Cohomological gauge theory, quiver matrix models and Donaldson–Thomas theory

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Abstract

We study the relation between Donaldson–Thomas theory of Calabi–Yau threefolds and a six-dimensional topological Yang–Mills theory. Our main example is the topological $U(N)$ gauge theory on flat space in its Coulomb branch. To evaluate its partition function we use equivariant localization techniques on its noncommutative deformation. As a result the gauge theory localizes on noncommutative instantons which can be classified in terms of N-coloured three-dimensional Young diagrams. We give to these noncommutative instantons a geometrical description in terms of certain stable framed coherent sheaves on projective space by using a higher-dimensional generalization of the ADHM formalism. From this formalism we construct a topological matrix quantum mechanics which computes an index of BPS states and provides an alternative approach to the six-dimensional gauge theory.

Contents

1 Introduction

Topological field and string theories that arise form a topological twist of a physical model capture the BPS sector of their physical counterparts. They have been used successfully in the last decades to compute certain classes of nonperturbative effects, thereby considerably improving our understanding of the quantum world. Topological theories also have a deep relation with mathematics as they relate BPS quantities to geometrical invariants of the underlying manifold. For example, topological string theory is equivalent to certain enumerative problems for Calabi–Yau threefolds and it computes invariants such as the Gopakumar–Vafa, Gromov–Witten and Donaldson–Thomas invariants. These invariants play a key role in the mirror symmetry conjecture which is by now one of the best understood examples of duality in string theory (see [1] for a review). On the other hand, these invariants enter directly in the computation of the entropy of a class of supersymmetric black holes that arises in string theory compactifications on a Calabi–Yau manifold via the Ooguri–Strominger–Vafa (OSV) conjecture and provide some computational control on the enumeration of quantum gravity microstates [2].

Donaldson–Thomas invariants count bound states of D0–D2 branes with a single D6-brane wrapping a Calabi–Yau threefold. The geometrical object that describes this configuration is called an ideal sheaf and the Donaldson–Thomas invariants can be roughly thought of as "volumes" of the moduli spaces of ideal sheaves. When the Calabi–Yau space is toric the topological string theory has a reformulation in terms of the classical statistical mechanics of a melting crystal [3]. In this setting the Donaldson–Thomas invariants enumerate atomic configurations in the melting process. Just like the Gopakumar–Vafa invariants, Donaldson–Thomas theory computes certain F-terms in the low energy effective action of the string theory compactification [4]–[6]. Indeed, it is conjectured that the Donaldson–Thomas invariants are equivalent to the Gopakumar–Vafa and Gromov–Witten invariants as they arise from different expansions of the same topological string amplitude. This conjecture is known to be true in the toric setting [7]. The melting crystal picture also has an interpretation as a sum over fluctuating Kähler geometries [8]. This sum can be explicitly rewritten as a path integral of an auxiliary six-dimensional topological gauge theory. This theory localizes on the moduli space of solutions of the Donaldson–Uhlenbeck–Yau equations which can be thought of as generalized higher-dimensional instantons. The proposal of [8] consists in identifying the instanton counting problem associated to this gauge theory with the Donaldson–Thomas enumeration of ideal sheaves.

Accordingly, Donaldson–Thomas theory is reduced to six-dimensional instanton calculus. To this end one can adopt the approach envisaged by Nekrasov in the four-dimensional case and apply the techniques of equivariant localization in topological field theories [9]. Roughly, Nekrasov's formalism consists of a noncommutative deformation of the gauge theory that resolves certain singularities of the instanton moduli space and a particular redefinition of the BRST charge that localizes the instanton measure on pointlike configurations. The BRST charge is now an equivariant differential with respect to the natural toric action on the $\mathbb{C}^2$ target space and as a result integration over the instanton collective coordinates is reduced to a sum over pointlike instantons via the equivariant localization formula. These ideas were applied to the six-dimensional gauge theory under consideration on a generic toric Calabi–Yau threefold in [8], with the main difference being the appearance of degenerate gauge field configurations which wrap rational curves. The gauge theory partition function can in this way be successfully matched with the melting crystal partition function.

The full picture is thus consistent, and gives strong support to the equivalence between the six-dimensional gauge theory on the D6-brane and Donaldson–Thomas theory. Nevertheless, a direct link is still lacking. The equivalence becomes clear only after the gauge theory partition function is matched with the topological vertex amplitude. It would be desirable to have a direct

understanding of Donaldson–Thomas theory in the gauge theoretic picture and in particular to investigate the ideal sheaf counting problem in terms of the gauge theory variables. Moreover, the fact that we are now dealing with a gauge theory poses two pressing questions. Firstly, it is natural to consider a higher rank generalization of the problem which corresponds to introducing an arbitrary number N of D6-branes in the picture. While this is conceptually simple the relation (if any) with the usual topological string theory is rather obscure. We expect that a resolution of this puzzle may shed some new light on the black hole microstate counting problem. Secondly, topological string amplitudes have modular transformation properties that allows one to carry the information about the enumerative invariants along the various regions of the Calabi–Yau moduli space. In particular, knowledge of the Gromov–Witten invariants at the large radius point is enough to compute them, say, at orbifold points [10]. One may wonder how this property is seen in Donaldson–Thomas theory given that the gauge theory is naturally defined at the large radius point.

The aim of this paper is to investigate in detail the aforementioned proposals and conjectures, and to make a first step towards the resolution of the outlined problems. In particular, we will explore the random partition combinatorics of the nonabelian theory and its formulation in terms of a topological vertex. Partial results in this direction were described in [11], as well as in [12]. We work mainly on flat space in the Coulomb branch of a $U(N)$ gauge theory. In this case we are able to set up a general formalism that we hope could be applied to more general settings. We attack the problem in two independent but intimately connected ways. As a start we define and evaluate the partition function of the noncommutative deformation of the gauge theory. The noncommutative gauge theory localizes onto a sum over critical points that are classified in terms of N-coloured three-dimensional Young diagrams. The fluctuation factor around each critical point has the form of a ratio of functional determinants that can be computed by direct evaluation. The result is a simple generalization of the MacMahon function which captures the counting of BPS states for a configuration of D6-branes widely separated in the transverse directions.

We give a purely geometric interpretation of these noncommutative instantons. Under a certain set of plausible assumptions we can relate the noncommutative gauge field configurations with the stable framed moduli space of certain coherent sheaves on $\mathbb{P}^3$, a compactification of the target space $\mathbb{C}^3$. The main technical tool is the use of Beilinson's spectral sequence to parametrize a coherent sheaf on $\mathbb{P}^3$ with a set of matrix equations. We show that a special class of such sheaves arise as the cohomology of certain nonlinear monads on $\mathbb{P}^3$. The matrix equations are naturally interpreted as a higher-dimensional generalization of the ADHM formalism. This construction provides a direct relationship between the gauge theory and Donaldson–Thomas theory, and should be compared to the recent construction by Diaconescu [13] for the local Donaldson–Thomas theory of curves in terms of ADHM quiver sheaves.

From this generalized ADHM language we construct a topological matrix quantum mechanics that dynamically describes the stable coherent sheaves. In the string theory language this model corresponds to the effective action on the gas of D0-branes that are bound to the D6-branes (in the presence of a suitable B-field) [14]. Equivalently, we may think of the matrix model as arising from the quantization of the collective coordinates around each instanton solution. In either way the matrix model recovers the classification of the critical instanton configurations in terms of N-coloured three-dimensional Young diagrams via the special properties of the linear maps in the generalized ADHM equations. The computation of the instanton fluctuation factors is reduced to the evaluation of an appropriate equivariant index associated with the topological matrix model. The results agree with the direct computation done in the original noncommutative gauge theory.

As an application of our formalism we compute the partition function of the rank N gauge theory in the Coulomb branch on a generic toric Calabi–Yau manifold. The result is the N-th power of the abelian partition function with an N dependent sign shift. This shift is consistent

with the expected shift in the topological string coupling constant when the amplitude is related to the D-brane charges at the attractor point of the BPS moduli space. In particular, our results should provide a testing ground for the OSV conjecture with many D6-branes. Although the gauge theory provides essentially the same Donaldson–Thomas invariants, the Gromov–Witten theory is new and involves an additional expansion parameter, the rank N of the gauge group.

It is instructive to compare at the outset with the analogous problem in four dimensions. Since the six-dimensional gauge theory we consider is maximally supersymmetric, it is tempting to believe that its dynamics are qualitatively similar to the well-studied $\mathcal{N} = 4$ supersymmetric $U(N)$ Yang-Mills theory in four dimensions which is a topological twist of the $\mathcal{N} = 2$ Seiberg–Witten theory. In that case the partition function is independent of the vevs of the scalar fields, and hence of the gauge symmetry breaking pattern, because it always computes the Euler characteristic of the (compactified) instanton moduli space. More precisely, for the $\mathcal{N} = 2$ Seiberg–Witten theory, which computes the low energy effective action, one can use holomorphy properties to argue that the prepotential is a generic solution [9], even when some of the Higgs field eigenvalues coincide. By using the equivalence between Donaldson–Witten theory and Seiberg–Witten theory [15], this shows (modulo subtleties associated to wall-crossing phenomena) that it is enough to localize the gauge theory onto the Cartan subgroup, and indeed a direct computation of Donaldson–Witten invariants for $SU(N)$ gauge group gives no newer information than the $U(1)^{N-1}$ Seiberg–Witten theory computation.

However, there is no reason to expect that the infrared dynamics of the six-dimensional gauge theory is similar. Moreover, at present it is not known what is the rigorous stability condition to place on higher-rank sheaves over three-dimensional Calabi–Yau manifolds in order to construct a well-behaved compactification of the moduli space. Due to our lack of understanding of this moduli space, there is no guarantee that the instanton configurations that we localize onto span the appropriate moduli space of nonabelian gauge field configurations. For the gauge theory on $\mathbb{C}^3$ we have good control over the instanton moduli space and an explicit construction (with the caveat that we have not rigorously defined the appropriate compactification of the instanton moduli space). We argue in the following that in this case the allowed sheaves are those provided by the noncommutative deformation, and give an explicit algebro-geometric description including appropriate stability conditions. For more general toric threefolds, our computations merely provide a localization of the full nonabelian Donaldson–Thomas theory onto those sheaves which are invariant under the equivariant action of the maximal torus of the gauge group. For more general symmetry breaking, the sheaves should carry a nontrivial framing (as proposed in [13]) and the fixed point locus need not consist of isolated points. Usage of the localization formulas in this case would require the much more complicated integration over degenerate fixed point submanifolds of moduli space weighted by the Euler class.

This paper is organized as follows. In Section 2 we review the geometrical setting of the six-dimensional gauge theory and its equivariant deformation. Section 3 is devoted to the non-commutative deformation of the gauge theory and the evaluation of the partition function on flat (noncommutative) space. Section 4 is devoted to geometrical aspects and contains the derivation of our ADHM-like formalism. This formalism is used in Section 5 to construct a topological matrix quantum mechanics that computes an index of BPS states. In Section 6 we compute the partition function of the rank N gauge theory in the Coulomb branch on a generic toric Calabi–Yau manifold. Section 7 contains some further discussion and applications of our results.

2   Topological $\mathcal{N}_T = 2$ Yang–Mills theory in six dimensions

In this section we will collect some known results concerning the six-dimensional gauge theory which captures the physics of the Kähler quantum foam. This theory has been studied in the

literature both as a cohomological field theory [16] and in the balanced formalism [17], i.e. with $\mathcal{N}_T = 2$ *topological* charges. In order to keep the discussion completely general we will consider the $U(N)$ gauge theory defined on a toric threefold, but for concrete applications we will often specify to an abelian $U(1)$ or $U(1)^N$ theory on $\mathbb{C}^3$. Some of the results reviewed here are also valid in the more general context of a generic Kähler manifold.

2.1 The cohomological gauge theory

The gauge theory in question can be defined through a topological twist of the maximally super-symmetric Yang–Mills theory in six dimensions. One starts with $\mathcal{N} = 1$ $U(N)$ supersymmetric Yang–Mills theory in ten dimensions and dimensionally reduces on a six-dimensional Kähler manifold X with $U(3)$ holonomy. After the twist the bosonic spectrum of the theory consists of a gauge field A_i, a complex Higgs field Φ and $(3,0)$-form $\rho = \rho^{3,0}$ along with their complex conjugates, all taking values in the adjoint representation of the $U(N)$ gauge group. In the fermionic sector we can twist the spin bundle with the canonical line bundle over X and obtain explicitly an isomorphism between fermions and differential forms [18]. The spectrum consists of a complex scalar η, one-forms $\psi^{1,0}$ and $\psi^{0,1}$, two-forms $\psi^{2,0}$ and $\psi^{0,2}$, and finally three-forms $\psi^{3,0}$ and $\psi^{0,3}$ for a total of 16 degrees of freedom matching those of the bosonic sector. The resulting supersymmetric gauge theory is cohomological and has two topological charges. It can therefore be studied as a balanced topological field theory as in [19]. The bosonic part of the action has the form

$$
\begin{aligned}
S \;=\;& \frac{1}{2} \int_X \mathrm{Tr} \left(\mathrm{d}_A \Phi \wedge *\mathrm{d}_A \overline{\Phi} + \left[\Phi, \overline{\Phi} \right]^2 + \left| F_A^{2,0} + \overline{\partial}_A^\dagger \rho \right|^2 + \left| F_A^{1,1} \right|^2 \right) \\
& + \frac{1}{2} \int_X \mathrm{Tr} \left(F_A \wedge F_A \wedge k_0 + \tfrac{\vartheta}{3} F_A \wedge F_A \wedge F_A \right) ,
\end{aligned}
\tag{2.1}
$$

where $\mathrm{d}_A = \mathrm{d} + \mathrm{i} \left[A, - \right]$ is the gauge-covariant derivative, $*$ is the Hodge operator with respect to the Kähler metric of X, $F_A = \mathrm{d}A + A \wedge A$ is the gauge field strength, k_0 is the background Kähler two-form of X, and ϑ is the six-dimensional theta-angle which will be identified later on with the topological string coupling g_s.

In [8] the $U(1)$ gauge theory has been given a suggestive interpretation as a Kähler quantum foam. One starts with a path integral which represents a sum over Kähler geometries with quantized Kähler class $k = k_0 + g_s F_A$. Expanding the Kähler gravity action $\int_X k \wedge k \wedge k$ and gauge fixing the residual symmetry gives precisely the $\mathcal{N}_T = 2$ topological gauge theory. In this interpretation it is essential that the gauge field curvature F_A is a representative of the Chern class of a complex line bundle over X. For higher rank gauge groups $U(N)$, $N > 1$ a characterization of the gauge theory as a gravitational theory is not known.

The gauge theory has a BRST symmetry and hence localizes onto the moduli space of solutions of the fixed point equations

$$
\begin{aligned}
F_A^{2,0} &= \overline{\partial}_A^\dagger \rho , \\
F_A^{1,1} \wedge k_0 \wedge k_0 + \left[\rho, \overline{\rho} \right] &= l \, k_0 \wedge k_0 \wedge k_0 , \\
\mathrm{d}_A \Phi &= 0 .
\end{aligned}
\tag{2.2}
$$

The right-hand side of the second equation is a quantum correction coming from the magnetic charge of the gauge bundle, where l is a constant. The solutions of these equations minimize the action (2.1) and we will therefore call them generalized instantons or just instantons. We will be interested in the set of minima where the field ρ is set to zero. This is always possible on a Calabi–Yau background, because of the uniqueness of the holomorphic three-form in that case. Then the

first two equations reduce to the Donaldson–Uhlenbeck–Yau (DUY) equations which are conditions of stability for holomorphic bundles over X with finite characteristic classes.

We will generically denote an appropriate compactification of the moduli space of solutions to the first two equations of (2.2) with $\mathcal{M}$, or with $\mathcal{M}(N, c_1, \mathrm{ch}_2, \mathrm{ch}_3)$ labelling the solutions by their rank and Chern classes when we want to be more explicit about their topology, or by $\mathcal{M}(X)$ when we want to stress the role of the underlying variety X on which the gauge theory is defined. We will also include in $\mathcal{M}$ those configurations that solve (2.2) where the gauge field is possibly singular on X. More precisely, we will not restrict attention to holomorphic bundles but in principle consider also coherent sheaves of $\mathcal{O}_X$-modules over X of rank N. It is not clear that this compactification always exists for a generic threefold as very little is known about these moduli spaces, and we may expect the singular loci to be intractable. This lack of a geometrical understanding is a major obstacle in carrying out the localization program. We will see in the rest of this paper how some progress can be made in specific situations.

2.2   Local geometry of the instanton moduli space

The moduli space $\mathcal{M}$ is a highly singular and badly behaved complex variety. When $N = 1$ and $c_1 = 0$ this problem has been cured in [7, 8], and with an appropriate compactification there is an isomorphism $\mathcal{M}(1, 0, \mathrm{ch}_2, \mathrm{ch}_3) \cong \mathcal{I}_k(X, \beta)$ with the moduli space of ideal sheaves over X with fixed two-homology class β and holomorphic Euler characteristic k as defined for example in [7]. This moduli space is also isomorphic to the Hilbert scheme of points and curves in X, $\mathsf{Hilb}^k(X, \beta)$. In the following we will keep the rank N arbitrary as we can formally consider a nonabelian gauge theory. In principle one could try to formulate the Donaldson–Thomas theory as a localization problem for a gauge theory in arbitrary rank. However, one should face the difficult technical problem of integrating over the appropriately compactified moduli space. We can hope to make computational progress in the Coulomb phase of the gauge theory where the gauge symmetry is completely broken down to the maximal torus $U(1)^N$ by the Higgs field vacuum expectation values and the moduli space essentially reduces to N copies of the Hilbert scheme where localization techniques can and have been successfully applied. A precise geometric characterization of $\mathcal{M}$ is not known and lies beyond the scope of this paper. In particular, integration over $\mathcal{M}$ is ill-defined and requires appropriate homological tools to be dealt with. For our practical purposes we will ignore these (important) issues and base our analysis on gauge theory techniques. It will be enough to know that for $N = 1$ a perfect obstruction theory can be developed and a virtual fundamental class of $\mathcal{M}$ has been defined [20]. In Section 4 we will explore a sheaf theoretical description of the moduli space $\mathcal{M}$ for $N \geq 1$ in the case $X = \mathbb{C}^3$.

From the gauge theory perspective we are dealing with holomorphic bundles and pairs (A, ρ). The first step in understanding the moduli space $\mathcal{M}$ is to characterize its local geometry. For this, we introduce the instanton deformation complex [8]

$$0 \longrightarrow \Omega^{0,0}(X, \mathrm{ad}\,\mathfrak{g}) \xrightarrow{\;C\;} \Omega^{0,1}(X, \mathrm{ad}\,\mathfrak{g}) \oplus \Omega^{0,3}(X, \mathrm{ad}\,\mathfrak{g}) \xrightarrow{\;D_A\;} \Omega^{0,2}(X, \mathrm{ad}\,\mathfrak{g}) \longrightarrow 0 \ , \qquad (2.3)$$

where $\Omega^{\bullet,\bullet}(X, \mathrm{ad}\,\mathfrak{g})$ denotes the bicomplex of complex differential forms taking values in the adjoint gauge bundle over X, and the maps C and D_A represent a linearized complexified gauge transformation and the linearization of the first equation in (2.2) respectively. A precise definition can be found in [8, 21]. This complex is elliptic and its first cohomology represents the holomorphic tangent space to $\mathcal{M}$ at a point (A, ρ), $T_{(A,\rho)}\mathcal{M}$. The degree zero cohomology represents reducible pairs (A, ρ), whereby the gauge field A is a reducible connection which we assume vanishes.

On the other hand, in general there is also a finite-dimensional second cohomology that measures obstructions and is customarily called the obstruction or normal bundle $\mathcal{N}$. It is associated with

the kernel of the conjugate operator $D_A^\dagger$. Since this is the operator that enters in the kinetic term for the antighost fields, the terminology "antighost bundle" is used for $\mathcal{N}$ in the physics literature as its fibres are spanned by the antighost zero modes. It is precisely this bundle which provides an "integration measure" on the moduli space. On general grounds the fermionic BRST symmetry localizes the partition function of the topological gauge theory onto the moduli space $\mathcal{M}$ (once a topological sector is chosen by fixing a critical point). There are nevertheless remaining fermionic terms to be integrated over. In particular, there is a four-fermion interaction schematically of the form $\mathcal{R}\,\psi\,\psi\,\psi\,\psi$ where ψ collectively denotes the Fermi fields. A careful analysis of the BRST transformations and of the on-shell action shows that this term produces an integral representative of the Euler characteristic of the antighost bundle over the moduli space [8, 19]. Equivalently, matching the Fermi zero modes with the fermionic path integral measure brings down the pfaffian of the curvature $\mathcal{R}$.

As the detailed analysis is quite involved, we will just denote this integral symbolically as

$$\int_{\mathcal{M}} \mathsf{e}(\mathcal{N}) \ . \tag{2.4}$$

It is difficult to give a precise definition of this integral and to evaluate it. For this one needs rather sophisticated tools such as a perfect obstruction theory and to deal with the relevant virtual fundamental class of $\mathcal{M}$. There is, however, one special case where we can make computational progress in the evaluation of (2.4) without developing this abstract formalism. This is the case when the ambient variety is $X = \mathbb{C}^3$. In this case one can use localization techniques to evaluate the integrals (2.4). Since β is necessarily trivial in this case (equivalently $\mathrm{ch}_2 = 0$), the relevant moduli space for $N = 1$ can be identified with the Hilbert scheme $\mathsf{Hilb}^k(\mathbb{C}^3) = (\mathbb{C}^3)^{[k]}$ consisting of zero-dimensional subschemes in $\mathbb{C}^3$. For $k > 3$ this moduli space generically contains branches of varying dimension and so is not even a manifold. However, the localization formulas of Section 2.3 below make sense nevertheless and we will use them to *define* the integrals (2.4).

2.3 The equivariant model

If the moduli space $\mathcal{M}$ were smooth and compact, then we could proceed to evaluate the integrals (2.4) by choosing an appropriate representative of the cohomology class $\mathsf{e}(\mathcal{N})$. However, this is not the case as moduli spaces of instantons suffer from non-compactness problems arising both from singularities where instantons shrink to zero size as well from the non-compactness of the ambient space X on which the gauge theory is defined. In field theoretical terms, we can think of divergences coming from the first problem as associated with small distances while the second problem reflects the need for an infrared regularization. The ultraviolet behaviour improves substantially by introducing a noncommutative deformation of the gauge theory that provides a natural compactification of the instanton moduli space. We will define and study the deformed instanton calculus in the ensuing sections, but for the time being we will assume that this problem has been solved.

A direct evaluation of the integrals (2.4) is still a formidable problem. A particularly powerful approach is to use equivariant localization. In our problem we will assume that the gauge theory is defined on a toric threefold X. It carries the action of a non-compact three-torus $\mathbb{T}^3$, and we further assume that this action lifts to the moduli space $\mathcal{M}$. Working $\mathbb{T}^3$-equivariantly means that we restrict our attention to critical gauge field configurations that are $\mathbb{T}^3$-invariant. A practical way to implement this scheme is to modify the gauge theory so that the BRST differential on the space of fields becomes an equivariant differential, so that an infinitesimal $\mathbb{T}^3$ rotation can be undone by a gauge transformation.

When $X = \mathbb{C}^3$, a physical realization of the equivariant modification with respect to the toric isometry is to put the gauge theory on the "Ω-background". For the present gauge theory this

procedure was developed in [8]. If we think of our cohomological gauge theory as arising from a topological twist of maximally supersymmetric Yang–Mills theory in six dimensions, then working equivariantly is equivalent to replacing the original scalar BRST operator Q used for the twist with a linear combination of the scalar and vector supercharges Q^i given by

$$Q_\Omega = Q + \epsilon_a \, \Omega_{ij}^a \, x^i \, Q^j \ . \tag{2.5}$$

Here ϵ_a, $a = 1, 2, 3$ are formal parameters of the $\mathbb{T}^3$ action and $\Omega^a = \Omega_{ij}^a \, x^j \, \frac{\partial}{\partial x_i}$ are vector fields which generate the $SO(6)$ rotational isometries of $\mathbb{R}^6 \cong \mathbb{C}^3$. We will restrict to the $U(3)$ holonomy subgroup that preserves the natural, translationally-invariant Kähler two-form of $\mathbb{C}^3$. Then the toric symmetry group $\mathbb{T}^3$ is the maximal torus of this $U(3)$. The result is a topological deformation of the gauge theory, augmenting the action by a Q_Ω-exact term, which depends explicitly on the vector fields Ω^a. Since the BRST charge Q_Ω is a linear combination of supercharges, perturbative bosonic and fermionic contributions cancel, and the partition function is saturated by instantons. The key point here is that, in the Coulomb phase, the Ω-deformation localizes the instanton measure onto point-like instanton configurations which are invariant under $\mathbb{T}^3$ rotations. The critical points of the deformed gauge theory action are thus isolated. Due to localization, the semiclassical approximation is exact and the full path integral reduces to a sum over contributions from isolated point-like instantons.

This deformation turns the evaluation of the integrals (2.4) into a problem that is both well-defined and computationally accessible by equivariant localization techniques. In addition, the Ω-background acts as an infrared regularization and the equivariant volumes of the instanton moduli spaces are finite. In the gauge theory action (2.1) the Ω-background modifies, among other things, the kinetic term for the Higgs field to

$$\mathrm{Tr} \left(\mathrm{d}_A \Phi - \imath_\Omega F_A \right) \wedge * \left(\mathrm{d}_A \overline{\Phi} - \imath_{\overline{\Omega}} F_A \right) \tag{2.6}$$

where $\imath_\Omega$ is contraction with the vector field that generates the toric isometries. This modifies the third equation of (2.2) to

$$\mathrm{d}_A \Phi = \imath_\Omega F_A \ , \tag{2.7}$$

which manifestly minimizes the action on the Ω-background. The Higgs potential and the fermionic terms are also affected by the Ω-background, but we will not need their precise forms.

The discussion of Section 2.2 above can be carried out in an equivariant setting with minor modifications and the suitable version of the integral (2.4) can be now evaluated with equivariant localization techniques. This will be done explicitly in the following sections for the Coulomb phase of the gauge theory. For the moment let us just sketch the idea behind the procedure, glossing over many details. In the equivariant model on $X = \mathbb{C}^3$ with the natural $\mathbb{T}^3$ action, the Euler class $\mathsf{e}(\mathcal{N})$ is an element of the $U(N) \times \mathbb{T}^3$ equivariant cohomology of the moduli space $\mathcal{M}$, where the gauge group $U(N)$ acts by rotating the trivialization of the instanton gauge bundle at infinity in $\mathbb{C}^3$. Its equivariant integral $\oint_{\mathcal{M}} \mathsf{e}(\mathcal{N})$ is the localization of the pushforward of $\mathsf{e}(\mathcal{N}) \in H^\bullet_{U(N) \times \mathbb{T}^3}(\mathcal{M})$ in $H^\bullet_{U(N) \times \mathbb{T}^3}(\mathrm{pt}) = \mathbb{C} \left[\mathrm{Sym}^N(\mathbb{C}); \epsilon_a \right]$ under the collapsing map $\mathcal{M} \to \mathrm{pt}$ onto a point, where the symmetric product $\mathrm{Sym}^N(\mathbb{C}) = \mathbb{C}^N / S_N$ parametrizes the complex conjugacy classes of $U(N)$. These classes can be labelled by elements $\boldsymbol{a} = (a_1, \dots, a_N)$ of the Cartan subalgebra $\mathfrak{u}(1)^{\oplus N} \cong \mathbb{R}^N$. Denoting $\xi = (\boldsymbol{a}, \epsilon_a)$, there is a moment map

$$\mu[\xi] = \mu_{U(N)}[\boldsymbol{a}] + \mu_{\mathbb{T}^3}[\epsilon_a] \ : \ \mathcal{M} \ \longrightarrow \ \left(\mathfrak{u}(1)^{\oplus N} \right)^* \oplus \mathrm{Lie}\left(\mathbb{T}^3 \right)^* \tag{2.8}$$

and an invariant symplectic two-form ω on $\mathcal{M}$ such that the $U(N) \times \mathbb{T}^3$ action on $\mathcal{M}$ is hamiltonian,

$$\mathrm{d}\mu[\xi] = -\imath_{V_\xi} \omega \ , \tag{2.9}$$

8

where V_ξ is the vector field on $\mathcal{M}$ representing the toric action generated by ξ.

Using (2.9) we can represent $\mathsf{e}(\mathcal{N})$ by a cohomologically equivalent equivariant differential form, and hence replace $\oint_\mathcal{M} \mathsf{e}(\mathcal{N})$ by an ordinary integral over the moduli space, $\int_\mathcal{M} \mathsf{e}(\mathcal{N}) \wedge \exp(\omega + \mu[\xi])$. We may then appeal to the Duistermaat–Heckman localization formula

$$\int_\mathcal{M} \frac{\omega^n}{n!}\ \mathrm{e}^{-\mu[\xi]} = \sum_{\substack{f \in \mathcal{M} \\ V_\xi(f)=0}} \frac{\mathrm{e}^{-\mu[\xi](f)}}{\prod_{i=1}^{n} w_i[\xi](f)}\ , \tag{2.10}$$

where $n = \dim_\mathbb{C} \mathcal{M}$ and the sum on the right-hand side runs over the fixed points of the toric action. The parameters $w_i[\xi]$ are the weights of the toric action on the tangent space to the critical points on $\mathcal{M}$. The Duistermaat–Heckman theorem assumes that the toric action on the ambient space X lifts to the moduli space and that its critical points are isolated. This is precisely what happens in the case where the gauge theory is defined on the Ω-background. One just needs to evaluate the integrand at each critical point and sum over all critical points with the appropriate weights. Thus in practice the evaluation of (2.4) in the equivariant model simply relies on being able to exhibit a complete classification of the critical points of the toric action. This will be explored in the following sections.

The assumptions made above on the instanton moduli space are not generally satisfied. To deal with such situations there is a powerful generalization of the Duistermaat–Heckman formula, the Atiyah–Bott localization formula for virtual classes. This formalism can be applied in the case at hand, and has been done in [7] for rank one sheaves. In the gauge theory setting, however, it is easier to deal with the equivariant localization as if the Duistermaat–Heckman formula and its supersymmetric generalizations were still reliable. In fact, the general story is a bit more involved than what we have been discussing thus far. When dealing with gauge theory on a toric manifold one must deal with the localization procedure more carefully. Now the $\mathbb{T}^3$-invariant configurations are not necessarily point-like and the partition function localizes into a sum of contributions coming from the points and curves which are fixed by the toric action. Each of these contributions can be expressed as an integral over the instanton moduli space, or more precisely over one of its connected components. Then one can deal with the integrals over the component moduli spaces as outlined above.

In [8] the cohomological gauge theory on a toric threefold X was expressed as a sum over contributions coming from the vertices of the underlying toric graph $\Delta(X)$, which correspond to point-like generalized instantons with action $\int_X \mathrm{Tr}\, F_A \wedge F_A \wedge F_A$. These contributions are connected by gauge field configurations fibred over the rational curves which connect the fixed points in the toric diagram, with action $\int_X \mathrm{Tr}\, k_0 \wedge F_A \wedge F_A$. This procedure can be summarized as a set of gluing rules which are completely analogous to those of the melting crystal reformulation of the topological A-model string theory, whereby the framing conditions to be imposed when matching the contributions from two different vertices of the crystal get mapped into a set of asymptotic boundary conditions for the gauge field configurations, or equivalently a framing of the generalized instanton gauge bundle at infinity.

