and  $(\pm|p, j)$  with  $p \in (0, 1)$  [44, 25]. In the following we will call them standard representations. States with  $p \geq 1$  belong to a different class of representations called spectral flow representations [17] in which  $L_0$  is not bounded from below.

### 4.2.2 Spectral flow representations

The name of this class of representations has its origin in the observation that the  $\hat{H}_4$  current algebra admits a family of spectral flow automorphisms  $\Sigma_\omega$ ,  $\omega \in \mathbb{Z}$  which acts on the modes as

$$\Sigma_\omega(P_n^\pm) = P_{n \mp \omega}^\pm, \quad \Sigma_\omega(J_n) = J_n, \quad \Sigma_\omega(K_n) = K_n - i\omega\delta_{n0}. \quad (4.13)$$

From this definition one also readily derives the action

$$\Sigma_\omega(L_n) = L_n - i\omega J_n, \quad (4.14)$$

on the Virasoro modes. Given a representation  $\mu$  implemented on the space  $\mathcal{H}_\mu$  via the map  $\rho_\mu : \hat{H}_4 \rightarrow \text{End}(\mathcal{H}_\mu)$ , we can define a new representation  $\mu_\omega$  which acts on the same space via the map  $\rho_{\mu_\omega} = \rho_\mu \circ \Sigma_{-\omega}$ . In view of its construction  $\mu_\omega$  is called a spectral flow representation. Spectral flow representations also exist for affine Lie algebras based on compact real forms of finite dimensional semi-simple Lie algebras but in this case  $L_0$  is still bounded from below and it can be shown that they are equivalent to ordinary highest-weight representations. This, however, is not the case for non-compact affine algebras and in particular for  $\hat{H}_4$ . The inclusion of the spectral flow representations in the spectrum allows to extend the range of the label  $p$  from the unit interval to the whole positive real axis.

Using the equations (4.13) and (4.14) one can easily relate the character of  $\mu_\omega$  to the character of the underlying standard representation  $\mu$ . Indeed, a simple algebraic manipulation within the trace yields

$$\chi_{\mu_\omega}^{\hat{H}_4}(q, z, w) = w^\omega \chi_\mu^{\hat{H}_4}(q, q^{-\omega}z, w). \quad (4.15)$$

In order to simplify the notation we will identify the label  $\mu_{\omega=0}$  with  $\mu$  whenever there is no chance of confusion.

### 4.3 Tensor product decompositions for the diagonal coset

In this section we analyse the decomposition of the  $\hat{H}_4 \times \hat{H}_4$  representations with respect to the diagonal subalgebras  $\epsilon(\hat{H}_4)$  which are relevant to the curved backgrounds discussed earlier in this paper. We explain why the standard character decompositions fail and provide a method which allows to circumvent these problems by using a mixture of character techniques and analytical input from tensor product decompositions of the horizontal subalgebra.

#### 4.3.1 General strategy

The affine representations relevant for the  $H_4$  WZW model are all induced from infinite dimensional unitary representations of the horizontal subalgebra. The modes, however, which are used to generate the remaining states transform in the finite dimensional adjoint representation which is *non-unitary*, a common feature of all WZW models based on non-compact groups. Yet, in the present case there is an additional complication because

the adjoint representation is *indecomposable but reducible*, reflecting the non-semi-simple nature of the Lie algebra  $H_4$ .

In this section we describe a method to derive the decomposition of the tensor products of standard affine representations. Our general strategy is to decompose the affine representations into representations of the horizontal subalgebra on each energy level first. Then we use the known tensor products for the horizontal subalgebra in order to determine the tensor product energy level by energy level. Finally we reorganise the result and express it in terms of affine characters again. This last step is in fact greatly simplified by the absence of singular vectors in the affine modules that are relevant here as we will explain below.

The main advantage of the method just described is that it allows to combine character techniques with the analytic knowledge about the tensor products of the horizontal subalgebra displayed in Eq. (4.5). This is very convenient for non-compact groups since the unitary representations are infinite dimensional. Unfortunately this method cannot be applied directly to the spectral flow representations, as discussed in more detail in section 4.3.3. It also fails if a given tensor product turns out not to be fully decomposable.

As already mentioned, the standard modules relevant for the WZW models are simply obtained by applying (properly symmetrised) combinations of negative modes to the ground states. Together with the absence of null vectors in the resulting Verma modules this allows us to represent the standard affine representations  $\hat{\mu}$  in the form

$$\hat{\mu}|_{H_4} = q^{h_\mu} \mu \otimes M(q) . \quad (4.16)$$

Here,  $\mu$  is the underlying representation of the horizontal subalgebra and  $M(q)$  denotes the universal enveloping algebra of the subalgebra generated by the negative modes of the  $\hat{H}_4$ -currents. The variable  $q$  keeps track of the energies of the states. Since all the modes of the affine currents transform in the adjoint representation we can write

$$M(q) = 1 + q \text{ad} + q^2 [\text{ad} + (\text{ad} \otimes \text{ad})_{\text{sym}}] + q^3 [\text{ad} + \text{ad} \otimes \text{ad} + (\text{ad} \otimes \text{ad} \otimes \text{ad})_{\text{sym}}] + \dots . \quad (4.17)$$

As discussed in appendix B, the tensor products of the adjoint representation contain indecomposable but reducible representations. However it is easy to see that the tensor product  $\text{ad} \otimes (\pm|p, j)$  is fully reducible. In fact indecomposable representations can only appear when the eigenvalue of  $K_0$  vanishes in the tensor product. The three examples relevant for us are the tensor products  $\text{ad}^{\otimes n}$ ,  $(0|s_1, j_1) \otimes (0|s_2, j_2)$  and  $(+|p, j_1) \otimes (-|p, j_2)$ . These cases are also analysed in more detail in appendix B.

From the previous paragraph we conclude that the product in (4.16) is fully reducible when  $\mu = (\pm|p, j)$  and we obtain

$$(\pm|p, j) \otimes M(q) = \bigoplus_{n \in \mathbb{Z}} N_{[M, n]}^\pm(q) (\pm|p, j + n) . \quad (4.18)$$

We can derive the explicit form of the multiplicity functions  $N_{[M, n]}^\pm(q)$  by writing the previous equation in terms of characters. The character of  $M(q)$  is given by

$$\chi_M(q, z) = \prod_{n=1}^{\infty} \left[ (1 - q^n)^2 (1 - zq^n)(1 - z^{-1}q^n) \right]^{-1} . \quad (4.19)$$

For  $|q| < |z| < 1$  a more convenient form is [45, 2]

$$\chi_M(q, z) = \sum_{n \in \mathbb{Z}} z^n \sum_{m=1}^{\infty} (-1)^{m+1} \frac{q^{\frac{m}{2}(m+2n-1) + \frac{1}{6}} (1 - q^m)}{\eta(q)^4} . \quad (4.20)$$

Since this function is symmetric with respect to the replacement  $z \mapsto 1/z$  we can also write

$$\chi_M(q, z) = \sum_{n \in \mathbb{Z}} z^n \sum_{m=1}^{\infty} (-1)^{m+1} \frac{q^{\frac{m}{2}(m+2|n|-1) + \frac{1}{6}} (1 - q^m)}{\eta(q)^4} . \quad (4.21)$$

Substituting the previous expressions for  $\chi_M(q, z)$  in Eq. (4.18) and comparing the coefficients of identical powers of the variable  $z$  on both sides of the equation we obtain

$$N_{[M,n]}^{\pm}(q) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{q^{\frac{m}{2}(m+2|n|-1) + \frac{1}{6}} (1 - q^m)}{\eta(q)^4} . \quad (4.22)$$

Note that, as anticipated by our notation, the result does not depend on  $j$ .

When  $\mu = (0|s, j)$  we cannot follow the same approach because the tensor products  $(0|s, j) \otimes \text{ad}$  are reducible but not fully decomposable (see appendix B). Similar problems can be expected for all non-compact groups and their non-abelian cosets. For instance in the case of  $\widetilde{SL}(2, \mathbb{R})$  the non-complete reducibility enters on sufficiently high energy levels in the discrete series of affine representations with half-integral spin.

