# Enumeration of the degree sequences of non–separable graphs and connected graphs


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**Abstract**

In 1962, S. L. Hakimi proved necessary and sufficient conditions for a given sequence of positive integers $d_1, d_2, \ldots, d_n$ to be the degree sequence of a non–separable graph or that of a connnected graph. Our goal in this note is to utilize these results to prove closed formulas for the functions $d_{ns}(2m)$ and $d_c(2m)$, the number of degree sequences with degree sum $2m$ representable by non–separable graphs and connected graphs (respectively). Indeed, we give both generating function proofs as well as bijective proofs of the following identities:

$$d_{ns}(2m) = p(2m) - p(2m-1) - \sum_{j=0}^{m-2} p(j)$$

and

$$d_c(2m) = p(2m) - p(m-1) - 2\sum_{j=0}^{m-2} p(j)$$

where $p(j)$ is the number of unrestricted integer partitions of $j$



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## 1 Introduction and Statement of Results

In this note, all graphs $G = (V, E)$ under consideration will be finite, undirected, and loopless but may contain multiple edges. We denote the *degree sequence* of the vertices $v_1, v_2, \ldots, v_m$ by $d_1, d_2, \ldots, d_m$ with the convention that $d_1 \geq d_2 \geq \cdots \geq d_m$. As usual, a graph is called *connected* if it has only one component. We say that a vertex $v$ is a *cut–vertex* of $G$ if $|E(G)| \geq 2$ and $G - v$ has more components than $G$. A graph is called *non–separable* if it is connected and has no cut–vertices.

In 1962, Hakimi [7] characterized those degree sequences for which there exists a non–separable graph realization and those for which there exists a connected graph realization. His results are the following:

**Theorem 1.1.** *Let $d_1 \geq d_2 \geq \cdots \geq d_n \geq 2$ be integers with $n \geq 2$. Then there exists a non–separable graph with degree sequence $d_1, d_2, \ldots, d_n$ if and only if*

- $d_1 + d_2 + \cdots + d_n$ *is even and*
- $d_1 \leq d_2 + d_3 + \cdots + d_n - 2n + 4$.

**Theorem 1.2.** *Let $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$ be integers with $n \geq 2$. Then there exists a connected graph with degree sequence $d_1, d_2, \ldots, d_n$ if and only if*

- $d_1 + d_2 + \cdots + d_n$ *is even,*
- $d_1 \leq d_2 + d_3 + \cdots + d_n$, *and*
- $d_1 + d_2 + d_3 + \cdots + d_n \geq 2(n - 1)$.

It should be noted that Hakimi's Theorem 1.1 appeared more recently in the work of Jackson and Jordán [8, Corollary 3.2] as a corollary to a more extensive theorem.

In this note, our goal is to enumerate all degree sequences of sum $2m$ for which there exists a realization via a non–separable graph and those for which there exists a connected realization. We will denote the number of degree sequences of sum $2m$ with a non–separable graph realization by $d_{ns}(2m)$. Similarly, we will let $d_c(2m)$ be the number of degree sequences of sum $2m$ for which there exists a connected graph realization. Then our ultimate goal in this note is to prove the following:

**Theorem 1.3.** *For all $m \geq 2$,*

$$d_{ns}(2m) = p(2m) - p(2m - 1) - \sum_{j=0}^{m-2} p(j)$$

*where $p(k)$ is the number of unrestricted integer partitions of $k$.*

**Theorem 1.4.** *For all $m \geq 1$,*

$$d_c(2m) = p(2m) - p(m - 1) - 2\sum_{j=0}^{m-2} p(j)$$

*where $p(k)$ is the number of unrestricted integer partitions of $k$.*

So, for example, the number of degree sequences of sum 6 with non–separable graph realizations is

$$d_{ns}(6) = p(6) - p(5) - p(0) - p(1) = 11 - 7 - 1 - 1 = 2.$$

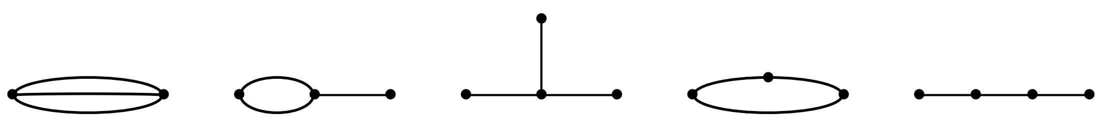
The two partitions in question, along with corresponding non–separable graph realizations, are shown below.