3   Noncommutative gauge theory

In this section we will consider another deformation of the six-dimensional maximally super-symmetric gauge theory which simplifies explicit computations. This deformation resolves the small instanton singularities of the moduli space $\mathcal{M}$ and enables explicit construction of instanton solutions. Moreover it provides a compactification of the instanton moduli space that is very natural from a gauge theoretical perspective. Its geometric interpretation will be elucidated in the

next section. For the time being we consider the $U(N)$ gauge theory defined on $X = \mathbb{C}^3 \cong \mathbb{R}^6$, and deform it on the noncommutative space $\mathbb{R}^6_\theta$. This is equivalent to regarding the gauge theory as an infinite-dimensional matrix model where the fields are replaced by operators acting on a separable Hilbert space [22]–[24]. In this case there is a single patch in the geometry and only six-dimensional point-like instantons contribute to the partition function. In particular, there is no contribution from four-dimensional instantons stretched over rational curves $\mathbb{P}^1$, which are absent in this geometry. As the detailed calculations in the following are somewhat technical, let us start by summarizing the main results of this section.

3.1   Statement of results

We localize the noncommutative gauge theory with respect to the equivariant action of the abelian group $\mathbb{T}^3 \times U(1)^N$ on $\mathbb{C}^3$. The resulting instanton expansion of the partition function is a generating function for Donaldson–Thomas invariants given by

$$Z^{U(1)^N}_{\mathrm{DT}}\left(\mathbb{C}^3\right) = \sum_{\vec{\pi}} (-1)^{(N+1)\,|\vec{\pi}|}\, q^{|\vec{\pi}|} \tag{3.1}$$

where $q = -\,\mathrm{e}^{\,\mathrm{i}\,\vartheta} = \mathrm{e}^{-g_s}$. The sum runs through N-coloured, three-dimensional random partitions $\vec{\pi} = (\pi_1, \ldots, \pi_N)$ with $|\vec{\pi}| = |\pi_1| + \cdots + |\pi_N|$ boxes. The set of components π_l of fixed size $k_l = |\pi_l|$ is in correspondence with the Hilbert scheme $\mathsf{Hilb}^{k_l}(\mathbb{C}^3)$ of k_l points in $\mathbb{C}^3$, which consists of ideals of codimension k_l in the polynomial ring $\mathbb{C}[z^1, z^2, z^3]$. The generating function (3.1) is a sum over torus invariant configurations of $U(N)$ noncommutative instantons, which in the equivariant model consists of contributions from $k = |\vec{\pi}|$ instantons on top of each other at the origin. In the abelian case $N = 1$ one recovers the anticipated MacMahon function

$$M(q) = \prod_{n=1}^{\infty} \frac{1}{\left(1 - q^n\right)^n} \;, \tag{3.2}$$

the generating series for ordinary three-dimensional partitions. Whenever N is odd, the sign is independent of the partitions and the partition function closely resembles the abelian result. On the other hand, when N is even the partition function counts invariants with an alternating sign. In Sections 4 and 5 we will connect this result to the counting of torsion-free sheaves on $\mathbb{C}^3$, while some physical applications will be described in Section 7.

3.2   Noncommutative instantons

The coordinates x^i of $\mathbb{R}^6_\theta$ satisfy the Heisenberg algebra

$$\left[x^i,\, x^j\right] = \mathrm{i}\,\theta^{ij}\,, \quad i,j = 1,\ldots,6\,, \tag{3.3}$$

where $\theta = (\theta^{ij})$ is a constant real antisymmetric 6×6 matrix which we assume is nondegenerate. Using an $SO(6)$ rotation, we can choose coordinates such that θ assumes its Jordan canonical form

$$\theta = \begin{pmatrix} 0 & \theta_1 & & & & \\ -\theta_1 & 0 & & & & \\ & & 0 & \theta_2 & & \\ & & -\theta_2 & 0 & & \\ & & & & 0 & \theta_3 \\ & & & & -\theta_3 & 0 \end{pmatrix}\,. \tag{3.4}$$

10

The algebra of "functions" on $\mathbb{R}^6_\theta$ will be denoted $\mathcal{A}$.

The instanton equations on the noncommutative background can be simplified by introducing the covariant coordinates

$$X^i = x^i + \mathrm{i}\,\theta^{ij} A_j \ , \tag{3.5}$$

and their complex combinations

$$Z^i = \frac{1}{\sqrt{2\theta_i}}\left(X^{2i-1} + \mathrm{i}\,X^{2i}\right) \qquad \text{for} \qquad i = 1, 2, 3 \ . \tag{3.6}$$

Using the Heisenberg commutation relations (3.3) to represent derivatives as inner derivations of the algebra $\mathcal{A}$, the instanton equations (2.2) can then be rewritten in the form

$$
\begin{aligned}
\left[Z^i,\, Z^j\right] + \epsilon^{ijk}\left[Z_k^\dagger,\, \rho\right] &= 0 \ , \\
\left[Z^i,\, Z_i^\dagger\right] + \left[\rho,\, \rho^\dagger\right] &= 3\, 1_{N\times N} \ , \\
\left[Z^i,\, \Phi\right] &= 0
\end{aligned}
\tag{3.7}
$$

where $i, j, k = 1, 2, 3$. The combination of the noncommutative deformation with the Ω-background parametrized by ϵ_i, $i = 1, 2, 3$ changes the last equation in (3.7) to

$$\left[Z^i,\, \Phi\right] = \epsilon_i\, Z^i \qquad \text{(no sum on } i\text{)} \ . \tag{3.8}$$

For the remainder of this paper we will always set the $(3,0)$-form field ρ to zero, as we work on a Calabi–Yau geometry. Then $\mathbb{T}^3$-invariance of the (unique) holomorphic three-form constrains the equivariant parameters of the Ω-background by the equation

$$\epsilon_1 + \epsilon_2 + \epsilon_3 = 0 \ . \tag{3.9}$$

These sets of equations can be solved by three-dimensional harmonic oscillator algebra. Defining $\alpha_i = \frac{1}{\sqrt{2\theta_i}}\left(x^{2i-1} + \mathrm{i}\,x^{2i}\right)$, the commutation relations (3.3) are equivalent to

$$[\alpha_i, \alpha_j] = 0 \qquad \text{and} \qquad \left[\alpha_i,\, \alpha_j^\dagger\right] = \delta_{ij} \ . \tag{3.10}$$

The unique irreducible representation of this algebra is provided by the Fock module

$$\mathcal{H} = \mathbb{C}\left[\alpha_1^\dagger,\, \alpha_2^\dagger,\, \alpha_3^\dagger\right]|0,0,0\rangle = \bigoplus_{n_1,n_2,n_3\in\mathbb{N}_0} \mathbb{C}|n_1, n_2, n_3\rangle \ , \tag{3.11}$$

where $|0,0,0\rangle$ is the Fock vacuum with $\alpha_i|0,0,0\rangle = 0$ for $i = 1, 2, 3$, and the orthonormal basis states $|n_1, n_2, n_3\rangle$ are connected by the action of the creation and annihilation operators subject to the commutation relations (3.10). The operators (3.6) may then be taken to act on the Hilbert space

$$\mathcal{H}_W = W \otimes \mathcal{H} \tag{3.12}$$

where $W \cong \mathbb{C}^N$ is a Chan–Paton multiplicity space of dimension N, the number of D6-branes (and the rank of the gauge theory). The space W carries the nonabelian degrees of freedom and we understand Z^i and Φ as $N \times N$ matrices of operators acting on $\mathcal{H}$, i.e. as elements of the algebra $M_{N\times N}(\mathbb{C})\otimes\mathcal{A}$. In the supersymmetric matrix model where the matrices are operators acting on the Hilbert space (3.12), we only need to focus on the operators Z^i, Φ and their hermitean conjugates since we are interested in the class of minima where all other fields, including the twisted fermions, are set to zero. For example, the vacuum solution with $F_A = 0$ is given by

$$Z^i = \alpha_i\, 1_{N\times N} \ ,$$

11

$$\Phi \;=\; \sum_{i=1}^{3} \epsilon_i\, \alpha_i^\dagger\, \alpha_i\, \mathbb{1}_{N\times N} \;. \tag{3.13}$$

For $U(1)$ gauge theory, other solutions are found with the solution generating technique, described for example in [25, 26]. Fix an integer $n \geq 1$ and consider a partial isometry U_n which projects all states $|i, j, k\rangle$ with $i + j + k < n$ out of the Fock space $\mathcal{H}$. It obeys

$$U_n^\dagger U_n = 1 - \Pi_n \qquad \text{and} \qquad U_n U_n^\dagger = 1 \tag{3.14}$$

where Π_n is the projector

$$\Pi_n = \sum_{i+j+k<n} |i, j, k\rangle\langle i, j, k| \;. \tag{3.15}$$

We may then build a solution from the vacuum (3.13) of the form

$$Z^i \;=\; U_n\, \alpha_i\, f(\mathcal{N})\, U_n^\dagger \;,$$

$$\Phi \;=\; U_n \sum_{i=1}^{3} \epsilon_i\, \alpha_i^\dagger\, \alpha_i\, U_n^\dagger \;, \tag{3.16}$$

where f is a real function of the total number operator

$$\mathcal{N} = \sum_{i=1}^{3} \alpha_i^\dagger\, \alpha_i \;. \tag{3.17}$$

The function $f(\mathcal{N})$ is found by substituting this ansatz into the instanton equations to generate a recursion relation. With the initial condition $f(0) = f(1) = \cdots = f(n-1) = 0$ and the finite action condition $f(r) \to 1$ as $r \to \infty$, one then finds the solution

$$f(\mathcal{N}) = \sqrt{1 - \frac{n\,(n+1)\,(n+2)}{(\mathcal{N}+1)\,(\mathcal{N}+2)\,(\mathcal{N}+3)}}\; (1 - \Pi_n) \;. \tag{3.18}$$

The topological charge of the noncommutative $U(1)$ instantons defined by

$$k := \mathrm{ch}_3 = -\tfrac{\mathrm{i}}{6}\, \theta_1\, \theta_2\, \theta_3\, \mathrm{Tr}\,_{\mathcal{H}}\, F_A \wedge F_A \wedge F_A = \tfrac{1}{6}\, n\,(n+1)\,(n+2) \tag{3.19}$$

is the number of states in $\mathcal{H}$ with $\mathcal{N} < n$, i.e., the number of states removed by the operator U_n, or equivalently the rank of the projector (3.15).

3.3 Nonabelian solutions

We will now make some comments on how to generalize the $U(1)$ solutions described above to generic $U(N)$ gauge group. One can start, for example, from the noncommutative $\mathfrak{u}(3)$-valued instanton gauge field configuration constructed in [27], which is a smooth deformation of the canonical connection on the Stiefel bundle over $\mathbb{P}^3$, written in local coordinates on a patch $\mathbb{C}^3$ of the projective space $\mathbb{P}^3$. To describe this solution, we set all $\theta_i := \theta$, $i = 1, 2, 3$ for simplicity, and consider the exterior derivative d as a vector space morphism $\mathrm{d} : \mathcal{A} \to \Omega^1_{\mathcal{A}}$, where the sheaf of one-forms $\Omega^1_{\mathcal{A}}$ over $\mathcal{A}$ is the bimodule $\mathcal{A}^{\oplus 6}$. Introduce elements

$$\psi_i = \sqrt{6\theta\, k}\, \left(\mathrm{d}\alpha_i - 2\theta\, \alpha_i \left(\gamma^2 + \gamma\, \sqrt{1 + 3\theta}\, \right)^{-1} \alpha_j^\dagger\, \mathrm{d}\alpha^j \right) \gamma^{-1} \tag{3.20}$$

12

of $\Omega^1_{\mathcal{A}}$ for $i = 1, 2, 3$, where $\gamma : \mathcal{H} \to \mathcal{H}$ is the invertible operator

$$\gamma = \sqrt{2\theta\,\mathcal{N} + 1 + 3\theta} \ . \tag{3.21}$$

From these operators one can construct a gauge field regarded as a morphism of $\mathcal{A}$-modules

$$F_A : \mathcal{H}_{W_0} \longrightarrow \mathcal{H}_{W_0} \otimes_{\mathcal{A}} \Omega^2_{\mathcal{A}} \ , \tag{3.22}$$

where $\mathcal{H}_{W_0}$ is the module $W_0 \otimes \mathcal{H}$ with $W_0 \cong \mathbb{C}^3$ the fibre space of the Stiefel bundle, and the sheaf of two-forms over $\mathcal{A}$ is the bimodule $\Omega^2_{\mathcal{A}} = \mathcal{A}^{\oplus 15}$. It is given by the 3×3 matrix of Fock space operators

$$F_A = \left(\psi_i \wedge \psi^\dagger_j \right) \tag{3.23}$$

which, in the basis of $\Omega^2_{\mathcal{A}}$ generated by (3.20), has components

$$F_A^{2,0} = 0 \qquad \text{and} \qquad \left(F_A^{1,1} \right)_{ij} = e_{ij} \tag{3.24}$$

where e_{ij} are the standard 3×3 matrix units. The corresponding covariant coordinates Z_0^i thus obey the commutation relations

$$\left[Z_0^i \,,\, Z_0^j \right] = 0 \qquad \text{and} \qquad \left[Z_{0,i} \,,\, Z^\dagger_{0,j} \right] = 3 e_{ij} \ , \tag{3.25}$$

and consequently they solve the noncommutative DUY equations in (3.7) for $U(3)$ gauge group with instanton charge k. See [27] for the detailed expressions for the covariant coordinates Z_0^i.

The desired $N \times N$ covariant coordinates Z^i may be realized in terms of the $U(3)$ solution Z_0^i above by appealing to the Hilbert hotel argument, following [28]. For this, we introduce a lexicographic ordering $\mathbb{N}_0^3 \sim \mathbb{N}_0$ on the Fock space $\mathcal{H}$ so that $|n_1, n_2, n_3\rangle = |q\rangle$ with $q \in \mathbb{N}_0$, and fix an orthonormal basis $\vec{\rho}_a$, $a = 0, 1, 2$ of the fibre space W_0. Then $\vec{\rho}_a \otimes |q\rangle$ is an orthonormal basis for $\mathcal{H}_{W_0}$ and there is a one-to-one correspondence $\vec{\rho}_a \otimes |q\rangle \leftrightarrow |3q + a\rangle$ of basis states. Similarly, by fixing an orthonormal basis $\vec{\lambda}_a$, $a = 0, 1, \ldots, N - 1$ of the $U(N)$ representation space $W \cong \mathbb{C}^N$, there is a one-to-one correspondence $\vec{\lambda}_a \otimes |q\rangle \leftrightarrow |N\,q + a\rangle$ for the corresponding orthonormal basis of $\mathcal{H}_W$. Let us now introduce the rectangular $N \times 3$ unitary isomorphism $U : \mathcal{H}_{W_0} \to \mathcal{H}_W$ by the formula

$$U = \sum_{a=0}^{2} \sum_{b=0}^{N-1} \sum_{\substack{q,r=0 \\ 3q+a=N\,r+b}}^{\infty} |N\,r + b\rangle\langle 3q + a| = \sum_{a=0}^{2} \sum_{b=0}^{N-1} \sum_{\substack{q,r=0 \\ 3q+a=N\,r+b}}^{\infty} \vec{\lambda}_b\,\vec{\rho}_a{}^\dagger \otimes |b\rangle\langle a| \ . \tag{3.26}$$

Starting from the $U(3)$ solution of the noncommutative gauge theory above, one then constructs the $U(N)$ solutions

$$Z^i = U\,Z_0^i\,U^\dagger \ . \tag{3.27}$$

Of course, these nonabelian solutions only constitute a subset of the full BPS solution space. More general solutions will contain appropriate versions of the function $f(\mathcal{N})$ which featured into the solution of the noncommutative DUY equations when $N = 1$, and which reflect extra free moduli such as the size and relative orientation of the D0-branes on top of each other at the origin of $X = \mathbb{C}^3$ (as exhibited explicitly in [25] for the four-dimensional case). However, in the following we will not need the details of this solution. We will only need to know that, even in the nonabelian case, the solution is parametrized by specific partial isometry operators as in Section 3.2 above. Some further aspects of the generic $U(N)$ noncommutative instantons in six dimensions will be described in the next section.

3.4 Three-dimensional coloured partitions

A key feature of the solution of the six-dimensional cohomological gauge theory will be its interpretation as a statistical theory of three-dimensional random partitions. To see random partitions emerging, let us diagonalize the field Φ using the $U(N)$ gauge symmetry to get

$$\Phi = \begin{pmatrix} \Phi_1 & & & \\ & \Phi_2 & & \\ & & \ddots & \\ & & & \Phi_N \end{pmatrix} . \tag{3.28}$$

This transformation induces a Vandermonde determinant $\det(\mathrm{ad}\,\Phi)$ in the path integral measure (3.39) below. One can now classify the fixed points of the nonabelian gauge theory by generalizing the arguments of [8, 29]. We are prescribed to compute the path integral over configurations of the Higgs field whose asymptotic limit is $\boldsymbol{a} = \mathrm{diag}(a_1, \ldots, a_N) \in \mathfrak{u}(1)^{\oplus N}$. With this choice of boundary condition the noncommutative field Φ has the form

$$\Phi = \boldsymbol{a} \otimes 1_{\mathcal{H}} + 1_{N \times N} \otimes \Phi_{\mathcal{H}} \tag{3.29}$$

on (3.12), up to a function that goes to zero at infinity faster that any power of x^i. Note that this boundary condition only holds on $X = \mathbb{C}^3$. Nontrivial geometries require a more involved gluing together of different $\mathbb{C}^3$ patches, as we discuss in Section 6. The degeneracies of the asymptotic Higgs vevs breaks the gauge group $U(N) \to \prod_l U(k_l)$ with

$$\sum_l k_l = N . \tag{3.30}$$

Correspondingly, the Chan–Paton multiplicity space W decomposes into irreducible representations $W = \bigoplus_l W_l$ with $\dim_{\mathbb{C}} W_l = k_l$.

With our choice of the equivariant action of $\mathbb{T}^3 \times U(1)^N$, the gauge theory will localize onto the maximal torus. Consequently the Higgs vevs a_l may all be assumed distinct and the vacuum is the one with maximal symmetry breaking $k_l = 1$ for $l = 1, \ldots, N$. This means that the full Hilbert space (3.12) splits into a sum of N "abelian" Hilbert spaces $\mathcal{H}$, each one "coloured" by the Higgs vev a_l. The theory localizes on noncommutative $U(1)$ instantons that are in correspondence with maps of the full Hilbert space onto the subspace

$$\mathcal{H}_{\mathcal{I}} = \bigoplus_{l=1}^{N} \mathcal{I}_{a_l}\big[\alpha_1^\dagger, \alpha_2^\dagger, \alpha_3^\dagger\big]|0,0,0\rangle , \tag{3.31}$$

where $\mathcal{I}_{a_l}$ are ideals in the polynomial ring $\mathbb{C}[z^1, z^2, z^3]$. Each partial isometry U_n satisfying (3.14) identifies the Fock space (3.11) with a subspace of the form in (3.31), with $\mathcal{I}$ the ideal of codimension k consisting of polynomials $f \in \mathbb{C}[z^1, z^2, z^3]$ for which

$$\Pi_n \cdot f\big(\alpha_1^\dagger, \alpha_2^\dagger, \alpha_3^\dagger\big)|0,0,0\rangle = 0 . \tag{3.32}$$

These ideals are generated by monomials $z^i\, z^j\, z^k$ and are in one-to-one correspondence with three-dimensional partitions. Thus in complete analogy with the four-dimensional case [9, 29, 30], the solutions correspond to three-dimensional (plane) partitions with the triples (i, j, k) in (3.15) corresponding to boxes of the partition. More precisely, the solution can be found in terms of coloured partitions

$$\vec{\pi} = (\pi_1, \ldots, \pi_N) , \tag{3.33}$$

which are rows of N ordinary three-dimensional partitions π_l labelled by a_l. We will explain this correspondence in more detail in Section 5.

A specific class of observables of the gauge theory is given by the trace of powers of the Higgs field Φ. Since the gauge theory is cohomological all the interesting observables (including the integral of the Chern character over $\mathbb{C}^3$) can be expressed through these quantities by means of descendent relations. In particular, for the solution associated to the sum of ideals $\mathfrak{I} = \mathfrak{I}_{a_1} \oplus \cdots \oplus \mathfrak{I}_{a_N}$ corresponding to the three-dimensional coloured partition (3.33), this produces the normalized character

$$
\begin{aligned}
\chi_{\mathfrak{I}}(t) &= \left(1 - \mathrm{e}^{t\,\epsilon_1}\right)\left(1 - \mathrm{e}^{t\,\epsilon_2}\right)\left(1 - \mathrm{e}^{t\,\epsilon_3}\right)\,\mathrm{Tr}\,_{\mathcal{H}_{\mathfrak{I}}}\,\mathrm{e}^{t\,\Phi} \\
&= \sum_{l=1}^{N} \mathrm{e}^{t\,a_l}\left(1 - \left(1 - \mathrm{e}^{t\,\epsilon_1}\right)\left(1 - \mathrm{e}^{t\,\epsilon_2}\right)\left(1 - \mathrm{e}^{t\,\epsilon_3}\right) \sum_{(i,j,k)\in\pi_l} \mathrm{e}^{t\,(\epsilon_1\,(i-1)+\epsilon_2\,(j-1)+\epsilon_3\,(k-1))}\right).
\end{aligned}
$$
(3.34)

This construction has a heuristic interpretation in terms of D-branes. Localizing the gauge theory onto the Cartan subalgebra is equivalent to displacing the N D6-branes in the four-dimensional extended space outside the Calabi–Yau manifold X. The vevs of the Higgs field correspond to the positions of neighbouring D-branes relative to one another. This means that there are N separated D6-branes, each one with its own bound state of D0-branes corresponding to a six-dimensional instanton through anomalous couplings to Ramond–Ramond fields on the D6-brane worldvolume. The D0-branes are indexed by the boxes of the three-dimensional partition π_l and the "colour" of the partition is the information relative to which D6-brane they are bound to. k_l D0-branes bound to a D6-brane labelled by a_l are described by a three-dimensional partition with k_l boxes in the l-th sector of the Hilbert space and correspond to a charge k_l instanton on the worldvolume of the D6-brane in position a_l. It should be stressed though that this geometric interpretation is somewhat naive and if the instanton measure has a nontrivial dependence on the Higgs vevs a_l, then this could reflect open string degrees of freedom stretching between different D6-branes.

3.5 Instanton weight

We will now examine the noncommutative instanton contributions to the partition function of the $U(N)$ topological gauge theory on $\mathbb{C}^3$ in the vacuum of maximal symmetry breaking. Proceeding with localization the contribution of an instanton associated to a collection of ideals $\mathfrak{I}$ comes with the weight factor

$$
\exp\left(-\frac{\vartheta}{48\pi^3}\,\mathrm{Tr}\,_{\mathcal{H}_{\mathfrak{I}}}\,F_A \wedge F_A \wedge F_A\right) = \exp\left(\frac{\mathrm{i}\,\vartheta\,\chi_{\mathfrak{I}}^{(3)}}{\epsilon_1\,\epsilon_2\,\epsilon_3}\right),
$$
(3.35)

where $\chi_{\mathfrak{I}}^{(3)}$ is the coefficient of t^3 in the power series expansion of the character (3.34) about $t = 0$. Using (3.9) one finds

$$
\chi_{\mathfrak{I}}^{(3)} = \sum_{l=1}^{N}\left(-\frac{a_l^3}{6} + \epsilon_1\,\epsilon_2\,\epsilon_3 \sum_{(i,j,k)\in\pi_l} 1\right).
$$
(3.36)

The first term is independent of the partitions π_l and can be dropped as a universal perturbative contribution. The second term yields, for each l, the total number of boxes $k_l = |\pi_l|$ in the partition π_l, which coincides with the topological charges (3.19) of the corresponding noncommutative $U(1)$ instantons. Thus for the weight of an instanton we obtain

$$
\mathrm{e}^{\,\mathrm{i}\,\vartheta\,|\vec{\pi}|}
$$
(3.37)

15

in the sector of instanton charge

$$k = |\vec{\pi}| = \sum_{l=1}^{N} |\pi_l| \tag{3.38}$$

given by the total number of boxes in the coloured partition.

3.6 Instanton measure

In addition to the weight there is also a contribution from the determinants representing quantum fluctuations around the instanton solutions. In the noncommutative gauge theory the ratio of fluctuation determinants is represented by [8]

$$\frac{\det\left(\mathrm{ad}\,\Phi\right)\det\left(\mathrm{ad}\,\Phi + \epsilon_1 + \epsilon_2\right)\det\left(\mathrm{ad}\,\Phi + \epsilon_1 + \epsilon_3\right)\det\left(\mathrm{ad}\,\Phi + \epsilon_2 + \epsilon_3\right)}{\det\left(\mathrm{ad}\,\Phi + \epsilon_1 + \epsilon_2 + \epsilon_3\right)\det\left(\mathrm{ad}\,\Phi + \epsilon_1\right)\det\left(\mathrm{ad}\,\Phi + \epsilon_2\right)\det\left(\mathrm{ad}\,\Phi + \epsilon_3\right)} \tag{3.39}$$

where Φ is in general nonabelian and given by (3.29). The origin of this ratio will be explained in more detail in Section 5. In the following we will compute the instanton measure explicitly. The ratio of determinants can be written as

$$\exp\left(-\int_0^\infty \frac{\mathrm{d}t}{t} \ \mathrm{Tr}_{\mathcal{H}_{\mathcal{I}}}\, \mathrm{e}^{\,t\,\Phi}\, \mathrm{Tr}_{\mathcal{H}_{\mathcal{I}}}\, \mathrm{e}^{-t\,\Phi}\left(1 - \mathrm{e}^{\,t\,\epsilon_1}\right)\left(1 - \mathrm{e}^{\,t\,\epsilon_2}\right)\left(1 - \mathrm{e}^{\,t\,\epsilon_3}\right)\right) \tag{3.40}$$

which can be recast as an index-like quantity

$$Z_I^{U(1)^N}(\vec{\pi}) = \exp\left(-\int_0^\infty \frac{\mathrm{d}t}{t}\, \frac{\chi_{\mathcal{I}}(t)\,\chi_{\mathcal{I}}(-t)}{\left(1 - \mathrm{e}^{\,t\,\epsilon_1}\right)\left(1 - \mathrm{e}^{\,t\,\epsilon_2}\right)\left(1 - \mathrm{e}^{\,t\,\epsilon_3}\right)}\right) =: \mathrm{e}^{-I^{U(1)^N}(\vec{\pi})}\,, \tag{3.41}$$

where $\vec{\pi}$ is the coloured partition corresponding to the given BPS state.

First, we review the abelian case $N = 1$ in some detail. Let us break up the integral $I^{U(1)}(\pi)$ in (3.41) into three contributions $I^{U(1)}(\pi) = I_{\mathrm{vac}}^{U(1)} + I_1^{U(1)}(\pi) + I_2^{U(1)}(\pi)$ given by

$$I_{\mathrm{vac}}^{U(1)} = \int_0^\infty \frac{\mathrm{d}t}{t}\, \frac{1}{\left(1 - \mathrm{e}^{\,t\,\epsilon_1}\right)\left(1 - \mathrm{e}^{\,t\,\epsilon_2}\right)\left(1 - \mathrm{e}^{\,t\,\epsilon_3}\right)}\,,$$

$$I_1^{U(1)}(\pi) = \sum_{(i,j,k)\in\pi} \int_0^\infty \frac{\mathrm{d}t}{t}\left(-\mathrm{e}^{\,t\,(\epsilon_1\,i + \epsilon_2\,j + \epsilon_3\,k)} + \mathrm{e}^{-t\,(\epsilon_1\,i + \epsilon_2\,j + \epsilon_3\,k)}\right)\,, \tag{3.42}$$

$$I_2^{U(1)}(\pi) = \sum_{\substack{(i,j,k)\in\pi \\ (i',j',k')\in\pi}} \int_0^\infty \frac{\mathrm{d}t}{t}\, \mathrm{e}^{\,t\,(\epsilon_1\,(i-i') + \epsilon_2\,(j-j') + \epsilon_3\,(k-k'))}$$

$$\times\left(\mathrm{e}^{\,t\,\epsilon_1} - \mathrm{e}^{-t\,\epsilon_1} + \mathrm{e}^{\,t\,\epsilon_2} - \mathrm{e}^{-t\,\epsilon_2} + \mathrm{e}^{\,t\,\epsilon_3} - \mathrm{e}^{-t\,\epsilon_3}\right)\,.$$

The integral $I_{\mathrm{vac}}^{U(1)}$ is the universal vacuum contribution from the empty partition and will be dropped in the following. Evaluating the remaining integrals we obtain

$$I_1^{U(1)}(\pi) = \mathrm{i}\,\pi \sum_{(i,j,k)\in\pi} \mathrm{sgn}(\epsilon_1\,i + \epsilon_2\,j + \epsilon_3\,k)\,, \tag{3.43}$$

$$I_2^{U(1)}(\pi) = \mathrm{i}\,\pi \sum_{\substack{(i,j,k)\in\pi \\ (i',j',k')\in\pi}} \Big(\mathrm{sgn}\big(\epsilon_1\,(i - i' + 1) + \epsilon_2\,(j - j') + \epsilon_3\,(k - k')\big)$$

$$+ \mathrm{sgn}\big(\epsilon_1\,(i - i') + \epsilon_2\,(j - j' + 1) + \epsilon_3\,(k - k')\big)$$

16

$$+ \operatorname{sgn}\!\big(\epsilon_1 \, (i - i') + \epsilon_2 \, (j - j') + \epsilon_3 \, (k - k' + 1)\big)\Big) \ .$$

Since we are interested only in the exponentials of these integrals, the value ± 1 of the signum function is irrelevant (we count $\mathrm{e}^{\pm\mathrm{i}\pi} = -1$). Thus we have

$$\mathrm{e}^{-I_1^{U(1)}(\pi)} = (-1)^{|\pi|-\mathrm{diag}} \ ,$$

$$\mathrm{e}^{-I_2^{U(1)}(\pi)} = (-1)^{3|\pi|^2-\mathrm{diag}} \ , \tag{3.44}$$

where diag denotes the "diagonal" terms for which the signum functions combine to give zero and have to be subtracted since they give a factor $+1$ instead of -1. The direct counting of how many diagonal terms are present for a given partition π is a bit complicated, and moreover one has to take into account the constraint (3.9). Instead, we will use induction to prove the result

$$\mathrm{e}^{-I_1^{U(1)}(\pi)-I_2^{U(1)}(\pi)} = (-1)^{|\pi|} \tag{3.45}$$

for any three-dimensional partition π.