We now apply the decomposition of the affine modules in (4.16) to the tensor product of two affine representations  $\hat{\mu} \otimes \hat{\nu}$ , in order to compute the branching functions  $b_{[\mu, \nu, \sigma]}(q)$  in the tensor product decomposition  $\hat{\mu} \otimes \hat{\nu} = \bigoplus_{\sigma} b_{[\mu, \nu, \sigma]}(q) \hat{\sigma}$ . We obtain

$$\hat{\mu} \otimes \hat{\nu}|_{H_4} = q^{h_{\mu} + h_{\nu}} \mu \otimes \nu \otimes M(q)^2 = (\mu \otimes \nu \otimes M(q)) \Big|_{H_4} , \quad (4.23)$$

where the hat over the tensor product representation on the right hand side indicates the affinization of the  $H_4$ -representation  $\mu \otimes \nu \otimes M(q)$ . This affinization should be understood as an induced  $\hat{H}_4$ -module based on the given representation of the horizontal subalgebra. We are assuming that the affine representations on both sides of the previous equation contain the *same* factor  $M(q)$  coming from the higher modes. This is true because the Verma modules are irreducible. To evaluate  $\mu \otimes \nu \otimes M(q)$  we first perform the tensor product  $\mu \otimes \nu = \bigoplus_{\sigma} N_{\mu\nu}^{\sigma} \sigma$  using eqs. (4.5) and then calculate energy level by energy level the tensor product of infinite dimensional representations with finite dimensional ones using character techniques. The final result is schematically given by

$$\mu \otimes \nu \otimes M(q) = \bigoplus_{\sigma, \rho} N_{\mu\nu}^{\sigma} N_{\sigma M}^{\rho}(q) \rho = \bigoplus_{\rho} b_{[\mu, \nu, \rho]}(q) \rho . \quad (4.24)$$

### 4.3.2 Decomposition of the tensor products of standard representations

We now apply the procedure just outlined to the decomposition of the  $\hat{H}_4 \oplus \hat{H}_4$  representations with respect to the diagonal  $\hat{H}_4$  subalgebras discussed in section 2. With no loss of generality we may assume that the embedding is of the form (2.21) with  $\eta_i = 1$ . All other choices can be reduced to this one by a suitable automorphism.

Before giving the general result, we derive in some detail the characters of the representations of the coset chiral algebra that appear in  $(+|p_1, j_1) \otimes (+|p_2, j_2)$ . We first compute the following tensor products

$$\begin{aligned}
(+|p_1, j_1) \otimes (+|p_2, j_2) \otimes M(q) &= \bigoplus_{n=0}^{\infty} (+|\nu_1^2 p_1 + \nu_2^2 p_2, j_1 + j_2 + n) \otimes M(q) \\
&= \bigoplus_{n=0}^{\infty} \bigoplus_{l \in \mathbb{Z}} (+, \nu_1^2 p_1 + \nu_2^2 p_2, j_1 + j_2 + n + l) \sum_{m=1} (-1)^{m+1} \frac{q^{\frac{m}{2}(m-2l-1) + \frac{1}{6}} (1 - q^m)}{\eta(q)^4} \\
&= \bigoplus_{l \in \mathbb{Z}} (+|\nu_1^2 p_1 + \nu_2^2 p_2, j_1 + j_2 + l) \sum_{m=1} (-1)^{m+1} \frac{q^{\frac{m}{2}(m-2l-1) + \frac{1}{6}}}{\eta(q)^4} . \tag{4.25}
\end{aligned}$$

Then we include all the factors  $q^{h_\mu}$  and  $q^{-\frac{c}{24}}$  required by the definitions (4.8) and (4.16). The final result for the coset character is

$$\chi_{[(+|p_1, j_1), (+|p_2, j_2); (+|p, j_1 + j_2 + n)]}^{\hat{H}_4 \times \hat{H}_4 / \hat{H}_4}(q) = \frac{q^{h_{(+|p_1, j_1)} + h_{(+|p_2, j_2)} - h_{(+|p, j_1 + j_2 + n)}}{\eta(q)^4} \sum_{m=0}^{\infty} (-1)^m q^{\frac{m}{2}(m+2n+1)} , \tag{4.26}$$

with  $p = \nu_1^2 p_1 + \nu_2^2 p_2$ . From the previous expression we can read off the conformal dimension of the coset primary fields

$$h_{[(+|p_1, j_1), (+|p_2, j_2); (+|p, j_1 + j_2 + n)]}^{\hat{H}_4 \times \hat{H}_4 / \hat{H}_4} = \begin{cases} h_{(+|p_1, j_1)} + h_{(+|p_2, j_2)} - h_{(+|p, j_1 + j_2 + n)} & , \text{ for } n \geq 0 \\ h_{(+|p_1, j_1)} + h_{(+|p_2, j_2)} - h_{(+|p, j_1 + j_2 + n)} - n & , \text{ for } n \leq 0 \end{cases} . \tag{4.27}$$

All other cases of the form  $\mu_1 \otimes \mu_2$  with  $\mu_1 = (\pm|p_1, j_1)$  and  $\mu_2 = (\pm|p_2, j_2)$  can be treated in exactly the same way. The result can be written in the following compact form

$$\chi_{[\mu_1, \mu_2; \mu_{12}(n)]}^{\hat{H}_4 \times \hat{H}_4 / \hat{H}_4}(q) = \frac{q^{h_{\mu_1} + h_{\mu_2} - h_{\mu_{12}(n)}}}{\eta(q)^4} \sum_{m=0}^{\infty} (-1)^m q^{\frac{m}{2}(m+2n+1)} , \tag{4.28}$$

$$h_{[\mu_1, \mu_2, \mu_{12}(n)]}^{\hat{H}_4 \times \hat{H}_4 / \hat{H}_4} = \begin{cases} h_{\mu_1} + h_{\mu_2} - h_{\mu_{12}(n)} & , \text{ for } n \geq 0 \\ h_{\mu_1} + h_{\mu_2} - h_{\mu_{12}(n)} - n & , \text{ for } n \leq 0 \end{cases} . \tag{4.29}$$

The label  $\mu_{12}(n)$  is specified by the following rules that simply reflect the tensor products of the horizontal algebra

| $\mu_1$        | $\mu_2$        | $p > 0$                     | $\mu_{12}(n)$          |
|----------------|----------------|-----------------------------|------------------------|
| $(+ p_1, j_1)$ | $(+ p_2, j_2)$ | $\nu_1^2 p_1 + \nu_2^2 p_2$ | $(+ p, j_1 + j_2 + n)$ |
| $(- p_1, j_1)$ | $(- p_2, j_2)$ | $\nu_1^2 p_1 + \nu_2^2 p_2$ | $(- p, j_1 + j_2 - n)$ |
| $(+ p_1, j_1)$ | $(- p_2, j_2)$ | $\nu_1^2 p_1 - \nu_2^2 p_2$ | $(+ p, j_1 + j_2 - n)$ |
| $(+ p_1, j_1)$ | $(- p_2, j_2)$ | $\nu_2^2 p_2 - \nu_1^2 p_1$ | $(- p, j_1 + j_2 + n)$ |

In the previous and in the following formulas the conformal dimension of the representations of the embedded algebra are given by (4.12), since one should use the induced metric on  $\epsilon(H_4)$ .



The only cases that require a different approach are the product  $(+|p_1, j_1) \otimes (-|p_2, j_2)$  with  $\nu_1^2 p_1 - \nu_2^2 p_2 = 0$  and the product  $(0|s_1, j_1) \otimes (0|s_2, j_2)$ . In both cases the full reducibility of the induced module  $(\mu \otimes \nu \otimes M(q))^\wedge$  is not guaranteed since the tensor products  $(0|s, j) \otimes \hat{a}$  are reducible but not fully decomposable. A priori one cannot exclude that indecomposable affine representations could play a role in the construction of the diagonal  $H_4$  cosets. This could be a rather common feature of non-compact cosets involving a non-abelian denominator and a closer investigation of this phenomenon and of its possible connections with logarithmic conformal field theories is left for future work. Let us mention that at least in the case of  $H_4$  these representations, if present, would not be part of the string spectrum since they will be removed by the Virasoro constraints.

Although we cannot provide a rigorous discussion of the decomposition of  $(+|p_1, j_1) \otimes (-|p_2, j_2)$  with  $\nu_1^2 p_1 - \nu_2^2 p_2 = 0$  and  $(0|s_1, j_1) \otimes (0|s_2, j_2)$ , we can derive a simple and plausible expression for the coset characters *assuming* the full reducibility of the tensor product of the affine representations and using the formal rules in Eq. (4.6). In the first case the full reducibility translates into the following character identity

$$\chi_{(+|p_1, j_1)}^{\hat{H}_4}(q, z) \chi_{(-|p_2, j_2)}^{\hat{H}_4}(q, z) = \int_0^\infty ds s \chi_{[(+|p_1, j_1), (+|p_2, j_2); (0|s, j_1 + j_2)]}^{\hat{H}_4 \times \hat{H}_4 / \hat{H}_4}(q) \chi_{(0|s, j_1 + j_2)}^{\hat{H}_4}(q, z) , \quad (4.30)$$

and after simplifying the common factors on both sides we obtain

$$\chi_{[(+|p_1, j_1), (+|p_2, j_2); (0|s, j_1 + j_2)]}^{\hat{H}_4 \times \hat{H}_4 / \hat{H}_4}(q) = \frac{q^{h_{(+|p_1, j_1)} + h_{(-|p_2, j_2)} - h_{(0|s, j_1 + j_2)}}}{\eta(q)^4} , \quad \nu_1^2 p_1 - \nu_2^2 p_2 = 0 . \quad (4.31)$$

As for the second case, we start from

$$\chi_{(0|s_1, j_1)}^{\hat{H}_4}(q, z) \chi_{(0|s_2, j_2)}^{\hat{H}_4}(q, z) = \int_0^{2\pi} \frac{d\psi}{2\pi} \chi_{[(0|s_1, j_1), (0|s_2, j_2); (0|s(\psi), j_1 + j_2)]}^{\hat{H}_4 \times \hat{H}_4 / \hat{H}_4}(q) \chi_{(0|s(\psi), j_1 + j_2)}^{\hat{H}_4}(q, z) , \quad (4.32)$$

and using again Eq. (4.6) we obtain

$$\chi_{[(0|s_1, j_1), (0|s_2, j_2); (0|s(\psi), j_1 + j_2)]}^{\hat{H}_4 \times \hat{H}_4 / \hat{H}_4}(q) = \frac{q^{h_{(0|s_1, j_1)} + h_{(0|s_2, j_2)} - h_{(0|s(\psi), j_1 + j_2)}}}{\eta(q)^4} , \quad (4.33)$$

where  $s^2(\psi) = s_1^2 + s_2^2 + 2s_1 s_2 \cos \psi$ .