$3+3$ $\quad$ $2+2+2$

Also by way of example, the number of degree sequences of sum 6 with connected graph realizations is

$$d_c(6) = p(6) - p(2) - 2p(0) - 2p(1) = 11 - 2 - 2 - 2 = 5.$$

The five partitions in question, along with corresponding connected graph realizations, are shown below.



$3+3$ $\quad$ $3+2+1$ $\quad$ $3+1+1+1$ $\quad$ $2+2+2$ $\quad$ $2+2+1+1$

The techniques necessary for proving Theorems 1.3 and 1.4 are elementary. First, we develop generating functions for functions closely related to $d_{ns}(2m)$ and $d_c(2m)$ where $n$, the number of vertices, is fixed. We then sum these generating functions over all possible values of $n$. Theorems 1.3 and 1.4 follow in straightforward fashion. We then close this work by providing alternative proofs of both results which are bijective in nature.

## 2 Degree Sequences of Non–Separable Graphs

We begin by focusing our attention on Theorem 1.3. We will first relax the "evenness" condition in the statement of Theorem 1.1; namely, we will not concern ourselves at this point with whether the sum of the integers $d_i$ is even. We will invoke this restriction at the end of the proof. Thus, we now consider a function $a_n(m)$, the number of partitions of $m$ into exactly $n$ parts satisfying the inequality in Theorem 1.1.

The generating function $A_n(q)$ for $a_n(m)$ is given by

$$A_n(q) = \sum_{m \geq 0} a_n(m) q^m = \sum_{\substack{d_1 \geq d_2 \geq \cdots \geq d_n \geq 2 \\ d_1 \leq d_2 + d_3 + \cdots + d_n - 2n + 4}} q^{d_1 + d_2 + \cdots + d_n}.$$

We will now show that

$$A_n(q) = q^{2n} \prod_{i=1}^{n} \frac{1}{1-q^i} - \frac{q^{2n+1}}{1-q} \prod_{i=1}^{n-1} \frac{1}{1-q^{2i}}. \tag{1}$$

There is some wisdom here in considering multivariable generating functions. Thus, for $k \geq 1$, let

$$G_{d,k}(q_1, \ldots, q_k) = \sum_{d_1 \geq d_2 \geq \cdots \geq d_k \geq d} q_1^{d_1} \cdots q_k^{d_k}, \tag{2}$$

where we shall specifically set $d = 2$ later in this section and $d = 1$ in the next section. For $k = 1$ we have

$$G_{d,1}(q_1) = \sum_{d_1=d}^{\infty} q_1^{d_1} = \frac{q_1^d}{1-q_1}.$$

We also see that

$$G_{d,k}(q_1, \ldots, q_k) = \frac{1}{1-q_1} G_{d,k-1}(q_1 q_2, q_3, \ldots, q_k) \quad \text{for } k \geq 2.$$

A straightforward induction on $k$ now proves that, for all $k \geq 1$,

$$G_{d,k}(q_1, \ldots, q_k) = \prod_{i=1}^{k} \frac{q_i^d}{1-(q_1 \cdots q_i)}. \tag{3}$$

With this information about $G_{d,k}$ in hand, we have

$$\begin{aligned}
A_n(q) &= \sum_{d_2 \geq \cdots \geq d_n \geq 2} \; \sum_{d_1=d_2}^{d_2+\cdots+d_n-2n+4} q^{d_1+\cdots+d_n} \\
&= \sum_{d_2 \geq \cdots \geq d_n \geq 2} q^{2d_2+d_3+\cdots+d_n} \frac{1-q^{d_3+\cdots+d_n-2n+5}}{1-q} \\
&= \frac{1}{1-q} G_{2,n-1}(q^2, q, \ldots, q) - \frac{q^{-2n+5}}{1-q} G_{2,n-1}(q^2, q^2, \ldots, q^2) \\
&= q^{2n} \prod_{i=1}^{n} \frac{1}{1-q^i} - \frac{q^{2n+1}}{1-q} \prod_{i=1}^{n-1} \frac{1}{1-q^{2i}} \quad \text{by (3).}
\end{aligned}$$

This proves equation (1) above.