Consider first the partition $\pi = (1,1,1)$ consisting of a single box, so that $|\pi| = 1$. In this case one has $\mathrm{e}^{I_1^{U(1)}}(\pi) = 1$, $\mathrm{e}^{I_2^{U(1)}}(\pi) = -1$ and $\mathrm{e}^{-I_1^{U(1)}(\pi)-I_2^{U(1)}(\pi)} = -1$, so this partition satisfies (3.45). Suppose now that for some partition π, eq. (3.45) is true. In the induction step we increase the number of boxes in this partition by one (in such a way that we still obtain a valid partition). Denote the new partition with the extra box at a given position (a,b,c) by π^*. One has $|\pi^*|-|\pi| = 1$, and so we have to prove that

$$\mathrm{e}^{-I_1^{U(1)}(\pi^*)-I_2^{U(1)}(\pi^*)+I_1^{U(1)}(\pi)+I_2^{U(1)}(\pi)} = -1 \ . \tag{3.46}$$

For the first integral we have

$$I_1^{U(1)}(\pi^*) - I_1^{U(1)}(\pi) = \mathrm{i}\,\pi \, \operatorname{sgn}(\epsilon_1 \, a + \epsilon_2 \, b + \epsilon_3 \, c) \ . \tag{3.47}$$

For the second integral, after some algebra we obtain

$$
\begin{aligned}
I_2^{U(1)}(\pi^*) &- I_2^{U(1)}(\pi) \\
&= 6|\pi| + 3 - \min(a, b-1, c-1) - \min(a-1, b, c) - \min(a-1, b, c-1) \\
&\quad - \min(a, b-1, c) - \min(a-1, b-1, c) - \min(a, b, c-1) \ .
\end{aligned} \tag{3.48}
$$

From these results it is easy to establish (3.46), proving the induction step.

Putting everything together, the abelian instanton measure is given by

$$Z_I^{U(1)}(\pi) = Z_{\mathrm{vac}}^{U(1)} \ (-1)^{|\pi|} \ . \tag{3.49}$$

Dropping the perturbative vacuum contribution and combining this result with the weight (3.37), the full instanton contribution to the $U(1)$ partition function is given by

$$\mathcal{Z}_{\mathrm{DT}}^{U(1)}(\mathbb{C}^3) = \sum_\pi \big(-\mathrm{e}^{\,\mathrm{i}\vartheta}\big)^{|\pi|} = M(q) \ . \tag{3.50}$$

This is the known Donaldson–Thomas partition function on $\mathbb{C}^3$.

This construction can be easily generalized to the case where the full nonabelian gauge symmetry is broken to $U(1)^N$. Using the character (3.34), the exponent $I^{U(1)^N}(\vec{\pi})$ of the fluctuation determinant (3.41) can again be written as the sum of three integrals

$$I_{\mathrm{vac}}^{U(1)^N} = \int_0^\infty \frac{\mathrm{d}t}{t} \, \frac{1}{\big(1 - \mathrm{e}^{\,t\epsilon_1}\big)\big(1 - \mathrm{e}^{\,t\epsilon_2}\big)\big(1 - \mathrm{e}^{\,t\epsilon_3}\big)} \sum_{l,n=1}^{N} \mathrm{e}^{\,t\,(a_l - a_n)} \ , \tag{3.51}$$

17

$$I_1^{U(1)^N}(\vec{\pi}) = \int_0^\infty \frac{\mathrm{d}t}{t} \sum_{l,n=1}^N \mathrm{e}^{t\,(a_l - a_n)}$$
$$\times \left(- \sum_{(i,j,k)\in\pi_l} \mathrm{e}^{t\,(\epsilon_1\,i + \epsilon_2\,j + \epsilon_3\,k)} + \sum_{(i,j,k)\in\pi_n} \mathrm{e}^{-t\,(\epsilon_1\,i + \epsilon_2\,j + \epsilon_3\,k)} \right) ,$$

$$I_2^{U(1)^N}(\vec{\pi}) = \int_0^\infty \frac{\mathrm{d}t}{t} \sum_{l,n=1}^N \mathrm{e}^{t\,(a_l - a_n)} \sum_{\substack{(i,j,k)\in\pi_l \\ (i',j',k')\in\pi_n}} \mathrm{e}^{t\,(\epsilon_1\,(i-i') + \epsilon_2\,(j-j') + \epsilon_3\,(k-k'))}$$
$$\times \left(\mathrm{e}^{t\,\epsilon_1} - \mathrm{e}^{-t\,\epsilon_1} + \mathrm{e}^{t\,\epsilon_2} - \mathrm{e}^{-t\,\epsilon_2} + \mathrm{e}^{t\,\epsilon_3} - \mathrm{e}^{-t\,\epsilon_3} \right)$$

where as before π_n, π_l denote components of the coloured partition (3.33) and a_n, a_l are components of the classical value of the Higgs field (3.28). Evaluating the integrals, the non-trivial contributions can be written as

$$I_1^{U(1)^N}(\vec{\pi})_{\mathrm{diag}} = \mathrm{i}\,\pi \sum_{l=1}^N \sum_{(i,j,k)\in\pi_l} \mathrm{sgn}(\epsilon_1\,i + \epsilon_2\,j + \epsilon_3\,k) ,$$

$$I_1^{U(1)^N}(\vec{\pi})_{\mathrm{offdiag}} = \mathrm{i}\,\pi \sum_{\substack{l,n=1 \\ l\neq n}}^N \sum_{(i,j,k)\in\pi_l} \mathrm{sgn}(\epsilon_1\,i + \epsilon_2\,j + \epsilon_3\,k + a_{ln}) ,$$

$$I_2^{U(1)^N}(\vec{\pi})_{\mathrm{diag}} = \mathrm{i}\,\pi \sum_{l=1}^N \sum_{\substack{(i,j,k)\in\pi_l \\ (i',j',k')\in\pi_l}} \Big(\mathrm{sgn}\big(\epsilon_1\,(i - i' + 1) + \epsilon_2\,(j - j') + \epsilon_3\,(k - k')\big)$$
$$+ \mathrm{sgn}\big(\epsilon_1\,(i - i') + \epsilon_2\,(j - j' + 1) + \epsilon_3\,(k - k')\big)$$
$$+ \mathrm{sgn}\big(\epsilon_1\,(i - i') + \epsilon_2\,(j - j') + \epsilon_3\,(k - k' + 1)\big) \Big) ,$$

$$I_2^{U(1)^N}(\vec{\pi})_{\mathrm{offdiag}} = \mathrm{i}\,\pi \sum_{\substack{l,n=1 \\ l<n}}^N \sum_{\substack{(i,j,k)\in\pi_l \\ (i',j',k')\in\pi_n}} \Big(\mathrm{sgn}\big(\epsilon_1\,(i - i' + 1) + \epsilon_2\,(j - j') + \epsilon_3\,(k - k') + a_{ln}\big)$$
$$+ \mathrm{sgn}\big(\epsilon_1\,(i - i') + \epsilon_2\,(j - j' + 1) + \epsilon_3\,(k - k') + a_{ln}\big)$$
$$+ \mathrm{sgn}\big(\epsilon_1\,(i - i') + \epsilon_2\,(j - j') + \epsilon_3\,(k - k' + 1) + a_{ln}\big)$$
$$- \mathrm{sgn}\big(\epsilon_1\,(i - i' - 1) + \epsilon_2\,(j - j') + \epsilon_3\,(k - k') + a_{ln}\big)$$
$$- \mathrm{sgn}\big(\epsilon_1\,(i - i') + \epsilon_2\,(j - j' - 1) + \epsilon_3\,(k - k') + a_{ln}\big)$$
$$- \mathrm{sgn}\big(\epsilon_1\,(i - i') + \epsilon_2\,(j - j') + \epsilon_3\,(k - k' - 1) + a_{ln}\big) \Big) , \qquad (3.52)$$

where we have broken up the integrals into diagonal and off-diagonal components in the summations over the colour indices, and denoted $a_{ln} := a_l - a_n$.

In the $U(1)^N$ phase one has $a_{ln} \neq 0$ for all $1 \leq l \neq n \leq N$, for otherwise the gauge symmetry would be enhanced. Then one can always choose the infrared regulators ϵ_1 and ϵ_2 such that $\epsilon_1\,M + \epsilon_2\,L + a_{ln} \neq 0$ for any pair of integers M, L, in addition to the Calabi–Yau condition (3.9). This ensures that the signum functions containing these vevs are never zero. It is easy to see that in this case one has

$$\mathrm{e}^{-I_1^{U(1)^N}(\vec{\pi})_{\mathrm{diag}}} = \mathrm{e}^{-\sum_l I_1^{U(1)}(\pi_l)} ,$$

18

$$e^{-I_1^{U(1)^N}(\vec{\pi})_{\text{offdiag}}} = \prod_{\substack{l,n=1 \\ l \neq n}}^{N} (-1)^{|\pi_l|} \,,$$

$$e^{-I_2^{U(1)^N}(\vec{\pi})_{\text{diag}}} = e^{-\sum_l I_2^{U(1)}(\pi_l)} \,,$$

$$e^{-I_2^{U(1)^N}(\vec{\pi})_{\text{offdiag}}} = 1 \,, \tag{3.53}$$

and hence the full contribution is given by

$$e^{-I_1^{U(1)^N}(\vec{\pi})-I_2^{U(1)^N}(\vec{\pi})} = e^{-\sum_l \left(I_1^{U(1)}(\pi_l)+I_2^{U(1)}(\pi_l) \right)} \left(\prod_{\substack{l,n=1 \\ l \neq n}}^{N} (-1)^{|\pi_l|} \right) \,. \tag{3.54}$$

Using the abelian result (3.45) to get

$$e^{-I_1^{U(1)}(\pi_l)-I_2^{U(1)}(\pi_l)} = (-1)^{|\pi_l|} \,, \tag{3.55}$$

we arrive finally at the instanton measure

$$e^{-I_1^{U(1)^N}(\vec{\pi})-I_2^{U(1)^N}(\vec{\pi})} = (-1)^{N\,|\vec{\pi}|} \,. \tag{3.56}$$

Combining this result with the weights (3.37) leads to the partition function (3.1).

4 Matrix equations and moduli of stable coherent sheaves

The purpose of this section is two-fold. Firstly, we will recast the BPS solutions of the non-commutative gauge theory explicitly in purely commutative terms, by relating the counting of noncommutative instantons to the counting of a special class of stable coherent sheaves in three dimensions. This gives a geometrical explanation for and generalizes the natural gauge theoretical identification between the compactified instanton moduli space $\mathcal{M}(X)$ and the Hilbert scheme of points $\mathsf{Hilb}^k(X)$ in the abelian case $N = 1$ when the underlying variety is $X = \mathbb{C}^3$. Secondly, we will exhibit an ADHM-like parametrization of the equations defining $\mathcal{M}(\mathbb{C}^3)$, analogous to the four-dimensional case. This will naturally bridge the noncommutative gauge theory formalism with the topological matrix quantum mechanics that we will introduce in the next section. This matrix model is equivalent to a quantization of the collective coordinates of the gauge theory around an instanton solution, and it will provide an alternative means for analysing the six-dimensional cohomological gauge theory. As before, we begin by summarizing the main results of this section before plunging into the detailed technical calculations.

4.1 Statement of results

There are natural mappings between isomorphism classes of the following three objects:

(A) $U(N)$ noncommutative instantons on $\mathbb{C}^3$ of topological charge $\text{ch}_3 = k$.

(B) Linear maps $B_i \in \text{End}(V)$, $i = 1, 2, 3$, $I \in \text{Hom}(W, V)$ and $J, K \in \text{Hom}(V, W)$ which solve the "ADHM-type" matrix equations

$$[B_1, B_2] + I\, J = 0 \,,$$

19

$$[B_1, B_3] + I\,K = 0\,,$$

$$[B_2, B_3] = 0\,,$$

$$\sum_{i=1}^{3} \big[B_i\,, B_i^\dagger \big] + I\,I^\dagger - J^\dagger\,J - K^\dagger\,K = 3\,1_V \tag{4.1}$$

on finite-dimensional hermitean vector spaces $V \cong \mathbb{C}^k$ and $W \cong \mathbb{C}^N$, modulo the natural action of $U(V)$ given by

$$B_i \longmapsto g\,B_i\,g^{-1}\,, \quad I \longmapsto g\,I\,, \quad J \longmapsto J\,g^{-1} \quad \text{and} \quad K \longmapsto K\,g^{-1}\,. \tag{4.2}$$

(C) Rank N torsion-free sheaves $\mathcal{E}$ on $\mathbb{P}^3$ with $\mathrm{ch}_3(\mathcal{E}) = k$, fixed trivializations on three lines at infinity, vanishing $H^1(\mathbb{P}^3, \mathcal{E}(-2))$, and satisfying certain stability conditions.

These three classes are not equivalent but rather represent alternative characterizations of one another. The map from (A) to (B) follows from a rewriting of the noncommutative DUY equations in (3.7) using special properties of projective modules over the noncommutative space $\mathbb{R}^6_\theta$ [22]. The map from (C) to (B) follows from detailed calculations in sheaf cohomology which rewrites the sheaves $\mathcal{E}$ as the cohomology of certain complexes of sheaves on $\mathbb{P}^3$. It canonically identifies both the D0-brane and D6-brane Chan–Paton spaces $V \subset \mathcal{H}$ and W as certain sheaf cohomology groups. In particular, W is associated with the framing of $\mathcal{E}$. The last equation of (4.1) is a stability condition. Combined with the linear maps $\phi \in \mathrm{End}(V)$, $P \in \mathrm{Hom}(W, V)$ solving the matrix equations

$$[B_1, \phi] = \epsilon_1\,B_1\,,$$

$$[B_2, \phi] - P\,J = \epsilon_2\,B_2\,,$$

$$[B_3, \phi] - P\,K = \epsilon_3\,B_3\,, \tag{4.3}$$

which come from the noncommutative Higgs fields Φ obeying (3.8), it will be used in the next section to formalize the connection between the equivariant model and the statistical mechanics of three-dimensional random partitions.

4.2 Linear algebra of noncommutative instantons

In the previous section we described charge k noncommutative instantons in six dimensions as elements of the algebra $M_{N \times N}(\mathbb{C}) \otimes \mathcal{A}$ acting on the free module $W \otimes \mathcal{A} \cong \mathcal{A}^{\oplus N}$ of rank N over the noncommutative algebra $\mathcal{A}$. The K-theory of the algebra $\mathcal{A}$ is the abelian group $\mathbb{Z} \oplus \mathbb{Z}$, with positive cone $\mathbb{N}_0 \oplus \mathbb{N}_0$. Thus every projective module $\mathcal{E}_{k,N}$ over $\mathcal{A}$ is labelled by a pair of positive integers k, N, which represent exactly the instanton number and rank respectively. Explicitly, one has [22]

$$\mathcal{E}_{k,N} = \mathcal{H}^{\oplus k} \oplus \mathcal{A}^{\oplus N} \tag{4.4}$$

where as before $\mathcal{H}$ is the Fock space (3.11). The key to obtaining a set of matrix equations which describes six-dimensional noncommutative instantons is the existence of a natural isomorphism $\mathcal{E}_{k,N} \cong \mathcal{A}^{\oplus N}$ of $\mathcal{A}$-modules.

A connection $\nabla : \mathcal{E}_{k,N} \to \mathcal{E}_{k,N} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{A}}$ induces a decomposition of the covariant coordinates $Z^i \in \mathrm{End}_{\mathcal{A}}(\mathcal{E}_{k,N})$ with respect to the splitting (4.4) as

$$Z^1 = \begin{pmatrix} B_1 & I_0 \\ I' & R_1 \end{pmatrix}\,, \quad Z^2 = \begin{pmatrix} B_2 & J' \\ J_0 & R_2 \end{pmatrix} \quad \text{and} \quad Z^3 = \begin{pmatrix} B_3 & K' \\ K_0 & R_3 \end{pmatrix}\,. \tag{4.5}$$

Using irreducibility of the Fock module $\mathcal{H}$, we note that the rank one free module $\mathcal{A}$ can be decomposed into countably many copies of $\mathcal{H}$ as $\mathcal{A} = \bigoplus_{n \in \mathbb{N}_0} P_n \cdot \mathcal{A}$, where $P_n = |n\rangle\langle n|$ is an orthogonal system of projectors onto one-dimensional subspaces. For each $n \in \mathbb{N}_0$, the mapping $P_n \cdot f \mapsto \langle n|f$ establishes an isomorphism $P_n \cdot \mathcal{A} \cong \mathcal{H}$ of $\mathcal{A}$-modules. Using this identification, in (4.5) we regard $B_i \in M_{k \times k}(\mathbb{C})$ as linear operators acting on a finite-dimensional hermitean vector space $V \cong \mathbb{C}^k$, while $R_i \in \mathrm{End}_{\mathcal{A}}(W \otimes \mathcal{A})$ with $W \cong \mathbb{C}^N$ the finite-dimensional Chan–Paton multiplicity space as in the previous section. The off-diagonal entries in eq. (4.5) are operators

$$I_0, J', K' \in \mathrm{Hom}_{\mathcal{A}}(W \otimes \mathcal{A}, V) \qquad \text{and} \qquad I', J_0, K_0 \in \mathrm{Hom}_{\mathcal{A}}(V, W \otimes \mathcal{A}) \ . \tag{4.6}$$

In the following we will use the gauge choice $I' = J' = K' = 0$.

Then the first instanton equation of (3.7) yields the sets of equations

$$\begin{aligned}
[B_1, B_2] + I_0 J_0 = 0 \qquad &\text{and} \qquad [R_1, R_2] - J_0 I_0 = 0 \quad , \\
[B_1, B_3] + I_0 K_0 = 0 \qquad &\text{and} \qquad [R_1, R_3] - K_0 I_0 = 0 \quad , \\
[B_2, B_3] = 0 \qquad &\text{and} \qquad [R_2, R_3] = 0
\end{aligned} \tag{4.7}$$

along with

$$R_1 J_0 - J_0 B_1 = 0 \ , \qquad R_1 K_0 - K_0 B_1 = 0 \qquad \text{and} \qquad I_0 R_2 - B_2 I_0 = 0 \tag{4.8}$$

plus

$$R_3 J_0 - J_0 B_3 = R_2 K_0 - K_0 B_2 \qquad \text{and} \qquad I_0 R_3 - B_3 I_0 = 0 \ . \tag{4.9}$$

The second instanton equation in (3.7) yields the sets of equations

$$\sum_{i=1}^{3} \left[B_i, B_i^\dagger \right] + I_0 I_0^\dagger - J_0^\dagger J_0 - K_0^\dagger K_0 = 3 \, 1_V \ ,$$

$$\sum_{i=1}^{3} \left[R_i, R_i^\dagger \right] - I_0^\dagger I_0 + J_0 J_0^\dagger + K_0 K_0^\dagger = 3 \, 1_{W \otimes \mathcal{A}} \ ,$$

$$I_0 R_1^\dagger - J_0^\dagger R_2 - K_0^\dagger R_3 = B_1^\dagger I_0 - B_2 J_0^\dagger - B_3 K_0^\dagger \ . \tag{4.10}$$

Let us now decompose the off-diagonal operators in eq. (4.5) as

$$I_0 = I \otimes \psi_I \ \in \ \mathrm{Hom}_{\mathcal{A}}\big(W \otimes \mathcal{A}^{\oplus N}, V\big) \cong \mathrm{Hom}(W, V) \otimes \mathrm{End}_{\mathcal{A}}\big(\mathcal{A}^{\oplus N}\big) \tag{4.11}$$

with $I \in M_{k \times N}(\mathbb{C})$ and

$$J_0 = J \otimes \psi_J, K_0 = K \otimes \psi_K \ \in \ \mathrm{Hom}_{\mathcal{A}}\big(V, W \otimes \mathcal{A}^{\oplus N}\big) \cong \mathrm{Hom}(V, W) \otimes \mathrm{End}_{\mathcal{A}}\big(\mathcal{A}^{\oplus N}\big) \tag{4.12}$$

with $J, K \in M_{N \times k}(\mathbb{C})$, where the elements of $M_{N \times N}(\mathbb{C}) \otimes \mathcal{A}$ satisfy $\psi_I \psi_J = \psi_I \psi_K = 1_{W \otimes \mathcal{A}}$. In this way we arrive, from (4.7) and (4.10), at the matrix equations (4.1).

We have thus found that the set of ADHM-type data

$$(B_i; I, J, K) \ \in \ \mathrm{Hom}(V, V)^{\oplus 3} \ \oplus \ \mathrm{Hom}(W, V) \ \oplus \ \mathrm{Hom}(V, W)^{\oplus 2} \tag{4.13}$$

can be used to characterize the noncommutative instantons. However, this set of data is not likely to be complete, i.e. a solution to the set of matrix equations (4.1) need not yield a solution $(R_i; \psi_I, \psi_J, \psi_K)$ to the remaining equations in (4.7)–(4.10). One can also reformulate the equivariant equations (3.8) of the Ω-background as matrix equations by writing the adjoint scalar field Φ in the block form

$$\Phi = \begin{pmatrix} \phi & P_0 \\ P' & \varrho \end{pmatrix} \tag{4.14}$$

with respect to the splitting of the projective module (4.4). Then the matrix equations for the $k \times k$ matrix ϕ in the gauge $P' = 0$ are given by (4.3) with $P_0 = P \otimes \psi_I$. In the next section these equations will be interpreted as the compensation of the toric action $\mathbb{T}^3$ by a gauge transformation in an associated topological matrix model.

Finally, let us see how to explicitly map solutions of the matrix equations (4.1) onto noncommutative instanton solutions on the free module $\mathcal{A}^{\oplus N}$. The partial isometry equations (3.14) identify the k-dimensional subspace $V := \ker U_n$. In particular, the linear operator $U_1^\dagger$ has a trivial kernel, while U_1 has a one-dimensional kernel which is spanned by the vacuum vector $|0, 0, 0\rangle$. It follows that as $\mathcal{A}$-modules there is a natural isomorphism $\ker U_n \cong \mathcal{H}^{\oplus k}$ and hence one can identify the projective module (4.4) with

$$\mathcal{E}_{k,N} \cong \ker(U_n) \oplus \mathcal{A}^{\oplus N} . \tag{4.15}$$

We may now define an invertible mapping $\mu : \mathcal{E}_{k,N} \xrightarrow{\approx} \mathcal{A}^{\oplus N}$ by

$$\mu(\xi, f) = \xi + U_n^\dagger \cdot f \qquad \text{and} \qquad \mu^{-1}(f) = \left(\Pi_n(f), \, U_n \cdot f \right) . \tag{4.16}$$

Under this isomorphism of projective $\mathcal{A}$-modules, the image of an arbitrary covariant coordinate $Z \in \mathrm{End}_\mathcal{A}(\mathcal{E}_{k,N})$ represented as

$$Z = \begin{pmatrix} B & I \\ J & R \end{pmatrix} \tag{4.17}$$

is the element of $\mathrm{End}_\mathcal{A}(\mathcal{A}^{\oplus N}) \cong M_{N \times N}(\mathbb{C}) \otimes \mathcal{A}$ given by

$$\mu(Z) = \Pi_n \, B \, \Pi_n + \Pi_n \, I^\dagger \, I \, U_n + U_n^\dagger \, J \, J^\dagger \, \Pi_n + U_n^\dagger \, R \, U_n . \tag{4.18}$$

An explicit construction of the data (4.13) for $U(3)$ gauge group may be done by an elementary extension of the construction of [31] for the four-dimensional noncommutative $U(2)$ ADHM data. The $U(3)$ noncommutative instanton solutions of [27] can be written in the form $A = \Psi^\dagger \, \mathrm{d}\Psi$, where $\Psi \in \mathrm{Hom}_\mathcal{A}(W_0 \otimes \mathcal{A}, \mathcal{E}_{k,3})$. These solutions can again be extended to generic $U(N)$ gauge group following the technique explained in Section 3.3. We will not explore further details of this construction in this paper.

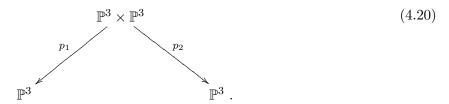### 4.3 Beilinson spectral sequence

In the remainder of this section we will show that the matrix equations (4.1) provide a geometrical interpretation of the noncommutative instantons satisfying the six-dimensional DUY equations on $\mathbb{C}^3$. For this, we will generalize an analogous construction in four dimensions exhibited by Nakajima in [32] (see also [33] and [34]). Since it is more convenient to work with compact spaces, we regard an instanton on $\mathbb{C}^3$ as a holomorphic bundle on $\mathbb{P}^3$ with a trivialization at infinity. To compactify the instanton moduli space we include all semistable torsion free sheaves on $\mathbb{P}^3$ with a trivialization condition at infinity and the appropriate topological quantum numbers of an instanton on $\mathbb{C}^3$. This allows for singularities of the instanton gauge field on $\mathbb{C}^3$, with the singularity locus the support where the torsion free sheaves fail to be holomorphic vector bundles.

We are thus interested in the framed moduli space of torsion free sheaves given by

$$\mathcal{M}_{N,k}\big(\mathbb{P}^3\big) = \left\{ \begin{array}{c} \mathcal{E} = \text{torsion free sheaf on } \mathbb{P}^3 \\ \mathrm{rank}(\mathcal{E}) = N , \quad c_1(\mathcal{E}) = 0 \\ \mathrm{ch}_2(\mathcal{E}) = 0 , \quad \mathrm{ch}_3(\mathcal{E}) = k \\ \mathcal{E}\big|_{p_\infty} \cong \mathcal{O}_{p_\infty}^{\oplus N} \end{array} \right\} \bigg/ \text{ isomorphisms} \tag{4.19}$$

22

where p_∞ is the plane at infinity. Since $\mathbb{C}^3 \cong \mathbb{P}^3/\mathbb{P}^2$, in projective coordinates one has explicitly $p_\infty = [0 : z_1 : z_2 : z_3] \cong \mathbb{P}^2$. The moduli space (4.19) corresponds to the counting of D0–D6 bound states on $\mathbb{C}^3$ in a suitable B-field background, as we demonstrate explicitly in the next section. Following [32], we will use the Beilinson spectral sequence to parametrize a generic torsion free sheaf $\mathcal{E}$ and then show that this spectral sequence degenerates into a complex of sheaves on $\mathbb{P}^3$ which is related to solutions of the matrix equations (4.1). The Beilinson theorem in our case implies that for any torsion free sheaf $\mathcal{E}$ on $\mathbb{P}^3$ there is a spectral sequence with s-th term $E_s^{p,q}$ which converges to $\mathcal{E}$ if $p + q = 0$ and to zero otherwise.

The Beilinson spectral sequence can be constructed by starting with the product space $\mathbb{P}^3 \times \mathbb{P}^3$ and the canonical projections onto the first and second factors

$$
\begin{array}{ccc}
 & \mathbb{P}^3 \times \mathbb{P}^3 & \\
 {}^{p_1}\swarrow & & \searrow^{p_2} \\
\mathbb{P}^3 & & \mathbb{P}^3 \ .
\end{array}
\tag{4.20}
$$

Then for any coherent sheaf $\mathcal{E}$ on $\mathbb{P}^3$ one has the projection formula

$$
p_{1\,*}\big(p_2^* \mathcal{E} \otimes \mathcal{O}_\Delta\big) = p_{1\,*}\big(p_2^* \mathcal{E}\big|_\Delta\big) = \mathcal{E}
\tag{4.21}
$$

where $\Delta \cong \mathbb{P}^3 \subset \mathbb{P}^3 \times \mathbb{P}^3$ is the diagonal. In the following we will use the notation

$$
\mathcal{E}_1 \boxtimes \mathcal{E}_2 := p_1^* \mathcal{E}_1 \otimes p_2^* \mathcal{E}_2
\tag{4.22}
$$

for the exterior product over $\mathbb{P}^3 \times \mathbb{P}^3$ of two sheaves $\mathcal{E}_1, \mathcal{E}_2$ on $\mathbb{P}^3$.

The next step consists in replacing the structure sheaf $\mathcal{O}_\Delta$ of the diagonal with an appropriate projective resolution, which we will take to be given by the Koszul complex. Consider the short exact sequence which defines the tangent bundle to $\mathbb{P}^3$ by

$$
0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 4} \longrightarrow T_{\mathbb{P}^3} \longrightarrow 0 \ .
\tag{4.23}
$$

If we tensor this sequence with $\mathcal{O}_{\mathbb{P}^3}(-1)$ we get the short exact sequence

$$
0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} \longrightarrow \mathcal{Q} \longrightarrow 0 \ ,
\tag{4.24}
$$

which defines the universal quotient bundle $\mathcal{Q} = T_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(-1)$ and its dual $\mathcal{Q}^* = \Omega_{\mathbb{P}^3}^1 \otimes \mathcal{O}_{\mathbb{P}^3}(1)$ where $\Omega_{\mathbb{P}^3}^1$ is the sheaf of one-forms on $\mathbb{P}^3$. Then the Koszul complex is given by [35]

$$
\begin{aligned}
0 \longrightarrow {\textstyle\bigwedge}^3\big(\mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathcal{Q}^*\big) &\longrightarrow {\textstyle\bigwedge}^2\big(\mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathcal{Q}^*\big) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathcal{Q}^* \longrightarrow \\
&\longrightarrow \mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^3} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0 \ .
\end{aligned}
\tag{4.25}
$$

From this complex we can construct a spectral sequence by taking an injective resolution of the hyperdirect image of eq. (4.21), and replacing $\mathcal{O}_\Delta$ with its Koszul resolution. More details can be found in [35, Chapter 2 §3.1] and [32, Chapter 2]. By introducing

$$
C^p := {\textstyle\bigwedge}^{-p}\big(\mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathcal{Q}^*\big)
\tag{4.26}
$$

the double complex can be expressed as the Fourier–Mukai transform

$$
\boldsymbol{R}^\bullet p_{1\,*}\big(p_2^* \mathcal{E} \otimes C^\bullet\big) \ ,
\tag{4.27}
$$

where $\boldsymbol{R}^{\bullet}$ is the right derived functor in the bounded derived category of coherent sheaves on $\mathbb{P}^3$. Note that each term of the Koszul resolution has the form

$$C^p = \mathcal{F}_1^p \boxtimes \mathcal{F}_2^p \,. \tag{4.28}$$

The Beilinson theorem then implies that for any coherent sheaf $\mathcal{E}$ on $\mathbb{P}^3$ there is a spectral sequence $E_s^{p,q}$ with E_1-term

$$E_1^{p,q} = \mathcal{F}_1^p \otimes H^q\left(\mathbb{P}^3, \, \mathcal{E} \otimes \mathcal{F}_2^p\right) \tag{4.29}$$

which converges to

$$E_\infty^{p,q} = \begin{cases} \mathcal{E}(-r) \,, & \text{if } p + q = 0 \,, \\ 0 \,, & \text{otherwise} \end{cases} \tag{4.30}$$

for each fixed integer $r \geq 0$, where we denote $\mathcal{E}(-r) := \mathcal{E} \otimes_{\mathcal{O}_{\mathbb{P}^3}} \mathcal{O}_{\mathbb{P}^3}(-r)$. Explicitly, the first term is given by

$$E_1^{p,q} = H^q\left(\mathbb{P}^3, \, \mathcal{E}(-r) \otimes \Omega_{\mathbb{P}^3}^{-p}(-p)\right) \otimes \mathcal{O}_{\mathbb{P}^3}(p) \tag{4.31}$$

for $p \leq 0$. The E_1-term complexes of the spectral sequence can be summarized in the diagram

$$\begin{array}{ccccccc}
E_1^{-3,3} & \xrightarrow{\ \ d_1\ \ } & E_1^{-2,3} & \xrightarrow{\ \ d_1\ \ } & E_1^{-1,3} & \xrightarrow{\ \ d_1\ \ } & E_1^{0,3} \\[2em]
E_1^{-3,2} & \xrightarrow{\ \ d_1\ \ } & E_1^{-2,2} & \xrightarrow{\ \ d_1\ \ } & E_1^{-1,2} & \xrightarrow{\ \ d_1\ \ } & E_1^{0,2} \\[2em]
E_1^{-3,1} & \xrightarrow{\ \ d_1\ \ } & E_1^{-3,1} & \xrightarrow{\ \ d_1\ \ } & E_1^{-1,1} & \xrightarrow{\ \ d_1\ \ } & E_1^{0,1} \\[2em]
E_1^{-3,0} & \xrightarrow{\ \ d_1\ \ } & E_1^{-2,0} & \xrightarrow{\ \ d_1\ \ } & E_1^{-1,0} & \xrightarrow{\ \ d_1\ \ } & E_1^{0,0}
\end{array} \tag{4.32}$$

where all other entries are zero for dimensional reasons and the only nonvanishing differential

$$d_1 \, : \, E_1^{p,q} \, \longrightarrow \, E_1^{p+1,q} \tag{4.33}$$

is determined by the morphisms in the Koszul complex (4.25). The double complex (4.32) is an object of the derived category. Our goal in the following is to reduce it to an object of the stable category of coherent sheaves on $\mathbb{P}^3$. Physically, this can be thought of as the stabilization of a topological B-model brane to a Type IIA D6-brane wrapping $\mathbb{P}^3$.