Also the case  $(\pm|p, j_1) \otimes (0|s, j_2)$  can easily be discussed using character techniques and the result is

$$\chi_{[(\pm|p, j_1), (0|s, j_2); (\pm|p, j_1 + j_2 + n)]}(q) = \frac{q^{h_{(\pm|p, j_1)} + h_{(0|s, j_2)} - h_{(\pm|p, j_1 + j_2 + n)}}}{\eta(q)^4} . \quad (4.34)$$

The conformal dimension is

$$h_{[(\pm|p, j_1), (0|s, j_2); (\pm|p, j_1 + j_2 + n)]}^{\hat{H}_4 \times \hat{H}_4 / \hat{H}_4} = h_{(\pm|p, j_1)} + h_{(0|s, j_2)} - h_{(\pm|p, j_1 + j_2 + n)} . \quad (4.35)$$

We stress again that the conformal dimension of the representations of the embedded algebra are computed with the induced metric and therefore are given by (4.12).

### 4.3.3 Remarks on the decomposition of tensor products with spectral flow representations

In the last part of this section we would like to comment on some aspects of the decomposition of tensor products involving spectral flow representations  $\mu_{\omega_1} \otimes \nu_{\omega_2}$ . Our first observation is that the spectral flow automorphisms (4.13) and the embeddings (2.21) satisfy the relation

$$\epsilon \circ (\Sigma_{\eta_1 \omega} \times \Sigma_{\eta_2 \omega}) = \Sigma_{\omega} \circ \epsilon . \quad (4.36)$$

Given a decomposition

$$\mu \otimes \nu = \bigoplus_{\lambda} N_{\mu\nu}^{\lambda} \lambda , \quad (4.37)$$

the previous relation implies

$$\mu_{\eta_1 \omega} \otimes \nu_{\eta_2 \omega} = \bigoplus_{\lambda} N_{\mu\nu}^{\lambda} \lambda_{\omega} , \quad \omega \in \mathbb{Z} . \quad (4.38)$$

The equivalence of coset characters resulting from Eq. (4.36) is just a manifestation of what is known as field identification in compact coset models. Indeed, it has been known for a long time that field identifications are implemented by the action of certain pairs of simple currents in the numerator and the denominator affine algebra. Simple currents in turn can be identified with spectral flow transformations and therefore Eq. (4.36) precisely singles out the pairs of spectral flows that induce field identifications. Since in contrast to the compact case here we have to identify an infinite number of coset representations, this leads to an infinite degeneracy in the coset decomposition which has to be removed by hand.

According to Eq. (4.36) and the corresponding field identification, it is sufficient to consider only tensor products of the form  $\mu \otimes \nu_{\omega}$ , with  $\mu$  a standard representation and  $\nu_{\omega}$  a spectral flow representation. From this point of view, the results obtained in the previous sections provide the decompositions of the tensor products with  $\omega = 0$  and  $\omega = \pm 1$ . The analysis of the decomposition of the other tensor product  $\mu \otimes \nu_{\omega}$  with  $|\omega| \geq 2$  is however a much more difficult problem and it would be interesting to develop rigorous methods to solve it. Character methods cannot be directly applied to this case. The characters in Eq. (4.10) are in fact formal power series that converge in different domains of the  $z$  complex plane. For instance, using Eq. (4.15), we can see that  $\chi_{(+|p,j)_{\omega}}^{\hat{H}_4}(q, z)$  converges in the annulus  $|q|^{\omega+1} < |z| < |q|^{\omega}$ . As discussed in [46] the formal character and the analytic expression coincide only up to contact terms that encode the unbounded part of the spectrum. It is not obvious that one can find a consistent way of computing with these formal series in order to extract the coset characters from their product.

## 5. The spectrum of the diagonal cosets

In the final section of this paper we determine the operator content of the diagonal cosets that have been discussed in section 3. If we combine these models with other CFTs such that the total central charge is the one required for a critical string theory background,

the coset vertex operators correspond to closed string states propagating in the curved space-time described by the  $\sigma$ -model.

The partition function of the WZW model based on  $H_4 \times H_4$  is given by the charge conjugation modular invariant which couples every representation of the affine algebra with its conjugate representation. Hence any multiplet of primary fields is completely specified by fixing its transformation properties under the holomorphic affine current algebra. As we reviewed in section 4 there are three types of representations of  $\hat{H}_4$ . For the derivation of the spectrum of the coset models it is convenient to divide the spectrum of  $H_4 \times H_4$  into sectors labeled by the representations of the two  $H_4$  factors. The spectrum then contains contributions from nine different sectors,

$$\begin{aligned}
\mathcal{H}_{++} &= [(+|p_1 j_1), (+|p_2 j_2)] & (0, 0) < (p_1, p_2) < (1, 1) & & (j_1, j_2) \in \mathbb{R}^2 \\
\mathcal{H}_{+-} &= [(+|p_1 j_1), (-|p_2 j_2)] & (0, 0) < (p_1, p_2) < (1, 1) & & (j_1, j_2) \in \mathbb{R}^2 \\
\mathcal{H}_{\pm 0} &= [(+|p_1 j_1), (0| s j_2)] & 0 < p_1 < 1, \quad s \geq 0 & & j_1 \in \mathbb{R}, \quad 0 \leq j_2 < 1 \\
\mathcal{H}_{00} &= [(0| s_1 j_1), (0| s_2 j_2)] & s_1 \geq 0, \quad s_2 \geq 0 & & (0, 0) \leq (j_1, j_2) < (1, 1) \quad (5.1)
\end{aligned}$$

and similar definitions for the sectors  $\mathcal{H}_{--}$ ,  $\mathcal{H}_{-+}$  and  $\mathcal{H}_{0\pm}$ . Moreover we have to take into account the images of all these sectors under independent amounts of spectral flow for the two  $H_4$  factors.

Our strategy for determining the spectrum of the coset theories discussed in this paper is as follows. For each of the sectors  $\mathcal{H}_{\mu\nu}$  of  $H_4 \times H_4$  we first calculate the modified tensor products  $\mu \otimes_\epsilon \nu$  and  $\mu \otimes_{\bar{\epsilon}} \nu$ , which are defined using the diagonal embeddings  $\epsilon$  and  $\bar{\epsilon}$  instead of the standard coproduct. Since the two embeddings are in general different, this gives rise to different types of representations for the left and the right movers. In order to identify the states of the coset we then impose the constraint

$$\epsilon(X) + \bar{\epsilon}(X) = 0, \quad (5.2)$$

for all generators  $X \in H_4$ . This implies that in each sector we are only allowed to keep those contributions for which the labels of the left-moving and right-moving representations that result from the decomposition coincide. Another consequence of the constraint (5.2) is that the operators in the spectrum of the coset models are completely identified by three labels, two for the representations of the original  $H_4 \times H_4$  model and one for either of the representations of the embedded  $H_4$ . The condition (5.2) with  $X = K_0$  will severely restrict the type of sectors that can contribute to the spectrum of the different cosets.

Since the decomposition of the tensor product of spectral flow representations is still an open problem, the discussion in this section will be restricted to the standard representations. We also introduce the short hand notation

$$c = \cos \alpha, \quad \bar{c} = \cos \bar{\alpha}, \quad s = \sin \alpha, \quad \bar{s} = \sin \bar{\alpha}, \quad (5.3)$$

and set  $\Lambda = 1$  for notational convenience.