By (1), the generating function $A(q)$ for $a(m)$, the number of integer partitions of $m$ into any number $n \geq 2$ parts which satisfy the inequality in Theorem 1.1, is given by

$$A(q) = \sum_{n \geq 2} A_n(q) = \sum_{n \geq 2} q^{2n} \prod_{i=1}^{n} \frac{1}{1-q^i} - \sum_{n \geq 2} \frac{q^{2n+1}}{1-q} \prod_{i=1}^{n-1} \frac{1}{1-q^{2i}}. \tag{4}$$

Now we wish to consider the two sums in $A(q)$ separately and interpret them as generating functions of well–known arithmetic functions. First, we recall a well-known identity of Euler which states that

$$1 + \sum_{n=1}^{\infty} t^n \prod_{i=1}^{n} \frac{1}{1-q^i} = \prod_{n=0}^{\infty} \frac{1}{1-tq^n}; \tag{5}$$

see Andrews [1, Corollary 2.2]. We will use this identity in key places in the work below.

We now focus our attention on the first sum on the right–hand side of (4). By (5) with $t = q^2$, and the fact that the generating function for $p(n)$ is

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}, \tag{6}$$

we have

$$\begin{aligned}
\sum_{n \geq 2} q^{2n} \prod_{i=1}^{n} \frac{1}{1-q^i} &= \prod_{n=2}^{\infty} \frac{1}{1-q^n} - 1 - \frac{q^2}{1-q} \\
&= (1-q) \sum_{n=0}^{\infty} p(n) q^n - 1 - \sum_{n=2}^{\infty} q^n \\
&= \sum_{m \geq 3} \left( p(m) - p(m-1) - 1 \right) q^m,
\end{aligned}$$

where we have used the facts that $p(0) = p(1) = 1$ and $p(2) = 2$.

Next, we consider the second sum on the right–hand side of (4). Note that

$$\begin{aligned}
\sum_{n\geq 2} q^{2n+1} \prod_{i=1}^{n-1} \frac{1}{1-q^{2i}} &= q^3 \sum_{n\geq 2} q^{2n-2} \prod_{i=1}^{n-1} \frac{1}{1-q^{2i}} \\
&= q^3 \sum_{n\geq 1} q^{2n} \prod_{i=1}^{n} \frac{1}{1-q^{2i}} \\
&= q^3 \prod_{i=1}^{\infty} \frac{1}{1-q^{2i}} - q^3.
\end{aligned}$$

The last line follows by first putting $t = q$ in (5), and thereafter replacing $q$ by $q^2$ throughout. By (6) and the last line above, we know

$$\begin{aligned}
\sum_{n\geq 2} q^{2n+1} \prod_{i=1}^{n-1} \frac{1}{1-q^{2i}} &= q^3 \sum_{m\geq 0} p(m) q^{2m} - q^3 \\
&= \sum_{m\geq 1} p(m) q^{2m+3} \qquad \text{again using } p(0) = 1.
\end{aligned}$$

In order to finish the analysis of the second sum in (4), we must multiply by the factor $\frac{1}{1-q}$. This yields

$$\begin{aligned}
\sum_{n\geq 2} \frac{q^{2n+1}}{1-q} \prod_{i=1}^{n-1} \frac{1}{1-q^{2i}} &= \frac{1}{1-q} \sum_{m\geq 1} p(m) q^{2m+3} \\
&= \sum_{k=0}^{\infty} q^k \sum_{m\geq 1} p(m) q^{2m+3} \\
&= \sum_{m\geq 1} \sum_{j=1}^{m} p(j) q^{2m+3} + \sum_{m\geq 1} \sum_{j=1}^{m} p(j) q^{2m+4}
\end{aligned}$$

by standard generating function manipulations. This last line can be rewritten as

$$\sum_{m\geq 3} \sum_{j=1}^{m-2} p(j) q^{2m-1} + \sum_{m\geq 3} \sum_{j=1}^{m-2} p(j) q^{2m}.$$