The last ingredient we will need comprises a few short exact sequences that will play an important role in explicit computations. They are given by

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\ \times z_0\ } \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\ z_0 = 0\ } \mathcal{O}_{p_\infty} \longrightarrow 0 \,, \tag{4.34}$$

$$0 \longrightarrow \Omega_{\mathbb{P}^3}^1 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 4} \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow 0 \,, \tag{4.35}$$

$$0 \longrightarrow \Omega_{\mathbb{P}^3}^2 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 6} \longrightarrow \Omega_{\mathbb{P}^3}^1 \longrightarrow 0 \,, \tag{4.36}$$

$$0 \longrightarrow \Omega_{\mathbb{P}^3}^3 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-3)^{\oplus 4} \longrightarrow \Omega_{\mathbb{P}^3}^2 \longrightarrow 0 \,. \tag{4.37}$$

The first sequence defines the plane at infinity $p_\infty = [0 : z_1 : z_2 : z_3] \cong \mathbb{P}^2$, while the other three sequences are the Euler sequences for differential forms on $\mathbb{P}^3$ obtained via truncation of the Koszul complex.

24

The final stage consists in imposing suitable boundary conditions on the sheaf $\mathcal{E}$. The appropriate boundary conditions were found in [8] and consist in imposing that the sheaf $\mathcal{E}$ restricted to the projective plane $\mathbb{P}^2$ at infinity corresponds to a four-dimensional instanton, viewed as an asymptote of the corresponding three-dimensional partition. By this we mean that it has the same cohomological properties as the torsion free sheaves on $\mathbb{P}^2$ studied in [32]. This condition should be imposed separately along each of the three complex "directions" in $\mathbb{C}^3$, i.e. the six-dimensional generalized instantons should behave in a different way when $z_i \to \infty$, $i = 1, 2, 3$, corresponding to generically distinct two-dimensional Young diagrams (partitions). The three corresponding projective planes at infinity are given in homogeneous coordinates by $p_\infty^1 = [z_0 : 0 : z_2 : z_3]$, $p_\infty^2 = [z_0 : z_1 : 0 : z_3]$ and $p_\infty^3 = [z_0 : z_1 : z_2 : 0]$, and each one contains a line ℓ_∞^i at infinity defined by $z_0 = 0$. We then impose the boundary condition that $\mathcal{E}$ is trivial *separately* on each one of these three lines. With this choice of trivializations one automatically has $c_1(\mathcal{E}) = 0$, and the sheaves $\mathcal{E}|_{p_\infty^i}$ have quantum numbers

$$c_1\big(\mathcal{E}|_{p_\infty^i}\big) = 0 \qquad \text{and} \qquad \mathrm{ch}_2\big(\mathcal{E}|_{p_\infty^i}\big) = c_2\big(\mathcal{E}|_{p_\infty^i}\big) = k_i \ . \tag{4.38}$$

These are torsion free sheaves on $\mathbb{P}^2$ of rank N trivialized on a line ℓ_∞^i which represent framed four-dimensional instantons of charge k_i. By T-duality, they correspond to D6–D2 bound states on $\mathbb{P}^3$ of D2-brane charge k_i.

The vacuum moduli space (4.19) is recovered by setting $k_i = \int_{p_\infty^i} \mathrm{ch}_2(\mathcal{E}) = 0$. These are the boundary conditions appropriate to the gauge theory on $\mathbb{C}^3$ and will be the case studied in this section and in Section 5. In Section 6 we will need the more general non-trivial asymptotics (4.38). We will now impose this boundary condition in three steps, dealing with the different terms of the spectral sequence.

4.4   Homological algebra

The first step is to determine $E_1^{-3,q}$ and $E_1^{0,q}$. These terms can be treated simultaneously because $\Omega_{\mathbb{P}^3}^3 \cong \mathcal{O}_{\mathbb{P}^3}(-4)$. Let us tensor the sequence (4.34) by $\mathcal{E}(-r)$ to get

$$0 \longrightarrow \mathcal{E}(-r-1) \longrightarrow \mathcal{E}(-r) \longrightarrow \mathcal{E}(-r)\big|_{p_\infty} \longrightarrow 0 \ , \tag{4.39}$$

where we have used the fact that $\mathcal{O}_{\mathbb{P}^2}$ is a locally free sheaf to set $\mathrm{Tor}^1(\mathcal{E}(-r)|_{p_\infty}, \mathcal{O}_{p_\infty}) = 0$. Applying the snake lemma one finds the associated long exact sequence in cohomology

$$0 \longrightarrow H^0\big(\mathbb{P}^3, \mathcal{E}(-r-1)\big) \longrightarrow H^0\big(\mathbb{P}^3, \mathcal{E}(-r)\big) \longrightarrow H^0\big(\mathbb{P}^2, \mathcal{E}(-r)\big|_{p_\infty}\big) \tag{4.40}$$

$$\longrightarrow H^1\big(\mathbb{P}^3, \mathcal{E}(-r-1)\big) \longrightarrow H^1\big(\mathbb{P}^3, \mathcal{E}(-r)\big) \longrightarrow H^1\big(\mathbb{P}^2, \mathcal{E}(-r)\big|_{p_\infty}\big)$$

$$\longrightarrow H^2\big(\mathbb{P}^3, \mathcal{E}(-r-1)\big) \longrightarrow H^2\big(\mathbb{P}^3, \mathcal{E}(-r)\big) \longrightarrow H^2\big(\mathbb{P}^2, \mathcal{E}(-r)\big|_{p_\infty}\big)$$

$$\longrightarrow H^3\big(\mathbb{P}^3, \mathcal{E}(-r-1)\big) \longrightarrow H^3\big(\mathbb{P}^3, \mathcal{E}(-r)\big) \longrightarrow 0 \ .$$

Since $\mathcal{E}|_{p_\infty}$ is a sheaf on $\mathbb{P}^2$ which is trivial when restricted on a projective line $\mathbb{P}^1 \subset \mathbb{P}^2$, we are in the same situation as in [32, Chapter 2]. It follows that

$$H^0\big(\mathbb{P}^2, \mathcal{E}(-r)\big|_{p_\infty}\big) = 0 \qquad \text{for} \qquad r \geq 1 \ , \tag{4.41}$$

25

$$H^2\left(\mathbb{P}^2,\,\mathcal{E}(-r)\big|_{p_\infty}\right)=0 \qquad \text{for} \qquad r\le 2\,. \tag{4.42}$$

By imposing this condition on the cohomology long exact sequence (4.40) we find

$$H^3\left(\mathbb{P}^3,\,\mathcal{E}(-r-1)\right)=H^3\left(\mathbb{P}^3,\,\mathcal{E}(-r)\right) \qquad \text{for} \qquad r\le 2\,, \tag{4.43}$$

$$H^0\left(\mathbb{P}^3,\,\mathcal{E}(-r-1)\right)=H^0\left(\mathbb{P}^3,\,\mathcal{E}(-r)\right) \qquad \text{for} \qquad r\ge 1\,. \tag{4.44}$$

By Serre's vanishing theorem [35], $H^q(\mathbb{P}^3,\mathcal{E}(m))=0$ is trivial for $m\gg 0$ and $q\ne 0$. Combined with Serre duality this implies

$$H^3\left(\mathbb{P}^3,\,\mathcal{E}(-r)\right)=0 \qquad \text{for} \qquad r\le 3\,, \tag{4.45}$$

$$H^0\left(\mathbb{P}^3,\,\mathcal{E}(-r)\right)=0 \qquad \text{for} \qquad r\ge 1\,. \tag{4.46}$$

The second step is to determine $E_1^{-1,q}$. Let us start by tensoring the exact sequence (4.34) with $\mathcal{E}(-r)\otimes\Omega^1_{\mathbb{P}^3}(1)$ to get

$$0\longrightarrow\Omega^1_{\mathbb{P}^3}(1)\otimes\mathcal{E}(-r-1)\longrightarrow\Omega^1_{\mathbb{P}^3}(1)\otimes\mathcal{E}(-r)\longrightarrow\left(\Omega^1_{\mathbb{P}^3}(1)\otimes\mathcal{E}(-r)\right)\big|_{p_\infty}\longrightarrow0\,. \tag{4.47}$$

Since $\left(\Omega^1_{\mathbb{P}^3}(1)\otimes\mathcal{E}(-r)\right)\big|_{p_\infty}$ is a sheaf defined on $p_\infty\cong\mathbb{P}^2$ which trivializes on a line $\mathbb{P}^1\subset\mathbb{P}^2$, we are again exactly in the situation of [32, Chapter 2] and hence

$$H^0\left(\mathbb{P}^2,\,\left(\Omega^1_{\mathbb{P}^3}(1)\otimes\mathcal{E}(-r)\right)\big|_{p_\infty}\right)=0 \qquad \text{for} \qquad r\ge 1\,,$$

$$H^2\left(\mathbb{P}^2,\,\left(\Omega^1_{\mathbb{P}^3}(1)\otimes\mathcal{E}(-r)\right)\big|_{p_\infty}\right)=0 \qquad \text{for} \qquad r\le 1\,. \tag{4.48}$$

From the corresponding long exact sequence in cohomology we have

$$H^2\left(\mathbb{P}^2,\,\left(\Omega^1_{\mathbb{P}^3}(1)\otimes\mathcal{E}(-r)\right)\big|_{p_\infty}\right)\longrightarrow H^3\left(\mathbb{P}^3,\,\Omega^1_{\mathbb{P}^3}(1)\otimes\mathcal{E}(-r-1)\right) \tag{4.49}$$

$$\longrightarrow H^3\left(\mathbb{P}^3,\,\Omega^1_{\mathbb{P}^3}(1)\otimes\mathcal{E}(-r)\right)\longrightarrow0\,.$$

By imposing the conditions (4.48) and (4.43) we get

$$H^3\left(\mathbb{P}^3,\,\Omega^1_{\mathbb{P}^3}(1)\otimes\mathcal{E}(-r)\right)=0 \qquad \text{for} \qquad r\le 2\,. \tag{4.50}$$

We get other conditions by tensoring the Euler sequence for one-forms (4.35) with $\mathcal{E}(-r)$ to get

$$0\longrightarrow\mathcal{E}(-r)\otimes\Omega^1_{\mathbb{P}^3}(1)\longrightarrow\mathcal{E}(-r)^{\oplus4}\longrightarrow\mathcal{E}(-r+1)\longrightarrow0\,. \tag{4.51}$$

The corresponding long exact cohomology sequence is

$$0\longrightarrow H^0\left(\mathbb{P}^3,\,\mathcal{E}(-r)\otimes\Omega^1_{\mathbb{P}^3}(1)\right)\longrightarrow H^0\left(\mathbb{P}^3,\,\mathcal{E}(-r)\right)^{\oplus4}\longrightarrow H^0\left(\mathbb{P}^3,\,\mathcal{E}(-r+1)\right) \tag{4.52}$$

$$\longrightarrow H^1\left(\mathbb{P}^3,\,\mathcal{E}(-r)\otimes\Omega^1_{\mathbb{P}^3}(1)\right)\longrightarrow H^1\left(\mathbb{P}^3,\,\mathcal{E}(-r)\right)^{\oplus4}\longrightarrow H^1\left(\mathbb{P}^3,\,\mathcal{E}(-r+1)\right)$$

$$\longrightarrow H^2\left(\mathbb{P}^3,\,\mathcal{E}(-r)\otimes\Omega^1_{\mathbb{P}^3}(1)\right)\longrightarrow H^2\left(\mathbb{P}^3,\,\mathcal{E}(-r)\right)^{\oplus4}\longrightarrow H^2\left(\mathbb{P}^3,\,\mathcal{E}(-r+1)\right)$$

$$\longrightarrow H^3\left(\mathbb{P}^3,\,\mathcal{E}(-r)\otimes\Omega^1_{\mathbb{P}^3}(1)\right)\longrightarrow H^3\left(\mathbb{P}^3,\,\mathcal{E}(-r)\right)^{\oplus4}\longrightarrow H^3\left(\mathbb{P}^3,\,\mathcal{E}(-r+1)\right)\longrightarrow0\,.$$

26

From the first line of this sequence, by imposing (4.46) we get

$$H^0\left(\mathbb{P}^3,\,\Omega^1_{\mathbb{P}^3}(1)\otimes\mathcal{E}(-r)\right)=0 \qquad \text{for} \qquad r\geq 1\,. \tag{4.53}$$

The third and final step is to determine $E_2^{-2,q}$. From the Euler sequence for three-forms (4.37) we get

$$0\longrightarrow\mathcal{E}(-r-2)\longrightarrow\mathcal{E}(-r-1)^{\oplus 4}\longrightarrow\mathcal{E}(-r)\otimes\Omega^2_{\mathbb{P}^3}(2)\longrightarrow 0\,. \tag{4.54}$$

Let us look at the corresponding long exact sequence in cohomology. It contains, in particular, the line

$$0\longrightarrow H^3\left(\mathbb{P}^3,\,\mathcal{E}(-r-2)\right)\longrightarrow H^3\left(\mathbb{P}^3,\,\mathcal{E}(-r-1)\right)^{\oplus 4} \tag{4.55}$$

$$\longrightarrow H^3\left(\mathbb{P}^3,\,\mathcal{E}(-r)\otimes\Omega^2_{\mathbb{P}^3}(2)\right)\longrightarrow 0\,.$$

By imposing the condition (4.45) one gets

$$H^3\left(\mathbb{P}^3,\,\mathcal{E}(-r)\otimes\Omega^2_{\mathbb{P}^3}(2)\right)=0 \qquad \text{for} \qquad r\leq 2\,. \tag{4.56}$$

The only short exact sequence that we haven't yet used is the Euler sequence for two-forms (4.36) from which it follows that

$$0\longrightarrow\mathcal{E}(-r)\otimes\Omega^2_{\mathbb{P}^3}\longrightarrow\mathcal{E}(-r-2)^{\oplus 6}\longrightarrow\mathcal{E}(-r)\otimes\Omega^1_{\mathbb{P}^3}\longrightarrow 0\,. \tag{4.57}$$

The corresponding long exact cohomology sequence yields

$$0\longrightarrow H^0\left(\mathbb{P}^3,\,\mathcal{E}(-r)\otimes\Omega^2_{\mathbb{P}^3}\right)\longrightarrow H^0\left(\mathbb{P}^3,\,\mathcal{E}(-r-2)\right)^{\oplus 6}\longrightarrow H^0\left(\mathbb{P}^3,\,\mathcal{E}(-r)\otimes\Omega^1_{\mathbb{P}^3}\right) \tag{4.58}$$

$$\longrightarrow H^1\left(\mathbb{P}^3,\,\mathcal{E}(-r)\otimes\Omega^2_{\mathbb{P}^3}\right)\longrightarrow H^1\left(\mathbb{P}^3,\,\mathcal{E}(-r-2)\right)^{\oplus 6}\longrightarrow H^1\left(\mathbb{P}^3,\,\mathcal{E}(-r)\otimes\Omega^1_{\mathbb{P}^3}\right)$$

$$\longrightarrow H^2\left(\mathbb{P}^3,\,\mathcal{E}(-r)\otimes\Omega^2_{\mathbb{P}^3}\right)\longrightarrow H^2\left(\mathbb{P}^3,\,\mathcal{E}(-r-2)\right)^{\oplus 6}\longrightarrow H^2\left(\mathbb{P}^3,\,\mathcal{E}(-r)\otimes\Omega^1_{\mathbb{P}^3}\right)$$

$$\longrightarrow H^3\left(\mathbb{P}^3,\,\mathcal{E}(-r)\otimes\Omega^2_{\mathbb{P}^3}\right)\longrightarrow H^3\left(\mathbb{P}^3,\,\mathcal{E}(-r-2)\right)^{\oplus 6}$$

$$\longrightarrow H^3\left(\mathbb{P}^3,\,\mathcal{E}(-r)\otimes\Omega^1_{\mathbb{P}^3}\right)\longrightarrow 0\,.$$

By using (4.46) and (4.53) we get

$$H^0\left(\mathbb{P}^3,\,\mathcal{E}(-r)\otimes\Omega^2_{\mathbb{P}^3}(2)\right)=0 \qquad \text{for} \qquad r\geq 1\,. \tag{4.59}$$

If we collect all the results obtained so far, we see that the values $r=1,2$ are rather special

since then all the cohomology groups H^0 and H^3 vanish and the spectral sequence becomes

$$
\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\[2mm]
E_1^{-3,2} \xrightarrow{\ d_1\ } & E_1^{-2,2} \xrightarrow{\ d_1\ } & E_1^{-1,2} \xrightarrow{\ d_1\ } & E_1^{0,2} \\[2mm]
E_1^{-3,1} \xrightarrow{\ d_1\ } & E_1^{-3,1} \xrightarrow{\ d_1\ } & E_1^{-1,1} \xrightarrow{\ d_1\ } & E_1^{0,1} \\[2mm]
0 & & 0 & & 0 & & 0
\end{array}
\tag{4.60}
$$

A great deal of information can now be obtained with the help of the Riemann–Roch theorem which computes the Euler character

$$
\chi(\mathcal{F}) = \sum_{q=0}^{3} (-1)^q \dim_{\mathbb{C}} H^q \left(\mathbb{P}^3 \,, \mathcal{F} \right) = \int_{\mathbb{P}^3} \operatorname{ch}(\mathcal{F}) \wedge \operatorname{td}\left(\mathbb{P}^3 \right)
\tag{4.61}
$$

for any sheaf $\mathcal{F}$ on $\mathbb{P}^3$. We are interested in the cases where $\mathcal{F}$ is $\mathcal{E}(-r)$, $\mathcal{E}(-r) \otimes \Omega^1_{\mathbb{P}^3}(1)$ and $\mathcal{E}(-r) \otimes \Omega^2_{\mathbb{P}^3}(2)$. Using (4.19) we parametrize the Chern character of $\mathcal{E}$ as

$$
\operatorname{ch}(\mathcal{E}) = N + k\,\xi^3
\tag{4.62}
$$

where $\xi = c_1(\mathcal{O}_{\mathbb{P}^3}(1))$ is the hyperplane class which generates the complex cohomology ring of the projective space $\mathbb{P}^3$.

The pertinent values of the Chern and Todd characteristic classes may be computed from the exact sequences (4.34)–(4.37) and are given by

$$
\begin{aligned}
\operatorname{ch}\left(\mathcal{O}_{\mathbb{P}^3}(-r) \right) &= \mathrm{e}^{-r\,\xi} \,, \\
\operatorname{ch}\left(\Omega^1_{\mathbb{P}^3} \right) &= 3 - 4\xi + 2\xi^2 - \tfrac{2}{3}\,\xi^3 \,, \\
\operatorname{ch}\left(\Omega^2_{\mathbb{P}^3} \right) &= 3 - 8\xi + 10\xi^2 - \tfrac{22}{3}\,\xi^3 \,, \\
\operatorname{td}\left(\mathbb{P}^3 \right) &= 1 + 2\xi + \tfrac{11}{6}\,\xi^2 + \xi^3 \,.
\end{aligned}
\tag{4.63}
$$

To apply the Riemann–Roch formula (4.61) we use the multiplicativity property of the Chern character $\operatorname{ch}(\mathcal{E}_1 \otimes \mathcal{E}_2) = \operatorname{ch}(\mathcal{E}_1) \wedge \operatorname{ch}(\mathcal{E}_2)$ and the fact that only the top form components ξ^3 survive the integration over $\mathbb{P}^3$. We will only need to deal with the case $r = 2$ explicitly below, for which the relevant results are

$$
\begin{aligned}
\chi\left(\mathcal{E}(-3) \right) &= k \,, \\
\chi\left(\mathcal{E}(-2) \otimes \Omega^2_{\mathbb{P}^3}(2) \right) &= 3k + N \,, \\
\chi\left(\mathcal{E}(-2) \otimes \Omega^1_{\mathbb{P}^3}(1) \right) &= 3k \,, \\
\chi\left(\mathcal{E}(-2) \right) &= k \,.
\end{aligned}
\tag{4.64}
$$

4.5   Nonlinear Beilinson monad

We will now impose an additional condition on the sheaf $\mathcal{E}$ which ensures that the spectral sequence becomes a four-term complex and that the Euler characteristics computed in eq. (4.64) above are

the stable dimensions (rather than virtual dimensions) of the vector spaces which appear in the complex. The condition we impose kills all the H^1 cohomology groups and simplifies the spectral sequence in the case $r = 2$. It is given by

$$H^1\left(\mathbb{P}^3,\, \mathcal{E}(-2)\right) = 0 \ . \tag{4.65}$$

This is the same condition which is placed on the holomorphic bundles over $\mathbb{P}^3$ that are used in the twistor construction of framed instantons in four dimensions [33]. It is also similar to the one defining the admissable sheaves of [36].

If we look at the long exact sequence (4.40) for $r = 2$ with the condition (4.41) and impose (4.65), we immediately see that

$$H^1\left(\mathbb{P}^3,\, \mathcal{E}(-3)\right) = 0 \ . \tag{4.66}$$

Moreover, if we consider the sequence (4.52) with $r = 2$ and impose both (4.46) and (4.65) we find

$$H^1\left(\mathbb{P}^3,\, \mathcal{E}(-2) \otimes \Omega^1_{\mathbb{P}^3}(1)\right) = 0 \ . \tag{4.67}$$

Finally, let us look at the cohomology sequence (4.58) for $r = 0$. By imposing (4.65) as well as (4.53) one finally finds

$$H^1\left(\mathbb{P}^3,\, \mathcal{E}(-2) \otimes \Omega^2_{\mathbb{P}^3}\right) = 0 \ . \tag{4.68}$$

The condition (4.65) thus automatically kills the $E_1^{p,1}$ line and the spectral sequence reduces to

This is our candidate for a generalized ADHM-like complex. The cohomology computations above also serve to show that the spectral sequence degenerates at the E_2-term. If we remember the Beilinson theorem, then we can write the cohomology of the differential complex in (4.69) as

Equivalently, we have the complex

$$0 \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}^3}(-3) \overset{a}{\longrightarrow} B \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \overset{b}{\longrightarrow} C \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \overset{c}{\longrightarrow} D \otimes \mathcal{O}_{\mathbb{P}^3}(0) \longrightarrow 0 \ , \tag{4.71}$$

where we have defined the complex vector spaces

$$
\begin{aligned}
V &= H^2\big(\mathbb{P}^3,\, \mathcal{E}(-3)\big) \,, \\
B &= H^2\big(\mathbb{P}^3,\, \mathcal{E}(-2) \otimes \Omega^2_{\mathbb{P}^3}(2)\big) \,, \\
C &= H^2\big(\mathbb{P}^3,\, \mathcal{E}(-2) \otimes \Omega^1_{\mathbb{P}^3}(1)\big) \,, \\
D &= H^2\big(\mathbb{P}^3,\, \mathcal{E}(-2)\big) \,.
\end{aligned}
\tag{4.72}
$$

It is straightforward to show that there is a natural identification $D = V$. Using the vanishing cohomology groups above, the long exact sequence (4.40) truncates for $r = 2$ to

$$
0 \longrightarrow H^1\big(\mathbb{P}^2,\, \mathcal{E}(-2)\big|_{p_\infty}\big) \longrightarrow H^2\big(\mathbb{P}^3,\, \mathcal{E}(-3)\big) \longrightarrow H^2\big(\mathbb{P}^3,\, \mathcal{E}(-2)\big) \longrightarrow 0 \,.
\tag{4.73}
$$

Since the four-dimensional instanton numbers in (4.38) are given by [32] $k_i = \dim_{\mathbb{C}} H^1(\mathbb{P}^2, \mathcal{E}(-2)|_{p^i_\infty})$, for the vacuum moduli space (4.19) one has $H^1(\mathbb{P}^2, \mathcal{E}(-2)|_{p_\infty}) = 0$ and hence

$$
H^2\big(\mathbb{P}^3,\, \mathcal{E}(-3)\big) = H^2\big(\mathbb{P}^3,\, \mathcal{E}(-2)\big) \,.
\tag{4.74}
$$

A similar argument provides a natural identification

$$
C = V \oplus V \oplus V \,.
\tag{4.75}
$$

Indeed, using (4.40) with $r = 1$ identifies $V = H^2(\mathbb{P}^3, \mathcal{E}(-1))$, since $H^2(\mathbb{P}^2, \mathcal{E}(-1)|_{p_\infty}) = 0$ exactly as above [32]. Then the exact sequence (4.52) with $r = 2$ implies (4.75).

By (4.70), the cohomology of the complex (4.71) gives the sheaf $\mathcal{E}(-2)$. To get a complex whose cohomology is simply $\mathcal{E}$ all we have to do is tensor by $\mathcal{O}_{\mathbb{P}^3}(2)$ to get the equivalent complex

$$
0 \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\ a\ } B \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\ b\ } V^{\oplus 3} \otimes \mathcal{O}_{\mathbb{P}^3}(1) \xrightarrow{\ c\ } V \otimes \mathcal{O}_{\mathbb{P}^3}(2) \longrightarrow 0 \,.
\tag{4.76}
$$

The crucial point is that the complex vector spaces appearing here are *finite dimensional.* If we look at eqs. (4.64), then one has $\dim_{\mathbb{C}} V = k$ and $\dim_{\mathbb{C}} B = 3k + N$.

Looking back at eq. (4.70), we see that the only non-trivial cohomology group of the complex (4.76) is the coherent sheaf

$$
\mathcal{E} = \underline{\ker}(b) \,/\, \underline{\mathrm{im}}(a) \,.
\tag{4.77}
$$

In particular, the map c is an epimorphism, a is a monomorphism, and $\underline{\ker}(c) = \underline{\mathrm{im}}(b)$. Using this last condition, we can truncate (4.76) to a three-term complex by replacing $V^{\oplus 3} \otimes \mathcal{O}_{\mathbb{P}^3}(1)$ with the locally free kernel sheaf $\mathcal{K}_c := \underline{\ker}(c)$, which describes the sheaf $\mathcal{E}$ as the cohomology of a *nonlinear monad*

$$
\mathcal{M}^\bullet \;:\; 0 \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\ a\ } B \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\ b\ } \mathcal{K}_c \longrightarrow 0
\tag{4.78}
$$

with $\dim_{\mathbb{C}} V = k$ and $\dim_{\mathbb{C}} B = 3k + N$, for which the map b is an epimorphism and a is a monomorphism with $b\,a = 0$.

Since $\mathcal{E}$ is the only nonvanishing cohomology of the complex (4.78), we can compute its Chern character through

$$
\mathrm{ch}(\mathcal{E}) = \mathrm{ch}(B \otimes \mathcal{O}_{\mathbb{P}^3}) - \mathrm{ch}\big(V \otimes \mathcal{O}_{\mathbb{P}^3}(-1)\big) - \mathrm{ch}(\mathcal{K}_c) \,.
\tag{4.79}
$$

By comparing with (4.62) and (4.63), we see that the bundle $\mathcal{K}_c$ on $\mathbb{P}^3$ has non-trivial characteristic classes in all degrees determined entirely by the instanton number k. In particular, one has

$$
\mathrm{rank}(\mathcal{K}_c) = 2k \qquad \text{and} \qquad c_1(\mathcal{K}_c) = k \,.
\tag{4.80}
$$

One also finds from (4.79) that $\mathcal{K}_c$ cannot be of the form $K \otimes \mathcal{O}_{\mathbb{P}^3}(1)$ for some $2k$-dimensional hermitean vector space K, reflecting the nonlinearity of the monad (4.78).

One can adapt the proof of [35, Lemma 4.1.3] to show that the complex (4.76) determines the sheaf $\mathcal{E}$ uniquely up to isomorphism. This is equivalent to the requirement that

$$\mathrm{Ext}^q\big(\mathcal{K}_c,\, \mathcal{O}_{\mathbb{P}^3}(-1)\big) = \mathrm{Ext}^q(\mathcal{K}_c, \mathcal{O}_{\mathbb{P}^3}) = 0 \tag{4.81}$$

for all q, which follows from the fact that the kernel sheaf is locally free and guarantees that the nonlinear monad (4.78) is unique up to isomorphism. In the following we will find it more convenient to work instead with the equivalent four term complex (4.76).

