A Unified Approach to Various Techniques for the Non–uniqueness of the Inverse Gravimetric Problem and Wavelet–Based Methods

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Abstract

This paper provides an overview of two topics. First, it presents a unified approach to various techniques addressing the non–uniqueness of the solution of the inverse gravimetric problem; alternative, simple proofs of some known results are also given. Second, it summarises in a concise and self–contained way a particular multiscale regularisation technique involving scaling functions and wavelets.

Key Words: inverse gravimetric problem, gravimetry, non–uniqueness, null space, ill–posed, regularisation, Newton potential, minimal norm, harmonic density, Fredholm integral equation, scaling function, wavelet, multiresolution analysis.

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1 Introduction

A typical example of an ill–posed inverse problem is the inversion of Newton's Law of Gravitation

$$V(y) = \gamma \int_B \frac{D(x)}{|x-y|} \, \mathrm{d}x \ ,$$

where V is the gravitational potential, which is given, for example at the Earth's surface or at satellite height, γ is the gravitational constant, B is the closed unit ball, and $D \in \mathrm{L}^2(B)$ is the unknown mass density function. The first known occurence of this inverse problem in the literature is in the works of Stokes in 1867 [52].
Note that we do not consider here the problem where the determination of the shape of the gravitating body is combined with or is chosen instead of the determination of the density function. For such problems or similar ones, we refer to [11, 40, 64, 65, 66, 67] and [28, Chapter III, § 1].
Actually, the inverse gravimetric problem discussed here violates all three of Hadamard's criteria for a well–posed problem.

1. The potential V must be harmonic outside B. Moreover, due to the Picard condition, a solution exists only if V belongs to an appropriate subset in the space of harmonic functions. However, this does not cause a serious problem since in practice the information of V is only finite–dimensional. In particular, an approximation or interpolation by an appropriate harmonic function is a natural ingredient of any practical method.

2. The most serious problem is the non–uniqueness of the solution: The associated Fredholm integral operator is of the first kind and has a null space which is the $\mathrm{L}^2(B)$–orthogonal space of the closed linear subspace of all harmonic functions on B. This orthogonal complement, whose elements are called anharmonic functions, is infinite–dimensional.

3. Restricting the operator to harmonic densities yields an injective mapping which, however has a discontinuous inverse leading to an unstable solution. If V is given at satellite height, this instability is exponential!

The problem of non–uniqueness has been discussed extensively in literature, starting with the paper [52] of Stokes. This problem can be bypassed by imposing some reasonable additional condition on the density. One such condition, suggested by the mathematical structure of the operator, is to require that the density is harmonic (the complementary anharmonic part must then be obtained from non–gravimetric data, such as seismic data).

The approximate calculation of the harmonic density has already been implemented in several papers (whereas the problem of determining the anharmonic part remains open).
Due to the lack of a sensible physical interpretation of the harmonic part of the density, various alternative constraints have been discussed in the literature. In general, one can observe that gravitational data yield significant information only about the uppermost part of the Earth, which is essentially laterally heterogeneous.
Wavelet–based multiscale methods for the approximation of the harmonic density have been presented in [23], [33]-[36]. In particular, the papers [35] and [36] also take into account the cases where a zeroth, first, or second order derivative of V is given at satellite height (cf. the satellite missions CHAMP, GRACE, and GOCE).
In this paper we present a unified approach to various techniques addressing the non–uniqueness of the solution of the inverse gravimetric problem; alternative, simple proofs of some known results are also given. Furthermore, we discuss the role of the harmonic density, which is known to be the minimal $\mathrm{L}^2(B)$–norm density. In this context, we note the similarity of the inverse gravimetric problem with the inverse problem of magnetoencephalography (MEG), where uniqueness can also be obtained by requiring that the solution minimises the L^2–norm, see [19] and [20]. In addition, we summarise in a concise and self–contained way a particular multiscale regularisation technique involving scaling functions and wavelets.
This paper is organised as follows: In Section 2, we derive a spectral relation between the density and the associated potential, which maps the non–uniqueness of the inverse problem to the radial dependence of the spherical harmonics coefficients of the density. This allows us to derive a well–known characterisation of the null space in Section 3. In Sections 4 to 8, we discuss known approaches for obtaining a unique solution, such as imposing the constraint of harmonicity or requiring the minimisation of certain functionals. In Section 9, we use the wavelet-based multiscale method of [22, 35, 36] to generate a regularised sequence of approximations which converges to the minimal density (i.e. to the density which minimises the L^2–norm). In this method each function is obtained pointwise and it involves the spherical convolution of a particularly constructed scaling function with V. In Section 10, we present calculations which assume that the potential is given at a point grid at the Earth's surface. The case where the second radial derivative of the potential is given at satellite height was analysed in [36]. Moreover, we study the effect of local noise on the data (the case of global noise was analysed in [36]).

Notation

The sets of real numbers and positive integers are denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively. Furthermore, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The unit sphere in $\mathbb{R}^3$ is denoted by $\mathbb{S}^2$. The corresponding ball is denoted by $B := \{x \in \mathbb{R}^3 \,|\, |x| \leq 1\}$. The Laplacian operator is denoted by Δ.
A well–known complete orthonormal system in $\mathrm{L}^2\left(\mathbb{S}^2\right)$ is the system of spherical harmonics $\{Y_{n,j}\}_{n\in\mathbb{N}_0;j=-n,\ldots,n}$, where each $Y_{n,j}$ is the restriction of a homogeneous harmonic polynomial of degree n to $\mathbb{S}^2$. Throughout this paper, we will make use of several properties of the spherical harmonics; for further details, including proofs, we refer to [21].

2 Derivation of a Spectral Relation Between the Potential and the Associated Density

The purpose of this section is to derive a relation between the spherical harmonics coefficients of the gravitational potential and those of the density function. Here, we will omit the gravitational constant since it is simply a factor. However, it will be taken into account in the numerical implementation of Section 10.

Theorem 2.1 *Let the mass density function $D \in \mathrm{L}^2(B)$ be given. This function admits the following representation*

$$D(x) = \sum_{n=0}^{\infty}\sum_{j=-n}^{n} D_{n,j}(|x|)Y_{n,j}\left(\frac{x}{|x|}\right) \tag{1}$$

which converges in $\mathrm{L}^2(B)$. Then the potential

$$V = \int_B \frac{D(x)}{|x-\cdot|}\,\mathrm{d}x$$

is given for $|y| > 1$ pointwise and for $|y| = 1$ in the sense of $\mathrm{L}^2\left(\mathbb{S}^2\right)$, by

$$V(y) = \sum_{n=0}^{\infty}\sum_{j=-n}^{n}\left(\int_0^1 r^{n+2}D_{n,j}(r)\,\mathrm{d}r\right)\frac{4\pi}{2n+1}\,|y|^{-n-1}Y_{n,j}\left(\frac{y}{|y|}\right). \tag{2}$$

Proof. Let P_m be the Legendre polynomial of degree m. Using the well–known formula (see, for instance, [21], p. 44)

$$\frac{1}{|x-y|} = \sum_{m=0}^{\infty}\frac{|x|^m}{|y|^{m+1}}P_m\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right) \quad , \quad |x| < |y| \tag{3}$$

and assuming $|y| > 1$, we find

$$\begin{aligned}
&\int_B \frac{D(x)}{|x-y|}\,\mathrm{d}x \\
&= \int_0^1 r^2 \int_{\mathbb{S}^2} D(r\xi)\frac{1}{|r\xi - y|}\,\mathrm{d}\omega(\xi)\,\mathrm{d}r \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=-n}^{n} \frac{1}{|y|^{m+1}} \int_0^1 r^2 D_{n,j}(r) r^m \,\mathrm{d}r \int_{\mathbb{S}^2} P_m\left(\xi\cdot\frac{y}{|y|}\right) Y_{n,j}(\xi)\;\mathrm{d}\omega(\xi) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=-n}^{n} |y|^{-m-1} \int_0^1 r^{m+2} D_{n,j}(r)\,\mathrm{d}r \\
&\qquad\qquad \times \frac{4\pi}{2m+1}\int_{\mathbb{S}^2} \frac{2m+1}{4\pi}\,P_m\left(\xi\cdot\frac{y}{|y|}\right) Y_{n,j}(\xi)\;\mathrm{d}\omega(\xi).
\end{aligned}$$

Since $\frac{2m+1}{4\pi}P_m$ represents the reproducing kernel of the linear space of all spherical harmonics of degree m with respect to $\langle .,.\rangle_{\mathrm{L}^2(\mathbb{S}^2)}$, the last integral equals $\delta_{nm} Y_{n,j}(\frac{y}{|y|})$. Hence,

$$\int_B \frac{D(x)}{|x-y|}\,\mathrm{d}x = \sum_{n=0}^{\infty}\sum_{j=-n}^{n} \int_0^1 r^{n+2} D_{n,j}(r)\,\mathrm{d}r\,\frac{4\pi}{2n+1}|y|^{-n-1} Y_{n,j}\left(\frac{y}{|y|}\right).$$

This series can be formally extended to $\mathbb{S}^2$ so that $V|_{\mathbb{S}^2} \in \mathrm{L}^2\left(\mathbb{S}^2\right)$ using the Cauchy–Schwarz–Bunjakovski–inequality:

$$\begin{aligned}
\|V|_{\mathbb{S}^2}\|^2_{\mathrm{L}^2(\mathbb{S}^2)} &= \sum_{n=0}^{\infty}\sum_{j=-n}^{n}\left(\int_0^1 r^{n+2}D_{n,j}(r)\,\mathrm{d}r\right)^2\left(\frac{4\pi}{2n+1}\right)^2 \\
&\leq \sum_{n=0}^{\infty}\sum_{j=-n}^{n}\left(\int_0^1 r^{2n+2}\,\mathrm{d}r\right)\left(\int_0^1 r^2\left(D_{n,j}(r)\right)^2\,\mathrm{d}r\right)\left(\frac{4\pi}{2n+1}\right)^2 \\
&\leq 16\pi^2\sum_{n=0}^{\infty}\sum_{j=-n}^{n}\int_0^1 r^2\left(D_{n,j}(r)\right)^2\,\mathrm{d}r \\
&= 16\pi^2\|D\|^2_{\mathrm{L}^2(B)} < +\infty.
\end{aligned}$$

■

We already observe here that the solution of the inverse problem of constructing D from the knowledge of V outside $\operatorname{int} B$ cannot have a unique solution, since the relation

$$\int_0^1 r^{n+2} D_{n,j}(r)\,\mathrm{d}r = \frac{2n+1}{4\pi}\left\langle V|_{\mathbb{S}^2}, Y_{n,j}\right\rangle_{\mathrm{L}^2(\mathbb{S}^2)} \tag{4}$$

admits an infinite number of choices for $D_{n,j}$. The relation (4) is well–known, see for example [39, 43, 44].