We are now in a position to finish the proof of Theorem 1.3. Since $a(2m) = d_{ns}(2m)$ for all $m \geq 2$, we see that the generating function for $d_{ns}(2m)$ is given by

$$\begin{aligned}
\sum_{m\geq 2} d_{ns}(2m) q^{2m} &= \sum_{m\geq 2} \left(p(2m) - p(2m-1) - 1\right) q^{2m} - \sum_{m\geq 3} \sum_{j=1}^{m-2} p(j) q^{2m} \\
&= \sum_{m\geq 2} \left(p(2m) - p(2m-1) - 1\right) q^{2m} - \sum_{m\geq 2} \sum_{j=1}^{m-2} p(j) q^{2m} \\
&\qquad \text{where the empty sum is set to zero} \\
&= \sum_{m\geq 2} (p(2m) - p(2m-1) - \sum_{j=0}^{m-2} p(j)) q^{2m} \text{ since } p(0) = 1.
\end{aligned}$$

Therefore, for all $m \geq 2$,

$$d_{ns}(2m) = p(2m) - p(2m-1) - \sum_{j=0}^{m-2} p(j)$$

and this completes the proof of Theorem 1.3. ∎

# 3 Degree Sequences of Connected Graphs

We now consider a proof of Theorem 1.4. As in the previous section, we first relax the "evenness" condition in the statement of Theorem 1.2. Thus, we consider a function $b_n(m)$ which is the number of partitions of $m$ into exactly $n$ parts satisfying the inequalities in Theorem 1.2.

Thus, the generating function $B_n(q)$ for $b_n(m)$ is given by

$$B_n(q) = \sum_{m \geq 0} b_n(m) q^m = \sum_{\substack{d_1 \geq d_2 \geq \cdots \geq d_n \geq 1 \\ d_1 \leq d_2 + \cdots + d_n \\ d_1 + d_2 + \cdots + d_n \geq 2(n-1)}} q^{d_1 + d_2 + \cdots + d_n}, \quad n \geq 2.$$

Now, we will show that

$$B_n(q) = q^n \prod_{i=1}^{n} \frac{1}{1-q^i} - \frac{q^{2n-1}}{1-q} \prod_{i=1}^{n-1} \frac{1}{1-q^{2i}} - \sum_{i=n}^{2n-3} p(i-n) q^i. \tag{7}$$

To prove (7), we first apply inclusion/exclusion to obtain

$$\begin{aligned} B_n(q) &= \sum_{\substack{d_1 \geq \cdots \geq d_n \geq 1 \\ d_1 \leq d_2 + \cdots + d_n}} q^{d_1 + \cdots + d_n} - \sum_{\substack{d_1 \geq \cdots \geq d_n \geq 1 \\ d_1 \leq d_2 + \cdots + d_n \\ d_1 + \cdots + d_n \leq 2n-3}} q^{d_1 + \cdots + d_n} \\ &= \sum_{\substack{d_1 \geq \cdots \geq d_n \geq 1 \\ d_1 \leq d_2 + \cdots + d_n}} q^{d_1 + \cdots + d_n} - \sum_{\substack{d_1 \geq \cdots \geq d_n \geq 1 \\ d_1 + \cdots + d_n \leq 2n-3}} q^{d_1 + \cdots + d_n} + \sum_{\substack{d_1 \geq \cdots \geq d_n \geq 1 \\ d_1 \geq d_2 + \cdots + d_n + 1 \\ d_1 + \cdots + d_n \leq 2n-3}} q^{d_1 + \cdots + d_n}. \end{aligned}$$