### 5.1 The cosets of type (--)

We now apply the procedure outlined above to the models in the class (--). The first step is to identify in which of the sectors  $\mathcal{H}_{\mu\nu}$  it is possible to solve the constraint (5.2) with  $X = K_0$ . Let us consider for instance the  $\mathcal{H}_{++}$  sector. Using the embeddings  $\epsilon$  and  $\bar{\epsilon}$  in (2.21), on the left we obtain representations of the form  $(+|p, j)$  with  $p = c^2 p_1 + s^2 p_2$  and on the right, due to the presence of the signs  $\bar{\eta}_i = -1$ , we obtain representations of the form  $(-|\bar{p}, \bar{j})$  with  $\bar{p} = \bar{c}^2 p_1 + \bar{s}^2 p_2$ . Given the difference in sign, these representations can never coincide and we conclude that the sector  $\mathcal{H}_{++}$  does not contribute to the spectrum of the coset. In the same way one can exclude also the sectors  $\mathcal{H}_{--}$ ,  $\mathcal{H}_{\pm 0}$  and  $\mathcal{H}_{0\pm}$ .

The sectors that contribute to the spectrum are  $\mathcal{H}_{+-}$ ,  $\mathcal{H}_{-+}$  and  $\mathcal{H}_{00}$ . The operator content of the coset depends on the range of  $\alpha$  and  $\bar{\alpha}$  and we can summarise the result of the analysis in the following schematic way

$$\begin{aligned}
\alpha < \bar{\alpha} : & \quad [(\pm|p_1, j_1), (\mp|p_2, j_2); (\pm|p, j)] \\
\alpha = \bar{\alpha} : & \quad [(\pm|p_1, j_1), (\mp|p_2, j_2); (0|s, j)] \\
\alpha > \bar{\alpha} : & \quad [(\pm|p_1, j_1), (\mp|p_2, j_2); (\mp|p, j)] \\
\text{any } \alpha, \bar{\alpha} : & \quad [(0|s_1, j_1), (0|s_2, j_2); (0|s, j)] \quad .
\end{aligned} \tag{5.4}$$

Here the terms in the square brackets are the three labels used to identify the coset characters. Note that there is a drastic change in the type of coset representations that appear in the spectrum as we move across the line  $\alpha = \bar{\alpha}$  which corresponds to the singular geometries discussed in section 3.

We now solve the constraints  $\epsilon(X) + \bar{\epsilon}(X) = 0$  associated with the other generators using the coset decompositions derived in section 4.3.2. In this way we can determine the relations among the parameters of the various representations in (5.4). We perform this analysis separately for the  $\mathcal{H}_{\pm\mp}$  and the  $\mathcal{H}_{00}$  sectors.

*The sector  $\mathcal{H}_{\pm\mp}$ :* Let us begin with the case  $\mathcal{H}_{\pm\mp}$  for  $\alpha \neq \bar{\alpha}$ . We consider first the labels  $p$  and  $\bar{p}$ . On the left we have  $p = \tau(c^2 p_1 - s^2 p_2)$  and on the right  $\bar{p} = -\tau(\bar{c}^2 p_1 - \bar{s}^2 p_2)$  where  $\tau = +1$  for  $\alpha < \bar{\alpha}$  and  $\tau = -1$  for  $\alpha > \bar{\alpha}$ . This corresponds to a coset representation of the form  $[(\pm|p_1, j_1), (\mp|p_2, j_2); (\pm\tau|p, j)]$ . The equation  $p = \bar{p}$  which follows from the coset constraints can be solved for  $p_2$ , giving

$$p_2 = \frac{c^2 + \bar{c}^2}{s^2 + \bar{s}^2} p_1 =: r p_1 \quad , \tag{5.5}$$

and therefore

$$p = \tau(c^2 p_1 - s^2 p_2) = \frac{c^2 \bar{s}^2 - s^2 \bar{c}^2}{s^2 + \bar{s}^2} \tau p_1 = -\frac{\sin(\alpha - \bar{\alpha}) \sin(\alpha + \bar{\alpha})}{\sin^2 \alpha + \sin^2 \bar{\alpha}} \tau p_1 \quad . \tag{5.6}$$

which is always positive, as required by the consistency of our decomposition. Now we can determine the allowed values of  $j$ . To do so we have to solve the equation

$$j := j_1 + j_2 \pm \mu p_1 \mp (\lambda - \mu) p_2 \mp \tau n = -(j_1 + j_2) \mp \bar{\mu} p_1 \pm (\lambda - \bar{\mu}) p_2 \mp \tau \bar{n} \quad . \tag{5.7}$$

The integers  $n, \bar{n}$  arise from the tensor product decomposition in Eq. (4.28). Given concrete values for  $p_1, j_1, n$  and  $\bar{n}$ , this equation can always be solved for  $j_2$ , resulting in

$$\begin{aligned} j_2 &= -j_1 \pm \tau(n - \bar{n})/2 \mp [(\mu + \bar{\mu})(1 + r) - 2\lambda r] p_1/2 \\ j &= \mp \tau(n + \bar{n})/2 \pm (\mu - \bar{\mu})(1 + r)p_1/2 . \end{aligned} \quad (5.8)$$

Therefore when  $\alpha \neq \bar{\alpha}$  all the parameters in (5.4) can be expressed in terms of the data  $(p_1, j_1, n, \bar{n})$ .

When  $\alpha = \bar{\alpha}$ , from (5.5) it follows that  $p_2 = \cot^2 \alpha p_1$  and that the sector  $\mathcal{H}_{\pm\mp}$  decomposes into continuous representations. We then find the conditions

$$\begin{aligned} j &= j_1 + j_2 \pm \mu p_1 \mp (\lambda - \mu)p_2 \pmod{1} , \\ \bar{j} &= -(j_1 + j_2) \mp \bar{\mu} p_1 \pm (\lambda - \bar{\mu})p_2 \pmod{1} , \end{aligned} \quad (5.9)$$

with  $j = \bar{j}$  or  $j = 1 - \bar{j}$ . In the first case the equation can be solved by

$$\begin{aligned} j_2 &= -j_1 + n/2 - [(\mu + \bar{\mu})(1 + r) - 2\lambda r] p_1/2 , \\ j &= n/2 + (\mu - \bar{\mu})(1 + r)p_1/2 \pmod{1} , \end{aligned} \quad (5.10)$$

with  $n \in \mathbb{Z}$ . In the second case if  $\mu = \bar{\mu}$  there is no restriction on  $j_1$  and  $j_2$  and  $j$  is given by (5.9), while if  $\mu \neq \bar{\mu}$  there are solutions only when  $(\mu - \bar{\mu})(1 + r)p_1 \in \mathbb{Z}$ . Finally there is no restriction on the allowed range of  $s$  as one can see from Eq. (4.31).

*The sector  $\mathcal{H}_{00}$ :* In this sector we have the constraint

$$j = j_1 + j_2 \equiv -(j_1 + j_2) \pmod{\mathbb{Z}} . \quad (5.11)$$

This equation can be solved by  $j_2 = \Upsilon - j_1$  with  $\Upsilon \in \{1/2, 1\}$ , implying  $j = \Upsilon$ . The range of  $s$  follows from the tensor product of the continuous representations in (4.5) and from the action of the embeddings. The result is  $s_{\min} \leq s \leq s_{\max}$  where the upper and lower bound are given by

$$s_{\min} = \max(|cs_1 - ss_2|, |\bar{c}s_1 - \bar{s}s_2|) , \quad s_{\max} = \min(cs_1 + ss_2, \bar{c}s_1 + \bar{s}s_2) . \quad (5.12)$$

This concludes our discussion of the spectrum of cosets of type  $(--)$ .

## 5.2 The cosets of type $(+-)$

The whole discussion for this class of models mimics the one in the previous subsection. In particular, we again have to distinguish three cases. Depending on the relative value of the parameters  $\alpha$  and  $\bar{\alpha}$  we find the sectors

$$\begin{aligned} \alpha < \bar{\alpha} : & [(\pm|p_1, j_1), (\mp|p_2, j_2); (\pm|p, j)] \\ \alpha = \bar{\alpha} : & [(\pm|p_1, j_1), (0|s, j_2); (\pm|s, j)] \\ \alpha > \bar{\alpha} : & [(\pm|p_1, j_1), (\pm|p_2, j_2); (\pm|p, j)] \\ \text{any } \alpha, \bar{\alpha} : & [(0|s_1, j_1), (0|s_2, j_2); (0|s, j)] . \end{aligned} \quad (5.13)$$

Note that in this case different sectors  $\mathcal{H}_{\mu\nu}$  of the  $H_4 \times H_4$  model contribute for different values of the parameters  $\alpha$  and  $\bar{\alpha}$ . More precisely, besides the  $\mathcal{H}_{00}$  sector that is always in the spectrum, we have the  $\mathcal{H}_{\pm\mp}$  sectors when  $\alpha < \bar{\alpha}$ , the  $\mathcal{H}_{\pm\pm}$  sectors when  $\alpha < \bar{\alpha}$  and finally the  $\mathcal{H}_{\pm 0}$  sectors when  $\alpha = \bar{\alpha}$ . In order to complete the description we have to impose the constraints and derive the relations between the different parameters in (5.13).