The non–uniqueness of the solution can also be inferred in many other ways. Already [52] mentions this fact, which was quantified more precisely later by other scientists. A precise description of the null space requires the discussion of harmonic and anharmonic functions which will be presented in Section 3.

For the sake of completeness, we mention that the solution would be unique if V was known in the whole space $\mathbb{R}^3$, because in this case (provided that D satisfies certain conditions) the Poisson equation

$$\Delta V = -4\pi D$$

would hold in B, see e.g. [39, 52].

3 Characterisation of the Null Space

There are several ways of characterising the null space of the operator

$$T : \mathrm{L}^2(B) \ni D \mapsto \int_B \frac{D(x)}{|x-\cdot|}\,\mathrm{d}x\bigg|_{\mathbb{S}^2} . \tag{5}$$

Note that since the potential is harmonic outside the Earth, it can be obtained as a solution of the corresponding outer Dirichlet problem from its values at $\mathbb{S}^2$; for this reason, in what follows we restrict the image of T to functions on $\mathbb{S}^2$.

Theorem 3.1 *The Hilbert space* $\mathrm{L}^2(B)$ *can be decomposed into two orthogonal spaces*

$$\mathrm{L}^2(B) = \mathrm{Harm}(B) \oplus \mathrm{Anharm}(B),$$

where

$$\begin{aligned} \mathrm{Harm}(B) &:= \left\{ H \in \mathrm{C}^{(2)}(B) \,\middle|\, \Delta H = 0 \right\}, \\ \mathrm{Anharm}(B) &:= \left\{ F \in \mathrm{L}^2(B) \,\middle|\, \langle F, H\rangle_{\mathrm{L}^2(B)} = 0 \,\forall\, H \in \mathrm{Harm}(B) \right\}. \end{aligned}$$

The null space of the operator T defined in (5) satisfies

$$\ker T = \mathrm{Anharm}(B).$$

Proof. We use an orthonormal basis for $\mathrm{L}^2(B)$ which is due to [7, 17, 33]:

$$B_{n,j,m}(x) = \gamma_{n,m} P_m^{(0,n+1/2)}\left(2|x|^2 - 1\right)|x|^n\, Y_{n,j}\left(\frac{x}{|x|}\right),$$

$m, n \in \mathbb{N}_0$, $j = -n, ..., n$, where $\{P_m^{(\alpha,\beta)}\}_{m\in\mathbb{N}_0}$ $(\alpha, \beta > -1)$ are the Jacobi polynomials satisfying (see e.g. [54])

1. Each $P_m^{(\alpha,\beta)}$ is a univariate polynomial of degree m,

2. $\int_{-1}^{1}(1-t)^\alpha(1+t)^\beta P_n^{(\alpha,\beta)}(t)P_m^{(\alpha,\beta)}(t)\,\mathrm{d}t = 0$ whenever $n \neq m$,

3. $P_m^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$

and the $\gamma_{n,m}$ are normalisation constants, which are not important here (see, for example, [37]). The use of this basis implies the following representation of $D_{n,j}$ in (1):

$$D_{n,j}(r) = r^n \sum_{m=0}^{\infty} d_{n,j,m}\gamma_{n,m}P_m^{(0,n+1/2)}\left(2r^2-1\right)$$

in the sense of the corresponding weighted L^2–space on $[0,1]$. We insert it in the left–hand side of (4) and obtain, using the substitution $r = \sqrt{(t+1)/2}$ the identity

$$\begin{aligned}
&\int_0^1 r^{n+2+n}\sum_{m=0}^{\infty} d_{n,j,m}\gamma_{n,m}P_m^{(0,n+1/2)}\left(2r^2-1\right)\,\mathrm{d}r\\
&= \int_{-1}^{1}\left[\frac{1}{2}(t+1)\right]^{n+1}\sum_{m=0}^{\infty} d_{n,j,m}\gamma_{n,m}P_m^{(0,n+1/2)}(t)\,\frac{1}{4}\left[\frac{1}{2}(t+1)\right]^{-1/2}\mathrm{d}t\\
&= \frac{1}{2^{n+5/2}}\int_{-1}^{1}(t+1)^{n+1/2}\left(\sum_{m=0}^{\infty} d_{n,j,m}\gamma_{n,m}P_m^{(0,n+1/2)}(t)\right)P_0^{(0,n+1/2)}(t)\,\mathrm{d}t\\
&= \frac{1}{2n+3}\,d_{n,j,0}\gamma_{n,0}.
\end{aligned}$$

Hence, $B_{n,j,m}$ is in the null space of T if and only if $m > 0$. Since the functions $B_{n,j,0}$ constitute the system of inner harmonics, which are a basis for the harmonic functions on a ball, we obtain the desired result. ∎

This result, which shows that all anharmonic functions, i.e. all functions orthogonal to all harmonic functions in the $\mathrm{L}^2(B)$–sense, are precisely these density functions which produce a vanishing potential outside the Earth, was proved (as pointed out by [32]) in [42, 43, 27] (in chronological order of the steps leading eventually to this result). This result was later also mentioned in [40]. There exist various ways of establishing this result, see also [9, 10, 23, 33, 55]. In the works of [14] and [63] it is shown, in addition, that the set of anharmonic functions can be characterised in the distributional sense as $\Delta\,\overset{\circ}{H}_2(B)$. Finally, for the discussion of the 2D–case, see e.g. [8, 52].

4 The Harmonic Solution is the Minimal L^2(B)–norm Solution

One way of treating the non–uniqueness is to look for a harmonic density function and to try to obtain the anharmonic complement using other type of data. Using Theorem 3.1 it is possible to derive already a formula for the harmonic solution of $TD = V$:

Corollary 4.1 (Harmonic Solution) *Let $V : \overline{\mathbb{R}^3 \setminus B} \to \mathbb{R}$ be an arbitrary function satisfying*

- $V|_{\mathbb{S}^2} \in \mathrm{L}^2\left(\mathbb{S}^2\right)$,
- $\sum_{n=0}^{\infty}\sum_{j=-n}^{n}\langle V|_{\mathbb{S}^2}, Y_{n,j}\rangle_{\mathrm{L}^2(\mathbb{S}^2)}^2 n^3 < +\infty$,
- $\Delta V = 0$ *in* $\mathbb{R}^3 \setminus B$.

Then the unique solution $D \in \mathrm{C}^{(2)}(B)$ with $V = \int_B \frac{D(x)}{|x-\cdot|}\,\mathrm{d}x$ and $\Delta D = 0$ in B is given by

$$D(x) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi}(2n+3)|x|^n \sum_{j=-n}^{n} \langle V|_{\mathbb{S}^2}, Y_{n,j}\rangle_{\mathrm{L}^2(\mathbb{S}^2)} Y_{n,j}\left(\frac{x}{|x|}\right) \tag{6}$$

provided that this series converges with respect to $\mathrm{L}^2(B)$.

Proof. To prove this result, we have to know that

$$H_{n,j}(x)\,(:= B_{n,j,0}(x)) = \sqrt{2n+3}\,|x|^n Y_{n,j}\left(\frac{x}{|x|}\right), \quad n \in \mathbb{N}_0,\ j = -n, ..., n,$$

is an orthonormal basis of $(\mathrm{Harm}(B), \langle\cdot,\cdot\rangle_{\mathrm{L}^2(B)})$, i.e. $\gamma_{n,0} = \sqrt{2n+3}$, $n \in \mathbb{N}_0$ (see e.g. [33, 45]). Hence,

$$D(x) = \sum_{n=0}^{\infty}\sum_{j=-n}^{n} d_{n,j}\,\sqrt{2n+3}\,|x|^n\, Y_{n,j}\left(\frac{x}{|x|}\right) \tag{7}$$

in the sense of $\mathrm{L}^2(B)$. Thus, Equation (4) with $D_{n,j}(r) = d_{n,j}\sqrt{2n+3}\,r^n$ yields

$$\int_0^1 r^{2n+2}\,\mathrm{d}r\, d_{n,j}\sqrt{2n+3} \;=\; \frac{2n+1}{4\pi}\langle V|_{\mathbb{S}^2}, Y_{n,j}\rangle_{\mathrm{L}^2(\mathbb{S}^2)} \ ,$$

which is equivalent to

$$d_{n,j} = \sqrt{2n+3}\;\frac{2n+1}{4\pi}\langle V|_{\mathbb{S}^2}, Y_{n,j}\rangle_{\mathrm{L}^2(\mathbb{S}^2)} \ .$$