Now, if $d_1 \geq \cdots \geq d_n \geq 1$ and $d_1 \geq d_2 + \cdots + d_n + 1$, then $d_1 + d_2 + \cdots + d_n \geq 2(d_2 + \cdots + d_n) + 1 \geq 2(n-1) + 1$. This means that the last sum above is empty and we have

$$B_n(q) = \sum_{\substack{d_1 \geq \cdots \geq d_n \geq 1 \\ d_1 \leq d_2 + \cdots + d_n}} q^{d_1 + \cdots + d_n} - \sum_{\substack{d_1 \geq \cdots \geq d_n \geq 1 \\ d_1 + \cdots + d_n \leq 2n-3}} q^{d_1 + \cdots + d_n}. \tag{8}$$

For the first sum on the right–hand side of (8), we have

$$\begin{aligned} \sum_{\substack{d_1 \geq \cdots \geq d_n \geq 1 \\ d_1 \leq d_2 + \cdots + d_n}} q^{d_1 + \cdots + d_n} &= \sum_{d_2 \geq \cdots \geq d_n \geq 1} \sum_{d_1 = d_2}^{d_2 + \cdots + d_n} q^{d_1 + \cdots + d_n} \\ &= \sum_{d_2 \geq \cdots \geq d_n \geq 1} q^{2d_2 + d_3 + \cdots + d_n} \frac{1 - q^{d_3 + \cdots + d_n + 1}}{1-q} \\ &= \frac{1}{1-q} G_{1,n-1}(q^2, q, \ldots, q) - \frac{q}{1-q} G_{1,n-1}(q^2, q^2, \ldots, q^2) \quad \text{from (2)} \\ &= q^n \prod_{i=1}^{n} \frac{1}{1-q^i} - \frac{q^{2n-1}}{1-q} \prod_{i=1}^{n-1} \frac{1}{1-q^{2i}} \quad \text{from (3).} \end{aligned}$$

We now consider the second sum on the right–hand side of (8). First, let $p_k(m)$ denote the number of partitions of $m$ into at most $k$ parts. (We know that $p_k(m)$ is also equal to the number of partitions of $m$ into parts no greater than $k$.) Then we have the generating function

$$\sum_{m=0}^{\infty} p_k(m)q^m = \prod_{i=1}^{k} \frac{1}{1-q^i}.$$

By (2) and (3), we know

$$\sum_{d_1 \geq \cdots \geq d_n \geq 1} q^{d_1+\cdots+d_n} = G_{1,n}(q,\ldots,q) = q^n \prod_{i=1}^{n} \frac{1}{1-q^i}.$$

Hence,

$$\sum_{d_1 \geq \cdots \geq d_n \geq 1} q^{d_1+\cdots+d_n} = \sum_{i=n}^{\infty} p_n(i-n)q^i,$$

so that the second sum on the right–hand side of (8) is given by

$$\sum_{\substack{d_1 \geq \cdots \geq d_n \geq 1 \\ d_1+\cdots+d_n \leq 2n-3}} q^{d_1+\cdots+d_n} = \sum_{i=n}^{2n-3} p_n(i-n)q^i.$$

It is well–known and easily seen that $p_k(m) = p(m)$ if $m \leq k$. Thus we have

$$\sum_{\substack{d_1 \geq \cdots \geq d_n \geq 1 \\ d_1+\cdots+d_n \leq 2n-3}} q^{d_1+\cdots+d_n} = \sum_{i=n}^{2n-3} p(i-n)q^i.$$

This completes the proof of equation (7).

Finally, we consider the generating function $B(q)$ for $b(m)$, the number of integer partitions of $m$ into any number $n \geq 2$ parts which satisfy the inequalities in Theorem 1.2.

We have

$$B(q) = \sum_{n=2}^{\infty} B_n(q) = \sum_{n=0}^{\infty} q^n \prod_{i=1}^{n} \frac{1}{1-q^i} - 1 - \frac{q}{1-q} \sum_{n=0}^{\infty} q^{2n} \prod_{i=1}^{n} \frac{1}{1-q^{2i}} - \sum_{n=2}^{\infty} \sum_{i=0}^{2n-3} p(i-n)q^i,$$

using the convention $p(m) = 0$ if $m < 0$. Applying (5), we further get,

$$B(q) = \prod_{n=1}^{\infty} \frac{1}{1-q^n} - \frac{q}{1-q} \prod_{n=1}^{\infty} \frac{1}{1-q^{2n}} - \sum_{n=2}^{\infty} \sum_{i=0}^{2n-3} p(i-n)q^i - 1,$$

or

$$B(q) = \sum_{m=0}^{\infty} p(m)q^m - \frac{q}{1-q} \sum_{m=0}^{\infty} p(m)q^{2m} - \sum_{n=2}^{\infty} \sum_{i=0}^{2n-3} p(i-n)q^i - 1.$$