*The sector  $\mathcal{H}_{\pm\pm}$ :* In the sector  $\mathcal{H}_{\pm\pm}$  one has  $p = c^2 p_1 + s^2 p_2$  and  $\bar{p} = \bar{c}^2 p_1 - \bar{s}^2 p_2$ . Solving the equation  $p = \bar{p}$  for  $p_2$  and  $p$  we obtain

$$p_2 = -\frac{c^2 - \bar{c}^2}{s^2 + \bar{s}^2} p_1 =: t p_2 \quad \text{and} \quad p = \frac{c^2 \bar{s}^2 + s^2 \bar{c}^2}{s^2 + \bar{s}^2} p_1 \quad . \quad (5.14)$$

This is indeed consistent with the requirement  $p_2 > 0$  as long as  $\alpha > \bar{\alpha}$ . Similarly we have to solve the equation

$$j := j_1 + j_2 \pm \mu p_1 \pm (\lambda - \mu) p_2 \pm n = j_1 - j_2 \pm \bar{\mu} p_1 \mp (\lambda - \bar{\mu}) p_2 \mp \bar{n} \quad (5.15)$$

for  $j_2$  and  $j$ . This yields

$$\begin{aligned} j_2 &= \mp (n + \bar{n})/2 \mp [(\mu - \bar{\mu}) - (\mu + \bar{\mu})t + 2\lambda t] p_1/2 \\ j &= j_1 \pm (n - \bar{n})/2 \pm [(\mu + \bar{\mu}) - (\mu - \bar{\mu})t] p_1/2 \end{aligned} \quad (5.16)$$

and completes the specification of the associated coset sector.

*The sector  $\mathcal{H}_{\pm\mp}$ :* In the next sector  $\mathcal{H}_{\pm\mp}$  one easily finds  $p = c^2 p_1 - s^2 p_2$  and  $\bar{p} = \bar{c}^2 p_1 + \bar{s}^2 p_2$ . The usual procedure of equating  $p$  and  $\bar{p}$  results in

$$p_2 = -t p_1 \quad \text{and} \quad p = \frac{c^2 \bar{s}^2 + s^2 \bar{c}^2}{s^2 + \bar{s}^2} p_1 \quad . \quad (5.17)$$

This time we see the consistency with the assumption  $\alpha < \bar{\alpha}$ . In addition to the previous equation we have to impose

$$j := j_1 + j_2 \pm \mu p_1 \mp (\lambda - \mu) p_2 \mp n = j_1 - j_2 \pm \bar{\mu} p_1 \pm (\lambda - \bar{\mu}) p_2 \pm \bar{n} \quad . \quad (5.18)$$

Hence we immediately conclude that

$$\begin{aligned} j_2 &= \pm (n + \bar{n})/2 \mp [(\mu - \bar{\mu}) - (\mu + \bar{\mu})t + 2\lambda t] p_1/2 \\ j &= j_1 \mp (n - \bar{n})/2 \pm [(\mu + \bar{\mu}) - (\mu - \bar{\mu})t] p_1/2 \quad . \end{aligned} \quad (5.19)$$

*The sector  $\mathcal{H}_{\pm 0}$ :* In the decomposition of the sectors  $\mathcal{H}_{\pm 0}$  one easily finds  $p = c^2 p_1$  and  $\bar{p} = \bar{c}^2 p_1$  and therefore these sectors can only arise when  $\alpha = \bar{\alpha}$ . The second constraint is

$$j := j_1 + j_2 \pm \mu p_1 + n = j_1 - j_2 \pm \bar{\mu} p_1 + \bar{n} \quad . \quad (5.20)$$

Solving this for  $j_2$  and plugging it back again results in

$$j_2 = -(n - \bar{n})/2 \mp (\mu - \bar{\mu}) p_1/2 \quad \text{and} \quad j = j_1 + (n + \bar{n})/2 \pm (\mu + \bar{\mu}) p_1/2 . \quad (5.21)$$

We also have to require that  $j_2$  lies in the interval  $[0, 1)$ . This restricts the parameters  $n$  and  $\bar{n}$ . For instance when  $\mu = \bar{\mu}$  this implies  $n = \bar{n}$  and  $j_2 = 0$  or  $n = \bar{n} + 1$  and  $j_2 = 1/2$ .

*The sector  $\mathcal{H}_{00}$ :* The discussion of the sector  $\mathcal{H}_{00}$  parallels the one in the previous subsection. The only difference is in the constraint

$$j := j_1 + j_2 = j_1 - j_2 \pmod{\mathbb{Z}} , \quad (5.22)$$

which has the solution

$$j_2 = \Upsilon \in \{0, 1/2\} \quad \text{and} \quad j = j_1 + \Upsilon \pmod{\mathbb{Z}} . \quad (5.23)$$

In addition  $s_{\min} \leq s \leq s_{\max}$  where  $s_{\min}$  and  $s_{\max}$  are given in Eq. (5.12).

### 5.3 The cosets of type $(++)$

We now determine the spectrum of the last class of models, the  $(++)$  cosets. When we considered their Lagrangian description in section 2 and 3 we found that the  $\sigma$ -model fields  $u_1$  and  $u_2$  satisfy the constraint

$$U(z, \bar{z}) := \nu_1^2 u_1(z, \bar{z}) + \nu_2^2 u_2(z, \bar{z}) = \rho , \quad (5.24)$$

with  $\rho \in \mathbb{R}$ . We need to find a way to impose this constraint on the spectrum of the original  $H_4 \times H_4$  WZW model. It is convenient to decompose the scalar field  $U(z, \bar{z})$  in its zero-mode, holomorphic and anti-holomorphic components

$$U(z, \bar{z}) = U_0 + U(z) + \bar{U}(\bar{z}) , \quad (5.25)$$

so that the previous constraint can be expressed in the form

$$U_0 = \rho , \quad \partial U(z) = 0 , \quad \bar{\partial} \bar{U}(\bar{z}) = 0 . \quad (5.26)$$

Only the condition on the zero-mode  $U_0$  correlates the left and right Hilbert spaces of the original WZW model, while the other two conditions can be imposed independently in the two Hilbert spaces. In fact the derivatives of  $U$  coincide with a linear combination of the affine currents. More precisely  $\partial U(z) = -\epsilon(K(z))$  and  $\bar{\partial} \bar{U}(\bar{z}) = \bar{\epsilon}(\bar{K}(\bar{z}))$  where  $K(z)$  and  $\bar{K}(\bar{z})$  are the affine currents of the  $H_4$  subalgebra and  $\epsilon(K(z)) = \nu_1^2 K^{(1)}(z) + \nu_2^2 K^{(2)}(z)$ . For simplicity in this section we consider only the case  $\mu = \bar{\mu} = \lambda = 0$ , which together with  $\alpha = \bar{\alpha}$  implies  $\epsilon = \bar{\epsilon}$ .

The most efficient way to impose the constraint  $\epsilon(K(z)) = 0$  in the holomorphic sector is to introduce ghost fields  $(b, c)$  with stress energy tensor  $T_{gh} = -b\partial c$  and conformal dimensions  $h_b = 1$  and  $h_c = 0$ . We then identify the physical Hilbert space with the cohomology of the BRST charge

$$Q = \oint \frac{dz}{2\pi i} c(z) \epsilon(K(z)) . \quad (5.27)$$

As a result, the physical states are the states  $|\psi\rangle$  in the Hilbert space of the original WZW model that satisfy the conditions

$$\epsilon(K_{-n})|\psi\rangle = 0, \quad n \geq 0 \quad \epsilon(J_{-n})|\psi\rangle = 0, \quad n \geq 1, \quad (5.28)$$

where  $\epsilon(J(z)) = J^{(1)}(z) + J^{(2)}(z)$ . We proceed in exactly the same way in the antiholomorphic sector, introducing ghost fields  $(\bar{b}, \bar{c})$  and a BRST charge  $\bar{Q}$ . This implies for the right modes of the currents precisely the same conditions satisfied by the left modes. Finally the constraint  $U_0 = \rho$  leads to an additional condition for the physical states

$$(\epsilon(J_0) - \bar{\epsilon}(\bar{J}_0))|\psi\rangle = 0, \quad (5.29)$$

since  $U_0$  and  $\epsilon(J_0) - \bar{\epsilon}(\bar{J}_0)$  form a couple of canonical variables. The constraints (5.28) can be solved only in the  $\mathcal{H}_{+-}$ ,  $\mathcal{H}_{-+}$  and  $\mathcal{H}_{00}$  sectors of the original WZW model and therefore the spectrum is given by

$$\begin{aligned} & [(\pm|p_1, j_1), (\mp|p_2, j_2); (0|s, j)] \\ & [(0|s_1, j_1), (0|s_2, j_2); (0|s, j)] \end{aligned} \quad (5.30)$$

with  $p_2 = \cot^2 \alpha p_1$ . Note that if we had followed exactly the same approach as in the previous subsections and imposed the constraint (5.2) with  $X = K_0$ , we would have reached the conclusion that every sector  $\mathcal{H}_{\mu\nu}$  contributes to the spectrum with no restrictions on the labels  $p_1$  and  $p_2$ .