■

The harmonic solution was discussed in numerous works such as [4], Section 5.5 as well as [18, 23, 24, 33, 34, 35, 36, 38, 45, 55, 61]. This solution lacks a convening physical interpretation, see also [55]. Furthermore, an additional drawback is the maximum principle, according to which the harmonic density is maximal (and minimal) at the surface (which is in contrast to the real density). However, it still has certain advantages. Namely, the search

for high frequency density anomalies is supported by exactly this maximum principle, since such structures primarily occur at the uppermost Earth layer and, indeed, can be derived in a remarkable qualitative resolution out of gravitational data, see [33, 34, 36, 38, 55]. In addition, the behaviour of r^n on $[0,1]$ for different n is consistent with the following relation: the higher the frequency viz degree (band) of a density phenomenon, the more it is concentrated towards the surface; this was pointed out in [39].
According to [32] there is an interpretation of the harmonic density with respect to all Stokesian quantities (see also [39]). Furthermore, the harmonic density plays a predominant role from the mathematical point of view due to the orthogonal decomposition of $\mathrm{L}^2(B)$. It is a projection of the real density to this part which influences the gravity. It has to be supplemented by an anharmonic function obtained from different, i.e. non–gravitational data such as seismic data (see [12, 55] for a detailed discussion of combining different types of data for this application).
Last but not least, the harmonic density is also the solution of minimal $\mathrm{L}^2(B)$–norm, which was observed in [4], p. 161 and [39, 45]. There exist several ways of proving this identity. The probably shortest one is the following simple functional analytic argumentation (see also [39]): a density $D \in \mathrm{L}^2(B)$ may be decomposed uniquely into a harmonic and an anharmonic part $D = D_{\mathrm{harm}} + D_{\mathrm{anharm}}$ such that $\|D\|^2_{\mathrm{L}^2(B)} = \|D_{\mathrm{harm}}\|^2_{\mathrm{L}^2(B)} + \|D_{\mathrm{anharm}}\|^2_{\mathrm{L}^2(B)}$ due to the orthogonality. Every change of D_{harm} changes the potential, whereas every change to D_{anharm} leaves the potential unchanged. Hence, the solution D of $TD = V$ with minimal $\mathrm{L}^2(B)$–norm is $D = D_{\mathrm{harm}}$.
Note that [45] used potential theoretic arguments to show the identity of the two concepts. This has the interesting side effect that one obtains that D_{harm} may be represented as a single layer potential. Single layer potentials as unique solutions of the inverse gravimetric problem were also discussed in [39]. It should also be noted that [45] in addition discusses a density solution with minimal (modified) $H^{1,2}$–norm, which also turns out to be the harmonic solution.
In the following, we present a straightforward derivation of a formula for the minimal $\mathrm{L}^2(B)$–norm solution. This derivation is similar with that of [19] and [20], where a formula for the minimal L^2–norm of the solution of the inverse problem of MEG was derived.

Theorem 4.2 (Minimal Norm Solution) *Let $V : \overline{\mathbb{R}^3 \setminus B} \to \mathbb{R}$ be an arbitrary function satisfying*

- $V|_{\mathbb{S}^2} \in \mathrm{L}^2\left(\mathbb{S}^2\right)$,
- $\sum_{n=0}^{\infty}\sum_{j=-n}^{n} \langle V|_{\mathbb{S}^2}, Y_{n,j}\rangle^2_{\mathrm{L}^2(\mathbb{S}^2)} n^3 < +\infty$,
- $\Delta V = 0$ *in* $\mathbb{R}^3 \setminus B$.

Then, among all $D \in \mathrm{L}^2(B)$ with $V = \int_B \frac{D(x)}{|x-\cdot|}\,\mathrm{d}x$, there is a unique minimiser of the functional

$$\mathcal{F}(D) := \int_B (D(x))^2 \,\mathrm{d}x$$

which is given in $\mathrm{L}^2(B)$ by

$$D(x) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi}\,(2n+3)|x|^n \sum_{j=-n}^{n} \langle V|_{\mathbb{S}^2}, Y_{n,j}\rangle_{\mathrm{L}^2(\mathbb{S}^2)} Y_{n,j}\left(\frac{x}{|x|}\right) \tag{8}$$

provided that this series converges with respect to $\mathrm{L}^2(B)$.

Proof. The conditions on V guarantee the solvability of the inverse problem (see [33, 36]). According to Theorem 2.1 any solution solving the inverse problem can be represented by the following series, which converges in $\mathrm{L}^2(B)$,

$$D(x) = \sum_{n=0}^{\infty}\sum_{j=-n}^{n} D_{n,j}(|x|)Y_{n,j}\left(\frac{x}{|x|}\right) \quad ,$$

where

$$\int_0^1 D_{n,j}(r) r^{n+2}\,\mathrm{d}r = \frac{2n+1}{4\pi}\,V_{n,j} \quad ,$$

with $V_{n,j} := \langle V|_{\mathbb{S}^2}, Y_{n,j}\rangle_{\mathrm{L}^2(\mathbb{S}^2)}$, such that

$$V(y) = \sum_{n=0}^{\infty}\sum_{j=-n}^{n} V_{n,j}\,|y|^{-n-1} Y_{n,j}\left(\frac{y}{|y|}\right), \quad y \in \mathbb{R}^3 \setminus B \quad .$$

Since

$$\int_B (D(x))^2 \,\mathrm{d}x \quad = \quad \sum_{n=0}^{\infty}\sum_{j=-n}^{n} \int_0^1 r^2\,(D_{n,j}(r))^2\,\mathrm{d}r \quad ,$$

we now have the following family of minimisation problems:

$$\begin{aligned} &\text{minimise} && \int_0^1 r^2\,(D_{n,j}(r))^2\,\mathrm{d}r \quad , \\ &\text{subject to} && \int_0^1 D_{n,j}(r) r^{n+2}\,\mathrm{d}r = \frac{2n+1}{4\pi}\,V_{n,j} \quad . \end{aligned}$$

We set $F_{n,j}(r) := rD_{n,j}(r)$. Then we have to solve

$$\begin{aligned} &\text{minimise} && \int_0^1 (F_{n,j}(r))^2\,\mathrm{d}r \quad , \\ &\text{subject to} && \int_0^1 F_{n,j}(r) r^{n+1}\,\mathrm{d}r = \frac{2n+1}{4\pi}\,V_{n,j} \quad . \end{aligned}$$

We decompose $F_{n,j}$ by $F_{n,j}(r) = \alpha r^{n+1} + G_{n,j}(r)$, where $G_{n,j}$ is $\mathrm{L}^2[0,1]$–orthogonal to $r \mapsto r^{n+1}$. Then the problem reduces to

$$\begin{aligned} &\text{minimise} && \alpha^2 \int_0^1 r^{2n+2}\,\mathrm{d}r + \|G_{n,j}\|^2_{\mathrm{L}^2[0,1]} \ , \\ &\text{subject to} && \alpha \int_0^1 r^{2n+2}\,\mathrm{d}r = \frac{2n+1}{4\pi}\, V_{n,j} \ . \end{aligned}$$

Obviously, $G_{n,j} \equiv 0$ and

$$\alpha = (2n+3)\,\frac{2n+1}{4\pi}\, V_{n,j} \ .$$

■

In what follows, we will derive a condition for the convergence of the series (8) (which is actually identical with the series (6)).

Theorem 4.3 *The series in (8) converges in* $\mathrm{L}^2(B)$ *if and only if*

$$\sum_{n=0}^{\infty} n^3 \sum_{j=-n}^{n} \langle V|_{\mathbb{S}^2}, Y_{n,j}\rangle^2_{\mathrm{L}^2(\mathbb{S}^2)} < +\infty \ . \tag{9}$$

Proof. The $\mathrm{L}^2(B)$–norm of D is given by

$$\begin{aligned} \|D\|^2_{\mathrm{L}^2(B)} &= \int_0^1 r^2 \int_{\mathbb{S}^2} (D(r\xi))^2\,\mathrm{d}\omega(\xi)\,\mathrm{d}r \\ &= \sum_{n=0}^{\infty} \left(\frac{2n+1}{4\pi}\,(2n+3)\right)^2 \int_0^1 r^2 \cdot r^{2n}\,\mathrm{d}r \sum_{j=-n}^{n} \langle V|_{\mathbb{S}^2}, Y_{n,j}\rangle^2_{\mathrm{L}^2(\mathbb{S}^2)} \\ &= \sum_{n=0}^{\infty} \left(\frac{2n+1}{4\pi}\right)^2 (2n+3) \sum_{j=-n}^{n} \langle V|_{\mathbb{S}^2}, Y_{n,j}\rangle^2_{\mathrm{L}^2(\mathbb{S}^2)} \ . \end{aligned}$$

■

Hence, the condition for the convergence of the series is precisely the condition for the solvability of the inverse problem. This is not surprising, since it corresponds to the principle of the Picard condition for inverse problems. Note that the real gravitational potential of the Earth certainly satisfies condition (9), since the (empirical) Kaula's rule states that

$$\sum_{j=-n}^{n} \langle V|_{\mathbb{S}^2}, Y_{n,j}\rangle^2_{\mathrm{L}^2(\mathbb{S}^2)} = O\left(\vartheta^{n+1} n^{-3}\right), \quad n \to \infty,$$

for some constant $\vartheta \in]0,1[$ related to the Bjerhammar sphere, see, for example, [46].

5 Quasi–harmonic Solution

A generalisation of the constraint of harmonicity is the constraint of quasi–harmonicity. In this case, the density satisfies the equation

$$\Delta_x \left(\frac{D(x)}{F(|x|)} \right) = 0,$$

where F is a given function without zeros. Such quasi–harmonic solutions were discussed in [24, 57, 58, 60, 61]. In [61] the case where the sphere is replaced by a spheroid is additionally treated. However, the phenomenon of a concentration of the density at the surface also occurs here according to [58, 60]. According to [58] "the use of these functions ... [was] not useful" and [57] reports "rather disappointing results".
In [60] the case of a monomial function F is analysed in detail and an associated spectral relation is derived. In what follows we will discuss a slight generalisation of this result.

Theorem 5.1 (Quasi–harmonic Solution) *Let V satisfy the conditions of Theorem 4.2. Then the unique solution $D \in \mathrm{C}^{(2)}(B)$ of*

$$\begin{aligned} \int_B \frac{D(x)}{|x - \cdot|} \, \mathrm{d}x &= V \text{ in } \overline{\mathbb{R}^3 \setminus B}, \\ \Delta_x \left(D(x)|x|^{-p} \right) &= 0 \text{ in } B, \end{aligned}$$

$p \in \mathbb{R}_0^+$ fixed, is given by

$$D(x) = \sum_{n=0}^{\infty} (2n + p + 3) \, \frac{2n + 1}{4\pi} \, |x|^{n+p} \sum_{j=-n}^{n} \langle V|_{\mathbb{S}^2} \, , Y_{n,j} \rangle_{\mathrm{L}^2(\mathbb{S}^2)} Y_{n,j} \left(\frac{x}{|x|} \right),$$

where the series converges in $\mathrm{L}^2(B)$.