4.6   Barth description of nonlinear monads

We will now give a description of the moduli space (4.19), under the conditions spelled out above, in terms of linear algebra by generalizing the Barth description of linear monads [33]. This will contain as a special case the matrix equations (4.1) describing the six-dimensional noncommutative instantons. For this, we regard the morphism c in (4.76) as an element of $(V^*)^{\oplus 3} \otimes V \otimes \widetilde{W}$, where $\widetilde{W} = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ is the vector space spanned by the homogeneous coordinates z_0, z_1, z_2, z_3 of $\mathbb{P}^3$. The map c can then be written as

$$c = c_0\, z_0 + c_1\, z_1 + c_2\, z_2 + c_3\, z_3 \,, \tag{4.82}$$

where $c_i : V^{\oplus 3} \to V$ are constant linear maps. Similarly, we can represent the morphisms a and b of the four-term complex (4.76) as

$$a = a_0\, z_0 + a_1\, z_1 + a_2\, z_2 + a_3\, z_3 \qquad \text{and} \qquad b = b_0\, z_0 + b_1\, z_1 + b_2\, z_2 + b_3\, z_3 \tag{4.83}$$

where $a_i : V \to B$ and $b_i : B \to V^{\oplus 3}$ are constant linear maps. The monadic condition $b\,a = 0$ then implies that

$$b_i\, a_i = 0 \qquad \text{and} \qquad b_i\, a_j + b_j\, a_i = 0 \tag{4.84}$$

for each $i, j = 0, 1, 2, 3$ with $i < j$. On the other hand, the condition $\underline{\ker}(c) = \underline{\mathrm{im}}(b)$ implies that

$$\ker(c_i) = \mathrm{im}(b_i) \qquad \text{and} \qquad c_i\, b_j + c_j\, b_i = 0 \,. \tag{4.85}$$

Restricting the complex (4.76) to the line at infinity $\ell_\infty = [z_0 : z_1 : 0 : 0] \cong \mathbb{P}^1 \subset \mathbb{P}^3$ we get

$$0 \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}^3}(-1)\big|_{\ell_\infty} \xrightarrow{\;a_\infty\;} B \otimes \mathcal{O}_{\mathbb{P}^3}\big|_{\ell_\infty} \xrightarrow{\;b_\infty\;} V^{\oplus 3} \otimes \mathcal{O}_{\mathbb{P}^3}(1)\big|_{\ell_\infty} \xrightarrow{\;c_\infty\;} V \otimes \mathcal{O}_{\mathbb{P}^3}(2)\big|_{\ell_\infty} \longrightarrow 0 \tag{4.86}$$

where $a_\infty = a_0\, z_0 + a_1\, z_1$, $b_\infty = b_0\, z_0 + b_1\, z_1$ and $c_\infty = c_0\, z_0 + c_1\, z_1$. The kernel sheaf $\mathcal{K}_b = \underline{\ker}(b)$ is locally free, and one has the exact sequence of sheaves

$$0 \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}^3}(-1)\big|_{\ell_\infty} \xrightarrow{\;a_\infty\;} \mathcal{K}_b\big|_{\ell_\infty} \longrightarrow \mathcal{E}\big|_{\ell_\infty} \longrightarrow 0 \tag{4.87}$$

where the third arrow comes from the projection onto (4.77). Using the associated long exact sequence in cohomology, along with $H^q(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$, $q = 1, 2$ and $\mathcal{E}|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty}^{\oplus N}$, we conclude that $H^1(\mathbb{P}^1, \mathcal{K}_b|_{\ell_\infty}) = 0$ and $H^0(\mathbb{P}^1, \mathcal{K}_b|_{\ell_\infty}) \cong H^0(\mathbb{P}^1, \mathcal{E}|_{\ell_\infty}) \cong \mathcal{E}_p$ where $\mathcal{E}_p$ is the fibre of $\mathcal{E}$ at some point $p \in \ell_\infty$. We set

$$W = H^0\big(\mathbb{P}^1, \, \mathcal{K}_b|_{\ell_\infty}\big) \,. \tag{4.88}$$

This is a complex vector space of dimension $\mathrm{rank}(\mathcal{E}) = N$ and a choice of basis for W corresponds to a choice of trivialization for $\mathcal{E}|_{\ell_\infty}$. More general framing data, for which $\mathcal{E}|_{\ell_\infty}$ can contain a sum of non-trivial line bundles, are considered in [13].

31

Similarly, from the exact sequence

$$0 \longrightarrow \mathcal{K}_b\big|_{\ell_\infty} \longrightarrow B \otimes \mathcal{O}_{\mathbb{P}^3}\big|_{\ell_\infty} \xrightarrow{b_\infty} V^{\oplus 3} \otimes \mathcal{O}_{\mathbb{P}^3}(1)\big|_{\ell_\infty} \xrightarrow{c_\infty} V \otimes \mathcal{O}_{\mathbb{P}^3}(2)\big|_{\ell_\infty} \longrightarrow 0 \ , \qquad (4.89)$$

along with $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$ and $H^1(\mathbb{P}^1, \mathcal{K}_b\big|_{\ell_\infty}) = 0$, we obtain

$$0 \longrightarrow W \longrightarrow B \xrightarrow{b_\infty} V^{\oplus 3} \otimes H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)\big) \xrightarrow{c_\infty} V \otimes H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)\big) \longrightarrow 0 \ . \qquad (4.90)$$

Using the identifications $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = \mathbb{C}z_0 \oplus \mathbb{C}z_1$ and $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) = \mathbb{C}z_0^2 \oplus \mathbb{C}z_0\,z_1 \oplus \mathbb{C}z_1^2$, we can rewrite (4.90) as

$$0 \longrightarrow W \longrightarrow B \xrightarrow{\binom{b_0}{b_1}} V^{\oplus 6} \xrightarrow{\gamma_{0,1}} V^{\oplus 3} \longrightarrow 0 \ , \qquad (4.91)$$

where generally we define

$$\gamma_{i,j} := \begin{pmatrix} c_i & 0 \\ c_j & c_i \\ 0 & c_j \end{pmatrix} : V^{\oplus 3} \oplus V^{\oplus 3} \longrightarrow V \oplus V \oplus V \qquad (4.92)$$

for $0 \le i < j \le 3$. It follows that $W = \ker(b_0) \cap \ker(b_1)$ and that $\gamma_{0,1}^\dagger : (V^*)^{\oplus 3} \to (V^*)^{\oplus 6}$ is injective.

Let us now apply the same argument to the dual complex

$$0 \longrightarrow V^* \otimes \mathcal{O}_{\mathbb{P}^3}(-2)\big|_{\ell_\infty} \xrightarrow{c_\infty^\dagger} (V^*)^{\oplus 3} \otimes \mathcal{O}_{\mathbb{P}^3}(-1)\big|_{\ell_\infty} \xrightarrow{b_\infty^\dagger} B^* \otimes \mathcal{O}_{\mathbb{P}^3}\big|_{\ell_\infty} \xrightarrow{a_\infty^\dagger} V^* \otimes \mathcal{O}_{\mathbb{P}^3}(1)\big|_{\ell_\infty} \longrightarrow 0 \qquad (4.93)$$

whose cohomologies before restriction to ℓ_∞ are $\underline{\mathcal{H}om}(\mathcal{E}, \mathcal{O}_{\mathbb{P}^3})$ and $\underline{\mathcal{E}xt}^1(\mathcal{E}, \mathcal{O}_{\mathbb{P}^3})$, where we have used $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^3}(2), \mathcal{O}_{\mathbb{P}^3}) = 0$. After tensoring with $\mathcal{O}_{\mathbb{P}^1}(1)$, this yields the exact sequence

$$0 \longrightarrow H^0\big(\mathbb{P}^1, \underline{\ker}(c_\infty^\dagger)\big) \longrightarrow (V^*)^{\oplus 3} \xrightarrow{\binom{b_0}{b_1}^\dagger} B^* \oplus B^* \xrightarrow{\alpha_{0,1}^\dagger} (V^*)^{\oplus 3} \longrightarrow 0 \qquad (4.94)$$

which implies that the linear map

$$\alpha_{0,1} := \begin{pmatrix} a_0 & a_1 & 0 \\ 0 & a_0 & a_1 \end{pmatrix} : V \oplus V \oplus V \longrightarrow B \oplus B \qquad (4.95)$$

is injective. Using the exact sequence (4.91) along with $\dim_{\mathbb{C}} B = 3 \dim_{\mathbb{C}} V + \dim_{\mathbb{C}} W$, we can thus identify

$$B = V \oplus V \oplus V \oplus W \ . \qquad (4.96)$$

Furthermore, since

$$\ker(b_0)\,/\,\mathrm{im}(a_0) = \mathcal{E}_{[1:0:0:0]} = W = \ker(b_0) \cap \ker(b_1) \qquad (4.97)$$

one has $\mathrm{im}(a_0) \cap \ker(b_1) = 0$. It follows that $b_0\,a_1 = -b_1\,a_0 : V \to V^{\oplus 3}$ is injective.

Using the monadic conditions $b_0\,a_1 + b_1\,a_0 = 0$ and $b_0\,a_0 = 0 = b_1\,a_1$, we can choose bases for the vector spaces V, W, B and $C = V^{\oplus 3}$ such that the injection $b_0\,a_1$ is given by

$$b_0\,a_1 = \begin{pmatrix} 0_{k \times k} \\ 0_{k \times k} \\ 1_{k \times k} \end{pmatrix} \qquad (4.98)$$

32

along with

$$a_0 = \begin{pmatrix} 0_{k\times k} \\ 1_{k\times k} \\ 0_{k\times k} \\ 0_{N\times k} \end{pmatrix} \quad \text{and} \quad b_0 = \begin{pmatrix} 1_{k\times k} & 0_{k\times k} & 0_{k\times k} & 0_{k\times N} \\ 0_{k\times k} & 0_{k\times k} & 0_{k\times k} & 0_{k\times N} \\ 0_{k\times k} & 0_{k\times k} & 1_{k\times k} & 0_{k\times N} \end{pmatrix} ,$$

$$a_1 = \begin{pmatrix} 0_{k\times k} \\ 0_{k\times k} \\ 1_{k\times k} \\ 0_{N\times k} \end{pmatrix} \quad \text{and} \quad b_1 = \begin{pmatrix} 0_{k\times k} & 0_{k\times k} & 0_{k\times k} & 0_{k\times N} \\ -1_{k\times k} & 0_{k\times k} & 0_{k\times k} & 0_{k\times N} \\ 0_{k\times k} & -1_{k\times k} & 0_{k\times k} & 0_{k\times N} \end{pmatrix} . \tag{4.99}$$

Using the conditions $\ker(c_0) = \mathrm{im}(b_0)$, $\ker(c_1) = \mathrm{im}(b_1)$ and $c_0\, b_1 + c_1\, b_0 = 0$ we can then write the maps

$$c_0 = \begin{pmatrix} 0_{k\times k} & 1_{k\times k} & 0_{k\times k} \end{pmatrix} \quad \text{and} \quad c_1 = \begin{pmatrix} 1_{k\times k} & 0_{k\times k} & 0_{k\times k} \end{pmatrix} . \tag{4.100}$$

Using the equations $b_2\, a_0 + b_0\, a_2 = 0 = b_2\, a_1 + b_1\, a_2$ and $b_3\, a_0 + b_0\, a_3 = 0 = b_3\, a_1 + b_1\, a_3$, together with $c_2\, b_0 + c_0\, b_2 = 0 = c_2\, b_1 + c_1\, b_2$ and $c_3\, b_0 + c_0\, b_3 = 0 = c_3\, b_1 + c_1\, b_3$, we can parametrize the remaining linear maps in the form

$$a_2 = \begin{pmatrix} B_1' \\ B_1 \\ B_2 \\ J \end{pmatrix} \quad \text{and} \quad b_2 = \begin{pmatrix} C_1 & -B_1' & 0_{k\times k} & 0_{k\times N} \\ C_2 & 0_{k\times k} & B_1' & 0_{k\times N} \\ C_3 & -B_2 & B_1 & I \end{pmatrix} ,$$

$$a_3 = \begin{pmatrix} B_3 \\ B_2' \\ B_3' \\ K \end{pmatrix} \quad \text{and} \quad b_3 = \begin{pmatrix} C_1' & -B_3 & 0_{k\times k} & 0_{k\times N} \\ C_2' & 0_{k\times k} & B_3 & 0_{k\times N} \\ C_3' & -B_3' & B_2' & I' \end{pmatrix} \tag{4.101}$$

with $B_i, B_i', C_i, C_i' \in M_{k\times k}(\mathbb{C})$, $J, K \in M_{N\times k}(\mathbb{C})$ and $I, I' \in M_{k\times N}(\mathbb{C})$, along with

$$c_2 = \begin{pmatrix} -C_2 & C_1 & -B_1' \end{pmatrix} \quad \text{and} \quad c_3 = \begin{pmatrix} -C_2' & C_1' & -B_3 \end{pmatrix} . \tag{4.102}$$

It remains to satisfy the remaining equations in (4.84) and (4.85). From the last three equations $b_2\, a_2 = 0 = b_3\, a_3$ and $b_2\, a_3 + b_3\, a_2 = 0$ of the monadic condition we get the respective sets of matrix relations

$$\begin{aligned}
B_1'\, B_1 &= C_1\, B_1' & \text{and} & & B_3\, B_2' &= C_1'\, B_3 , \\
B_1'\, B_2 &= -C_2\, B_1' & \text{and} & & B_3\, B_3' &= -C_2'\, B_3 , \\
[B_1, B_2] + I\, J &= -C_3\, B_1' & \text{and} & & [B_2', B_3'] + I'\, K &= -C_3'\, B_3
\end{aligned} \tag{4.103}$$

along with

$$\begin{aligned}
B_3\, B_1 + B_1'\, B_2' &= C_1\, B_3 + C_1'\, B_1' , \\
B_3\, B_2 + B_1'\, B_3' &= -C_2\, B_3 - C_2'\, B_1' , \\
[B_2, B_2'] + [B_3', B_1] - I\, K - I'\, J &= C_3\, B_3 + C_3'\, B_1' .
\end{aligned} \tag{4.104}$$

On the other hand, from the remaining conditions $c_2\, b_2 = 0 = c_3\, b_3$ we get the respective additional matrix equations

$$\begin{aligned}
[C_1, C_2] &= B_1'\, C_3 & \text{and} & & [C_1', C_2'] &= B_3\, C_3' , \\
B_1'\, B_2 &= -C_2\, B_1' & \text{and} & & B_3\, B_3' &= -C_2'\, B_3 , \\
B_1'\, I &= 0 & \text{and} & & B_3\, I' &= 0 ,
\end{aligned} \tag{4.105}$$

while from $c_2 b_3 + c_3 b_2 = 0$ we find

$$\begin{aligned}
[C_1, C_2'] + [C_1', C_2] &= B_3 C_3 + B_1' C_3' \,, \\
B_3 B_2 + B_1' B_3' &= -C_2 B_3 + C_2' B_1' \,, \\
B_3 B_1 + B_1' B_2' &= C_1 B_3 + C_1' B_1' \,, \\
B_3 I &= -B_1' I' \,.
\end{aligned} \tag{4.106}$$

To investigate the solution space of the somewhat complicated system of matrix equations (4.103)–(4.106), we should recall that the boundary conditions appopriate to the six-dimensional generalized instantons on $\mathbb{C}^3$ require trivializations on *three* independent lines $\mathbb{P}^1 \subset \mathbb{P}^3$. We will choose the remaining two lines to be given in homogeneous coordinates by $[z_0 : 0 : z_2 : 0]$ and $[0 : z_1 : z_2 : 0]$. We then restrict the complex (4.76) to these lines exactly as before. By an identical analysis to that given above one arrives at analogous constraints on the kernels and ranges of the linear maps involved, which then imposes additional restrictions on the matrices satisfying eqs. (4.103)–(4.106) above.

For instance, the injectivity of the linear maps $b_0 a_2$ and $b_1 a_2$ lead respectively to the kernel constraints

$$\ker(B_1') \cap \ker(B_2) = 0 \qquad \text{and} \qquad \ker(B_1') \cap \ker(B_1) = 0 \,, \tag{4.107}$$

which automatically guarantees all other vanishing conditions on the maps a_i and b_i. Likewise, injectivity of $\gamma_{0,2}^\dagger$ and $\gamma_{1,2}^\dagger$ lead respectively to the range constraints

$$\mathrm{im}(B_1') \cap \mathrm{im}(C_2) = 0 \qquad \text{and} \qquad \mathrm{im}(B_1') \cap \mathrm{im}(C_1) = 0 \,. \tag{4.108}$$

Finally, the framing data determine isomorphisms between the Chan–Paton space (4.88) and the two subspaces of vectors:

- $(v, w) \in \big(\ker(B_1') \cap \ker(C_2) \big) \oplus W$ for which $B_2(v) = I(w)$; and

- $(v, w) \in V \oplus W$ for which $B_1(v) = -I(w)$.

We interpret these extra constraints as stability conditions on the linear maps satisfying the matrix equations (4.103)–(4.106). The issue of stability will be discussed further in Section 4.7 below and in Section 5.

To illustrate the use of these extra conditions, suppose that a vector $v \in V$ satisfies the equation $B_1' B_1(v) = C_1 B_1'(v)$. Then by (4.108) one has $B_1(v) = B_1'(v) = 0$, and hence $v = 0$ by (4.107). This implies that the linear map $B_1' B_1 - C_1 B_1' : V \to V$ is bijective, which contradicts the first equation of (4.103). Thus we set $B_1' = 0$. Then (4.103) implies the first equation of the set (4.1). It is straightforward to find that the next two equations of (4.1) follow similarly, after some identifications amongst the various matrices. As in Section 4.2, there are generically more moduli and equations than needed to identify a torsion free sheaf on $\mathbb{C}^3$ with the set of matrix equations (4.1). The role of these extra moduli will be explained in Section 5.2.

4.7 Stability

There is a natural free action of the group $GL(k, \mathbb{C})$ on the data $(B_i, B_i', C_i, C_i', J, K, I, I')$ of Section 4.6 above, of the form in (4.2) with $g \in GL(k, \mathbb{C})$. As in the case of linear monads [35], one can show that any isomorphism of a complex (4.76) which preserves the trivializations on the lines ℓ_∞^i and the choice of bases for the vector spaces V, W, B and C made above has this form. We can

restrict the complex (4.76) to $\mathbb{C}^3 = \mathbb{P}^3 \setminus \mathbb{P}^2$ by setting $z_0 = 1$ and replacing $\mathcal{O}_{\mathbb{P}^3}(-r)$ by $\mathcal{O}_{\mathbb{C}^3}$ everywhere. Then on the fibre at $z = (z_1, z_2, z_3) \in \mathbb{C}^3$, the morphisms a, b and c induce homomorphisms of vector spaces

$$V \xrightarrow{\ \sigma_z\ } \begin{matrix} V \oplus V \oplus V \\ \oplus \\ W \end{matrix} \xrightarrow{\ \tau_z\ } V \oplus V \oplus V \xrightarrow{\ \eta_z\ } V \ . \tag{4.109}$$

Following the analog statement for linear monads on $\mathbb{P}^d$ [36], one can show that whenever $N \geq 3$ the cohomology sheaf (4.77) is nondegenerate, and hence that the cohomology of a generic complex (4.76) is a torsion free sheaf on $\mathbb{P}^3$ with the stated properties. Then from (4.107), along with suitable identifications of the various matrices, it follows that the localized maps $a_z \in \mathrm{Hom}(V, B)$ are injective for all $z \in \mathbb{P}^3$. On the other hand, surjectivity of $c_z \in \mathrm{Hom}(C, V)$ and exactness $\ker(c_z) = \mathrm{im}(b_z)$ for all $z \in \mathbb{P}^3$, together with (4.108) and appropriate matrix identifications, presumably implies a certain algebraic stability condition analogous to the four-dimensional case [32]. Using stability along with the fact that B_2 and B_3 commute, one can repeat the proof of [32, Proposition 2.7] step by step to show that in the abelian case the maps

$$J = K = 0 \qquad \text{for} \quad N = 1 \tag{4.110}$$

and the cohomology sheaf (4.77) is isomorphic to the ideal $\mathcal{I}$ given in the description of the Hilbert scheme $\mathsf{Hilb}^k(\mathbb{C}^3)$ in terms of instantons in noncommutative gauge theory on $\mathbb{C}^3$ presented in the previous section. The algebraic stability condition also follows from the last matrix equation in (4.1) which encodes the noncommutative deformation. We will see this explicitly in the next section wherein we shall make the instanton complex (4.109) equivariant with respect to the toric action following [32, Chapter 5] and [37], and utilize the powerful localization techniques of a suitably defined topological matrix model. This is tantamount to introducing the matrix equations (4.3) of the Ω-background.

5 Topological matrix quantum mechanics

In this section we will carry on with the case $X = \mathbb{C}^3$ to avoid complications coming from a non-trivial topology of the ambient space. In this context the gauge theory computes bound states of D6 and D0 branes with a suitable B-field turned on. As in the four-dimensional case there are two ways of doing the instanton counting computation. One way is to write down the noncommutative gauge theory directly, solve for the critical points and compute the fluctuation determinants around each critical point, as we did in Section 3. The other way is to consider the effective field theory of a gas of k D0-branes coupled to the D6-branes. This is equivalent to a topological matrix quantum mechanics on the resolved moduli space. For instance, we may think of this field theory as arising from quantization of the collective coordinates around a fixed point of the original gauge theory. In the static limit, relevant for considerations involving the BPS ground states, this gives rise to an ADHM-like formalism which provides a dynamical realization of the matrix equations describing the torsion free sheaves of the previous section. This will nicely tie the computation in terms of noncommutative instantons into one involving the abelian category of holomorphic D-branes (coherent sheaves).

The set of fields and equations of motion involved in the matrix quantum mechanics can be nicely interpreted as a representation of a quiver with relations, an oriented graph consisting of nodes and arrows together with linear combinations of paths in the graph. To provide a representation of the quiver diagram in the category of complex vector spaces, one associates a set of vector spaces to the nodes and linear maps ("fields") to the arrows which respect the relations ("constraints"). In the large volume limit one can represent a D-brane configuration, viewed as an object in the

abelian category of coherent sheaves, as a representation of a quiver. This ties in nicely with the varieties naturally associated to quivers that were introduced by Nakajima to study ordinary instanton moduli spaces.[1] One may regard the quiver construction presented in the following as a generalization of this formalism to study moduli spaces of solutions of the DUY equations. Physically we are left with a supersymmetric quantum mechanics that lives on the moduli space of the quiver variety, whose supersymmetric ground states correspond to BPS bound states in the original D-brane picture. The quiver quantum mechanics that describes the BPS states of the D6–D0 system with an appropriate B-field turned on was introduced in [12]. The connection between the supersymmetric quantum mechanics and the noncommutative gauge theory description of the D6–D0 system was suggested originally in [14], where it was also shown that this system is non-supersymmetric in the absence of a B-field. This means that the noncommutative deformation is crucial for obtaining stable instanton solutions and that a description involving torsion free sheaves is unavoidable in this context.

Ultimately the quiver approach and the noncommutative gauge theory give equivalent descriptions of the same physical problem, but it is important to bear in mind that in the noncommutative computation one is expanding the gauge theory path integral around critical points and computing the quantum fluctuations around the classical solutions. On the other hand, the matrix quantum mechanics already contains the fluctuation factors. By fixing the number of D0-branes to be k (and likewise the rank of the matrices), we are automatically working in what from the gauge theory perspective would be the charge k instanton sector. The quiver matrix model computes directly the equivariant volume of this moduli space. Finally, one takes back this information to the gauge theory. In the gauge theory, by expanding around any critical point one is left with a matrix model on the instanton moduli space. This is what the quiver matrix model computes.

In this section we will begin by briefly reviewing the typical computation of ratios of fluctuation determinants in cohomological field theory, following the treatment of [38], and then apply this formalism to the quiver matrix quantum mechanics. To compute the quiver partition function we will also give a classification of the critical points of the matrix model. The path integral localizes onto fixed points of the toric action and each fixed point is given by a solution of the equations (4.1). The equivariant character of the complex (4.109) at a fixed point represents the linearized contribution of the fixed point to the partition function. Finally, the computation will be completed by using the localization formula. We will obtain exact agreement with the results coming from noncommutative gauge theory, reproducing in particular the partition function (3.1).

5.1 *Cohomological field theory formalism*

Let us start with a set of equations $\vec{\mathcal{E}} = \vec{0}$. In our applications these will either be the ADHM-like equations of the previous section or the six-dimensional DUY equations of Section 3. These equations will be functions of some complex fields which we denote collectively by X_i and are assumed to transform in the adjoint representation of some $U(k)$ gauge symmetry group. To construct a supersymmetric field theory one needs to supplement these fields with superpartners to form multiplets (X_i, Ψ_i) with BRST transformations

$$QX_i = \Psi_i \qquad \text{and} \qquad Q\Psi_i = [\phi, X_i] \ . \tag{5.1}$$

The X_i are coordinates on the field space, which after imposing the constraints and gauge invariance becomes the moduli space, and Ψ_i are their differentials.

---

[1]The original motivation was to understand moduli spaces of four-dimensional instantons on singular varieties, but the formalism can be extended to smooth spaces as well and was indeed used in [32].

To these fields we add the Fermi multiplet of antighosts and auxiliary fields $(\vec{\chi}, \vec{H})$ associated with the equations $\vec{\mathcal{E}} = \vec{0}$. Schematically, the bosonic part of the action contains a term

$$S_{\text{bos}} = \text{i Tr } \vec{\mathcal{E}} \cdot \vec{H} + \text{Tr } \vec{H}^2 , \qquad (5.2)$$

which on-shell gives $\vec{H} = -\frac{\text{i}}{2} \vec{\mathcal{E}}$ and $S_{\text{bos}} = \frac{1}{4} \text{Tr } \vec{\mathcal{E}}^2$. Because of this, the multiplet $(\vec{\chi}, \vec{H})$ has the same quantum numbers as the equations $\vec{\mathcal{E}}$. Finally, one adds the gauge multiplet $(\phi, \overline{\phi}, \eta)$ necessary to close the BRST algebra, where ϕ is the generator of gauge transformations which is related to the Higgs field Φ of the noncommutative gauge theory via the decomposition (4.14). Their BRST transformations are given by

$$Q\phi = 0 , \quad Q\overline{\phi} = \eta \quad \text{and} \quad Q\eta = [\phi, \overline{\phi}] . \qquad (5.3)$$

We will split the set of equations $\vec{\mathcal{E}}$ into two sets $(\vec{\mathcal{E}}_c = \vec{0}, \mathcal{E}_r = 0)$ of complex and real equations, respectively, which play a role analogous to the F-term and D-term conditions. The latter condition can be regarded as a stability condition.

We will also want to work equivariantly with respect to some toric action $\mathbb{T}^d$. Let us choose the rotations $X_i \rightarrow X_i \, \text{e}^{-\text{i}\epsilon_i}$ for some parameters ϵ_i that generate the toric action. The BRST transformations are then modified to

$$QX_i = \Psi_i \qquad \text{and} \qquad Q\Psi_i = [\phi, X_i] - \epsilon_i X_i , \qquad (5.4)$$

where as usual the transformation of the fermions reflects the infinitesimal transformation of X_i under the symmetry group $U(k) \times \mathbb{T}^d$. Let us consider now the Fermi multiplet $(\vec{\chi}_c, \chi_r, \vec{H}_c, H_r)$ associated with the equations $(\vec{\mathcal{E}}_c, \mathcal{E}_r)$. The equations $\vec{\mathcal{E}}_c$ transform as an adjoint field under $U(k)$ and as $\text{e}^{\,\text{i}\epsilon_c}$ under the toric action, where ϵ_c denotes some linear combination of the toric parameters ϵ_i that may be different for each equation. The second equation $\mathcal{E}_r$ again transforms in the adjoint representation of $U(k)$ but is invariant under the toric action. These conditions determine the BRST transformation rules for the Fermi multiplets to be

$$\begin{aligned} Q\chi_{c,i} &= H_{c,i} & \text{and} \quad QH_{c,i} &= [\phi, \chi_{c,i}] - \epsilon_{c,i} \chi_{c,i} , \\ Q\chi_r &= H_r & \text{and} \quad QH_r &= [\phi, \chi_r] , \end{aligned} \qquad (5.5)$$

where the label i runs over the set of equations of $\vec{\mathcal{E}}_c$. Finally, the gauge multiplet transforms again as in (5.3).

The action of the cohomological gauge theory can be represented as

$$S = Q \text{ Tr } \left(\eta [\phi, \overline{\phi}] - \vec{\chi} \cdot \vec{\mathcal{E}} + g \vec{\chi} \cdot \vec{H} + \Psi_i [X_i, \overline{\phi}] \right) \qquad (5.6)$$

and the path integral localizes onto the solutions of the equations $\vec{\mathcal{E}} = \vec{0}$. Note that the path integral is independent of the coupling constant g, as usual in cohomological gauge theories. Let us now explicitly evaluate the path integral. The first step is to use the $U(k)$ gauge invariance to diagonalize the gauge generator ϕ. This produces a Vandermonde determinant $\det(\text{ad}\,\phi)$ in the path integral measure. Then one would like to integrate out the fields $\vec{\chi}$, which appear quadratically in the action. However, this is not immediately possible. By looking at the BRST transformations (5.5) we see that the mass matrix of χ_r, $\text{Tr } \vec{\chi}_c \cdot [\phi, \vec{\chi}_c]$, can have zero modes (while this is not the case for $\vec{\chi}_c$ if $\epsilon_{c,i}$ are generic). To cure this problem we add the term

$$t_1 \, Q \text{ Tr } \chi_r \, \overline{\phi} \qquad (5.7)$$

to the action and take the limits $t_1 \to \infty$ and $g \to \infty$ with $g \ll t_1$. This is a legitimate procedure since the quantum field theory is topological and independent of t_1. But in this way the resulting action has no kinetic term for the fermions, and so we add it by hand via the term

$$t_2 \, Q \operatorname{Tr} \left(X_i \, \Psi_i^\dagger - X_i^\dagger \, \Psi_i \right) . \tag{5.8}$$

Again this is legitimate since we are adding a BRST-exact term to the action and so the path integral will not change.

We can now proceed in three steps:

- Take the limit $t_1 \to \infty$. The relevant part of the action is

$$t_1 \operatorname{Tr} \left(H_r \, \overline{\phi} + \chi_r \, \eta \right) \tag{5.9}$$

  and these fields can be trivially integrated out. This means that in the following expressions we can neglect them since their contributions to the path integral are suppressed in the $t_1 \to \infty$ limit.

- Now take the limit $g \to \infty$. The relevant part of the action is quadratic in $\vec{\chi}_c$ (and in $\vec{H}_c$). We can integrate them out and the result is a factor

$$\det \left(\operatorname{ad} \phi - \epsilon_{c,i} \right) \tag{5.10}$$

  for each of the fields $\chi_{c,i}$. Note that we get a determinant and not its square root since $\chi_{c,i}$ is a complex field. The determinant is in the numerator since the field $\chi_{c,i}$ obeys Fermi statistics.

- Finally, we take the limit $t_2 \to \infty$ and integrate out the fields X_i (and Ψ_i) which now appear quadratically in the action. We obtain a factor

$$\frac{1}{\det \left(\operatorname{ad} \phi - \epsilon_i \right)} \tag{5.11}$$

  for each of the fields. Again we get determinants since X_i are complex fields, but now in the denominator since they are bosonic.

We have dropped various normalization factors coming from the gaussian integrals that involve the couplings g, t_1 and t_2, which as expected cancel between the bosonic and fermionic integrations, and some ratio of the toric parameters ϵ_i which depends on the choice of equivariant action. It is important to remember that we still have the Cartan subalgebra integral over the diagonal field ϕ left to do.

When the equations are the ADHM-like or the DUY equations the resulting integral represents the instanton fluctuation factor in the charge k instanton sector. In this approach k is fixed and is the rank of the matrices X_i. The fluctuation determinant has the following structure. In the numerator there appear the Vandermonde determinant and a determinant due to the equations $\vec{\mathcal{E}}_c = \vec{0}$. In the denominator the factors are related to the fields X_i and reflect their quantum numbers. The determinant is of the form (constraint)/(fields) and this structure generalizes to any topological matrix quantum mechanics [39]. We will make extensive use of this in the following.[2]

---

[2]The Vandermonde determinant is not really a constraint but in a certain sense is related to the equation $\mathcal{E}_r = 0$. Recall that it arises when one uses gauge invariance to diagonalize ϕ.