Since

$$\frac{q}{1-q} = \frac{q}{1-q^2} + \frac{q^2}{1-q^2},$$

it is easy to pick out from $B(q)$ all terms with even exponents on $q$, so that

$$\sum_{m=1}^{\infty} b(2m)q^{2m} = \sum_{m=0}^{\infty} p(2m)q^{2m} - \frac{q^2}{1-q^2} \sum_{m=0}^{\infty} p(m)q^{2m} - \sum_{n=2}^{\infty} \sum_{j=0}^{n-2} p(2j-n)q^{2j} - 1;$$

that is,

$$\sum_{m=1}^{\infty} b(2m)q^m = \sum_{m=1}^{\infty} p(2m)q^m - \frac{q}{1-q}\sum_{m=0}^{\infty} p(m)q^m - \sum_{n=2}^{\infty}\sum_{j=0}^{n-2} p(2j-n)q^j,$$

or

$$\sum_{m=1}^{\infty} b(2m)q^m = \sum_{m=1}^{\infty} p(2m)q^m - \sum_{m=1}^{\infty}\sum_{j=0}^{m-1} p(j)q^m - \sum_{m=0}^{\infty}\sum_{n=m+2}^{\infty} p(2m-n)q^m,$$

where, in fact,

$$\sum_{n=m+2}^{\infty} p(2m-n) = \sum_{j=0}^{m-2} p(j).$$

Equating coefficients of $q^m$, we have for $m \geq 1$,

$$b(2m) = d_c(2m) = p(2m) - \sum_{j=0}^{m-1} p(j) - \sum_{j=0}^{m-2} p(j),$$

and the proof of Theorem 1.4 is complete.

# 4 Bijective Proofs

In this section we give bijective proofs of Theorems 1.3 and 1.4. Let us take the first one first.

Let $\mathcal{P}(N)$ be the set of all partitions $(d_1, \ldots, d_n)$ of the integer $N$,

$$N = d_1 + \cdots + d_n, \qquad d_1 \geq \cdots \geq d_n \geq 1.$$

Then $|\mathcal{P}(N)| = p(N)$.

We want to determine the number of partitions in the subset $\mathcal{P}^*(2m)$ of $\mathcal{P}(2m)$, satifying the hypotheses of Theorem 1.1. We do this by removing the nonadmissible partitions from $\mathcal{P}(2m)$.

First we remove the unique partition of $2m$ with $n = 1$. We are then left with the set

$$\mathcal{P}_1 = \{(d_1, \ldots, d_n) \in \mathcal{P}(2m) \mid n \geq 2\}.$$

Next, we remove the subset

$$\mathcal{Q}_1 = \{(d_1, \ldots, d_n) \in \mathcal{P}_1 \mid d_n = 1\}.$$

Clearly, we have a bijection

$$\mathcal{Q}_1 \longrightarrow \mathcal{P}(2m-1),$$

given by $(d_1, \ldots, d_{n-1}, 1) \longmapsto (d_1, \ldots, d_{n-1})$. Thus the set $\mathcal{P}_2 = \mathcal{P}_1 \backslash \mathcal{Q}_1$ contains $p(2m) - 1 - p(2m-1)$ partitions of $2m$.