Proof. Obviously, the functions $x \mapsto |x|^{n+p} Y_{n,j}\left(\frac{x}{|x|}\right)$ provide an orthogonal basis for such quasi–harmonic functions with respect to the $\mathrm{L}^2(B)$–inner product. The square of the normalisation constant is given by

$$\begin{aligned} \int_B \left[|x|^{n+p} Y_{n,j} \left(\frac{x}{|x|} \right) \right]^2 \mathrm{d}x &= \int_0^1 r^{2n+2p+2} \, \mathrm{d}r \int_\Omega Y_{n,j}(\xi)^2 \, \mathrm{d}\omega(\xi) \\ &= \frac{1}{2n + 2p + 3} \; . \end{aligned}$$

Hence, using $D_{n,j}(r) = d_{n,j} \sqrt{2n + 2p + 3} \, r^{n+p}$ in (4), we obtain

$$\begin{aligned} \frac{2n + 1}{4\pi} \langle V|_{\mathbb{S}^2} \, , Y_{n,j} \rangle_{\mathrm{L}^2(\mathbb{S}^2)} &= \int_0^1 r^{2n+2+p} \, \mathrm{d}r \, d_{n,j} \sqrt{2n + 2p + 3} \\ &= d_{n,j} \, \frac{\sqrt{2n + 2p + 3}}{2n + p + 3}. \end{aligned}$$

Consequently,

$$\begin{aligned} D(x) &= \sum_{n=0}^{\infty}\sum_{j=-n}^{n} d_{n,j}\sqrt{2n+2p+3}\,|x|^{n+p}\,Y_{n,j}\left(\frac{x}{|x|}\right) \\ &= \sum_{n=0}^{\infty}(2n+p+3)\,\frac{2n+1}{4\pi}\,|x|^{n+p}\sum_{j=-n}^{n}\langle V|_{\mathbb{S}^2}\,, Y_{n,j}\rangle_{\mathrm{L}^2(\mathbb{S}^2)} Y_{n,j}\left(\frac{x}{|x|}\right) \end{aligned}$$

in the sense of $\mathrm{L}^2(B)$. The convergence of the series is guaranteed by the conditions on V, since the Fourier coefficients of D satisfy

$$d_{n,j} = \langle V|_{\mathbb{S}^2}\,, Y_{n,j}\rangle_{\mathrm{L}^2(\mathbb{S}^2)} \cdot O\left(n^{3/2}\right)$$

as $n \to \infty$. ■

In the case of $p = 0$ the above representation leads to the harmonic solution.

6 Biharmonic Solution

A biharmonic constraint for the density, i.e. $\Delta\Delta D = 0$, is discussed in [51] and [60]. Clearly, the knowledge of $V|_{\mathbb{S}^2}$ is now insufficient for obtaining a unique solution, since the harmonic solution represents only a particular case of a biharmonic solution. For this reason, in the literature it is assumed that $D|_{\mathbb{S}^2}$ is given. According to [51], the minimisation of the $\mathrm{L}^2(B)$–norm of ∇D, provided that $D|_{\mathbb{S}^2}$ and $V|_{\mathbb{S}^2}$ are given, implies that the solution D is biharmonic. A spectral relation for the spherical harmonics coefficients is given in [51].

However, these results have to be slightly corrected, since the knowledge of V and D at $\mathbb{S}^2$ is *not* sufficient for a unique solution, as we show in the following theorem.

Theorem 6.1 (Biharmonic Solution) *All solutions $D \in \mathrm{C}^{(4)}(B)$ of the problem*

$$\begin{aligned} \Delta\Delta D &= 0 \text{ in } B, \\ \int_B \frac{D(x)}{|x-\cdot|}\,\mathrm{d}x &= V \text{ given on } \mathbb{S}^2, \\ & D \text{ given on } \mathbb{S}^2 \end{aligned}$$

can be represented by

$$\begin{aligned} D(x) &= \sum_{n=0}^{1}\sum_{j=-n}^{n}\left(a_{n,j}|x|^n + b_{n,j}|x|^{n+2} + c_{n,j}|x|^{-n+1}\right) Y_{n,j}\left(\frac{x}{|x|}\right) \\ &\quad + \frac{1}{8\pi}\sum_{n=2}^{\infty}(2n+1)(2n+3)(2n+5)\left(|x|^n - |x|^{n+2}\right) \end{aligned}$$

$$\times \sum_{j=-n}^{n} \langle V|_{\mathbb{S}^2}, Y_{n,j}\rangle_{\mathrm{L}^2(\mathbb{S}^2)} Y_{n,j}\left(\frac{x}{|x|}\right)$$
$$+\frac{1}{2}\sum_{n=2}^{\infty}\left[(2n+5)|x|^{n+2}-(2n+3)|x|^{n}\right]$$
$$\times \sum_{j=-n}^{n} \langle D|_{\mathbb{S}^2}, Y_{n,j}\rangle_{\mathrm{L}^2(\mathbb{S}^2)} Y_{n,j}\left(\frac{x}{|x|}\right),$$

provided that these series converge at least in the $\mathrm{L}^2(B)$*–sense, where the coefficients* $a_{0,0},...,c_{1,1}$ *are given by the underdetermined system of linear equations*

$$\begin{array}{ccccccc} \frac{1}{2n+3}\,a_{n,j} & + & \frac{1}{2n+5}\,b_{n,j} & + & \frac{1}{4}\,c_{n,j} & = & \frac{2n+1}{4\pi}\,\langle V|_{\mathbb{S}^2}, Y_{n,j}\rangle_{\mathrm{L}^2(\mathbb{S}^2)} \\ a_{n,j} & + & b_{n,j} & + & c_{n,j} & = & \langle D|_{\mathbb{S}^2}, Y_{n,j}\rangle_{\mathrm{L}^2(\mathbb{S}^2)} \end{array}$$

for $n=0,1,\ j=-n,...,n$.

Proof. We first need a biharmonic basis for D. It is easy to verify, by using the decomposition $\Delta_x = \frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\Delta^*_\xi$, $r=|x|$, $\xi = x/|x|$ (see, for example, [21], p. 14 and 36 for further details), that $\Delta_x\Delta_x(r^\alpha Y_{n,j}(\xi))=0$ has exactly four solutions for fixed $n\in\mathbb{N}_0$:

$$\alpha\in\{-n-1,-n+1,n,n+2\},$$

where $-n-1$ and n correspond to the harmonic case and the non–negative solutions n and $n+2$ yield bounded functions. However, for $n=0$ and $n=1$ the choice $\alpha=-n+1$ gives another bounded basis function. We, thus, use $D_{n,j}(r)=a_{n,j}r^n+b_{n,j}r^{n+2}$ for $n\geq 2$ and $D_{n,j}(r)=a_{n,j}r^n+b_{n,j}r^{n+2}+c_{n,j}r^{-n+1}$ for $n\leq 1$ in (4).
We first investigate the case $n\geq 2$, where we get

$$\begin{aligned} \frac{2n+1}{4\pi}\langle V|_{\mathbb{S}^2}, Y_{n,j}\rangle_{\mathrm{L}^2(\mathbb{S}^2)} &= a_{n,j}\int_0^1 r^{2n+2}\,\mathrm{d}r + b_{n,j}\int_0^1 r^{2n+4}\,\mathrm{d}r \\ &= \frac{a_{n,j}}{2n+3}+\frac{b_{n,j}}{2n+5} \end{aligned} \tag{10}$$

and

$$\langle D|_{\mathbb{S}^2}, Y_{n,j}\rangle_{\mathrm{L}^2(\mathbb{S}^2)} = D_{n,j}(1) = a_{n,j}+b_{n,j}. \tag{11}$$

This is uniquely solvable:

$$\begin{aligned} a_{n,j} &= \frac{1}{8\pi}(2n+1)(2n+3)(2n+5)\langle V|_{\mathbb{S}^2}, Y_{n,j}\rangle_{\mathrm{L}^2(\mathbb{S}^2)} \\ &\quad -\frac{1}{2}(2n+3)\langle D|_{\mathbb{S}^2}, Y_{n,j}\rangle_{\mathrm{L}^2(\mathbb{S}^2)}, \\ b_{n,j} &= -\frac{1}{8\pi}(2n+1)(2n+3)(2n+5)\langle V|_{\mathbb{S}^2}, Y_{n,j}\rangle_{\mathrm{L}^2(\mathbb{S}^2)} \\ &\quad +\frac{1}{2}(2n+5)\langle D|_{\mathbb{S}^2}, Y_{n,j}\rangle_{\mathrm{L}^2(\mathbb{S}^2)} \ . \end{aligned}$$

If $n \leq 1$, the two equations (10) and (11) become:

$$\begin{aligned} \frac{a_{n,j}}{2n+3} + \frac{b_{n,j}}{2n+5} + \frac{c_{n,j}}{4} &= \frac{2n+1}{4\pi} \left\langle V|_{\mathbb{S}^2}, Y_{n,j} \right\rangle_{\mathrm{L}^2(\mathbb{S}^2)} \\ \left\langle D|_{\mathbb{S}^2}, Y_{n,j} \right\rangle_{\mathrm{L}^2(\mathbb{S}^2)} = D_{n,j}(1) &= a_{n,j} + b_{n,j} + c_{n,j}, \end{aligned}$$

which leaves exactly one degree of freedom for every pair (n, j). ■

The formula presented in [51] corresponds to the choice $c_{n,j} = 0$.
The case of a harmonic solution is consistent with this result, since harmonicity implies $c_{n,j} = 0$ and $b_{n,j} = 0$, which is equivalent to

$$\left\langle D|_{\mathbb{S}^2}, Y_{n,j} \right\rangle_{\mathrm{L}^2(\mathbb{S}^2)} = \frac{1}{4\pi} (2n+1)(2n+3) \left\langle V|_{\mathbb{S}^2}, Y_{n,j} \right\rangle_{\mathrm{L}^2(\mathbb{S}^2)} ,$$

which is the result of Corollary 4.1.
However, it should be noted that the assumption that $D|_{\mathbb{S}^2}$ is known cannot be realised in practice on a sufficiently dense point grid.