We are now ready to systematically construct the topological quiver quantum mechanics of [12] and compute its path integral. We introduce two vector spaces V and W of complex dimensions $\dim_{\mathbb{C}} V = k$ and $\dim_{\mathbb{C}} W = N$. In the noncommutative gauge theory the space V is an "internal" finite-dimensional subspace of the Fock module $\mathcal{H}$, represented geometrically by sheaf cohomology groups in (4.72), while W is an "external" Chan–Paton space determined geometrically by framing data through the sheaf cohomology group (4.88). In the quiver description they represent the spaces where the fields of the matrix quantum mechanics take values. In the D-brane picture V is spanned by the gas of k D0-branes, while W represents the N (spectator) D6-branes. In this description we fix the topological sector and restrict attention to instantons of charge k. As in the previous sections we will keep the number N of D6-branes arbitrary for formal considerations, but concrete computations will require an explicit choice of gauge symmetry breaking pattern.

The fields of the quiver are given by

$$\begin{aligned} X_i &= (B_1, B_2, B_3, \varphi, I) \;, \\ \Psi_i &= (\psi_1, \psi_2, \psi_3, \zeta, \rho) \;. \end{aligned} \tag{5.12}$$

The matrices B_i arise from 0–0 strings and represent the position of the coincident D0-branes inside the D6-branes. One may regard the fields (B_1, B_2, B_3, φ) as arising from the reduction to zero dimensions of the six-dimensional Yang–Mills multiplet (Z_1, Z_2, Z_3, ρ). On the other hand, the field I describes open strings stretching from the D6-branes to the D0-branes and is characteristic of the quiver formalism. It characterizes the size and orientation of the D0-branes inside the D6-branes, and is required to make the system supersymmetric. Thus the bosonic fields are defined as linear maps

$$\begin{aligned} (B_1, B_2, B_3, \varphi) &\in \operatorname{Hom}(V, V) \;, \\ I &\in \operatorname{Hom}(W, V) \;. \end{aligned} \tag{5.13}$$

The matrices B_i and I originate in the noncommutative gauge theory through the decompositions of the covariant coordinates (4.5) and of the Higgs field (4.14). As we will localize below onto the maximal torus of the $U(N)$ gauge group, we can neglect the remaining fields J and K using (4.110).

The fields B_i and φ all lie in the adjoint representation of $U(k)$ where k is the number of D0-branes (or the instanton number). From the dimensional reduction, we can identify their transformation properties under the toric $\mathbb{T}^3$ action as

$$\begin{aligned} B_i &\longmapsto B_i \, \mathrm{e}^{-\mathrm{i}\,\epsilon_i} \;, \\ \varphi &\longmapsto \varphi \, \mathrm{e}^{-\mathrm{i}\,(\epsilon_1 + \epsilon_2 + \epsilon_3)} \;. \end{aligned} \tag{5.14}$$

We will frequently use the notation $\epsilon = \epsilon_1 + \epsilon_2 + \epsilon_3$ (with $\epsilon = 0$ when we wish to make the Calabi–Yau condition (3.9) explicit). On the other hand, I is a $U(k) \times U(N)$ bifundamental field where N is the number of D6-branes (or the rank of the six-dimensional gauge theory). Under the full symmetry group $U(k) \times U(N) \times \mathbb{T}^3$ it transforms as

$$I \longmapsto g_{U(k)} \, I \, g_{U(N)}^{\dagger} \, \mathrm{e}^{-\mathrm{i}\varepsilon} \;. \tag{5.15}$$

The transformation of I under the toric action $\mathbb{T}^3$ is not fixed by any constraint. In the following we will argue that $\varepsilon = 0$ (although the precise value of the toric parameter ε is not important for the evaluation of the path integral).

The corresponding BRST transformations read

$$
\begin{aligned}
QB_i &= \psi_i && \text{and} && Q\psi_i = [\phi, B_i] - \epsilon_i B_i \ , \\
Q\varphi &= \zeta && \text{and} && Q\zeta = [\phi, \varphi] - \epsilon\, \varphi \ , \\
QI &= \rho && \text{and} && Q\rho = \phi\, I - I\,\boldsymbol{a} - \varepsilon\, I \ ,
\end{aligned}
\tag{5.16}
$$

where $\boldsymbol{a} = \mathrm{diag}(a_1, \ldots, a_N)$ is a background field which parametrizes an element of the Cartan subalgebra $\mathfrak{u}(1)^{\oplus N}$. In the noncommutative gauge theory on the D6-branes, $\boldsymbol{a}$ plays the role of the vev of the Higgs field $\mu(\Phi)$, defined by a mapping analogous to (4.18). In the present approach the fields $\boldsymbol{a}$ and ϕ parametrize distinct D6 and D0 brane gauge transformations. From (5.16) it follows that they are only related to each other on-shell at the BRST fixed points. Figure 1 depicts the relevant fields that will enter into the quiver description of the Hilbert scheme below.
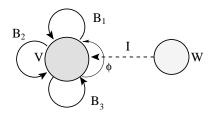

Figure 1: Quiver description of the Hilbert scheme $\mathsf{Hilb}^k(\mathbb{C}^3)$.

For the bosonic fields we will consider the equations of motion

$$
\begin{aligned}
\mathcal{E}_i \ &: \ [B_i, B_j] + \epsilon_{ijk}\,[B_k^\dagger, \varphi] = 0 \ , \\
\mathcal{E}_r \ &: \ \sum_{i=1}^{3} [B_i, B_i^\dagger] + [\varphi, \varphi^\dagger] + I\, I^\dagger = r \ , \\
\mathcal{E}_I \ &: \ I^\dagger\, \varphi = 0 \ .
\end{aligned}
\tag{5.17}
$$

The equations $\mathcal{E}_i$ and $\mathcal{E}_I$ are relations for the quiver depicted in Fig. 1, while the equation $\mathcal{E}_r$ is a cyclic vector for the representation of the quiver in the moduli space of coherent sheaves. The Fayet–Iliopoulos parameter $r > 0$ is determined by the noncommutative deformation of the original gauge theory and it determines the mass of the D0–D6 fields in terms of the asymptotic B-field required to preserve supersymmetry in the D6–D0 bound states.

The extra field φ and equation $\mathcal{E}_I$ can in fact be seen to arise from the extra moduli and matrix equations that we found in (4.103)–(4.106), e.g. by identifying $B_1' = \varphi^\dagger$. Its appearence there is natural since the projective space $\mathbb{P}^3$ is not Calabi–Yau. They can also be seen to arise from the noncommutative instanton equations (3.7) by decomposing the field ρ similarly to (4.14) with $\varphi \in \mathrm{End}_{\mathcal{A}}(\mathcal{H}^{\oplus k})$. For the quiver matrix model appropriate to the dynamics on $\mathbb{C}^3$, however, one should set $\varphi = 0$ and arrive at the system of matrix equations (4.1). These extra moduli can also play the role of the extra fields required when considers non-trivial asymptotic boundary conditions, such as those of (4.38), as we do in the next section.

We now have to add the Fermi multiplets $(\vec{\chi}, \vec{H})$, which contain the antighost and auxiliary fields $\vec{\chi} = (\chi_1, \chi_2, \chi_3, \chi_r, \xi)$ and $\vec{H} = (H_1, H_2, H_3, H_r, h)$. As we stressed earlier, the auxiliary fields are determined by the equations $\vec{\mathcal{E}}$ on-shell and so must carry the same quantum numbers. This implies that the antighosts are defined as maps

$$
(\chi_1, \chi_2, \chi_3, \chi_r) \ \in \ \mathrm{Hom}(V, V) \ ,
$$

40

$$\xi \ \in \ \mathrm{Hom}(V, W) \ , \tag{5.18}$$

since ξ corresponds to the equation $\mathcal{E}_I$ which maps a vector in V to a vector in W.

Let us take a closer look at the defining equations (5.17). As in Section 5.1 above, the complex equations $\mathcal{E}_i$ live in the adjoint representation of $U(k)$ but transform under the toric action with a factor $\mathrm{e}^{-\mathrm{i}(\epsilon - \epsilon_i)}$. The real equation $\mathcal{E}_r$ lives in the adjoint representation of $U(k)$ and is invariant under the toric action. Finally, the equation $\mathcal{E}_I$ transforms under $U(k) \times U(N) \times \mathbb{T}^3$ as

$$I^\dagger \, \varphi \ \longmapsto \ \mathrm{e}^{\,\mathrm{i}(\varepsilon - \epsilon)} \, g_{U(N)} \, I^\dagger \, g_{U(k)}^\dagger \, g_{U(k)} \, \varphi \, g_{U(k)}^\dagger = \ \mathrm{e}^{\,\mathrm{i}(\varepsilon - \epsilon)} \, g_{U(N)} \, I^\dagger \, \varphi \, g_{U(k)}^\dagger \ . \tag{5.19}$$

We now have all the ingredients necessary to write down the BRST transformations for the remaining fields as

$$
\begin{aligned}
Q\chi_i &= H_i && \text{and} & QH_i &= [\phi, \chi_i] - (\epsilon - \epsilon_i)\,\chi_i \ , \\
Q\chi_r &= H_r && \text{and} & QH_r &= [\phi, \chi_r] \ , \\
Q\xi &= h && \text{and} & Qh &= \boldsymbol{a}\,\xi - \xi\,\phi + (\varepsilon - \epsilon)\,\xi \ ,
\end{aligned}
\tag{5.20}
$$

to which we add the gauge multiplet to close the algebra (5.3).

The action that corresponds to this system of fields and equations is given by

$$
\begin{aligned}
S \ = \ & Q\,\mathrm{Tr}\,\Big(\chi_i^\dagger\,(H_i - \mathcal{E}_i) + \chi_r\,(H_r - \mathcal{E}_r) + \xi^\dagger\,(h - \mathcal{E}_I) + \psi_i\,\big[\,\overline{\phi}, B_i^\dagger\,\big] \\
& + \zeta\,\big[\,\overline{\phi},\, \varphi^\dagger\,\big] + \rho\,\overline{\phi}\,I^\dagger + \eta\,[\phi,\, \overline{\phi}\,] + \mathrm{h.c.} \Big) \ .
\end{aligned}
\tag{5.21}
$$

This action is topological and the path integral can be treated as we did in the Section 5.1 above. The critical points are determined by the zeroes of the BRST charge. We are interested in the class of minima where φ vanishes (and the fermions are set to zero). The fixed point equations are then

$$
\begin{aligned}
(B_i)_{ab}\,(\phi_a - \phi_b - \epsilon_i) &= \ 0 \ , \\
I_{a,l}\,(\phi_a - a_l - \varepsilon) &= \ 0
\end{aligned}
\tag{5.22}
$$

where we have diagonalized both ϕ (producing a Vandermonde determinant $\det(\mathrm{ad}\,\phi)$ in the path integral measure) and $\boldsymbol{a}$ by $U(k)$ and $U(N)$ gauge transformations, respectively. We will give a more precise classification of the fixed points in terms of three-dimensional partitions in Section 5.3 below.

Regardless of what the structure of the fixed point set is, we can write down directly the fluctuation determinants with the general rules outlined in Section 5.1 above. The fields give a contribution in the denominator determined by their quantum numbers, while the constraints similarly appear in the numerator. The main difference from the noncommutative gauge theory is that now we have an additional field and an additional constraint. Putting everything together we get the partition function

$$
\begin{aligned}
Z \ = \ & \oint \ \prod_{i=1}^k \mathrm{d}\phi_i \ \frac{\det\,(\mathrm{ad}\,\phi)\,\det\,(\mathrm{ad}\,\phi + \epsilon_1 + \epsilon_2)\,\det\,(\mathrm{ad}\,\phi + \epsilon_1 + \epsilon_3)\,\det\,(\mathrm{ad}\,\phi + \epsilon_2 + \epsilon_3)}{\det\,(\mathrm{ad}\,\phi + \epsilon)\,\det\,(\mathrm{ad}\,\phi + \epsilon_1)\,\det\,(\mathrm{ad}\,\phi + \epsilon_2)\,\det\,(\mathrm{ad}\,\phi + \epsilon_3)} \\
& \times \ \frac{\det\big(-\phi \otimes 1_W + 1_V \otimes \boldsymbol{a} + (\varepsilon - \epsilon)\big)}{\det\,(\phi \otimes 1_W - 1_V \otimes \boldsymbol{a} - \varepsilon)}
\end{aligned}
\tag{5.23}
$$

where we have again dropped several factors including the volume of the gauge group. As in the noncommutative gauge theory, the ratio of determinants formally cancels up to a sign when $\epsilon = 0$. Thus the integration in (5.23) is ill-defined as a Lebesgue integral and must be defined via an appropriate contour integration.

At this point one should proceed to evaluate the integral of ϕ over the Cartan subalgebra $\mathfrak{u}(1)^{\oplus k}$. This is the main difference from the noncommutative gauge theory, even though the ratios of determinants formally look alike. In the noncommutative gauge theory the ratio of determinants is a function of only the equivariant parameters, since there we compute the determinants by taking products of eigenvalues of operators over the Hilbert space $\mathcal{H}$. Here things are different, as we are really dealing with a finite-dimensional $k \times k$ matrix model, and the ratio is a function of the eigenvalues of ϕ. The integral over the maximal torus of the group $U(k)$ still has to be performed and it requires an appropriate prescription to pick an integration contour which encircles the relevant poles. Note that the fixed points of the toric action appear as poles in the denominator. The evaluation of (5.23) by residues is in fact equivalent to the use of the localization formula applied to the equivariant Euler characteristic of the complex (4.109), a fact that will be exploited in the following.

It is amusing to note that the analogous problem was solved in four dimensions by lifting the theory to five dimensions (with the fifth dimension compactified), where a natural prescription for the contour integral can be found [39, 40]. The rationale behind this procedure is that an instanton counting problem in four dimensions can be lifted to a soliton counting problem in five dimensions. It is tempting to speculate that the ratio of determinants in (5.23) (and in more general instances) may have a clearer origin in the context of topological M-theory on the product of a Calabi–Yau threefold with $\mathbb{S}^1$.

5.3 Fixed points and three-dimensional partitions

We will now clarify how the matrix quantum mechanics equations (5.22) can be solved in terms of plane partitions. We will focus on the abelian case where the parametrization of the Hilbert scheme allows for an explicit classification of the fixed points. In this framework we can show explicitly how the melting crystal picture is encoded in the matrix model formalism and thus how the gravitational quantum foam emerges naturally from the gauge theory variables. The following analysis is a generalization of the arguments of [32]. We will go through the argument in two steps. First we will recover the Hilbert scheme of points by the matrix quantum mechanics equations of motion. Then we will show how to construct a three-dimensional partition given a fixed point in the Hilbert scheme and viceversa.

Consider the Hilbert scheme of k points in $\mathbb{C}^3$ given by

$$\mathsf{Hilb}^k\big(\mathbb{C}^3\big) = \big\{ \text{ideals } \mathcal{I} \subset \mathbb{C}[B_1, B_2, B_3] \;\big|\; \dim_\mathbb{C} \mathbb{C}[B_1, B_2, B_3]/\mathcal{I} = k \big\} . \tag{5.24}$$

Then we claim that

$$\mathsf{Hilb}^k\big(\mathbb{C}^3\big) \cong \left\{ (B_1, B_2, B_3, I) \;\middle|\; \begin{array}{c} [B_1, B_2] = [B_1, B_3] = [B_2, B_3] = 0 \\ \text{Stability: there is no proper} \\ \text{subspace } S \subset \mathbb{C}^k \text{ such that} \\ B_i(S) \subset S \text{ and } \mathrm{im}(I) \subset S \end{array} \right\} \Big/ GL(k, \mathbb{C}) \tag{5.25}$$

where $B_i \in \mathrm{End}(\mathbb{C}^k)$ and $I \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}^k)$. In the quiver language one has $\mathbb{C} = W$ and $\mathbb{C}^k = V$. The action of the gauge group $GL(k, \mathbb{C})$ is given in (4.2).

The correspondence can be seen as follows. Suppose we are given an ideal $\mathcal{I} \in \mathsf{Hilb}^k(\mathbb{C}^3)$. Then we can define $V = \mathbb{C}[B_1, B_2, B_3]/\mathcal{I}$, and $B_i \in \mathrm{End}(V)$ to be given as multiplication by B_i mod $\mathcal{I}$ and $I \in \mathrm{Hom}(\mathbb{C}, V)$ by $I(1) = 1 \mod \mathcal{I}$. Then all the B_i commute since they are realized as multiplication and the stability condition holds since products of the B_i's times 1 span the

whole of the polynomial ring $\mathbb{C}[B_1, B_2, B_3]$. Conversely, suppose we are given a quadruple of maps (B_1, B_2, B_3, I). We introduce the map

$$
\begin{aligned}
\mu : \mathbb{C}[B_1, B_2, B_3] &\longrightarrow \mathbb{C}^k \\
f &\longmapsto f(B_1, B_2, B_3)I(1)
\end{aligned}
\tag{5.26}
$$

which is well-defined since the B_i's commute. Consider now the subspace $\mathrm{im}(\mu) \subset \mathbb{C}^k$. This subspace is B_i-invariant since it is the subspace spanned by the B_i themselves and $\mathrm{im}(I) \subset \mathrm{im}(\mu)$. Then the stability condition implies $\mathrm{im}(\mu) = \mathbb{C}^k$. This means that the map μ is surjective. Then $\mathfrak{I} := \ker \mu$ is an ideal with $\dim_{\mathbb{C}} \mathbb{C}[B_1, B_2, B_3]/\mathfrak{I} = k$. Explicitly, one has

$$
\mathfrak{I} = \left\{ f(z) \in \mathcal{O}_{\mathbb{C}^3} \cong \mathbb{C}[B_1, B_2, B_3] \mid f(B_1, B_2, B_3)I(1) = 0 \right\} .
\tag{5.27}
$$

This ideal is isomorphic to the rank one cohomology sheaf (4.77). By restricting the complex (4.76) with $N = 1$ to $\mathbb{C}^3 = \mathbb{P}^3 \setminus \mathbb{P}^2$ as before, the image of (4.77) in $\mathcal{O}_{\mathbb{C}^3}$ induced by the localized maps a_z and b_z with $J = K = 0$, and suitable matrix identifications in Section 4.6, is precisely the ideal (5.27). The proof parallels [32, Proposition 2.7]. As an explicit example, the charge 2 abelian instanton moduli space is $\mathcal{M}_{1,2}(\mathbb{C}^3) \cong \mathbb{C}^3 \times \mathcal{O}_{\mathbb{P}^2}(-1)$, where the first factor is the space of 2×2 matrices (B_1, B_2, B_3) which parametrize the center of mass of the instantons, while the second factor is the resolution of the relative position singularity at the origin which gives the size and orientation of the instanton configuration.

We have thus constructed an explicit correspondence between elements of the Hilbert scheme and commuting matrices (with a stability condition) which correspond to the quiver quantum mechanics. We will now consider a fixed point and show that it can be parametrized by a three-dimensional partition. We consider a torus $\mathbb{T}^3$ acting on $\mathbb{C}^3$ with generators (t_1, t_2, t_3). This action lifts to the Hilbert scheme. A fixed point given by (B_1, B_2, B_3, I) is characterized by the condition that an equivariant rotation is equivalent to a gauge transformation

$$
\begin{aligned}
t_1\, B_1 &= g\, B_1\, g^{-1} , \\
t_2\, B_2 &= g\, B_2\, g^{-1} , \\
t_3\, B_3 &= g\, B_3\, g^{-1} , \\
I &= g\, I
\end{aligned}
\tag{5.28}
$$

with $g \in GL(k, \mathbb{C})$. We use the weight decomposition

$$
V = \bigoplus_{i,j,k \in \mathbb{Z}} V(i-1, j-1, k-1)
\tag{5.29}
$$

with

$$
V(i-1, j-1, k-1) = \left\{ v \in V \mid g^{-1}\, v = t_1^{i-1}\, t_2^{j-1}\, t_3^{k-1}\, v \right\} .
\tag{5.30}
$$

The notation has been chosen such that $V(0,0,0)$ denotes the subspace spanned by gauge-invariant vectors.

Consider a generic triple of integers (i, j, k). According to the definition (5.30) the only non-vanishing components of the maps (B_1, B_2, B_3, I) with respect to the splitting (5.29) are given by

$$
\begin{aligned}
B_1 : V(i, j, k) &\longrightarrow V(i-1, j, k) , \\
B_2 : V(i, j, k) &\longrightarrow V(i, j-1, k) ,
\end{aligned}
$$

43

$$B_3 : V(i,j,k) \longrightarrow V(i,j,k-1) ,$$
$$I : W \longrightarrow V(0,0,0) . \tag{5.31}$$

This can be seen as follows. By the stability condition the vector space V is spanned by elements of the form $B_1^p B_2^q B_3^m I(1)$ with $p, q, m \in \mathbb{N}_0$. In particular, the subspace $V(0,0,0)$ is spanned by $I(1)$ and is one-dimensional. Consider now acting on the vector $I(1)$ that generates this space with, say, B_1. This gives the vector $B_1 I(1)$. Now we see that

$$g^{-1} \left(B_1 I(1) \right) = g^{-1} B_1 \, g \, g^{-1} \, I(1) = t_1^{-1} B_1 \, I(1) \tag{5.32}$$

due to (5.28). This means that $B_1 I(1) \in V(-1,0,0)$. The more general cases in (5.31) are treated similarly.

To make contact with the fixed point equation we parametrize the gauge group by the $k \times k$ matrix ϕ and diagonalize it at the fixed points. We also write $t_i = \mathrm{e}^{-\mathrm{i}\epsilon_i}$. The defining equation in (5.30) can be satisfied by picking the k eigenvalues of ϕ and setting them equal to

$$\phi_{(i,j,k);l} = a_l + \epsilon_1 \, (i-1) + \epsilon_2 \, (j-1) + \epsilon_3 \, (k-1) . \tag{5.33}$$

Using (5.31), this reproduces the expected result from (5.22). Then this weight decomposition of the vector space V is equivalent to the classification of the fixed points as done in the previous sections. At each fixed point the eigenvalues of ϕ are determined by the weights of the toric action. Following [32] we will now see how the definition (5.30) implies that the allowed values of ϕ (i.e. the allowed non-trivial spaces $V(i-1,j-1,k-1)$) are in correspondence with plane partitions.

We note that $V(i,j,k) = 0$ if one of (i,j,k) is strictly positive. Only negative or zero values are allowed. If we generalize this reasoning to vectors of the form $B_i^p I(1)$ we conclude that

$$\begin{aligned}
\dim_{\mathbb{C}} V(i,0,0) &\geq \dim_{\mathbb{C}} V(i-1,0,0) , \\
\dim_{\mathbb{C}} V(0,j,0) &\geq \dim_{\mathbb{C}} V(0,j-1,0) , \\
\dim_{\mathbb{C}} V(0,0,k) &\geq \dim_{\mathbb{C}} V(0,0,k-1) ,
\end{aligned} \tag{5.34}$$

and that these dimensions can only be either zero or one. Intuitively, we are "constructing" the exterior boxes of the three-dimensional partition and each box is represented by a non-trivial vector space. Now we will go to the boxes in the interior by using the commutativity relations $[B_i, B_j] = 0$. We will proceed by induction.

The commutativity relation $[B_1, B_2] = 0$ ensures that the configurations

$$\begin{array}{ccc}
V(i,j-1,k) \cong \mathbb{C} \xrightarrow{\ B_1\ } V(i-1,j-1,k) \neq 0 \\
{\scriptstyle B_2}\Big\uparrow \qquad\qquad\qquad \Big\uparrow \\
V(i,j,k) \cong \mathbb{C} \xrightarrow{\hspace{3cm}} 0
\end{array} \tag{5.35}$$

and

$$\begin{array}{ccc}
0 \xrightarrow{\hspace{2cm}} V(i-1,j-1,k) \neq 0 \\
\Big\uparrow \qquad\qquad\qquad {\scriptstyle B_2}\Big\uparrow \\
V(i,j,k) \cong \mathbb{C} \xrightarrow{\ B_1\ } V(i-1,j,k) \cong \mathbb{C}
\end{array} \tag{5.36}$$

44

are impossible, where the notation $V(i, j, k) \cong \mathbb{C}$ stands for the *assumption* that these spaces are one-dimensional. On the other hand, the diagram

$$
\begin{array}{ccc}
V(i, j-1, k) \cong \mathbb{C} & \xrightarrow{B_1} & V(i-1, j-1, k) \\
{\scriptstyle B_2} \big\uparrow & & \big\uparrow {\scriptstyle B_2} \\
V(i, j, k) \cong \mathbb{C} & \xrightarrow[B_1]{} & V(i-1, j, k) \cong \mathbb{C}
\end{array}
\tag{5.37}
$$

implies that $V(i-1, j-1, k)$ has dimension 0 or 1. We can think of B_1 and B_2 as the "directions" of the base of the three-dimensional partitions. Then these conditions tell us that the base is a usual Young tableau oriented as in [32].

Now we have to put boxes on top of this "base" Young tableau. What are the allowed configurations? The equation $[B_1, B_3] = 0$ implies that the two diagrams

$$
\begin{array}{ccc}
V(i, j, k-1) \cong \mathbb{C} & \xrightarrow{B_1} & V(i-1, j, k-1) \neq 0 \\
{\scriptstyle B_3} \big\uparrow & & \big\uparrow \\
V(i, j, k) \cong \mathbb{C} & \xrightarrow{\hspace{2cm}} & 0
\end{array}
\tag{5.38}
$$

and

$$
\begin{array}{ccc}
0 & \xrightarrow{\hspace{1cm}} & V(i-1, j, k-1) \neq 0 \\
\big\uparrow & & \big\uparrow {\scriptstyle B_3} \\
V(i, j, k) \cong \mathbb{C} & \xrightarrow[B_1]{} & V(i-1, j, k) \cong \mathbb{C}
\end{array}
\tag{5.39}
$$

are forbidden. Again the commutativity of

$$
\begin{array}{ccc}
V(i, j, k-1) \cong \mathbb{C} & \xrightarrow{B_1} & V(i-1, j, k-1) \\
{\scriptstyle B_3} \big\uparrow & & \big\uparrow {\scriptstyle B_3} \\
V(i, j, k) \cong \mathbb{C} & \xrightarrow[B_1]{} & V(i-1, j, k) \cong \mathbb{C}
\end{array}
\tag{5.40}
$$

implies that $\dim_{\mathbb{C}} V(i-1, j, k-1)$ is either 1 or 0. Finally, the equation $[B_2, B_3] = 0$ forbids the two configurations

$$
\begin{array}{ccc}
V(i, j, k-1) \cong \mathbb{C} & \xrightarrow{B_2} & V(i, j-1, k-1) \neq 0 \\
{\scriptstyle B_3} \big\uparrow & & \big\uparrow \\
V(i, j, k) \cong \mathbb{C} & \xrightarrow{\hspace{2cm}} & 0
\end{array}
\tag{5.41}
$$

and

$$
\begin{array}{ccc}
0 & \xrightarrow{\hspace{1cm}} & V(i, j-1, k-1) \neq 0 \\
\big\uparrow & & \big\uparrow {\scriptstyle B_3} \\
V(i, j, k) \cong \mathbb{C} & \xrightarrow[B_2]{} & V(i, j-1, k) \cong \mathbb{C} \ ,
\end{array}
\tag{5.42}
$$

while the commutativity of

$$
\begin{array}{ccc}
V(i, j, k-1) \cong \mathbb{C} & \xrightarrow{B_2} & V(i, j-1, k-1) \\
{\scriptstyle B_3} \big\uparrow & & \big\uparrow {\scriptstyle B_3} \\
V(i, j, k) \cong \mathbb{C} & \xrightarrow[B_2]{} & V(i, j-1, k) \cong \mathbb{C}
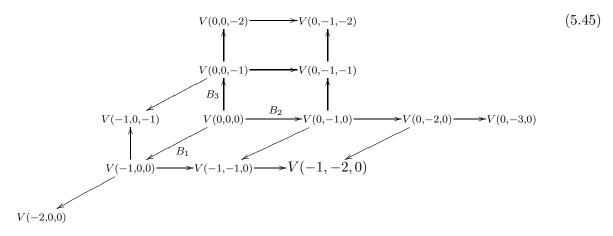\end{array}
\tag{5.43}
$$

implies that $\dim_{\mathbb{C}} V(i, j-1, k-1)$ is either 1 or 0.

By induction this implies that each vector space $V(i, j, k)$ in (5.29) is either one-dimensional or zero-dimensional. Moreover, the allowed configurations correspond to three-dimensional partitions where we put each box in the position of a one-dimensional vector space. For example, we can construct the partition starting from a two-dimensional Young tableau lying on the plane (B_1, B_2). On top of each position (i, j) corresponding to a box of the tableau we put $\pi_{i,j}$ boxes. The diagrams above corresponding to the equations $[B_1, B_3] = [B_2, B_3] = 0$ tell us that we can pile the boxes only in the "correct" way, so that

$$\pi_{i,j} \geq \pi_{i+r,j+s} \qquad \text{for} \qquad r, s \geq 0 \tag{5.44}$$

with $k = \sum_{i,j \leq 0} \pi_{i,j}$. Pictorially, the diagram

$$\tag{5.45}$$

is a simple plane partition. Note that one has to specify from the beginning which vector spaces are non-trivial in order to get a partition.

This construction easily generalizes to the nonabelian theory in the broken phase $U(1)^N$. In this situation we can assume that the relevant moduli space splits into a direct sum of sectors labeled by the Higgs vevs a_l, each one essentially identical to the Hilbert scheme of points. Consequently, the classification above of the fixed points can be easily generalized by simply adding a label a_l (colour) to each vector space, $V_{a_l}(i, j, k)$. This agrees with the classification suggested in [11], while in [12] the equations of the topological matrix model have been interpreted in terms of skew partitions with N "corners" on the interior boundary. In the string picture this corresponds to a situation in which the N D6-branes are well separated in the transverse space. The D0-brane bound states with each of the D6-branes give N copies of the melting crystal configuration.

Note that in this framework the role played by I (representing the 0–6 open strings) is to simply label the colour of the three-dimensional partition. In our approach we use the N non-trivial components $I(1)_{(0,0,0);l}$ labelled by the Higgs vevs a_l to build N individual three-dimensional partitions. Notice also that the k non-vanishing components of B_i and I at a fixed point, corresponding to the number of boxes of the associated three-dimensional Young diagram, are fixed completely by the matrix equations (4.1). This follows from the fact that the matrices in the last equation of (4.1) (the D-term constraint) are all $k \times k$ and the diagonal components yield exactly k constraints. Thus the fixed points are indeed isolated.

Geometrically, these coloured partitions can be understood as fixed points in the framed moduli space (4.19) as follows. There is a natural action of the torus $\mathbb{T}^3 \times U(1)^N$ on $\mathcal{M}_{N,k}(\mathbb{P}^3)$ induced by the $\mathbb{T}^3$-action on the toric manifold $\mathbb{P}^3$ and the action of the maximal torus $U(1)^N$ on $W \otimes \mathcal{O}_{p_\infty}$, where $W = \bigoplus_l W_l$ decomposes into irreducible representations $W_l \cong \mathbb{C}$ of $U(1)$. Then the $\mathbb{T}^3 \times U(1)^N$ fixed points on $\mathcal{M}_{N,k}(\mathbb{P}^3)$ are the coherent, torus invariant sheaves $\mathcal{E}_{\vec{\pi}} = \mathcal{I}_{a_1} \oplus \cdots \oplus \mathcal{I}_{a_N}$ with pointlike

support $Z = Z_{a_1} \sqcup \cdots \sqcup Z_{a_N}$ at the origin of $\mathbb{C}^3$, where $\mathfrak{I}_{a_l}$ is a $\mathbb{T}^3$-invariant ideal sheaf supported on the $\mathbb{T}^3$-fixed zero-dimensional subscheme $Z_{a_l} \subset \mathbb{C}^3 = \mathbb{P}^3 \setminus p_\infty$ such that $\mathfrak{I}_{a_l}|_{p_\infty} \cong W_l \otimes \mathcal{O}_{p_\infty}$. In particular, the weight decomposition (5.29) coincides with $H^0(\mathcal{O}_Z)$.