Finally, to arrive at $\mathcal{P}^*(2m)$, we remove the set

$$\mathcal{Q}_2 = \{(d_1, \ldots, d_n) \in \mathcal{P}_2 \mid d_1 > d_2 + \cdots + d_n - 2n + 4\}.$$

If $(d_1, \ldots, d_n) \in \mathcal{Q}_2$, then

$$d_1 + d_2 + \cdots + d_n > 2(d_2 + \cdots + d_n) - 2n + 4,$$

or, equivalently,

$$m - 2 \geq (d_2 - 1) + \cdots + (d_n - 1).$$

Hence we may define a map

$$\varphi : \mathcal{Q}_2 \longrightarrow \mathcal{P}(1) \cup \ldots \cup \mathcal{P}(m-2),$$

by putting $\varphi(d_1, d_2, \ldots, d_n) = (d_2 - 1, \ldots, d_n - 1)$. We now show that $\varphi$ is a bijection.

Since $d_1$ is uniquely determined by $d_2, \ldots, d_n$ (and the fixed $m$), the map $\varphi$ is injective. On the other hand, suppose that $(x_2, \ldots, x_n) \in \mathcal{P}(1) \cup \ldots \cup \mathcal{P}(m-2)$. Set

$$x_1 + 1 = 2m - ((x_2 + 1) + \cdots + (x_n + 1)).$$

Since $m - 1 > x_2 + \cdots + x_n$, we then have

$$x_1 + 1 > (x_2 + 1) + \cdots + (x_n + 1) - 2n + 4.$$

Moreover,

$$x_1 - x_2 > (x_3 + 1) + \cdots + (x_n + 1) - 2n + 4 \geq 0,$$

so that $x_1 + 1 \geq x_2 + 1 \geq \cdots \geq x_n + 1 \geq 2$. Thus $(x_1 + 1, x_2 + 1, \ldots, x_n + 1) \in \mathcal{Q}_2$, and $\varphi$ is a bijection.

Since $\mathcal{P}(1), \ldots, \mathcal{P}(m-2)$ are pairwise disjoint, and $|\mathcal{P}(j)| = p(j)$, we have

$$|\mathcal{P}(1) \cup \ldots \cup \mathcal{P}(m-2)| = \sum_{j=1}^{m-2} p(j),$$

and

$$|\mathcal{P}^*(2m)| = |\mathcal{P}_2| - |\mathcal{Q}_2| = p(2m) - p(2m-1) - \sum_{j=0}^{m-2} p(j),$$

where we used that $p(0) = 1$. This completes the bijective proof of Theorem 1.3.

Next, we turn to the proof of Theorem 1.4. Again we will start with the set $\mathcal{P}(2m)$ and successively remove nonadmissible partitions, to arrive at the set $\mathcal{P}^{**}(2m)$ consisting of all partitions of $2m$ satisfying the hypotheses of Theorem 1.2.

Also now we remove the unique partition with $n = 1$ to get the set $\mathcal{P}_1$. Next, we set

$$\mathcal{Q}_3 = \{(d_1, \ldots, d_n) \in \mathcal{P}_1 \mid d_1 > d_2 + \cdots + d_n\}.$$

If $(d_1, \ldots, d_n) \in \mathcal{Q}_3$, we have $2d_1 > d_1 + \cdots + d_n = 2m$; hence $d_1 \geq m + 1$. Thus we have $2m = d_1 + \cdots + d_n \geq m + 1 + d_2 + \cdots + d_n$, so that

$$d_2 + \cdots + d_n \leq m - 1.$$

We may therefore define a map

$$\psi : \mathcal{Q}_3 \longrightarrow \mathcal{P}(1) \cup \ldots \cup \mathcal{P}(m-1),$$

by putting $\psi(d_1, \ldots, d_n) = (d_2, \ldots, d_n)$. Since $d_1 = 2m - (d_2 + \cdots + d_n)$, the map $\psi$ is injective. We go on to show that $\psi$ also is surjective.