7 Discussion of the Radial Mean

In what follows we discuss a particular case of a non–radially dependent density. In this section, we write $V_{n,j} := \left\langle V|_{\mathbb{S}^2}, Y_{n,j} \right\rangle_{\mathrm{L}^2(\mathbb{S}^2)}$.

Theorem 7.1 (Spectral Relation for a Layer Density) *Let a spherical shell be given by the constraints*

$$0 \leq \tau \leq |x| \leq \tau + \varepsilon \leq 1, \quad \varepsilon > 0 .$$

If D^{L} is a square–integrable density function which has the form

$$D^{\mathrm{L}}(x) = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} D^{\mathrm{L}}_{n,j} Y_{n,j}\left(\frac{x}{|x|}\right)$$

inside the shell and which vanishes outside the shell and V is the corresponding gravitational potential, then

$$D^{\mathrm{L}}_{n,j} = \frac{(2n+1)(n+3)}{4\pi} \left((\tau+\varepsilon)^{n+3} - \tau^{n+3} \right)^{-1} V_{n,j} .$$

Proof. Using again (3) we have

$$\begin{aligned} V(y) &= \sum_{n=0}^{\infty} \int_{\tau}^{\tau+\varepsilon} r^2 \int_{\mathbb{S}^2} D^{\mathrm{L}}(r\xi) \frac{r^n}{|y|^{n+1}} P_n\left(\xi \cdot \frac{y}{|y|}\right) \mathrm{d}\omega(\xi)\, \mathrm{d}r \\ &= \sum_{n=0}^{\infty} \int_{\tau}^{\tau+\varepsilon} r^{n+2}\, \mathrm{d}r\, |y|^{-n-1} \frac{4\pi}{2n+1} \sum_{j=-n}^{n} D^{\mathrm{L}}_{n,j} Y_{n,j}\left(\frac{y}{|y|}\right) \\ &= \sum_{n=0}^{\infty} \frac{4\pi}{(2n+1)(n+3)} \left((\tau+\varepsilon)^{n+3} - \tau^{n+3} \right) |y|^{-n-1} \sum_{j=-n}^{n} D^{\mathrm{L}}_{n,j} Y_{n,j}\left(\frac{y}{|y|}\right). \end{aligned}$$

This holds pointwise for $|y| > 1$. Due to the square–integrability of D^{L} this can be extended to $\mathbb{S}^2$ in the sense that we get a function in $\mathrm{L}^2\left(\mathbb{S}^2\right)$:

$$\sum_{n=0}^{\infty} \left(\frac{4\pi}{(2n+1)(n+3)} \left((\tau+\varepsilon)^{n+3} - \tau^{n+3} \right) \right)^2 \sum_{j=-n}^{n} \left(D^{\mathrm{L}}_{n,j} \right)^2 \leq 16\pi^2 \sum_{n=0}^{\infty} \sum_{j=-n}^{n} \left(D^{\mathrm{L}}_{n,j} \right)^2 < +\infty \ .$$

Hence, we have the desired relation. ■

8 Other Constraints

There are several other constraints discussed in the literature. For reasons of completeness, these constraints are mentioned below.
The functional $\int_B (|x| D(x))^2 \, \mathrm{d}x$ is minimised in [51] and a formula for the spherical harmonics expansion of D is derived.
The gravitational potential energy as functional of D is used in [44, 45].
An approach which postulates a fluid mantle is discussed in [50, 52].
In [31], in addition to the constraint of harmonicity, it is postulated that the lateral density variations in the interior of the Earth's mantle and the density contrast on the undulated core mantle boundary do not influence the gravitational field in the core. This ansatz leads to a system of linear equations of which the minimal energy solution is chosen.
A measure theoretic approach is discussed in [3, 4, 49]. Note that [3] also contains a list of references on historical remarks regarding the inverse gravimetric problem.
An interesting idea is to use the boundedness of the density $0 \leq D \leq D_{\max}$ given by physical considerations. The reduction of the non–uniqueness problem using such a constraint is investigated in [41, 49, 67] (see also the references in [25]). The condition of positive solutions of linear inverse problems is discussed in [13]. It is the opinion of the authors that such constraints for the inverse gravimetric problem should be investigated further.

9 A Multiscale Method

The exist several methods for the numerical calculation of the density inside the Earth. They range from purely local approaches, like a block ansatz, to completely global tools like a representation in orthogonal polynomials. In the case of a block ansatz, the Earth, or a part of it, is subdivided into predefined (overlapping or non–overlapping) blocks with a very simple density structure, such as a constant density, per block, see, for instance [24, 26, 30, 56]. Also within this category of purely local methods are those

approaches which use point masses, see, for example, [9, 53]. Another approach of that kind is the use of basis functions with local support, see [24, 45].
A typical example of a global approach is a truncated singular value decomposition (TSVD), see, for instance, [55, 60]. As discussed earlier most constraints lead to an explicit singular value decomposition of the inverse operator. Thus, calculating the expansion coefficients of the given gravity function and then inserting these coefficients into the truncated series representation of the density yields an approximation of the unknown solution. In this context, the corresponding orthonormal basis functions are always polynomials. Thus, this method is purely global, since Fourier coefficients with respect to a polynomial basis are global means with no local preference.
For the sake of completeness, it should be mentioned that there also exist additional numerical approaches including a Monte Carlo method [55], a finite element ansatz [10], and the use of cubic and exponential splines for the 2D problem [6]. Moreover, the measure theoretic approach in [49] also includes the discussion of a corresponding numerical method. Furthermore, the theory of a regularisation method (going back to Tykhonov [62]) for the calculation of the harmonic density is discussed in [63].
Both purely global and purely local methods, have certain drawbacks. A disadvantage of a global method using polynomials is its inability to model local heterogeneity, such as an inhomogeneous distribution of the data grid points, a locally varying accuracy of the data, or a spatially restricted change of the data. The only parameter for controlling the resolution is the degree of truncation, which, however, has a global effect. It controls the resolution everywhere in the Earth, although we cannot expect that gravity data yield an equally accurate description of the Earth's deep structures on the one hand and the crust on the other hand — this drawback is also mentioned in [55]. Moreover, the coefficients describing the solution have no spatial reference, i.e., we are not able to say that the gravity data over, say, the Southern Atlantic Ocean primarily influence a certain subset of these coefficients. Any change to the data and any noise disturbing part of the data contribute almost equally to all coefficients in the expansion of the potential and therefore in the same manner to the coefficients of the solution in the TSVD.
In this respect, a good compromise appears to be the use of functions which are globally defined, but which concentrate on a certain subdomain. Such so–called localised functions can be obtained by using product kernels and they appear in different contexts.
On the one hand, product kernels can be used to construct basis functions for the solution. In contrast to polynomials, their graphs look approximately like peaks, where their area of localisation depends on the given basis function. Hence, this allows to treat differently different subdomains. A spatially varying quality and density of the data can be taken into account by produc-

ing a solution which is globally coarse but which gets sharper in areas with more and better data. This is demonstrated in [38] for the inverse gravimetric problem and in [2] for the seismic travel time tomography problem. In the context of the inverse gravimetric problem, such approaches have also been employed as collocation methods with a more statistical background (see [24, 45, 47, 56, 58, 60]) and as spline methods based on the theory of reproducing kernel Hilbert spaces (see [18, 38] and the historical references in [1] where the use of similar methods for different applications is discussed). It should be noted that since the model space is of finite dimensions the ill–posedness is circumvented.
On the other hand, product kernels can also be used as scaling functions and wavelets. In this context, they are equipped with a parameter J, i.e. now we consider a family, usually a sequence, of such kernels. The main idea is to convolve separately the given right–hand side with the scaling functions for each J. In a certain limit of the parameter such as $J \to \infty$ the relevant sequence converges to the exact solution. An essential feature is that every element of the sequence depends continuously on the right–hand side (no matter how close it is already to the exact solution), whereas the exact solution depends discontinuously on the right–hand side. This yields a regularisation, even if the model space has an infinite dimension. This approach allows a multiresolution analysis, i.e. it allows to obtain the result at different resolutions, e.g. ranging from continental size to the structures of a small island. The detail information representing the step from one resolution to the next can be calculated by using the wavelet kernels. The use of such methods for the inverse gravimetric problem was already proposed in [5]. They were eventually developed and numerically investigated in [23] and [33]-[36].
A particular method of the above type is a multiscale method, which will be described below, in analogy to the general principle described in [35] (see also [22, 36]).

Theorem 9.1 *Let $\sigma \geq 1$ and let V satisfy the conditions of Theorem 4.2 such that $D \in \mathrm{L}^2(B)$ is the corresponding harmonic density. Moreover, let $\Phi_J : B \times (\sigma\mathbb{S}^2) \to \mathbb{R}$, $J \in \mathbb{N}_0$, be a sequence of functions, here called scaling functions, given by*

$$\Phi_J(x, \sigma\eta) \quad := \quad \sum_{n=0}^{\infty} \varphi_J(n) \sigma^n (2n+3)|x|^n \left(\frac{2n+1}{4\pi}\right)^2 \frac{1}{\sigma} P_n\left(\frac{x}{|x|} \cdot \eta\right) \quad ,$$

where

- $0 \leq \varphi_J(n) \leq \varphi_{J+1}(n)$ *for all* $n, J \in \mathbb{N}_0$,
- $\lim_{J\to\infty} \varphi_J(n) = 1$ *for all* $n \in \mathbb{N}_0$,
- $\sum_{n=0}^{\infty} (\varphi_J(n))^2 \sigma^{2n} n^5 < +\infty$ *for all* $J \in \mathbb{N}_0$.