We still have to require the invariance of the physical spectrum with respect to the residual gauge transformations generated by the modes of the affine currents  $P_i(z)$ ,  $i = 1, 2$  and by the zero mode of the current  $\epsilon(J(z)) + \bar{\epsilon}(\bar{J}(\bar{z}))$ . This is equivalent to the requirement that the constraints in (5.2) are satisfied for this restricted set of generators.

Before imposing these conditions, we would like to make a few comments about the energy-momentum tensor of the coset model. Let us start with the energy-momentum tensor of the  $H_4 \times H_4$  model and of the ghost fields

$$T_{\text{tot}}(z) = T_{H_4}^{(1)}(z) + T_{H_4}^{(2)}(z) + T_{gh}(z). \quad (5.31)$$

This energy-momentum tensor has central charge  $c = 6$  and since

$$\{Q, \epsilon(J(z))b(z)\} = T_{gh}(z) + \epsilon(J(z))\epsilon(K(z)), \quad (5.32)$$

when restricted to the cohomology it can be written as

$$T_{\text{tot}}(z) \sim T_{H_4}^{(1)}(z) + T_{H_4}^{(2)}(z) - \epsilon(J(z))\epsilon(K(z)). \quad (5.33)$$

The central charge is further reduced to  $c = 4$  by the gauging of the affine currents  $P_i(z)$ ,  $i = 1, 2$ . This can be accomplished subtracting from (5.33) the  $c = 2$  energy-momentum tensor

$$T_r(z) = \frac{1}{2} [\epsilon(P_1(z))^2 + \epsilon(P_2(z))^2 + \epsilon(K(z))^2]. \quad (5.34)$$



The result is that in the physical subspace the stress-energy tensor of this class of models coincides with the one given by the standard coset construction [9, 10]

$$T_{(++)} = T_{H_4}^{(1)} + T_{H_4}^{(2)} - T_{\epsilon(H_4)} . \quad (5.35)$$

Let us now determine the relations among the labels of the representations in (5.30)

*The sector  $\mathcal{H}_{\pm\mp}$ :* This sector decomposes into continuous representations. Here we find the constraint

$$j = j_1 + j_2 \pmod{1} , \quad \bar{j} = j_1 + j_2 \pmod{1} , \quad (5.36)$$

with  $j = 0$  and  $\bar{j} = 0$ . The equation can be solved by

$$j_2 = -j_1 + n/2 , \quad j = n/2 \pmod{1} , \quad (5.37)$$

with  $n \in \mathbb{Z}$ .

*The sector  $\mathcal{H}_{00}$ :* In this sector we have the constraint  $j_1 + j_2 = 0 \pmod{1}$  with solution  $j_2 = 1 - j_1$  and  $j = 0$ . The range of  $s$  is  $|\nu_1 s_1 - \nu_2 s_2| \leq s \leq \nu_1 s_1 + \nu_2 s_2$ .

## 6. Conclusions

In this article we studied in detail the diagonal cosets of the Heisenberg group. They form a large and interesting class of curved string theory backgrounds and provide an example of a coset construction where both the numerator and the denominator group are non-compact and non-abelian. We classified all possible diagonal cosets and derived the metric, dilaton and antisymmetric tensor that specify the corresponding curved space-times. We found three classes of models, thereby generalising the results of [33]. The resulting models are all particular examples of a family of string backgrounds related to plane waves by abelian T-duality transformations [42].

Our three classes of models are labeled by two signs and are called  $(++)$ ,  $(+-)$  and  $(--)$ , respectively. A minus sign means that the right embedding includes the outer automorphism  $\Omega_{-1}^{(0,1)}$  acting on the corresponding  $H_4$  factors in  $H_4 \times H_4$ . All three classes depend on several continuous parameters, in particular on two angles  $\alpha$ ,  $\bar{\alpha}$ . The models considered in [33] correspond to the symmetric case  $\alpha = \bar{\alpha}$  and describe singular space-times. The general models with  $\alpha \neq \bar{\alpha}$  introduced in this paper are generically non-singular and interpolate between singular and non-singular space-times.

We also derived the spectrum of the diagonal cosets using conformal field theory techniques. In order to achieve this we first studied the decomposition of the tensor products of affine  $H_4$  representations with respect to the embedded  $H_4$  algebra. We described a method to derive the branching functions for representations with zero spectral flow which avoided the problems that would have arisen if we had tried to use the standard technique of character decompositions. We expect that the approach followed in this paper will be useful for the treatment of other non-compact and non-abelian cosets such as  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})/SL(2, \mathbb{R})$ .

Several aspects of the models discussed in this paper deserve further investigation. First of all we would like to find a rigorous method to study the decomposition of the tensor products of spectral flow representations, for which we could only present partial results. In this way one would obtain a complete description of the spectrum of the diagonal cosets and could also study their one-loop partition functions.

Following the work of Antoniadis and Obers [33], we would also like to investigate the effect of T-duality transformations on the space of the diagonal cosets of the Heisenberg group and in particular to consider their action on the spectrum of the models described in this paper. In this way one could establish under which conditions the duality symmetries of the curved backgrounds reflect exact symmetry of the underlying coset conformal field theories.

In this paper we have seen in a particular example that when the coset construction is applied to a non-semisimple group there is a significant amount of freedom, to the point that the resulting coset conformal field theories usually come in continuous families. This should be a generic feature of non-semisimple cosets and it would be worth exploiting it to construct other models of this type. For instance it would be very interesting to find other four-dimensional models that could be identified with families of curved string backgrounds, as it is the case for the diagonal cosets of the Heisenberg group. A possible class of this type are the cosets  $(H_4)^{n+1}/(H_4)^n$ .

Finally, another valuable line of research would be the study of string interactions in these backgrounds. With the information gathered in this paper it should be possible to construct correlation functions of coset primary fields. In order to determine the three- and four-point couplings one should use the structure constants of the  $H_4$  WZW model derived in [25] and properly generalise the analysis performed recently in [2] for the abelian cosets.

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## A. Geometries for the fully asymmetric cosets of type $(--)$ and $(+-)$

For the sake of completeness we reserved this appendix to summarise the geometric data that arise for the fully asymmetric versions of the cosets of type  $(--)$  and  $(+-)$ .

### A.1 Cosets of type $(--)$

Before we spell out the background data for the fully asymmetric coset of type  $(--)$  it is convenient to define the quantity

$$r(u) = 2c^2\bar{c}^2 - c^2 - \bar{c}^2 + 2c\bar{c}s\bar{s}\cos(2u) \quad , \quad (\text{A.1})$$

where we used the notation introduced in Eq. (5.3). The function  $r(u)$  is a generalisation of the functions  $r^\pm(u)$  and  $R^\pm(u)$  defined in the main text. It completely specifies the dilation

$$\Phi = -\frac{1}{2}\ln(r(u)) \quad . \quad (\text{A.2})$$

Employing the previous definition one can also simplify the metric which assumes the form

$$\begin{aligned} ds^2 = & 4dudv + \frac{1}{r(u)} \left\{ A_{xx}dx^2 - 2A_{xy}dxdy + A_{yy}dy^2 + 2(B_x x + B_y y) \sin(u) dx du \right. \\ & \left. + 2(C_x x + C_y y) \sin(u) dy du + (D_{xx}x^2 + D_{xy}xy + D_{yy}y^2) du^2 \right\} + D du^2 \quad . \end{aligned} \quad (\text{A.3})$$

In order to keep this expression short we list the auxiliary functions appearing in this expression separately. The first group of functions is given by

$$\begin{aligned} A_{xx} &= \left\{ c^2 + \bar{c}^2 - 2c^2\bar{c}^2 + 2c\bar{c}s\bar{s} - 2 - 2(c\bar{c} + s\bar{s} + 2c\bar{c}s\bar{s}\cos(u))\cos(u) \right\} \\ A_{xy} &= \left\{ c^2 + 3\bar{c}^2 - 2 + 2(c\bar{c}^3 - s\bar{s}^3)\cos(u) \right\} \\ A_{yy} &= -\left\{ 2 + 4\bar{c}^4 - 5\bar{c}^2 + 2c\bar{c}s\bar{s} + c^2(2\bar{c}^2 - 1) \right. \\ & \quad \left. + 2(2\bar{c}^2 - 1)(c\bar{c} - s\bar{s})\cos(u) - 4c\bar{c}s\bar{s}\cos^2(u) \right\} \quad . \end{aligned} \quad (\text{A.4})$$