5.4 Quiver variety for Donaldson–Thomas data

At this stage we have provided a classification of the critical points in the abelian theory that can be easily generalized to the $U(1)^N$ picture. To complete the localization program we have to now compute the quantum fluctuation factor around each critical point. This can be done explicitly in our ADHM-type formalism since it provides a direct parametrization of the compactified instanton moduli space. The heuristic structure of "fields" and "constraints" that we have exploited above to write down the ratio of determinants in (5.23) can be made precise in terms of an equivariant index that counts BPS states. Geometrically, we will provide a description of the (virtual) tangent space around each fixed point (the "fields"), and on top of it we build the normal bundle (the "constraints"). This finally gives the last missing piece of information needed to use the localization formula. We will assume that the gauge symmetry is broken down to the maximal torus $U(1)^N$ and recover the abelian theory as a particular case.

Recall that in the quiver description above the two vector spaces V and W with $\dim_\mathbb{C} V = k$ and $\dim_\mathbb{C} W = N$ represent respectively the gas of k D0-branes and the N D6-branes. Naively, the bosonic fields are elements (5.13), while the fermionic fields associated with the equations of motion live in (5.18). However, this is not really what we need since we have to compute contributions coming from the fixed points and each fixed point is characterized by the fact that the equivariant transformation mixes with the linear transformations of the vector spaces V and W. To make this apparent we introduce a three-dimensional $\mathbb{T}^3$-module Q that acts on V. At a fixed point $f \in \mathcal{M}$ we have to supplement the conditions (5.13) and (5.18) with the information that a gauge transformation is equivalent to an equivariant rotation. Let us take the three-torus to be $\mathbb{T}^3 = (t_1 = e^{i\epsilon_1}, t_2 = e^{i\epsilon_2}, t_3 = e^{i\epsilon_3})$, and introduce the following notation. T_i is the one-dimensional module generated by t_i, $T_i T_j$ is generated by $t_i t_j$ and $T_1 T_2 T_3$ by $t_1 t_2 t_3$, and similarly for the dual modules $T_i^* := T_i^{-1}$. We write E_l for the module over $\mathbb{T}^3 \times U(1)^N$ generated by $e_l = e^{i a_l}$. For brevity we omit tensor product symbols between the T-modules.

With this notation, at the fixed points of the $\mathbb{T}^3 \times U(1)^N$ action, which correspond from above to coloured partitions $\vec{\pi} = (\pi_1, \ldots, \pi_N)$, we decompose the vector spaces

$$V_{\vec{\pi}} = \sum_{l=1}^{N} e_l \sum_{(i,j,k) \in \pi_l} t_1^{i-1} t_2^{j-1} t_3^{k-1} ,$$

$$W_{\vec{\pi}} = \sum_{l=1}^{N} e_l \tag{5.46}$$

as $\mathbb{T}^3 \times U(1)^N$ representations viewed as polynomials in t_1, t_2, t_3 and e_l, $l = 1, \ldots, N$. For each l, the sum over boxes of π_l in $V_{\vec{\pi}}$ is the trace of the $\mathbb{T}^3$-action (i.e. the $\mathbb{T}^3$-character) on $\mathbb{C}[B_1, B_2, B_3]/\mathfrak{I}_{a_l}$. Taking into account the action of the torus on the vector space V we can write

$$B_1 \in \mathrm{Hom}(V_{\vec{\pi}}, V_{\vec{\pi}}) \otimes T_1^{-1} ,$$

$$B_2 \in \mathrm{Hom}(V_{\vec{\pi}}, V_{\vec{\pi}}) \otimes T_2^{-1} ,$$

$$B_3 \in \mathrm{Hom}(V_{\vec{\pi}}, V_{\vec{\pi}}) \otimes T_3^{-1} ,$$

$$\varphi \in \mathrm{Hom}(V_{\vec{\pi}}, V_{\vec{\pi}}) \otimes (T_1 T_2 T_3)^{-1} ,$$

$$I \;\in\; \operatorname{Hom}(W_{\vec{\pi}}, V_{\vec{\pi}}) \; . \tag{5.47}$$

It is important to stress that these decompositions only hold at the fixed points and not generically on the instanton moduli space. We have chosen I as before to be invariant under the toric action. This amounts to setting $\varepsilon = 0$, which we will see later on is consistent with the known abelian case. Similarly for the constraints (again at the fixed points)

$$
\begin{aligned}
\chi_1 &\in \operatorname{Hom}(V_{\vec{\pi}}, V_{\vec{\pi}}) \otimes T_1^{-1} T_2^{-1} \;, \\
\chi_2 &\in \operatorname{Hom}(V_{\vec{\pi}}, V_{\vec{\pi}}) \otimes T_1^{-1} T_3^{-1} \;, \\
\chi_3 &\in \operatorname{Hom}(V_{\vec{\pi}}, V_{\vec{\pi}}) \otimes T_2^{-1} T_3^{-1} \;, \\
\chi_r &\in \operatorname{Hom}(V_{\vec{\pi}}, V_{\vec{\pi}}) \;, \\
\xi &\in \operatorname{Hom}(V_{\vec{\pi}}, W_{\vec{\pi}}) \otimes T_1^{-1} T_2^{-1} T_3^{-1} \;.
\end{aligned}
\tag{5.48}
$$

We will call an element

$$(B_1, B_2, B_3, \varphi, I) \;\in\; \left(Q \otimes \operatorname{Hom}(V,V)\right) \;\oplus\; \left(\textstyle\bigwedge^3 Q \otimes \operatorname{Hom}(V,V)\right) \;\oplus\; \operatorname{Hom}(W,V) \tag{5.49}$$

a Donaldson–Thomas datum, where

$$
\begin{aligned}
Q &= T_1^{-1} + T_2^{-1} + T_3^{-1} \;, \\
\textstyle\bigwedge^2 Q &= T_1^{-1} T_2^{-1} + T_1^{-1} T_3^{-1} + T_2^{-1} T_3^{-1} \;, \\
\textstyle\bigwedge^3 Q &= \det Q = T_1^{-1} T_2^{-1} T_3^{-1} \;.
\end{aligned}
\tag{5.50}
$$

Recall that there is a natural $GL(k, \mathbb{C})$ action on this data. If we impose the stability condition on (5.49), then this group action is free. Then we may define the geometric invariant theory quotient of the subspace of (5.49) given by $\mu_c^{-1}(0)/\!/ GL(k, \mathbb{C})$, where $\mu_c = (\mathcal{E}_i, \mathcal{E}_I)$ is a complex moment map. This is the *quiver variety* for Donaldson–Thomas data. This can also presumably be defined by relaxing stability and taking instead a hyper-Kähler quotient of the data (5.49) given by $\mu_c^{-1}(0) \cap \mu_r^{-1}(r)/\!/ U(k)$, where $\mu_r = \mathcal{E}_r$ is a real moment map. However, like the other moduli spaces considered in this paper, we are not aware of any scheme (or stack) construction on this set.

5.5 Localization formula and character

Let $(B_1, B_2, B_3, \varphi, I)$ be a Donaldon–Thomas datum corresponding to the fixed point $\vec{\pi}$. Let us study the local geometry of the instanton moduli space around this fixed point. Consider the complex

$$
\operatorname{Hom}(V_{\vec{\pi}}, V_{\vec{\pi}}) \;\xrightarrow{\;\sigma\;}\;
\begin{array}{c}
\operatorname{Hom}(V_{\vec{\pi}}, V_{\vec{\pi}}) \otimes Q \\
\oplus \\
\operatorname{Hom}(W_{\vec{\pi}}, V_{\vec{\pi}}) \\
\oplus \\
\operatorname{Hom}(V_{\vec{\pi}}, V_{\vec{\pi}}) \otimes \bigwedge^3 Q
\end{array}
\;\xrightarrow{\;\tau\;}\;
\begin{array}{c}
\operatorname{Hom}(V_{\vec{\pi}}, V_{\vec{\pi}}) \otimes \bigwedge^2 Q \\
\oplus \\
\operatorname{Hom}(V_{\vec{\pi}}, W_{\vec{\pi}}) \otimes \bigwedge^3 Q \; .
\end{array}
\tag{5.51}
$$

The map σ is an infinitesimal (complex) gauge transformation

$$
\sigma(\phi) = \begin{pmatrix} \phi\,B_1 - B_1\,\phi \\ \phi\,B_2 - B_2\,\phi \\ \phi\,B_3 - B_3\,\phi \\ \phi\,I - I\,\boldsymbol{a} \\ \phi\,\varphi - \varphi\,\phi \end{pmatrix} \;,
\tag{5.52}
$$

48

while the map τ is the differential of the equations that define the moduli space $[B_i, B_j] = 0$ given by

$$\tau \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ s \\ Y_4 \end{pmatrix} = \begin{pmatrix} [B_1, Y_2] + [Y_1, B_2] \\ [B_1, Y_3] + [Y_1, B_3] \\ [B_2, Y_3] + [Y_2, B_3] \\ s^\dagger \varphi + I^\dagger Y_4 \end{pmatrix} , \tag{5.53}$$

where one can think of Y_i as δB_i and so on.

The complex (5.51) is the matrix quantum mechanics analog of the instanton deformation complex (2.3). In a similar way, its first cohomology is a local model of the Zariski tangent space to the moduli space, while its second cohomology parametrizes obstructions. This is exactly the information we need to integrate (2.4). We want to apply the Duistermaat–Heckman localization formula (2.10), or better its supersymmetric generalizations [37], to the integral (2.4). This involves the ratio of the top Chern class of the obstruction bundle over the weights coming from the tangent space. For what concerns the computation of the Chern classes we can decompose the tangent and normal bundles over the moduli space as Whitney sums of line bundles by using the splitting principle as $T\mathcal{M} = \bigoplus_i \mathcal{L}_i$ and $\mathcal{N} = \bigoplus_i \mathcal{Q}_i$. Accordingly, the equivariant Chern polynomials are given by

$$c(T\mathcal{M}) = \prod_{i=1}^{n} \left(c_1(\mathcal{L}_i) + w_i[T\mathcal{M}] \right) ,$$

$$c(\mathcal{N}) = \prod_{i=1}^{n} \left(c_1(\mathcal{Q}_i) + w_i[\mathcal{N}] \right) , \tag{5.54}$$

where the local weights w_i in general depend on the toric parameters $(\epsilon_1, \epsilon_2, \epsilon_3, \boldsymbol{a})$. The same information is contained in the equivariant Chern characters

$$\mathrm{ch}(T\mathcal{M}) = \sum_{i=1}^{n} \mathrm{e}^{c_1(\mathcal{L}_i) + w_i[T\mathcal{M}]} ,$$

$$\mathrm{ch}(\mathcal{N}) = \sum_{i=1}^{n} \mathrm{e}^{c_1(\mathcal{Q}_i) + w_i[\mathcal{N}]} . \tag{5.55}$$

The Chern classes of the line bundles do not contribute to the localization formula since they have to be evaluated at a point (and the critical points are isolated). Then we can use directly the above expansions to extract the relevant weights to be used in the localization formula. In practise this is accomplished via the transform [9, 37]

$$\sum_{i=1}^{n} n_i \, \mathrm{e}^{w_i} \longmapsto \prod_{i=1}^{n} w_i^{n_i} . \tag{5.56}$$

This means that all the relevant data that enter in the localization formula are already contained in the equivariant index of the complex (5.51). To be precise the index computes the virtual sum $H^1 \ominus H^0 \ominus H^2$ of cohomology groups. We assume that H^0 vanishes, which is equivalent to restricting attention to irreducible connections. Using (5.56) we see then that the equivariant index computes exactly the inverse of the ratio of the weights that enter in the localization formula. The equivariant index is given in terms of the characters of the representation evaluated at the fixed point as

$$\chi_{\vec{\pi}}(\mathbb{C}^3)^{[k]} = V_{\vec{\pi}}^* \otimes V_{\vec{\pi}} \otimes \left(T_1^{-1} + T_2^{-1} + T_3^{-1} + T_1^{-1} T_2^{-1} T_3^{-1} \right) + W_{\vec{\pi}}^* \otimes V_{\vec{\pi}}$$

$$-V_{\vec{\pi}}^* \otimes V_{\vec{\pi}} \otimes \left(1 + T_1^{-1} T_2^{-1} + T_1^{-1} T_3^{-1} + T_2^{-1} T_3^{-1}\right) - V_{\vec{\pi}}^* \otimes W_{\vec{\pi}} \otimes T_1^{-1} T_2^{-1} T_3^{-1}$$
$$= W_{\vec{\pi}}^* \otimes V_{\vec{\pi}} - \frac{V_{\vec{\pi}}^* \otimes W_{\vec{\pi}}}{t_1 \, t_2 \, t_3} + V_{\vec{\pi}}^* \otimes V_{\vec{\pi}} \frac{(1 - t_1)\,(1 - t_2)\,(1 - t_3)}{t_1 \, t_2 \, t_3} \ . \tag{5.57}$$

In the abelian case, $W \cong \mathbb{C}$ and we can formally set $W = 1$. Then the character (5.57) reproduces exactly the vertex character computed in [7]. It is not clear what is the precise relation of the above construction with the more geometric approach of [7], since in the gauge theory picture there are several matter fields which do not figure into the computations of [7]. Nevertheless, it is reassuring that we can reproduce the result of [7] without resorting to the evaluation of virtual fundamental classes but only with arguments well rooted in our physical intuition. It would be very interesting to understand in the language of [7] what the meaning is of the vector space W. A final remark about the equivariant transformation of I, i.e. the choice $\varepsilon = 0$. This can be now justified since it is the choice that reproduces the abelian character.

From this discussion we can immediately write down the partition function

$$\mathcal{Z}_{\mathrm{DT}}^{U(1)^N}\left(\mathbb{C}^3\right) = \sum_{\vec{\pi}} \mathcal{Z}_{\vec{\pi}} \; e^{\,i\,\vartheta\,|\vec{\pi}|} \ , \tag{5.58}$$

where $\mathcal{Z}_{\vec{\pi}}$ is what the matrix integral (5.23) computes and is given by (5.57) through the rule (5.56). However, its explicit form is not very illuminating. To obtain a more manageable form let us simplify the character a bit. We begin by looking at the abelian theory with $N = 1$. Then

$$\chi_{\pi,\mathrm{ab}}\left(\mathbb{C}^3\right)^{[k]} = V_\pi - \frac{V_\pi^*}{t_1 \, t_2 \, t_3} + V_\pi^* \otimes V_\pi \frac{(1 - t_1)\,(1 - t_2)\,(1 - t_3)}{t_1 \, t_2 \, t_3} \tag{5.59}$$

where at a fixed point

$$V_\pi = \sum_{(i,j,k) \in \pi} t_1^{i-1}\, t_2^{j-1}\, t_3^{k-1} \ . \tag{5.60}$$

One can easily see that

$$\chi_{\pi,\mathrm{ab}}\left(\mathbb{C}^3\right)^{[k]} = \mathsf{T}_\pi^+ + \mathsf{T}_\pi^- \tag{5.61}$$

where

$$\mathsf{T}_\pi^+ = V_\pi - V_\pi \otimes V_\pi^* \frac{(1 - t_1)\,(1 - t_2)}{t_1 \, t_2} \ ,$$
$$\mathsf{T}_\pi^- = -\frac{V_\pi^*}{t_1 \, t_2 \, t_3} + V_\pi \otimes V_\pi^* \frac{(1 - t_1)\,(1 - t_2)}{t_1 \, t_2 \, t_3} \ . \tag{5.62}$$

With the dual operation $t \mapsto t^* = t^{-1}$, this splitting of the character has the remarkable property

$$\left(\mathsf{T}_\pi^+\right)^* \big|_{t_1\, t_2\, t_3 = 1} = -\mathsf{T}_\pi^- \big|_{t_1\, t_2\, t_3 = 1} \ . \tag{5.63}$$

Note that this property is true only when one imposes the Calabi–Yau condition $t_1\, t_2\, t_3 = 1$.

What is remarkable about the property (5.63) is that due to (5.56) the contribution to the full fluctuation determinant is a minus sign to some power and this power can be computed to be $\mathsf{T}_\pi^+(t_1 = t_2 = t_3 = 1)$. Explicitly, one has

$$\chi_{\pi,\mathrm{ab}}\left(\mathbb{C}^3\right)^{[k]} \big|_{t_1\, t_2\, t_3 = 1} = \mathsf{T}_\pi^+ \big|_{t_1\, t_2\, t_3 = 1} + \mathsf{T}_\pi^- \big|_{t_1\, t_2\, t_3 = 1}$$
$$= \mathsf{T}_\pi^+ \big|_{t_1\, t_2\, t_3 = 1} - \left(\mathsf{T}_\pi^+\right)^* \big|_{t_1\, t_2\, t_3 = 1}$$

$$= \sum_{i=1}^{n} n_i \; e^{\,w_i[\epsilon_1, \epsilon_2, \epsilon_3]} - \sum_{i=1}^{n} n_i \; e^{-w_i[\epsilon_1, \epsilon_2, \epsilon_3]} \; . \tag{5.64}$$

By using the transform (5.56) we can write the contribution to the partition function as

$$\mathcal{Z}_\pi = \prod_{i=1}^{n} \frac{w_i[\epsilon_1, \epsilon_2, \epsilon_3]^{n_i}}{\left(- w_i[\epsilon_1, \epsilon_2, \epsilon_3]\right)^{n_i}} = \prod_{i=1}^{n} (-1)^{n_i} = (-1)^{\sum_i n_i} \; , \tag{5.65}$$

and the sum over the multiplicities n_i can be obtained by taking $\mathsf{T}_\pi^+ \big|_{t_1\, t_2\, t_3 = 1}$ and setting the weights w_i to zero, i.e. setting ϵ_i to zero or t_i to one. It is easy to see that $\mathsf{T}_\pi^+(t_1 = t_2 = t_3 = 1) = |\pi|$ since the second term of T_π^+ in (5.62) vanishes at $t_i = 1$. Thus we conclude that the partition function in this case is given by

$$\mathcal{Z}_{\mathrm{DT}}^{U(1)}(\mathbb{C}^3) = \sum_\pi (-1)^{|\pi|} \; e^{\,\mathrm{i}\,\vartheta\,|\pi|} \; , \tag{5.66}$$

which reproduces the MacMahon function (3.50) with the usual redefinition $-e^{\,\mathrm{i}\,\vartheta} = e^{-g_s} = q$.

These arguments carry over to the $U(1)^N$ theory with only minor modifications. Now we have to consider the full character (5.57). The splitting

$$\mathsf{T}_{\vec{\pi}}^+ = V_{\vec{\pi}} \otimes W_{\vec{\pi}}^* - V_{\vec{\pi}} \otimes V_{\vec{\pi}}^* \frac{(1 - t_1)\,(1 - t_2)}{t_1\, t_2} \; ,$$

$$\mathsf{T}_{\vec{\pi}}^- = -\frac{V_{\vec{\pi}}^* \otimes W_{\vec{\pi}}}{t_1\, t_2\, t_3} + V_{\vec{\pi}} \otimes V_{\vec{\pi}}^* \frac{(1 - t_1)\,(1 - t_2)}{t_1\, t_2\, t_3} \tag{5.67}$$

is helpful in simplifying the computation. At the fixed points $\vec{\pi}$ the vector spaces V and W decompose as in (5.46). The property (5.63) still holds but now the dual involution is defined as $(\epsilon_1, \epsilon_2, \epsilon_3, \boldsymbol{a}) \mapsto (-\epsilon_1, -\epsilon_2, -\epsilon_3, -\boldsymbol{a})$. By a similar argument as above we need only evaluate $\mathsf{T}_{\vec{\pi}}^+$ at $(\epsilon_1, \epsilon_2, \epsilon_3, \boldsymbol{a}) = (0, 0, 0, \boldsymbol{0})$. The second term in (5.67) again drops out and the first one gives

$$\sum_{l=1}^{N} \sum_{l'=1}^{N} \sum_{(i,j,k) \in \pi_{l'}} 1 = N \sum_{l'=1}^{N} |\pi_{l'}| \; . \tag{5.68}$$

To complete the computation of the partition function, the only missing ingredient now is the instanton action. Analogously to [34, 40], we write the universal sheaf $\mathcal{E}$ on the moduli space $\mathcal{M}_{N,k}(\mathbb{P}^3)$ as

$$\mathcal{E} = W \; \oplus \; V \otimes \left(S^- \ominus S^+\right) \tag{5.69}$$

where $S^\pm$ are the positive/negative chirality spinor bundles over $\mathbb{P}^3$, localized at a point of the fibre of $\mathbb{C}^3 = \mathbb{P}^3/\mathbb{P}^2$. At a critical point $\vec{\pi}$ we regard (5.69) as a virtual $\mathbb{T}^3 \times U(1)^N$ representation. By using the correspondence between spinors and differential forms given by twisting the spinor bundles to get $S^\pm \cong \Omega_{\mathbb{P}^3}^{\mathrm{even/odd},0}$ [18], we can derive the Chern character

$$\mathrm{ch}(\mathcal{E}_{\vec{\pi}}) = W_{\vec{\pi}} + (t_1 + t_2 + t_3 + t_1\, t_2\, t_3 - 1 - t_1\, t_2 - t_2\, t_3 - t_1\, t_3)\; V_{\vec{\pi}}$$

$$= W_{\vec{\pi}} - (1 - t_1)\,(1 - t_2)\,(1 - t_3)\; V_{\vec{\pi}} \; . \tag{5.70}$$

In the quiver formalism we work in an instanton sector with fixed charge $k = \dim_{\mathbb{C}} V_{\vec{\pi}}$ (so that the instanton action is proportional to k), and this corresponds to the total number of boxes of the partition by the same arguments used in the classification of the fixed points. Each subspace in the weight decomposition of the vector space $V_{\vec{\pi}}$ is one-dimensional and corresponds to a box in the partition $\vec{\pi}$. The generating function for the nonabelian invariants is thus

$$\mathcal{Z}_{\mathrm{DT}}^{U(1)^N}(\mathbb{C}^3) = \sum_{\vec{\pi}} (-1)^{N\,|\vec{\pi}|} \; e^{\,\mathrm{i}\,\vartheta\,|\vec{\pi}|} \; , \tag{5.71}$$

which coincides with eq. (3.1).

The results described in this section have a clear interpretation in terms of the noncommutative gauge theory of Section 3 that was spelled out in detail in Section 4. We can formally take this a step further and stress some close similarities between the two approaches. Recall that for the classification of the fixed points in the noncommutative gauge theory one simply needs to construct the Hilbert space on which the noncommutative algebra $\mathcal{A}$ is represented. In particular, the ratio of fluctuation determinants has the same structure, the main difference being that in the noncommutative gauge theory we are dealing with the determinants of operators acting on a separable Hilbert space which gives directly the result, while in the matrix model after the computation of the determinants there is still the integral over the Cartan subalgebra to do. The evaluation of the contour integral (5.23) was finally sidestepped by constructing an explicit local model for the instanton moduli space. The computation of the associated index is equivalent to a direct use of the localization formula.

We can build a formal dictionary to go back and forth between the two approaches. This could prove very helpful in extending our general formalism to other setups. The equivariant Chern character (5.70) can be derived precisely in the noncommutative gauge theory. If we use the notation

$$\chi_{\mathcal{I}}(t) := \mathrm{ch}_{\vec{\pi}}(t) = W_{\vec{\pi}} - \left(1 - \mathrm{e}^{t\,\epsilon_1}\right)\left(1 - \mathrm{e}^{t\,\epsilon_2}\right)\left(1 - \mathrm{e}^{t\,\epsilon_3}\right) V_{\vec{\pi}} \tag{5.72}$$

along with the redefinitions $t_i = \mathrm{e}^{t\,\epsilon_i}$ and $e_l = \mathrm{e}^{t\,a_l}$, then the integrand of (3.41) formally reproduces the character (5.57) up to the perturbative contribution $W_{\vec{\pi}} \otimes W_{\vec{\pi}}^*$ and an irrelevant overall sign. The sign mismatch was explained above in terms of the alternating sign in the definition of the index. The role of the exponentiation and of the integral over t is to reproduce the transform (5.56). Altogether, we can take this as a rule to compute the equivariant index from the ratio of fluctuation determinants as computed in the noncommutative gauge theory. This is perhaps not surprising as field theoretically both approaches are just two different ways to handle the ill-defined localization in the original cohomological gauge theory. An application of this formalism will be presented in the next section where we compute the partition function of the $U(1)^N$ model on a generic toric Calabi–Yau manifold.

6   Partition function on a toric manifold

In the previous sections we have derived a precise dictionary between the noncommutative gauge theory and an auxiliary matrix quantum mechanics. We will now apply our formalism in a controlled setup, switching between the two approaches when convenient. In particular, we will write down the partition function of the $U(1)^N$ gauge theory on an arbitrary toric Calabi–Yau threefold, extending the gauge theory prescription of [8] for handling the instanton counting on a generic toric manifold and the geometric results of [7] to compute the quantum fluctuation determinants. Although our final result does not provide any new geometrical invariants of threefolds, as it does not capture the full nonabelian structure of the Donaldson–Thomas invariants, it does compute the number of BPS bound states of branes in this specific regime of the theory.

6.1   Instanton action

Let X be a nonsingular toric threefold with Kähler two-form k_0 and Newton polyhedron $\Delta(X)$, the image of X under the moment map associated to the toric action on X. The vertices f of $\Delta(X)$ correspond to the fixed points of the $\mathbb{T}^3$-action on X. For each f there is a $\mathbb{T}^3$-invariant open $\mathbb{C}^3$ chart centred at the fixed point. On each patch we can choose coordinates corresponding to the

directions (t_1, t_2, t_3), where $(t_1, t_2, t_3) = (\mathrm{e}^{\mathrm{i}\epsilon_1}, \mathrm{e}^{\mathrm{i}\epsilon_2}, \mathrm{e}^{\mathrm{i}\epsilon_3})$ are the generators of the toric action. The edges of the polyhedron $\Delta(X)$ correspond to the $\mathbb{T}^3$-invariant lines of X. They represent generic projective lines $\mathbb{P}^1$ which join two fixed points f_1 and f_2 in $\Delta(X)$. Instantons of $U(N)$ noncommutative gauge theory on each $\mathbb{C}^3$ patch in the Coulomb phase correspond to sums of monomial ideals $\mathcal{I}_f = \mathcal{I}_{a_{1,f}} \oplus \cdots \oplus \mathcal{I}_{a_{N,f}}$ in $\mathcal{O}_{\mathbb{C}^3}$ associated to coloured three-dimensional partitions $\vec{\pi}_f = (\pi_{1,f}, \ldots, \pi_{N,f})$. Such collections of ideals correspond globally to $\mathbb{T}^3 \times U(1)^N$-invariant torsion free sheaves $\mathcal{E}$ of rank N on X with associated subscheme Z supported on the fixed points in $\Delta(X)$ and the lines connecting them, together with a framing $\mathcal{E}_\infty \cong W \otimes \mathcal{O}_X$.

We need to compute $\operatorname{Tr}_{\mathcal{H}_{\mathcal{I}_f}} \left(\mathrm{e}^{t\,\Phi} \right)$, where Φ is a nonabelian Higgs field and the Hilbert space $\mathcal{H}_{\mathcal{I}_f}$ corresponds to a three-dimensional partition $\vec{\pi}_f$ with fixed asymptotic behaviour at infinity. One can write down directly the Chern character at a fixed point $f \in \Delta(X)$ corresponding to the generalized instanton configuration with fixed asymptotics as

$$
\begin{aligned}
\chi_{\mathcal{I}_f}(t) \;=\; & \sum_{l=1}^{N} e_{l,f} \Bigg(1 - (1-t_1)(1-t_2) \sum_{(i,j)\in\lambda_{3,l,f}} t_1^{i-1} t_2^{j-1} - (1-t_1)(1-t_3) \sum_{(i,k)\in\lambda_{2,l,f}} t_1^{i-1} t_3^{k-1} \\
& - (1-t_2)(1-t_3) \sum_{(j,k)\in\lambda_{1,l,f}} t_2^{j-1} t_3^{k-1} \\
& - (1-t_1)(1-t_2)(1-t_3) \sum_{(i,j,k)\in\pi_{l,f}} t_1^{i-1} t_2^{j-1} t_3^{k-1} \Bigg) .
\end{aligned}
\tag{6.1}
$$

The first set of terms represent the vacuum contribution which is fixed asymptotically by the $3N$ two-dimensional Young tableaux $(\lambda_{1,l,f}, \lambda_{2,l,f}, \lambda_{3,l,f})$, the asymptotics of $\pi_{l,f}$ in the coordinate directions labelling the corresponding edges emanating from the vertex f. We are covering the toric manifold X with $\mathbb{C}^3$ patches and solving the noncommutative gauge theory in each patch. The asymptotic boundary conditions are necessary to glue the patches together. The ordinary two-dimensional partition which is the asymptotic condition in the i-th direction t_i is denoted by $\lambda_{i,l,f}$, with the index $l = 1, \ldots, N$ reminding us from which sector of the Hilbert space $\mathcal{H}_{\mathcal{I}_f}$ they come from. The last term corresponds to the three-dimensional partitions $\pi_{l,f}$ which should now be understood as properly renormalized for $n \gg 0$ with volume

$$
|\pi_{l,f}| = \Bigg(\sum_{\substack{(i,j,k)\in\pi_{l,f} \\ i,j,k\leq n}} 1 \Bigg) - (n+1) \left(|\lambda_{1,l,f}| + |\lambda_{2,l,f}| + |\lambda_{3,l,f}| \right) .
\tag{6.2}
$$

In the following we will impose the Calabi–Yau condition (3.9) and define $x = \frac{\epsilon_2}{\epsilon_1}$. The equivariant parameters are defined differently in each $\mathbb{C}^3$ patch and in such a way that they match when gluing the patches together as in the topological vertex gluing rules. The contribution of the third Chern character to the instanton action, i.e. the coefficient $\chi_{\mathcal{I}_f}^{(3)}$ of t^3 in the small t expansion of (6.1), gives

$$
\begin{aligned}
\frac{\mathrm{i}}{48\pi^3} \int_X \operatorname{Tr} F_A \wedge F_A \wedge F_A \;=\; & \sum_{f\in\Delta(X)} \sum_{l=1}^{N} \Bigg(-\frac{a_{l,f}^3}{6\epsilon_{1,f}\, x_f\, (1+x_f)} + \sum_{(i,j,k)\in\pi_{l,f}} 1 \\
& + \sum_{(i,j)\in\lambda_{3,l,f}} \left(-\frac{1}{2} + j + \frac{i-j}{1+x_f} + \frac{a_{l,f}}{\epsilon_{1,f}(1+x_f)} \right) \\
& + \sum_{(i,k)\in\lambda_{2,l,f}} \left(-\frac{1}{2} + k + \frac{k-i}{x_f} - \frac{a_{l,f}}{\epsilon_{1,f}\, x_f} \right) + \sum_{(j,k)\in\lambda_{1,l,f}} \left(-\frac{1}{2} + k - j\, x_f + k\, x_f - \frac{a_{l,f}}{\epsilon_{1,f}} \right) \Bigg)
\end{aligned}
\tag{6.3}
$$

where we have written down explicitly the sum over fixed points f associated with the vertices of the toric diagram $\Delta(X)$. Each vertex contribution has a proper factor associated with the vertex itself, but it also comes with edge factors associated with the asymptotic two-dimensional partitions. To properly treat the edge factors one has to consider both contributions associated to an edge coming from the two adjacent vertices. This will be done explicitly in the following.