Then

$$\lim_{J\to\infty}\left\|D-\int_{\sigma\mathbb{S}^2}\Phi_J(\cdot,y)V(y)\,\mathrm{d}\omega(y)\right\|_{\mathrm{L}^2(B)}=0.$$

Proof. The addition theorem for spherical harmonics implies

$$\frac{2n+1}{4\pi\sigma}\,P_n\left(\frac{x}{|x|}\cdot\eta\right)=\sum_{j=-n}^{n}Y_{n,j}\left(\frac{x}{|x|}\right)\frac{1}{\sigma}Y_{n,j}(\eta),$$

where $\{\frac{1}{\sigma}Y_{n,j}(\frac{\cdot}{\sigma})\}_{n\in\mathbb{N}_0;\,j=-n,\ldots,n}$ is a complete orthonormal system in the Hilbert space $\mathrm{L}^2(\sigma\mathbb{S}^2)$. For fixed $x\in B$, it follows that

$$\begin{aligned}&\sum_{n=0}^{\infty}(\varphi_J(n))^2\,\sigma^{2n}(2n+3)^2|x|^{2n}\left(\frac{2n+1}{4\pi}\right)^2\sum_{j=-n}^{n}\left(Y_{n,j}\left(\frac{x}{|x|}\right)\right)^2\\ &\quad=\sum_{n=0}^{\infty}(\varphi_J(n))^2\,\sigma^{2n}(2n+3)^2|x|^{2n}\left(\frac{2n+1}{4\pi}\right)^3\\ &\quad\le\sum_{n=0}^{\infty}(\varphi_J(n))^2\,\sigma^{2n}\,\frac{(2n+3)^5}{64\pi^3}.\end{aligned}$$

Consequently, the series representation of $\Phi_J(x,\cdot)$ converges with respect to $\mathrm{L}^2(\sigma\mathbb{S}^2)$. Thus, for all $x\in B$,

$$\begin{aligned}&\int_{\sigma\mathbb{S}^2}\Phi_J(x,y)V(y)\,\mathrm{d}\omega(y)\\ &\quad=\sum_{n=0}^{\infty}\varphi_J(n)\sigma^n(2n+3)|x|^n\,\frac{2n+1}{4\pi}\\ &\qquad\times\sum_{j=-n}^{n}Y_{n,j}\left(\frac{x}{|x|}\right)\left\langle V|_{\sigma\mathbb{S}^2},\frac{1}{\sigma}Y_{n,j}\left(\frac{\cdot}{\sigma}\right)\right\rangle_{\mathrm{L}^2(\sigma\mathbb{S}^2)}.\end{aligned}\tag{12}$$

Note that

$$\left\langle V|_{\sigma\mathbb{S}^2},\frac{1}{\sigma}Y_{n,j}\left(\frac{\cdot}{\sigma}\right)\right\rangle_{\mathrm{L}^2(\sigma\mathbb{S}^2)}=\sigma^{-n}\left\langle V|_{\mathbb{S}^2},Y_{n,j}\right\rangle_{\mathrm{L}^2(\mathbb{S}^2)}.$$

Since $\{x\mapsto\sqrt{2n+3}|x|^nY_{n,j}(\frac{x}{|x|})\}_{n\in\mathbb{N}_0;j=-n,\ldots,n}$ represents an orthonormal system in $\mathrm{L}^2(B)$, we get

$$\begin{aligned}&\left\|D-\int_{\sigma\mathbb{S}^2}\Phi_J(\cdot,y)V(y)\,\mathrm{d}\omega(y)\right\|^2_{\mathrm{L}^2(B)}\\ &\quad=\sum_{n=0}^{\infty}\sum_{j=-n}^{n}(1-\varphi_J(n))^2\left(\frac{2n+1}{4\pi}\right)^2(2n+3)\left\langle V|_{\mathbb{S}^2},Y_{n,j}\right\rangle^2_{\mathrm{L}^2(\mathbb{S}^2)}.\end{aligned}$$

We know that $D \in \mathrm{L}^2(B)$ exists and, thus, due to the Parseval identity and the requirement $0 \leq \varphi_J(n) \leq 1$, this series is uniformly convergent with respect to $J \in \mathbb{N}_0$. Consequently,

$$\begin{aligned}
&\lim_{J\to\infty} \left\| D - \int_{\sigma\mathbb{S}^2} \Phi_J(\cdot, y) V(y) \, \mathrm{d}\omega(y) \right\|^2_{\mathrm{L}^2(B)} \\
= &\sum_{n=0}^{\infty} \sum_{j=-n}^{n} \lim_{J\to\infty} (1-\varphi_J(n))^2 \left(\frac{2n+1}{4\pi}\right)^2 (2n+3) \left\langle V|_{\mathbb{S}^2}, Y_{n,j}\right\rangle^2_{\mathrm{L}^2(\mathbb{S}^2)} \\
= &\ 0.
\end{aligned}$$

■

In the usual way linear wavelets can be established for this constellation.

Theorem 9.2 *Let the conditions of Theorem 9.1 be satisfied. If the kernels $\Psi_J : B \times (\sigma\mathbb{S}^2) \to \mathbb{R}$, $J \in \mathbb{N}_0$, here called wavelets, are defined by*

$$\Psi_J(x, \sigma\eta) \ := \ \sum_{n=0}^{\infty} \psi_J(n) \sigma^n (2n+3) |x|^n \left(\frac{2n+1}{4\pi}\right)^2 \frac{1}{\sigma} P_n\left(\frac{x}{|x|}\cdot\eta\right),$$

with

$$\psi_J(n) := \varphi_{J+1}(n) - \varphi_J(n)$$

for all $J, n \in \mathbb{N}_0$ and $\psi_{-1}(n) := \varphi_0(n)$ for all $n \in \mathbb{N}_0$, then

$$\int_{\sigma\mathbb{S}^2} \Phi_{J+1}(\cdot, y) V(y) \, \mathrm{d}\omega(y) = \int_{\sigma\mathbb{S}^2} \Phi_J(\cdot, y) V(y) \, \mathrm{d}\omega(y) + \int_{\sigma\mathbb{S}^2} \Psi_J(\cdot, y) V(y) \, \mathrm{d}\omega(y)$$

and

$$D = \sum_{J=-1}^{\infty} \int_{\sigma\mathbb{S}^2} \Psi_J(\cdot, y) V(y) \, \mathrm{d}\omega(y)$$

with respect to $\mathrm{L}^2(B)$.

We will show now that the described method represents a regularisation of the inverse problem.

Theorem 9.3 *Let Φ_J, $J \in \mathbb{N}$, satisfy the requirements formulated in Theorem 9.1. Then all operators*

$$\begin{aligned}
R_J : \mathrm{L}^2\left(\sigma\mathbb{S}^2\right) &\to \mathrm{L}^2(B) \\
V &\mapsto \int_{\sigma\mathbb{S}^2} \Phi_J(\cdot, y) V(y) \, \mathrm{d}\omega(y) \ ,
\end{aligned}$$

$J \in \mathbb{N}_0$, are continuous.

Proof. We prove the continuity by showing that the linear operator R_J, where $J \in \mathbb{N}_0$ is arbitrary, is bounded due to the third requirement on the generating sequence $(\varphi_J(n))_{n\in\mathbb{N}_0}$:

$$\begin{aligned}
\|R_J V\|^2_{\mathrm{L}^2(B)} &= \left\| \int_{\sigma\mathbb{S}^2} \Phi_J(\cdot, y) V(y) \, \mathrm{d}\omega(y) \right\|^2_{\mathrm{L}^2(B)} \\
&= \left\| \sum_{n=0}^{\infty} \varphi_J(n) \sigma^n (2n+3) |\cdot|^n \frac{2n+1}{4\pi} \right. \\
&\qquad \left. \times \sum_{j=-n}^{n} \left\langle V|_{\sigma\mathbb{S}^2}, \frac{1}{\sigma} Y_{n,j}\left(\frac{\cdot}{\sigma}\right) \right\rangle_{\mathrm{L}^2(\sigma\mathbb{S}^2)} Y_{n,j}\left(\frac{\cdot}{|\cdot|}\right) \right\|^2_{\mathrm{L}^2(B)} \\
&= \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (\varphi_J(n))^2 \sigma^{2n} (2n+3) \left(\frac{2n+1}{4\pi}\right)^2 \left\langle V|_{\sigma\mathbb{S}^2}, \frac{1}{\sigma} Y_{n,j}\left(\frac{\cdot}{\sigma}\right) \right\rangle^2_{\mathrm{L}^2(\sigma\mathbb{S}^2)} \\
&\le \frac{1}{16\pi^2} \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (\varphi_J(n))^2 \sigma^{2n} (2n+3)^3 \left\langle V|_{\sigma\mathbb{S}^2}, \frac{1}{\sigma} Y_{n,j}\left(\frac{\cdot}{\sigma}\right) \right\rangle^2_{\mathrm{L}^2(\sigma\mathbb{S}^2)} \\
&\le \sup_{m\in\mathbb{N}_0} \left((\varphi_J(m))^2 \sigma^{2m} (2m+3)^3 \right) \frac{1}{16\pi^2} \|V\|^2_{\mathrm{L}^2(\sigma\mathbb{S}^2)}.
\end{aligned}$$

■

10 Numerical Results

We will show here some numerical results. We have used the EGM96 gravity potential [29] from degree 3 to 360, evaluated at a 720×720 equiangular point grid at the surface[1] of the Earth, see Figure 1. The scaling functions Φ_J were defined via the cubic polynomial symbol, which is generated out of

$$\varphi_0(x) := \begin{cases} (1-x)^2(1+2x), & \text{if } 0 \le x \le 1 \\ 0, & \text{if } x > 1 \end{cases}$$

via the dilation $\varphi_J(n) := \varphi_0(2^{-J} n)$, $n \in \mathbb{N}_0$. The resulting expansion of Φ_J in terms of Legendre polynomials up to degree $2^J - 1$ was calculated via the Clenshaw algorithm [15] as finite sum of Legendre polynomials. Finally, the convolution of the scaling function and the potential, which is an integration over the spherical surface of the Earth (i.e. the unit sphere) was realised by the Driscoll–Healy method [16]. Each integration yields an approximation to the density at one point in the Earth. For plotting the approximate solution of Theorem 9.1 we have calculated this solution at point grids on