The second and the third group have the following form,

$$\begin{aligned} B_x &= (c^2 + \bar{c}^2 - 1)(c\bar{c} - s\bar{s}) \\ B_y &= \left\{ c\bar{c}(-2\bar{c}^4 + 2\bar{c}^2 + c^2) + s\bar{s}(2\bar{c}^4 - 2\bar{c}^2 + c^2 - 1) + 2c\bar{c}s\bar{s}(2c^2 - 1)\cos(u) \right\} \\ C_x &= \left\{ \left(1 - 2\bar{c}^4 + \bar{c}^2 + c^2(2\bar{c}^2 - 1)\right)(c\bar{c} + s\bar{s}) + 4c\bar{c}s\bar{s}(2\bar{c}^4 + (2c^2 - 3)\bar{c}^2 - c^2)\cos(u) \right\} \\ C_y &= (2\bar{c}^2 - 1)\left\{ c^2(\bar{c}c - s\bar{s}) + s\bar{s} - 2c\bar{c}s\bar{s}\cos(u) \right\} \quad . \end{aligned} \quad (\text{A.5})$$

Finally we have

$$\begin{aligned}
D_{xx} &= -\frac{1}{4} (2c^2 - 1) (2\bar{c}^2 - 1) \left\{ 2 + 2c^2\bar{c}^2 - c^2 - \bar{c}^2 - 2c\bar{c}s\bar{s} \right. \\
&\quad \left. - 2(c\bar{c} + s\bar{s}) \cos(u) + 4c\bar{c}s\bar{s} \cos^2(u) \right\} \\
D_{xy} &= \frac{1 - 2c^2}{2} \left\{ 2 - 2c^4 + \bar{c}^2 + c^2 (2\bar{c}^2 - 1) - 2 \left( c\bar{c}^3 (3 - 2\bar{c}^2) + s\bar{s}^3 (3 - 2\bar{s}^2) \right) \cos(u) \right\} \\
D_{yy} &= -\frac{1 - 2c^2}{4} \left\{ 2 + 8\bar{c}^6 + 12c^2\bar{c}^4 - 18\bar{c}^4 - 12c^2\bar{c}^2 - c^2 + 7\bar{c}^2 + 2c\bar{c}s\bar{s} (1 - 2\bar{c}^2) \right. \\
&\quad \left. + 2(c\bar{c} - s\bar{s}) (4\bar{c}^2\bar{s}^2 + 1) \cos(u) - 4c\bar{c}s\bar{s} (1 - 2\bar{c}^2) \cos^2(u) \right\} \\
D &= 2\Lambda \left\{ \lambda (4c^2\bar{c}^2 - c^2 - \bar{c}^2) + \mu (1 - 2\bar{c}^2) + \bar{\mu} (1 - 2c^2) \right\} . \tag{A.6}
\end{aligned}$$

For a general choice of parameters the background also support a three-form flux. Using the same conventions as above it may be expressed as

$$\begin{aligned}
H &= \frac{2c\bar{s}}{r(u)^2} (c^2 - \bar{c}^2) \left\{ \bar{c}s (4\bar{c}^2 c^2 - c^2 - \bar{c}^2 + 2s^2) + c\bar{s} (4\bar{c}^2 c^2 - c^2 - \bar{c}^2 + 2\bar{s}^2) \right. \\
&\quad \left. + 2cs (2\bar{c}^2 c^2 + 2 - c^2 - \bar{c}^2 - 2c\bar{c}s\bar{s}) \cos(u) + 4c\bar{c}s\bar{s} (\bar{c}s + c\bar{s}) \cos^2(u) \right. \\
&\quad \left. + 8c^2\bar{c}s^2\bar{s} \cos^3(u) \right\} \sin(u) dx \wedge dy \wedge du . \tag{A.7}
\end{aligned}$$

The background described here may be cast into the standard form (3.1) of a gravitational plane wave by a suitable change of coordinates. Since to find the explicit coordinate transformation one has to follow a rather cumbersome procedure, we refrain from doing so.

## A.2 Cosets of type (+-)

Let us turn our attention to the third type of cosets of class (+-) now. This time a crucial ingredient of the metric and the other background fields are the following functions

$$\begin{aligned}
r_c^\pm(u) &= c^2 + \bar{c}^2 \pm 2c\bar{c} \cos(u) \\
r_s^\pm(u) &= s^2 + \bar{s}^2 \pm 2s\bar{s} \cos(u) . \tag{A.8}
\end{aligned}$$

Here we used the same abbreviations as in (5.3). Like before this abbreviation is useful in order to express the dilaton which is given by

$$\Phi = -\frac{1}{2} \ln(r_c^-(u)) \tag{A.9}$$

The metric is also easily derived. Its shape resembles the one for the  $(--)$ -gauging and its explicit form is given by

$$\begin{aligned}
ds^2 = & \frac{2dudv}{s^2} + \frac{1}{r_c^-(u)} \left\{ A_{xx}dx^2 - 2A_{xy}dxdy + A_{yy}dy^2 + (B_x x + B_y y) \sin(u) dxdu \right. \\
& \left. + (C_x x + C_y y) \sin(u) dydu + (D_{xx}x^2 + D_{xy}xy + D_{yy}y^2) du^2 \right\} + D du^2 .
\end{aligned} \tag{A.10}$$

The main difference to the  $(--)$ -case can be found in the auxiliary functions needed to express eq. (A.10). For the first set of functions one obtains

$$\begin{aligned}
A_{xx} &= r_s^+(u) \\
A_{xy} &= \left\{ 2 + (2\bar{c}^2 - 1)c^2 + 2c\bar{c}s\bar{s} - 3\bar{c}^2 - 2\bar{s}^2(c\bar{c} - s\bar{s}) \cos(u) - 4c\bar{c}s\bar{s} \cos^2(u) \right\} \\
A_{yy} &= -\left\{ -2 + 8c^2\bar{c}^4 - 4\bar{c}^4 - 8c^2\bar{c}^2 + 5\bar{c}^2 + 4c\bar{c}s\bar{s}(2\bar{c}^2 - 1) + c^2 \right. \\
&\quad \left. - 2\left( s\bar{s}(1 - 2\bar{c}^2) - 2c\bar{c}\bar{s}^2 \right) \cos(u) - 8c\bar{c}s\bar{s}(2\bar{c}^2 - 1) \cos^2(u) \right\} .
\end{aligned} \tag{A.11}$$

The second set is given by

$$\begin{aligned}
B_x &= \frac{1}{s^2} \left\{ (1 - 2\bar{c}^2)c^3\bar{c} + (3 - 2\bar{c}^2)s\bar{s}\bar{c}^2 + \bar{c}^3c + s\bar{s}(\bar{c}^2 - 2) \right\} \\
B_y &= \frac{1}{s^2\bar{s}^2} \left\{ \left( -4\bar{c}^4 + 7\bar{c}^2 + c^2(4\bar{c}^4 - 8\bar{c}^2 + 3) - 2 \right) (c\bar{c} - s\bar{s}) + 4c\bar{c}s\bar{s}^3(2c^2 - 1) \cos(u) \right\} \\
C_x &= -\frac{1}{s^2\bar{s}^2} \left\{ \left( -2\bar{c}^4 + 3\bar{c}^2 + c^2(4\bar{c}^4 - 6\bar{c}^2 + 3) - 2 \right) (c\bar{c} + s\bar{s}) + 4c\bar{c}s^3\bar{s} \cos(u) \right\} \\
C_y &= -\frac{1}{s^2\bar{s}^2} \left\{ c\bar{c}(2\bar{c}^4 - 3\bar{c}^2 + c^2) + s\bar{s} \left( 2 + 6\bar{c}^4 - 9\bar{c}^2 + c^2(-8\bar{s}^4 + 4\bar{s}^2 + 1) \right) \right. \\
&\quad \left. + 4c\bar{c}s\bar{s} \left( -6\bar{c}^4 + 9\bar{c}^2 + c^2(8\bar{c}^4 - 12\bar{c}^2 + 3) - 2 \right) \cos(u) \right\} .
\end{aligned} \tag{A.12}$$

Finally, the remaining ones assume the form

$$\begin{aligned}
D_{xx} &= \frac{(1 - 2c^2) r_s^-(u)}{4s^2 \bar{s}^2} \\
D_{xy} &= -\frac{1 - 2c^2}{2s^2} \left\{ 2 + 4\bar{c}^4 - 7\bar{c}^2 - 2c\bar{c}s\bar{s} + c^2(2\bar{c}^2 - 1) \right. \\
&\quad \left. - 2(2\bar{c}^4 - 3\bar{c}^2 + 1)(c\bar{c} + s\bar{s}) \cos(u) + 4c\bar{c}s\bar{s} \cos^2(u) \right\} \\
D_{yy} &= \frac{1 - 2c^2}{4s^2} \left\{ 2 - c^2 - 13\bar{c}^2 + 16c^2\bar{c}^6 - 8\bar{c}^6 - 32c^2\bar{c}^4 + 20\bar{c}^4 + 16c^2\bar{c}^2 \right. \\
&\quad \left. - 4c\bar{c}s\bar{s}(4\bar{c}^4 - 6\bar{c}^2 + 1) - 2(2c\bar{c}(2\bar{c}^4 - 3\bar{c}^2 + 1) + s\bar{s}(4\bar{c}^4 - 6\bar{c}^2 + 1)) \cos(u) \right. \\
&\quad \left. + 8c\bar{c}s\bar{s}(4\bar{c}^4 - 6\bar{c}^2 + 1) \cos^2(u) \right\} \\
D &= \frac{\Lambda}{s^2 \bar{s}^2} \left\{ \lambda(2s^2\bar{c}^2 - c^2 - \bar{c}^2) + \mu - \bar{\mu}(1 - 2c^2) \right\} . \tag{A.13}
\end{aligned}$$

The gauged WZW model also comes with a non-trivial three-form flux which is needed in order to ensure conformal invariance. A straightforward calculation yields

$$H = \frac{dx \wedge dy \wedge du}{s^2 \bar{s}^2 r_c^-(u)^2} (c^2 - \bar{c}^2) (2 + 2c^2\bar{c}^2 - r_c^+(u)) (c\bar{c}\bar{s}^2 - \bar{c}^2 s\bar{s} + 2c\bar{c}s\bar{s} \cos(u)) \sin(u) . \tag{A.14}$$

## B. Tensor products of $H_4$ -representations

In this appendix we summarise a few tensor products of finite and infinite dimensional representations of the Heisenberg Lie algebra  $H_4$ . We shall show that whenever the generator  $K$  vanishes, reducible but indecomposable representations can appear in the tensor product.