The first contribution in (6.3) is independent of the partitions $\vec{\pi}_f$, and as such can be factored out as a "perturbative" contribution as before. The second contribution gives a factor of $|\vec{\pi}_f|$ associated to each vertex of the toric diagram and generalizes the instanton action. The remaining three terms come from the contribution of the asymptotics and have to be combined with the analogous contributions coming from other fixed points. For example, let us choose a fixed point f and consider the term where we sum over boxes of $\lambda_{3,l,f}$. The vertex we are considering is joined to another vertex along an edge and the partitions on the edge have the structure given by $\lambda_{3,l,f}$. The edge represents a $\mathbb{T}^3$-invariant rational curve $\mathbb{P}^1$ with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-m_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-m_2)$ determining the local geometry of $\Delta(X)$ near the edge. The Calabi–Yau condition implies

$$m_1 + m_2 = 2 . \tag{6.4}$$

The two contributions we have to consider are exactly the same but each one is expressed in terms of the equivariant parameters that are associated with the local coordinates in each $\mathbb{C}^3$ patch. The relation between the equivariant parameters is then given by the transition function between the two coordinate charts as [7]

$$
\begin{aligned}
\epsilon_{1,f_2} &= \epsilon_{1,f_1} + m_1 \epsilon_{3,f_1} , \\
\epsilon_{2,f_2} &= \epsilon_{2,f_1} + m_2 \epsilon_{3,f_1} , \\
\epsilon_{3,f_2} &= -\epsilon_{3,f_1} , \\
x_{f_2} &= \frac{\epsilon_{2,f_2}}{\epsilon_{1,f_2}} = \frac{x_{f_1} - m_2 - m_2 x_{f_1}}{1 - m_1 - m_1 x_{f_1}} ,
\end{aligned}
\tag{6.5}
$$

where (f_1, f_2) labels the two fixed points joined by the edge.

After a bit of algebra we find

$$
\begin{aligned}
\sum_{l=1}^{N} \sum_{(i,j)\in\lambda_{3,l,f_1}} &\left(-\frac{1}{2} + j + \frac{i-j}{1+x_{f_1}} + \frac{a_{l,f_1}}{\epsilon_{1,f_1}(1+x_{f_1})} - \frac{1}{2} + j + \frac{i-j}{1+x_{f_2}} + \frac{a_{l,f_2}}{\epsilon_{1,f_2}(1+x_{f_2})} \right) \\
&= \sum_{l=1}^{N} \sum_{(i,j)\in\lambda_{3,l,f_1}} \left(-1 + 2j + i\,m_1 - j\,m_1 \right) \\
&= \sum_{l=1}^{N} \sum_{(i,j)\in\lambda_{3,l,f_1}} \left(m_1\,(i-1) + m_2\,(j-1) + 1 \right)
\end{aligned}
\tag{6.6}
$$

where in the last equality we have used (6.4) to express the edge contribution in a universal form. The Higgs vevs cancel since we require $a_{l,f_1} = a_{l,f_2}$ as part of the gluing conditions,[3] and the edge contribution along the "direction" t_3 is the sum of N terms which all have the same form as in the abelian $N = 1$ theory. After performing similar computations along the other two "directions" t_2 and t_1, the ch$_3$ term of the instanton action gives

$$
I(\vec{\pi}_f) = \sum_{f\in\Delta(X)} \sum_{l=1}^{N} |\pi_{l,f}| + \sum_{e\in\Delta(X)} \sum_{l=1}^{N} \sum_{(i,j)\in\lambda_{l,e}} \left(m_{1,e}\,(i-1) + m_{2,e}\,(j-1) + 1 \right) , \tag{6.7}
$$

[3]Each $a_{l,f}$ gives the asymptotic boundary condition on the instanton labelled by f, and we can glue together two instantons at infinity if and only if they have the same value of $a_{l,f}$.

54

where the sum over f runs through the vertices of the toric diagram $\Delta(X)$ while the sum over e runs through the edges.

To complete the evaluation of the instanton action one needs to also consider the second Chern character, i.e. the coefficient $\chi^{(2)}_{\mathfrak{I}_f}$ of t^2 in the small t expansion of (6.1). One finds

$$-\frac{1}{8\pi^2} \int_X k_0 \wedge \operatorname{Tr} F_A \wedge F_A \;=\; \sum_{f\in\Delta(X)} \frac{H_f \, \chi^{(2)}_{\mathfrak{I}_f}}{\epsilon_{1,f}\,\epsilon_{2,f}\,\epsilon_{3,f}} \tag{6.8}$$

$$= \sum_{f\in\Delta(X)} \sum_{l=1}^{N} \left(\frac{a^2_{l,f}\,H_f}{2\epsilon_{1,f}\,\epsilon_{2,f}\,\epsilon_{3,f}} - \sum_{(i,j)\in\lambda_{3,l}} \frac{H_f}{\epsilon_{3,f}} - \sum_{(i,k)\in\lambda_{2,l}} \frac{H_f}{\epsilon_{2,f}} - \sum_{(j,k)\in\lambda_{1,l}} \frac{H_f}{\epsilon_{1,f}} \right)$$

where H_f is the value at the fixed point f of the hamiltonian H associated with the vector field Ω that generates the equivariant rotations, i.e., $\mathrm{d}H = \imath_\Omega k_0$. The term depending on the Higgs vevs can again be dropped and we can analyse the edge contributions as before with the result

$$- \sum_{e\in\Delta(X)} \sum_{l=1}^{N} t_e \, |\lambda_{l,e}| \tag{6.9}$$

where $t_e = \frac{H_{f_1}-H_{f_2}}{\epsilon_e}$, for each pair of fixed points (f_1, f_2) connected by the edge e, is the Kähler parameter of the line $\mathbb{P}^1$ associated to e. Altogether, the instanton weight is given by

$$\mathrm{e}^{\,\mathrm{i}\,\vartheta\,\mathsf{I}(\vec\pi_f)}\; \mathrm{e}^{-\sum_{e\in\Delta(X)}\,\sum_{l=1}^{N} t_e\,|\lambda_{l,e}|}\;. \tag{6.10}$$

6.2 Fluctuation determinants

The next step in the evaluation of the partition function is to determine the ratio of quantum fluctuation determinants. According to our rule of the previous section, we can write down the ratio in the noncommutative gauge theory and read off from the integrand the equivariant Euler characteristic of the complex (5.51) using the techniques of [7]. The ratio at a fixed point $f \in \Delta(X)$ comes out to be

$$\frac{\chi_{\mathfrak{I}_f}(t)\,\chi_{\mathfrak{I}_f}(-t)}{(1-t_1)\,(1-t_2)\,(1-t_3)} \tag{6.11}$$

as in (3.41), with $\chi_{\mathfrak{I}_f}(t)$ given by (6.1). The character at the fixed point f decomposes as

$$\chi_f(X) \;=\; \Big(W_f - (1-t_1)\,(1-t_2)\,V_{12,f} - (1-t_1)\,(1-t_3)\,V_{13,f}$$
$$- (1-t_2)\,(1-t_3)\,V_{23,f} - (1-t_1)\,(1-t_2)\,(1-t_3)\,V_f \Big)$$
$$\otimes \Big(W_f^* - \frac{(1-t_1)\,(1-t_2)}{t_1\,t_2}\,V_{12,f}^* - \frac{(1-t_1)\,(1-t_3)}{t_1\,t_3}\,V_{13,f}^* \tag{6.12}$$
$$- \frac{(1-t_2)\,(1-t_3)}{t_2\,t_3}\,V_{23,f}^* + \frac{(1-t_1)\,(1-t_2)\,(1-t_3)}{t_1\,t_2\,t_3}\,V_f^* \Big)\,\frac{1}{(1-t_1)\,(1-t_2)\,(1-t_3)}$$

where the $\mathbb{T}^3 \times U(1)^N$ modules are given by

$$W_f \;=\; \sum_{l=1}^{N} e_{l,f}\;,$$

$$V_f = \sum_{l=1}^{N} e_{l,f} \sum_{(i,j,k) \in \pi_{l,f}} t_1^{i-1} t_2^{j-1} t_3^{k-1} ,$$

$$V_{\alpha\beta,f} = \sum_{l=1}^{N} e_{l,f} \sum_{(i,j) \in \lambda_{\gamma,l,f}} t_\alpha^{i-1} t_\beta^{j-1} \qquad (6.13)$$

with (α, β, γ) a cyclic permutation of $(1, 2, 3)$.

The computation of the ratio of determinants from (6.12) is now just a tedious but completely straightforward algebraic exercise. By using splittings of modules with properties analogous to those of Section 5.5, one easily shows that all contributions independent of the Chan–Paton space W_f vanish at $(\epsilon_1, \epsilon_2, \epsilon_3, \boldsymbol{a}) = (0, 0, 0, \boldsymbol{0})$. After dropping the perturbative contribution $W_f \otimes W_f^*$, one thus finds that the only non-vanishing contributions which survive at $(\epsilon_1, \epsilon_2, \epsilon_3, \boldsymbol{a}) = (0, 0, 0, \boldsymbol{0})$ are given by

$$\frac{W_f \otimes V_f^*}{t_1 t_2 t_3} - V_f \otimes W_f^* - \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1 t_2 t_3} V_f \otimes V_f^* \qquad (6.14)$$

$$+ \frac{1}{1-t_3} \left(-V_{12,f} \otimes W_f^* - \frac{W_f \otimes V_{12,f}^*}{t_1 t_2} + \frac{(1-t_1)(1-t_2)}{t_1 t_2} V_{12,f} \otimes V_{12,f}^* \right)$$

$$+ \frac{1}{1-t_2} \left(-V_{13,f} \otimes W_f^* - \frac{W_f \otimes V_{13,f}^*}{t_1 t_3} + \frac{(1-t_1)(1-t_3)}{t_1 t_3} V_{13,f} \otimes V_{13,f}^* \right)$$

$$+ \frac{1}{1-t_1} \left(-V_{23,f} \otimes W_f^* - \frac{W_f \otimes V_{23,f}^*}{t_2 t_3} + \frac{(1-t_2)(1-t_3)}{t_2 t_3} V_{23,f} \otimes V_{23,f}^* \right) .$$

After applying the transformation (5.56), each of the four lines in (6.14) gives a minus sign to some power.

The first line of (6.14) coincides with the character (5.57) computed in Section 5.5 and gives a factor $(-1)^N \sum_{l=1}^{N} |\pi_{l,f}|$ for each fixed point $f \in \Delta(X)$. The remaining terms are a nonabelian generalization of the edge character computed in [7] (reproduced formally by setting W to 1), and can be understood from the point of view of the matrix quantum mechanics as follows. By T-duality, the D6–D2 system corresponds to a four-dimensional instanton problem over each rational curve of X. These asymptotics are each described by an ADHM quiver [32], with associated vector spaces W_f of dimension N and $V_{\alpha\beta,f}$ of dimension $k_{\gamma,f} = |\vec{\lambda}_{\gamma,f}|$. The full quiver is the modification of the D6–D0 quiver of Fig. 1 obtained by inserting the vector spaces $V_{\alpha\beta,f}$ plus all additional open string fields [12]. This modifies the complex (5.51) by including terms from the four-dimensional ADHM deformation complex [32, 37]. The additional contributions in (6.14) then arise from the usual characters in four-dimensions. They may be computed in the present case by carefully matching the edge contributions with the partner terms coming from other vertices of the toric diagram $\Delta(X)$ as in Section 6.1 above.

For example, consider the contribution $\mathsf{L}_{23,f}$ from the last line of (6.14), which is oriented in the t_1 "direction". To this term we have to add its partner term coming from the vertex which shares the same edge e, and which has the same two-dimensional partition structure of $V_{23,f}$ as its asymptotic behaviour. Using (6.5) with $t_1 \leftrightarrow t_3$ the full edge contribution is then

$$\mathsf{E}_{23,e} = \mathsf{L}_{23,f}(t_1, t_2, t_3, e_{l,f}) + \mathsf{L}_{23,f}\left(t_1^{-1}, t_2 t_1^{m_1}, t_3 t_1^{m_2}, e_{l,f}\right) . \qquad (6.15)$$

We use the splitting $\mathsf{L}_{23,f} = \mathsf{L}_{23,f}^+ + \mathsf{L}_{23,f}^-$ with

$$\mathsf{L}_{23,f}^+ = \frac{1}{1-t_1} \left(-V_{23,f} \otimes W_f^* + \frac{1-t_2}{t_2} V_{23,f} \otimes V_{23,f}^* \right) \qquad (6.16)$$

56

and $\left(\mathsf{L}_{23,f}^{+}\right)^{*}\big|_{t_1 t_2 t_3=1} = -\mathsf{L}_{23,f}^{-}\big|_{t_1 t_2 t_3=1}$. Together with the condition (6.4), after some algebra one finds that the corresponding splitting $\mathsf{E}_{23,e}^{+}$ in the limit $(\epsilon_1, \epsilon_2, \epsilon_3, \boldsymbol{a}) = (0,0,0,\boldsymbol{0})$ is given by

$$\sum_{l,l'=1}^{N} \sum_{(j,k)\in\lambda_{1,l,f}} (j\, m_1 + k\, m_2 - 1) + \sum_{l,l'=1}^{N} \sum_{(j,k)\in\lambda_{1,l,f}} \sum_{(j',k')\in\lambda_{1,l',f}} m_1 \tag{6.17}$$

$$= N \sum_{l=1}^{N} \sum_{(j,k)\in\lambda_{1,l,f}} \left(m_1 (j-1) + m_2 (k-1) + 1\right) + \sum_{l,l'=1}^{N} |\lambda_{1,l,f}|\, |\lambda_{1,l',f}|\, m_1 \,.$$

In this way one finds that the final result for the ratio of fluctuation determinants can be expressed as $(-1)^{\mathsf{J}(\vec{\pi}_f)}$, where

$$\mathsf{J}(\vec{\pi}_f) = \sum_{f\in\Delta(X)} N \sum_{l=1}^{N} |\pi_{l,f}| + \sum_{e\in\Delta(X)} N \sum_{l=1}^{N} \sum_{(i,j)\in\lambda_{l,e}} \left(m_{1,e}\,(i-1) + m_{2,e}\,(j-1) + 1\right)$$

$$+ \sum_{e\in\Delta(X)} \sum_{l,l'=1}^{N} |\lambda_{l,e}|\, |\lambda_{l',e}|\, m_{1,e} \tag{6.18}$$

generalizes the abelian $N=1$ result of [7, 8].

6.3 Partition function

We can now collect all the ingredients and write down the Donaldson–Thomas partition function on any toric Calabi–Yau manifold X in the $U(1)^N$ phase of the six-dimensional topological Yang–Mills theory on X. One finds

$$\mathsf{Z}_{\mathrm{DT}}^{U(1)^N}(X) = \sum_{\vec{\pi}_f} (-1)^{\mathsf{J}(\vec{\pi}_f)} \; \mathrm{e}^{\,\mathrm{i}\vartheta\,\mathsf{I}(\vec{\pi}_f)} \; \mathrm{e}^{-\sum_{e\in\Delta(X)}\,\sum_{l=1}^{N}|\lambda_{l,e}|\,t_e} \,. \tag{6.19}$$

After some rewriting we have

$$\mathsf{Z}_{\mathrm{DT}}^{U(1)^N}(X) = \sum_{\vec{\pi}_f} q^{\mathsf{I}(\vec{\pi}_f)} \, (-1)^{(N+1)\,\mathsf{I}(\vec{\pi}_f)} \prod_{e\in\Delta(X)} (-1)^{\sum_{l,l'=1}^{N}|\lambda_{l,e}|\,|\lambda_{l',e}|\,m_{1,e}} \; \mathrm{e}^{-\sum_{l=1}^{N}|\lambda_{l,e}|\,t_e} \,, \tag{6.20}$$

where the 1 in the $N+1$ factor arises from the minus sign in the definition $q = -\,\mathrm{e}^{\,\mathrm{i}\vartheta}$ and $\mathsf{I}(\vec{\pi}_f)$ is the ch_3 contribution to the instanton action given by (6.7). The m_1-dependent signs in (6.20) are naturally interpreted as framings of the corresponding edges, as in the topological vertex formalism.

As a simple example, let us consider the resolved conifold $X = X_{\mathrm{con}}$, the total space of the rank two holomorphic bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{P}^1$ (viewed as the normal bundle to the local Calabi–Yau curve $\mathbb{P}^1$). In this case one has

$$\mathsf{Z}_{\mathrm{DT}}^{U(1)^N}(X_{\mathrm{con}}) = \sum_{\substack{\pi_{1,f_1},\dots,\pi_{N,f_1} \\ \pi_{1,f_2},\dots,\pi_{N,f_2}}} q^{\sum_{l=1}^{N}(|\pi_{l,f_1}|+|\pi_{l,f_2}|)+\sum_{l=1}^{N}\,\sum_{(i,j)\in\lambda_l}(i+j+1)}$$

$$\times (-1)^{(N+1)\left(\sum_{l=1}^{N}(|\pi_{l,f_1}|+|\pi_{l,f_2}|)+\sum_{l=1}^{N}\,\sum_{(i,j)\in\lambda_l}(i+j+1)\right)}$$

$$\times (-1)^{\sum_{l,l'=1}^{N}|\lambda_l|\,|\lambda_{l'}|} \; \mathrm{e}^{-\sum_{l=1}^{N} t\,|\lambda_l|} \,. \tag{6.21}$$

This formula gives

$$\mathsf{Z}_{\mathrm{DT}}^{U(1)^N}(X_{\mathrm{con}}) = \left(\sum_{\pi_f} \left((-1)^{N+1}\, q\right)^{|\pi_f|+\sum_{(i,j)\in\lambda}(i+j+1)} (-1)^{|\lambda|} \; \mathrm{e}^{-t\,|\lambda|}\right)^{N}$$

$$= \prod_{n=1}^{\infty} \frac{\left(1 - (-1)^{N\,n}\, q^n\; \mathrm{e}^{-t}\right)^{N\,n}}{\left(1 - (-1)^{N\,n}\, q^n\right)^{N\,n}} \; . \tag{6.22}$$

Up to an alternating sign, this is just the N-th power of the abelian result which coincides with the topological string amplitude computed in the melting crystal reformulation of the topological vertex [3, 8]. The alternating sign will be explained in the next section.

7 Summary and applications

In this final section we will discuss some properties of the $U(1)^N$ partition functions (6.20). We will also discuss some open questions and relations to other models. In particular, we summarize our results and describe how they could be applied to other settings.

7.1 The OSV conjecture

Let us consider for simplicity the threefold $X = \mathbb{C}^3$. The partition function (3.1) can be rewritten in the form

$$\mathcal{Z}_{\mathrm{DT}}^{U(1)^N}\left(\mathbb{C}^3\right) = \sum_{\pi_1,\ldots,\pi_N} \mathrm{e}^{\,(N\,\mathrm{i}\,\pi - g_s)\, \sum_{l=1}^{N} |\pi_l|} = M(\tilde{q})^N \; , \tag{7.1}$$

where $M(\tilde{q})$ is the MacMahon function (3.2) of $\tilde{q} = \mathrm{e}^{-\tilde{g}_s}$ and

$$\tilde{g}_s = g_s - N\,\mathrm{i}\,\pi \; . \tag{7.2}$$

The partition function is just a power of the abelian $N = 1$ result. This is reasonable since the torus fixed points of the moduli space (4.19) essentially broke up accordingly (so that no new non-trivial instantons are present). On the other hand, the Donaldson–Thomas string coupling constant g_s is modified to (7.2) as well. As we now explain, this modification is natural from the point of view of the OSV conjecture [2].

Consider Type IIA string theory on $X \times \mathbb{R}^4$, where D6, D2 and D0 branes wrap holomorphic cycles of the Calabi–Yau threefold X. The mixed ensemble partition function for BPS bound states with fixed chemical potentials ϕ_2^a and ϕ_0 for the D2 and D0 brane charges, and magnetic charge p^0 of the D6-branes, is denoted $Z_{\mathrm{BH}}(p^0; \phi_2^a, \phi_0)$. It factorizes in the limit of large charge p^0 and small string coupling G_s as [2]

$$Z_{\mathrm{BH}}\left(p^0\,; \phi_2^a, \phi_0\right) = \left| Z_{\mathrm{top}}(G_s, t^a) \right|^2 \; , \tag{7.3}$$

where $Z_{\mathrm{top}}(G_s, t^a)$ is the A-model topological string partition function on X evaluated at the attractor point [41, 42] of the moduli space given by

$$G_s = \frac{4\pi\,\mathrm{i}}{X^0} = \frac{4\pi\,\mathrm{i}}{p^0 + \mathrm{i}\,\frac{\phi_0}{\pi}} \qquad \text{and} \qquad t^a = -\frac{2\phi_2^a}{p^0 + \mathrm{i}\,\frac{\phi_0}{\pi}} \tag{7.4}$$

with t^a the Kähler parameters of the two-cycles wrapped by the D2-branes.

If we suppose now that the six-dimensional cohomological gauge theory on X computes the topological A-model partition function, then the $U(N)$ generalization means including multiple D6-branes from the point of view of the OSV formula (7.3). However, the Calabi–Yau crystal description is valid in the strong coupling regime [3] and so we expect the relation

$$g_s = \frac{1}{G_s} \; . \tag{7.5}$$

58

Rewriting the OSV formula (7.3) in the S-dual string couplings $\tilde{g}_s$ and g_s with and without the D6-branes, one finds

$$\tilde{g}_s = g_s - \frac{i\,p^0}{4\pi} \ . \tag{7.6}$$

Taking into account the relation $p^0 = \frac{N}{4\pi}$,[4] we see that the string coupling in the Donaldson–Thomas description changes in the way expected from the OSV conjecture.[5]

According to the generic prediction (7.3), the general formula (6.20) should factorize in the large N limit. It would be interesting to further check the OSV factorization for our $\mathbb{C}^3$ example. Factorization including D6-branes has only been checked for the single example $X = K3 \times \mathbb{T}^2$ [43]. Furthermore, the conformal field theory derivation of the OSV formula of [44] does not readily generalize for multiple D6-branes. Given that the string coupling in our case lives in the strong coupling regime, it would be interesting to see if and how the OSV factorization works here. A refined version of the OSV formula was found in [45] using the $U(1)$ Donaldson–Thomas theory for a D6 brane-antibrane pair on compact Calabi–Yau manifolds. It was also shown there that in certain limits of the background one can identify the abelian Donaldson–Thomas partition function with the BPS index for stable D6–D2–D0 bound states with unit D6-brane charge.

7.2 Enumerative invariants

Let us study the small q expansion of the partition function (7.1), with the aim of understanding its role in enumerative geometry. For example, for $U(2)$ gauge group one gets

$$
\begin{aligned}
Z_{\mathrm{DT}}^{U(1)^2}(\mathbb{C}^3) &= \sum_{\pi_1,\pi_2} (-1)^{3(|\pi_1|+|\pi_2|)}\, q^{|\pi_1|+|\pi_2|} \\
&= \sum_{\pi_1,\pi_2} (-1)^{|\pi_1|+|\pi_2|}\, q^{|\pi_1|+|\pi_2|} \\
&= \left(\sum_\pi (-q)^{|\pi|} \right)^2 \\
&= \left(\prod_{n=1}^\infty \left(1 - (-q)^n\right)^{-n} \right)^2 \\
&= 1 - 2q + 7q^2 - 18q^3 + 47q^4 - 110q^5 + 258q^6 - 568q^7 + 1237q^8 + O\left(q^9\right)\ .
\end{aligned} \tag{7.7}
$$

Similarly, for the $U(N)$ gauge theory one finds

$$
\begin{aligned}
Z_{\mathrm{DT}}^{U(1)^N}(\mathbb{C}^3) &= \sum_{\pi_1,\dots,\pi_N} (-1)^{(N+1)\,(|\pi_1|+\cdots+|\pi_N|)}\, q^{|\pi_1|+\cdots+|\pi_N|} \\
&= \left(\sum_\pi (-1)^{(N+1)\,|\pi|}\, q^{|\pi|} \right)^N \\
&= \left(\prod_{n=1}^\infty \left(1 - ((-1)^{N+1}\, q)^n\right)^{-n} \right)^N \\
&= 1 - (-1)^N\, N\, q + \left(\tfrac{1}{2}(-1)^{2N}\, N^2 + \tfrac{5}{2}(-1)^{2N}\, N\right)\, q^2
\end{aligned} \tag{7.8}
$$

---

[4]The 4π factor is a matter of convention.

[5]The dependence on N does not spoil the S-duality conjecture. In fact, the instanton measure does not depend on the Higgs vevs. Nevertheless, it would be interesting to check this prediction against a computation in some Chern–Simons theory.

$$+ \left(-\tfrac{1}{6} \, (-1)^{3N} \, N^3 - \tfrac{5}{2} \, (-1)^{3N} \, N^2 - \tfrac{10}{3} \, (-1)^{3N} \, N \right) \, q^3$$
$$+ \left(\tfrac{1}{24} \, (-1)^{4N} \, N^4 + \tfrac{5}{4} \, (-1)^{4N} \, N^3 + \tfrac{155}{24} \, (-1)^{4N} \, N^2 + \tfrac{21}{4} \, (-1)^{4N} \, N \right) \, q^4 + O \left(q^5 \right) \ .$$

The numerical invariants computed by the small q-expansion (7.8) are all integer-valued.

As the nonabelian $U(1)^N$ partition function on $\mathbb{C}^3$ is given by the N-th power of the MacMahon function, up to sign factors, the numerical values of the Donaldson–Thomas invariants are the same. But the continuation of the MacMahon function in (7.1) to complex values of the string coupling suggests a non-trivial interpretation in terms of Gromov–Witten theory. By using the correspondence between the Donaldson–Thomas and Gromov–Witten partition functions [7], one sees that the Gromov–Witten invariants change non-trivially. This change can be encoded in the topological string coupling constant associated with the D6-brane charge. In terms of the usual Gromov–Witten theory, this may arise through geometric transition via a connection with Chern–Simons gauge theory with complexified coupling constant, as arises for complex gauge groups. It would be very interesting to understand the meaning of the new parameter N in terms of the underlying closed topological string theory.

However, we should reiterate that our computations are only valid in the Coulomb phase, where the gauge group is completely broken to its maximal torus $U(1)^N$. Furthermore, we considered only the target space $X = \mathbb{C}^3$ properly, wherein the instanton moduli space could be characterized by a stable framed representation of the quiver with relations of Fig. 1 in the category of complex vector spaces. The precise characterization of the full moduli space and the construction of the full nonabelian Donaldson–Thomas theory is a much more difficult task. For a generic threefold X it should involve a stable twisted representation of the quiver with relations depicted in Fig. 1 (including ADHM quivers over the rational curves) in the abelian category of coherent sheaves of $\mathcal{O}_X$-modules [46], with *non-trivial* framing. A recent proposal for such a construction in the case that X is a local curve can be found in [13]. Understanding of the moduli space of the full nonabelian Donaldson–Thomas theory, exploring its combinatorial nature in terms of random partitions, and searching for new geometric invariants are all interesting and challenging tasks, which are however beyond the scope of the present work. In this paper we have only probed a small corner of the full moduli space, and developed some potentially useful starting techniques in this direction.

7.3 Relations with other models

It is instructive to compare our results with those obtained on other backgrounds, some of which can be related to ours through duality transformations. It was argued in [11, 47] that the fact that perturbative A-model topological string amplitudes capture non-perturbative information about D-brane bound states can be understood as a consequence of S-duality, when embedding the topological string theory into the physical Type IIB superstring theory. This S-duality was used in [48] to relate the abelian Donaldson–Thomas invariants on a Type IIA compactification with the Gopakumar–Vafa BPS invariants of M2-branes in M-theory, establishing another point of view on the relationship between four and five dimensional black holes of [49]. One can start with a Type IIA compactification on a ten-dimensional background given by the product of a Calabi–Yau threefold and the four-dimensional space $\mathbb{R}^3 \times \mathbb{S}^1$. Then the charge one D6-brane background can be transformed into a charge one Taub–NUT geometry by applying a $T\,S\,T$ duality transformation, where the D2 and D0 branes become respectively fundamental strings and Kaluza–Klein momentum modes along the $\mathbb{S}^1$ of the Taub–NUT geometry. More precisely, after a T-duality transformation along the $\mathbb{S}^1$ of the space $\mathbb{R}^3 \times \mathbb{S}^1$, the Type IIA D6-brane gets mapped into the Type IIB D5-brane. S-duality of the physical Type IIB string theory transforms this D5-brane into an NS5-brane, and finally after the last T-duality transformation one arrives at the Taub–NUT geometry. This configuration can then be lifted to M-theory. See [48] for further details.

It is reasonable to expect that this set of dualities continues to hold for the nonabelian D6-brane configuration, though more work would need to be done at the supergravity level to establish this. However, if this is the case then it is interesting to speculate that our computation predicts a relation between topological A-model amplitudes on the Taub–NUT geometry and the partition function of N Type IIB NS5-branes, as a consequence of S-duality. It was pointed out in [47] that a "mirror" version of this duality is already implied in [50], where it was argued that B-model amplitudes compute the partition function of N Type IIA NS5-branes. In the setup of [50] one starts with N NS5-branes on a Calabi–Yau threefold X and relates this configuration through T-duality with Type IIB string theory on $X \times \mathbb{M}^4$, where $\mathbb{M}^4$ is a Taub–NUT space (or better an ALE space of type A_{N-1} when we let the size of the compactified circle grow to infinity). In concrete computations they take $X = K3 \times \mathbb{T}^2$ and its $\mathbb{Z}_2$ orbifold.

By a supergravity analysis and careful matching of F-terms on both sides, one can compute the nonperturbative contribution to the partition function of N *well-separated* NS5-branes, which turns out to be equal to the $(N-1)$-th power of the B-model topological string amplitude.[6] Reversing the statement, one may say that perturbative topological string amplitudes only compute the partition function of NS5-branes in the Coulomb branch. Thus even though it is not completely clear how to explicitly translate this set of dualities to our setup, our results are compatible with the findings of [50].

A similar picture arises in [51] where a one-parameter generalization of topological string theory was proposed. Although it seems natural following the reasonings of [51] to identify this parameter with the rank N of the nonabelian Donaldson–Thomas theory, it is not clear to us how to make precise contact with their proposal. Other multiparameter extensions of topological string theory are found in [52] and [53]. It would also be interesting to compare our results with the $U(N)$ extension of the construction of [54] which considered the equivariant reduction to lower dimension of the six-dimensional gauge theory on a local surface.

Acknowledgments

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