[1]Note that such a multiscale method can also be used to regularise the exponential ill–posedness in the case of $\sigma > 1$. This has been rigorously established and numerically implemented in [36] for the case of the harmonic density.

different surfaces. The results are shown in Figures 2 to 5. Moreover, we use a locally perturbed potential, see Figure 7, and reconstruct the density at the higher sphere in the same way, see Figure 6. It can clearly be seen that the perturbation remains local.
The figures illustrate the multiresolution analysis that the method provides. The higher the scale, the better the resolution and the more local the added details. Comparing the different spheres one observes a smoothing and decreasing tendency towards the Earth's centre. This is not unusual for the inverse gravimetric problems since the radially symmetric structures of the deep Earth are only represented via their constant global mean value in the gravitational field [33, 55]. Moreover, as discussed earlier the maximum principle for harmonic functions shows that the concentration of the large density variations at the surface, as one can see in Figure 5, is a feature of the harmonic density approach.
In summary, the minimal norm density which equals the harmonic density reveals topographical structures related to surface–near strong mass density variations, such as mountains, islands, and tectonic faults. Some of those features are already visible in the potential. The density recovery seems to bring them out clearly. No matter what additional constraint is used to obtain uniqueness, the methods are limited by the fact that certain essential parts of the Earth's mass density distribution are not captured by the solution of the inverse problem, which include coarse radially symmetric structures of the mantle and the core. There are probably slight angular variations of those structures, e.g. concerning the topography of the core–mantle–boundary. However, the present accuracy of gravity measurements appears insufficient to reveal such anomalies.

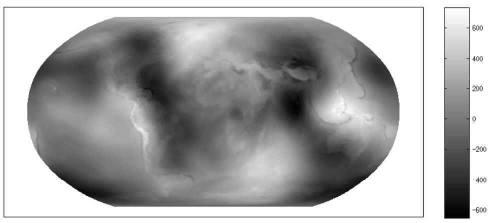

Figure 1: Used gravitational potential (from degree 3 to 360) in m^2/s^2

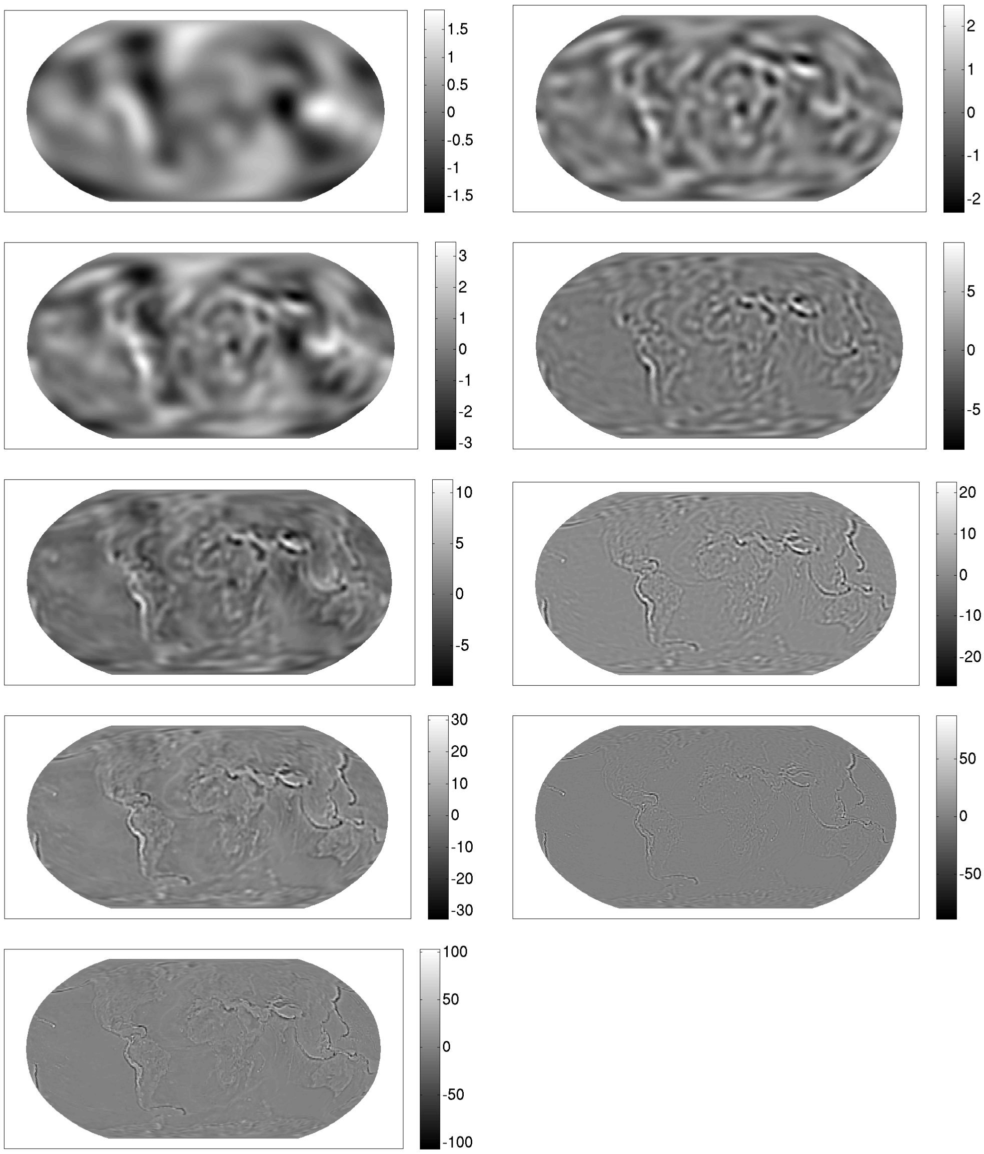

Figure 2: Calculated density (from degree 3 to 360) of minimal norm (in kg/m^3) at a sphere of radius $\rho = 0.999$, i.e. close to the surface: The left column shows the convolution with the scaling functions of scales $J = 4$ (top) to $J = 8$ (bottom), whereas the right column shows the difference of the approximations of two consecutive scales which could also be calculated by convolving the potential with the corresponding wavelet (see Theorem 9.2).

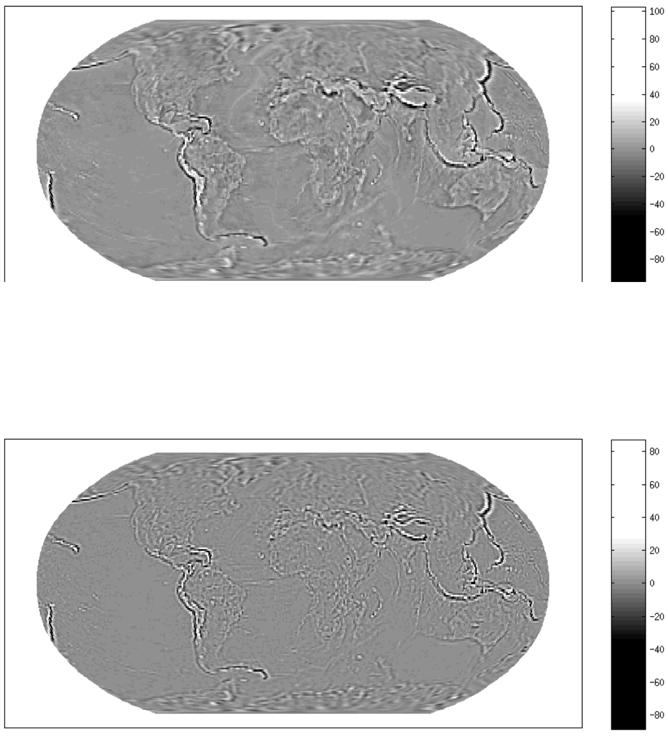

Figure 3: Similar to Figure 2, density at scale 8 (top) and scale step, i.e. wavelet approximation, at scale 7 (bottom), both with different colour scales that reveal more details.