### B.1 Adjoint times adjoint

The adjoint representation of  $H_4$  mirrors the non-semi-simplicity of the Lie algebra. Its structure may be read off from the composition series

$$\text{ad} : \quad \begin{array}{ccc} & [1] & \\ \nearrow & & \searrow \\ [0] & & [0] \\ \searrow & & \nearrow \\ & [-1] & \end{array} . \tag{B.1}$$

In this diagram  $[j]$  denotes a one-dimensional representation on which  $-iJ$  acts as the scalar  $j$  while  $K$  acts trivially. To the right we find an irreducible invariant subspace  $[0]$ , given by the span of  $K$ . If we divide out this subspace we find two new invariant subspaces,  $[1]$  and  $[-1]$ , represented by  $P^\pm$ . Taking again the corresponding quotient we finally end up with a second  $[0]$ , the span of  $J$ . Similar diagrams will be used for the more complicated representations discussed below.

We are interested in (symmetrised) tensor products of the adjoint representation with itself since they are relevant for the construction of the affine modules. The main contribution to  $\text{ad} \otimes \text{ad}$  is schematically given by

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & P^+ \otimes P^+ & & \\
 & \nearrow & & \searrow & \\
 J \otimes P^+ + P^+ \otimes J & & & & K \otimes P^+ + P^+ \otimes K \\
 \nearrow & & \dots & & \searrow \\
 J \otimes J & & & & K \otimes K , \\
 \searrow & & & & \\
 J \otimes P^- + P^- \otimes J & & & & K \otimes P^- + P^- \otimes K \\
 \searrow & & & & \nearrow \\
 & & P^- \otimes P^- & & 
 \end{array}
 \end{array} \tag{B.2}$$

which is a nine-dimensional indecomposable representation. It is part of an infinite series of indecomposable representations which arise in higher tensor products of the adjoint representation with itself. In fact if we start with the state  $J \otimes J \otimes \dots \otimes J$  on the left hand side, we expect to find a representation of dimension  $n^2$  whose schematic description is

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & [n] & & \\
 & \nearrow & & \searrow & \\
 \dots & & & & \dots \\
 \nearrow & & \dots & & \searrow \\
 [0] & & & & [0] \\
 \searrow & & & & \nearrow \\
 \dots & & & & \dots \\
 & & [-n] & & 
 \end{array}
 \cong
 \begin{array}{ccccc}
 & & (P^+)^{\otimes n} & & \\
 & \nearrow & & \searrow & \\
 \dots & & & & \dots \\
 \nearrow & & J^{\otimes n} & & \searrow \\
 & & & & K^{\otimes n} \\
 \searrow & & & & \nearrow \\
 \dots & & & & \dots \\
 & & (P^-)^{\otimes n} & & 
 \end{array}
 .
 \end{array} \tag{B.3}$$

The product  $\text{ad} \otimes \text{ad}$  also contains a singlet given by

$$[0] \cong 2(K \otimes J + J \otimes K) + (P^+ \otimes P^- + P^- \otimes P^+) , \tag{B.4}$$

which cannot be reached from any other state. The remaining six vectors belong to the antisymmetric part of the tensor product  $\text{ad} \otimes \text{ad}$ . They form two three-dimensional indecomposable representations which have the structure

$$\begin{array}{c}
 \begin{array}{ccc}
 & & [1] \\
 & \nearrow & \\
 & [0] & \\
 \nearrow & & \\
 [-1] & & 
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 [1] & & \\
 \searrow & & \\
 [0] & & \\
 \searrow & & \\
 & & [-1]
 \end{array}
 .
 \end{array} \tag{B.5}$$

## B.2 Continuous times adjoint

We would like to show that the tensor product  $(0|s, j) \otimes \text{ad}$  is indecomposable. This observation is not particularly surprising since the generator  $K$  acts trivially in both constituents and consequently on the whole module. Nevertheless the statement is non-trivial and has to be checked thoroughly. We will argue that the Casimir operator is not diagonalizable on the tensor product, thus proving our assertion.

Let us consider the four-dimensional subspace of vectors with  $-iJ = j + n$  ( $n \in \mathbb{Z}$ ). Denote by  $|v\rangle$  the vector in  $(0|s, j)$  with  $-iJ|v\rangle = (j + n)|v\rangle$ . A convenient basis is then

given by the vectors

$$v_1 = P^+|v\rangle \otimes P^- + P^-|v\rangle \otimes P^+ \quad v_2 = -4s^2|v\rangle \otimes K \quad (\text{B.6})$$

$$v_3 = -|v\rangle \otimes K - i(P^+|v\rangle \otimes P^- - P^-|v\rangle \otimes P^+) \quad v_4 = |v\rangle \otimes J \quad (\text{B.7})$$

The total quadratic Casimir may be expressed as

$$\begin{aligned} C &= \frac{1}{2}(P^+P^- + P^-P^+) + 2JK \\ &= \frac{1}{2}[(P_1^+ + P_2^+)(P_1^- + P_2^-) + (P_1^- + P_2^-)(P_1^+ + P_2^+)] + 2(J_1 + J_2)(K_1 + K_2) \quad (\text{B.8}) \\ &= C_1 + C_2 + P_1^+P_2^- + P_1^-P_2^+ + 2J_1K_2 + 2K_1J_2 \end{aligned}$$

in terms of the Casimirs and of the generators of the individual algebras. For the vectors above we have  $C_1 = s^2$  and  $K_1 = K_2 = 0$ . This simplifies the calculations considerably and leads to the following matrix form,

$$C = \begin{pmatrix} s^2 & 0 & 0 & 0 \\ 0 & s^2 & 1 & 0 \\ 0 & 0 & s^2 & 1 \\ 0 & 0 & 0 & s^2 \end{pmatrix} . \quad (\text{B.9})$$

We thus proved that the affine continuous representation is not completely reducible with respect to its horizontal subalgebra. Similar indecomposable representations will appear on higher energy levels but we leave the complete analysis for future work. Note that the occurrence of indecomposable representations in the CFT should not affect the string theory spectrum, since the states created by the negative modes of the currents  $J$  and  $K$  are not physical, i.e. they are removed by the Virasoro constraints.

As we said indecomposable representations only appear when  $K = 0$ , so we expect that the tensor product  $(+|p, j) \otimes \text{ad}$  (with  $p \neq 0$ ) is completely reducible. It is easy to prove that in fact

$$(+|p, j) \otimes \text{ad} = (+|p, j+1) \oplus 2(+|p, j) \oplus (+|p, j-1) \quad (\text{B.10})$$

To see this let us assume that the infinite dimensional discrete representation is generated by a vector  $|v\rangle$  with

$$P^+|v\rangle = 0 \quad K|v\rangle = ip|v\rangle \quad J|v\rangle = ij|v\rangle \quad (\text{B.11})$$

Then it is not difficult to find four highest weight, none of which leads to non-trivial invariant subspaces.<sup>1</sup> The corresponding highest weight vectors read

$$|v\rangle \otimes P^+, \quad |v\rangle \otimes K, \quad P^-|v\rangle \otimes P^+ + 2pi|v\rangle \otimes J, \quad P^-|v\rangle \otimes K - ip|v\rangle \otimes P^- \quad (\text{B.12})$$

For convenience we ordered the highest weight states by their eigenvalues with respect to  $-iJ$ . In the given order the latter read  $j+1, j, j$  and  $j-1$ .

---

<sup>1</sup>Note that in this respect  $H_4$  differs from the case of  $AdS_3$  or  $SL(2, \mathbb{R})$  where representations arise that are not fully reducible, e.g. in the tensor product  $(+, 1) \otimes 1$ .



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