Let $(x_2, \ldots, x_n) \in \mathcal{P}(1) \cup \ldots \cup \mathcal{P}(m-1)$, and put $x_1 = 2m - (x_2 + \cdots + x_n)$. If we can show that $(x_1, \ldots, x_n) \in \mathcal{Q}_3$, then $\psi$ is surjective. We have $x_1 = 2m - (x_2 + \cdots + x_n) \geq 2m - (m-1) = m + 1$. Clearly, $x_2 \leq m - 1$, so that we have the monotonicity $x_1 \geq x_2 \geq \cdots \geq x_n \geq 1$. Moreover, since

$x_2 + \cdots + x_n < m$, we have $x_1 = 2m - (x_2 + \cdots + x_n) > m > x_2 + \cdots + x_n$. Thus $\psi$ is surjective; hence a bijection. Therefore, we have

$$|\mathcal{Q}_3| = |\mathcal{P}(1) \cup \ldots \cup \mathcal{P}(m-1)| = \sum_{j=1}^{m-1} p(j).$$

Next, set

$$\mathcal{Q}_4 = \{(d_1, \ldots, d_n) \in \mathcal{P}_1 \mid d_1 + \cdots + d_n < 2(n-1)\}.$$

Then $\mathcal{P}^{**}(2m) = \mathcal{P}_1 \setminus (\mathcal{Q}_3 \cup \mathcal{Q}_4)$. If $(d_1, \ldots, d_n) \in \mathcal{Q}_3$, then $d_1 + \cdots + d_n > 2(d_2 + \cdots + d_n) \geq 2(n-1)$. Hence $\mathcal{Q}_3 \cap \mathcal{Q}_4 = \emptyset$, and $|\mathcal{P}^{**}(2m)| = |\mathcal{P}_1| - |\mathcal{Q}_3| - |\mathcal{Q}_4|$.

Let $(d_1, \ldots, d_n) \in \mathcal{Q}_4$. Then $2m = d_1 + \cdots + d_n < 2(n-1)$, so that $n \geq m+2$. Suppose that $d_1 \geq \cdots \geq d_r \geq 2$ and $d_{r+1} = \cdots = d_n = 1$. Then we define a map

$$\vartheta : \mathcal{Q}_4 \longrightarrow \mathcal{P}(0) \cup \ldots \cup \mathcal{P}(m-2),$$

by putting $\vartheta(d_1, \ldots, d_n) = (d_1 - 1, d_2 - 1, \ldots, d_r - 1)$. In particular, $\vartheta(1, 1, \ldots, 1) = 0$ and $\mathcal{P}(0) = \{\emptyset\}$. Clearly, the map $\vartheta$ is injective. On the other hand, let $(x_1, \ldots, x_r)$ be a partition of a nonnegative integer at most equal to $m-2$. Determine $n$ such that $(x_1 + \cdots + x_r) + n = 2m$. Since $x_1 + \cdots + x_r \leq m-2$, we have $n \geq m+2$. Thus $(x_1 + 1, \ldots, x_r + 1, 1, \ldots, 1) \in \mathcal{Q}_4$, the map $\vartheta$ is surjective, and the proof of Theorem 1.4 is easily completed.

# 5 Closing Thoughts

It is clear that those degree sequences enumerated by $d_{ns}(2m)$ are also among those enumerated by $d_c(2m)$ (by definition). Hence, the difference of these two functions may be of interest. For completeness' sake, we define $d_{cs}(2m)$ to be the number of degree sequences of sum $2m$ which have connected graph realizations but no non-separable graph realizations. Then Theorems 1.3 and 1.4 imply that, for all $m \geq 2$,

$$d_{cs}(2m) = p(2m-1) - p(m-1) - \sum_{j=0}^{m-2} p(j).$$

It is important to note that partitions whose parts satisfy certain inequalities (as we see in Hakimi's characterizations above in Theorems 1.1 and 1.2) have been studied in many other contexts. For example, see the work of Andrews, Paule, and Riese [2] for a very similar result to Theorem 1.3; there, MacMahon's partition analysis is used heavily. Andrews, Paule, and Riese have completed other projects of a similar nature using partition analysis; the interested reader may wish to see [3, 4] and the bibliographic reference lists therein for additional examples. (Although we could have also used partition analysis in this paper, we chose a much more elementary approach in the proof above, one which accomplishes the work of partition analysis but does not require as much mathematical machinery.) For additional examples of work done on partitions whose parts satisfy specific inequalities, see the works of Uppuluri and Carpenter [10], Sellers [9], Corteel and Savage [5], and Corteel, Savage, and Wilf [6].

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