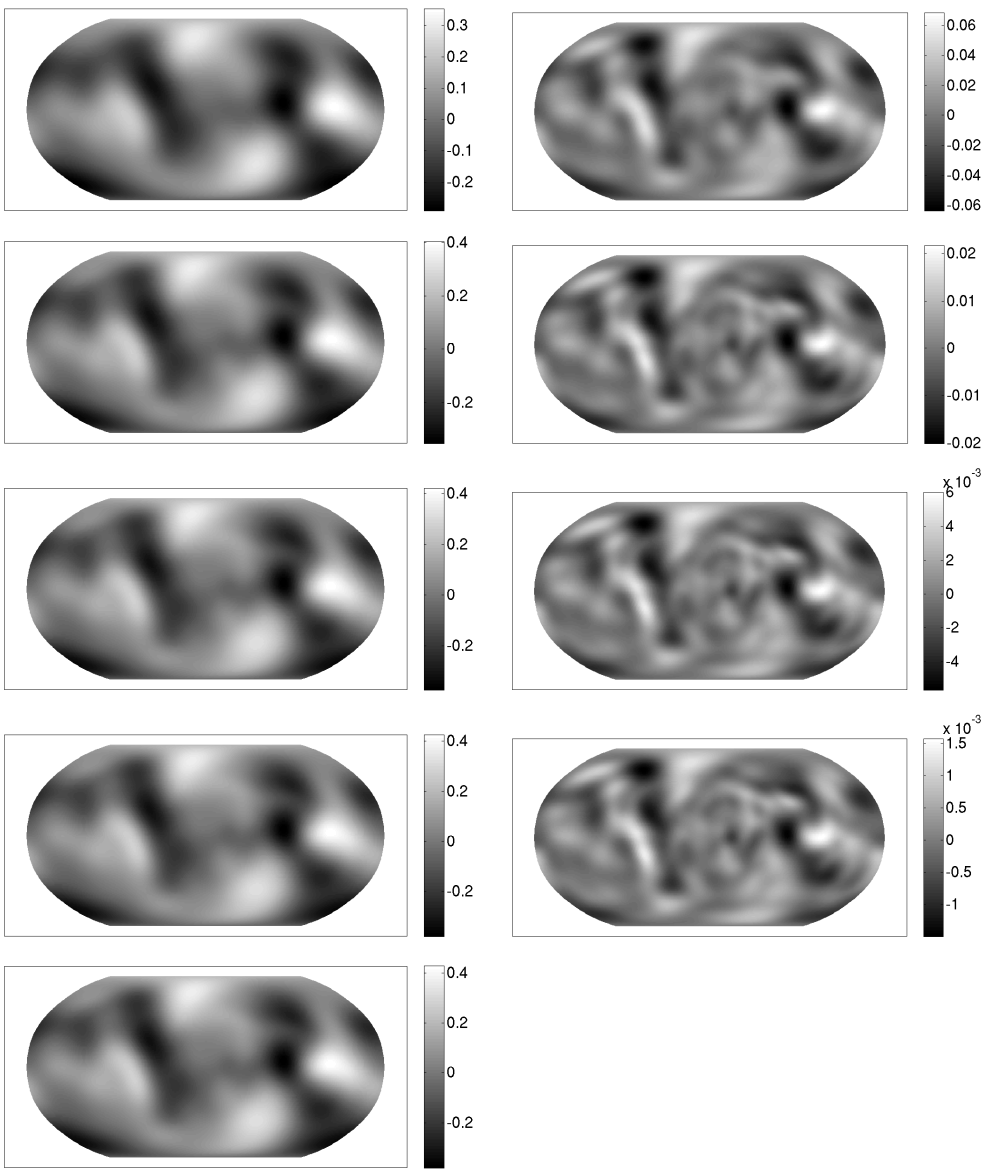

Figure 4: Similar to Figure 2 but for $\rho = 0.7$.

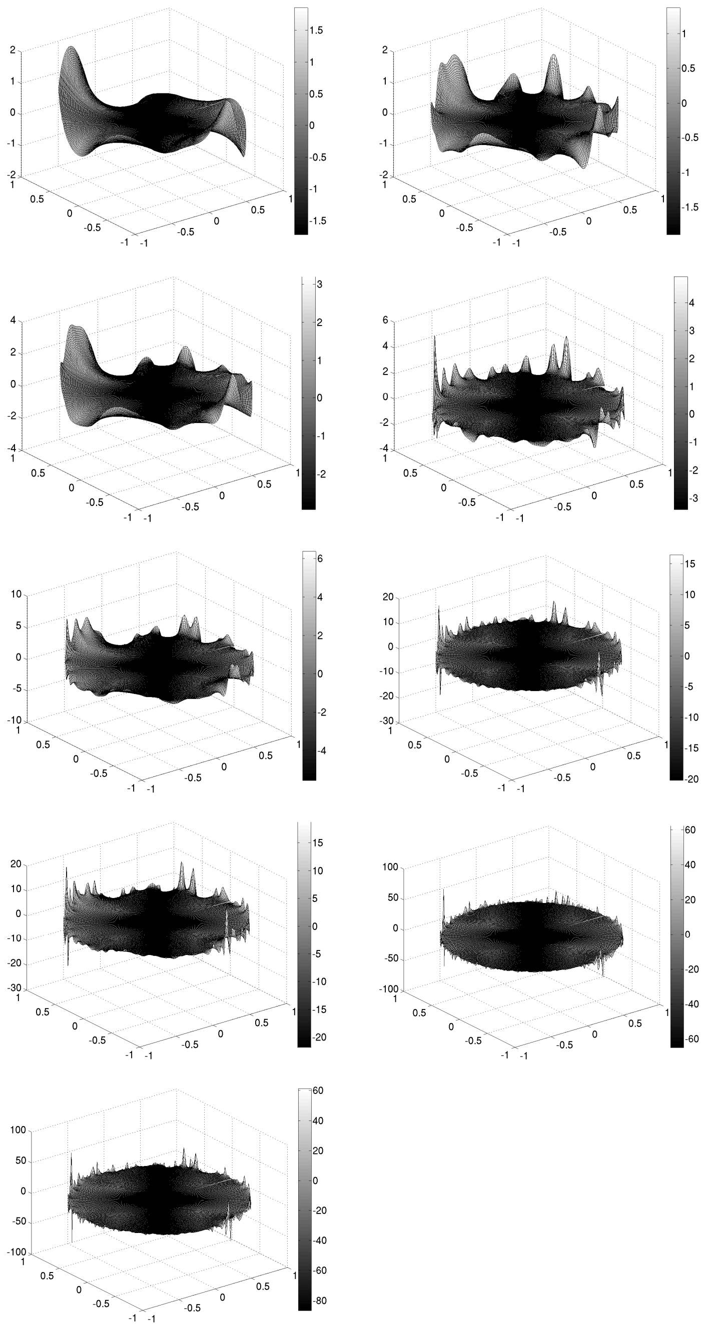

Figure 5: Calculated density (from degree 3 to 360) of minimal norm (in kg/m^3) at a cross section of a great circle: Again, the left column shows the convolution with the scaling functions of scales $J = 4$ (top) to $J = 8$ (bottom) whereas the right column shows the difference of the approximations of two consecutive scales.

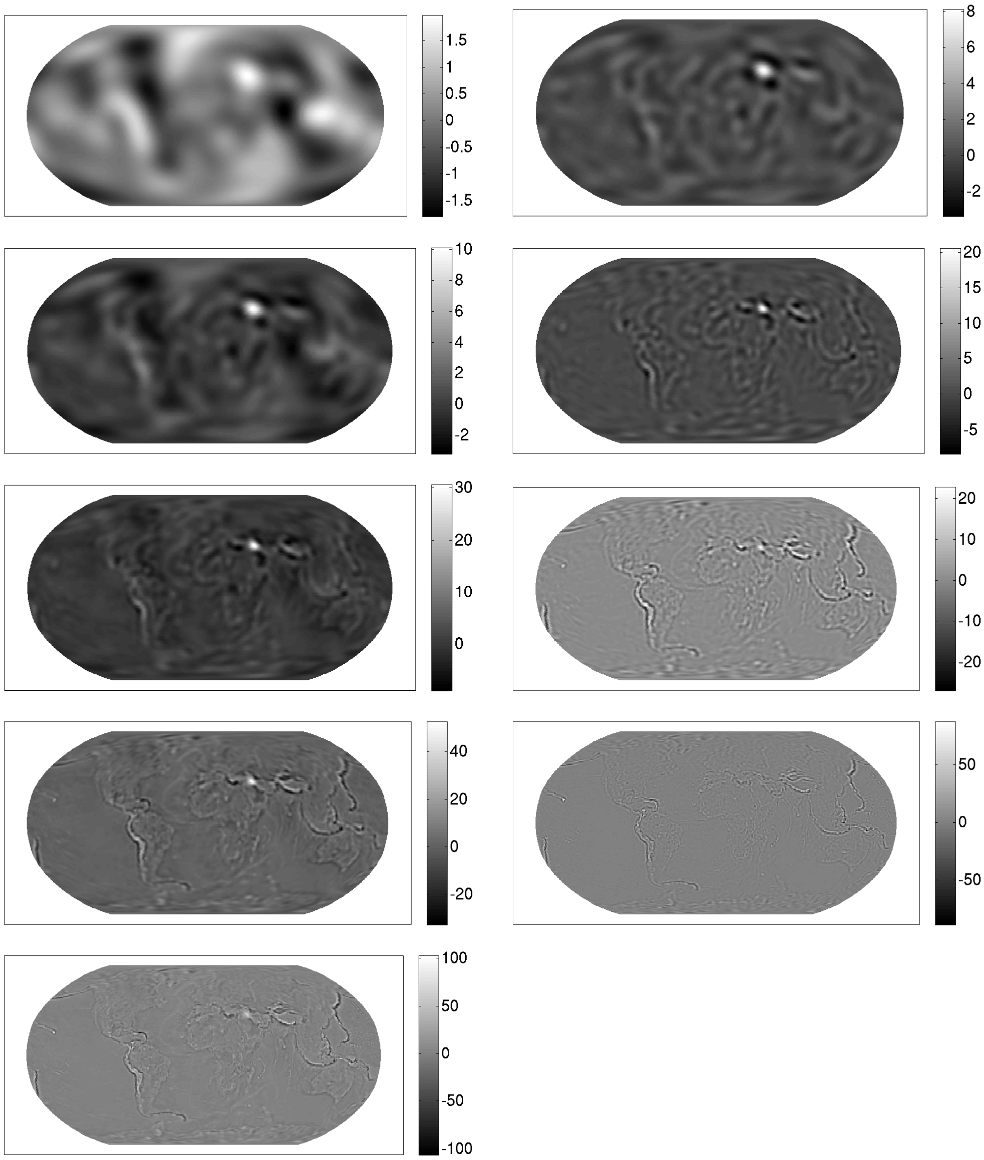

Figure 6: Calculated harmonic density like in Figure 2 but calculated out of the perturbed potential in Figure 7.

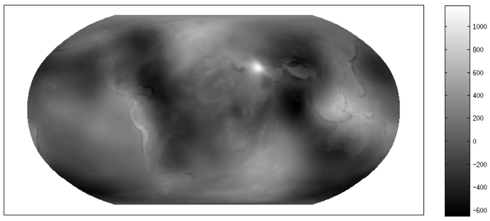

Figure 7: Perturbed gravitational potential

11 Conclusions

The characterisation of the problem of non–uniqueness for the inverse gravimetric problem and different constraints yielding uniqueness have been discussed. Apparently, none of the constraints yields a solution with a satisfactory physical interpretation. In particular, the deep structures of the Earth definitely cannot be recovered from gravity data (perhaps seismic data can be used for this purpose). It appears that a reasonable approach is to calculate a harmonic density function from gravity data and to combine it with an anharmonic density function obtained from non–gravitational data.
We have shown that a known multiscale method using product kernels, is able to yield a satisfactory approximation to the harmonic density. These results also demonstrate a well–known observation: the harmonic density alone already yields interesting qualitative information on density anomalies in the uppermost layer of the Earth.
Future research should investigate density recovery using seismic data (normal mode tomography might be a good choice for this purpose).

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