Counting 2-Connected Deletion-Minors of Binary Matroids

Bill Jackson
School of Mathematical Sciences
Queen Mary, University of London
Mile End Road
London E1 4NS, England
B.JACKSON@QMUL.AC.UK

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Abstract

We introduce a new invariant for a binary matroid M and use it to prove upper bounds on the number of circuits and, more generally, the number of 2-connected deletion minors of M containing a fixed element. In addition, we conjecture that the invariant can be used to bound the roots of the characteristic polynomial of M.

1 Introduction

The purpose of this paper is to introduce a new invariant for a binary matroid M and use it to prove upper bounds on the number of circuits and, more generally, the number of 2-connected deletion minors of M containing a fixed element. The matroid invariant extends a graph invariant previously introduced in [3].

Given a binary matroid M, let $\mathcal{B}(M)$ be the set of bases of the cocyle space of M and put

$$\Lambda(M) = \min_{B \in \mathcal{B}(M)} \max_{K \in B} \{|K|\}.$$

For example, if M is the cycle matroid of the wheel with s spokes, we have $\Lambda(M) = 3$ and the minimum is obtained by taking the basis of the cocycle space of G which consists of the sets of edges incident with each vertex on the rim of the wheel.

For a graph G, we put $\Lambda(G) = \Lambda(M_G)$ where M_G is the cycle matroid of G. As in the above example, the stars centred on all but one of the vertices of G span the cocycle space of G (and form a basis whenever G is connected). Thus $\Lambda(G)$ is bounded above by the second largest degree of G. Our interest in $\Lambda(G)$ was sparked initially by the study of the roots of chromatic polynomials of graphs in [7]. It is an elementary fact that all of the *integer* chromatic roots of G lie in the interval $[0, \Delta(G)]$ where $\Delta(G)$ denotes the maximum degree of G, i.e. the chromatic number of G is at most $\Delta(G) + 1$. Sokal [7, Corollary 6.4] showed that *all* the chromatic roots (real or complex) can be bounded in terms of the second-largest degree $\Delta_2(G)$: they lie in the disc $|q| < 7.963907\Delta_2(G) + 1$. Furthermore, he conjectured, following a suggestion of Shrock and Tsai [5, 6], that it might be possible to bound all the chromatic roots in terms of $\Lambda(G)$. An important step in [7] is to show that the number of connected m-edge subgraphs containing a fixed vertex of G is at most $e^m\Delta(G)^m$. It is not possible to obtain a similar bound in terms of $\Lambda(G)$. Consider for example the case when G is the wheel with s spokes. We saw above that $\Lambda(G) = 3$. The number of connected 1-edge subgraphs of G containing the central vertex of G is s and this can be arbitrarily large compared to $\Lambda(G)$. On the other hand, the following result shows that the number of 2-connected m-edge subgraphs containing a fixed edge of a graph G can be bounded by an exponential function of $\Lambda(G)$.

Theorem 1.1 *[3] Let G be a graph and f be an edge of G. Then the number of 2-connected m-edge subgraphs of G containing f is at most $(2\Lambda(G)/\ln 2)^m$.*

The main result of this paper is a partial extension of Theorem 1.1 to binary matroids.

Theorem 1.2 *Let M be a binary matroid and f be an element of M. Then:*
(a) the number of circuits of M containing f is at most $\Lambda(M)^{m-1}$;
(b) the number of 2-connected deletion minors of M containing f is at most $2^{m^2}\Lambda(M)^m$.

Theorem 1.2 will follow from Theorems 2.5 and 2.8, below. It is an open problem to decide if the number of 2-connected deletion minors of a binary matroid M containing a fixed element can be bounded above by $\alpha^m\Lambda(M)^m$ for some constant α.

An anonymous referee suggested that Theorem 1.2 could be extended to all matroids M by replacing $\Lambda(M)$ by another invariant $\Theta(M)$ defined to be the smallest integer k such that every element of M belongs to a cocircuit of size k. Unfortunately, this is false even for graphic matroids. Consider the graph G consisting of two vertices joined by p internally disjoint paths of length two, and let M_G be the cycle matroid of G. We have $\Lambda(M_G) = p$ and $\Theta(M_G) = 2$. The number of 2-connected 4-edge subgraphs of G containing any fixed edge e is $p - 1$, which can be arbitrarily large compared to $\Theta(M_G)$.

Our long term aim is to adapt the methods of [7] to bound the roots of characteristic polynomials of binary matroids. In particular, by restricting to the special case of cographic matroids, we would like to obtain an analogous result to [7, Corollary 6.4] for the roots of flow polynomials of graphs. We will return to this in Section 3.

We close this section with some remarks on the complexity of computing $\Lambda(M)$ for a binary matroid M. We can compute $\Lambda(M)$ in polynomial time when M is graphic by using maximum flow calculations, see [3].[1] We may also determine $\Lambda(M)$ for a cographic matroid by using an algorithm for finding a 'shortest cycle basis' of a graph due to Horton [1].[2] We do not know if $\Lambda(M)$ can be determined in polynomial time for an arbitrary binary matroid M. However, the related problem of finding a minimum size cocircuit in a binary matroid is known to be NP-hard, see [10].

2 Counting 2-Connected Deletion Minors

Given a matroid M, let $E(M)$ denote the ground set of M, $\mathcal{C}(M)$ the set of circuits of M, $\mathcal{K}(M)$ the set of cocircuits of M, and $r(M)$ the rank of M. A matroid N is a *deletion minor* of M if $N = M\backslash S$ for some $S \subset E(M)$, a *contraction minor* of M if $N = M/T$ for some $T \subset E(M)$, and a *minor* of M if $N = (M\backslash S)/T$ for some disjoint subsets $S, T \subset E(M)$. The matroid M is *2-connected* if every pair of elements of $E(M)$ are contained in a common circuit. The following lemma is due to Tutte [9] (see also [4, Theorem 4.3.1]).

Lemma 2.1 *If M is a 2-connected matroid and $e \in E(M)$, then at least one of the matroids $M\backslash e$ and M/e is 2-connected.*

A matroid M is *binary* if there exists a vector space V over GF(2) and a map $f\colon E(M) \to V$ such that, for each $S \subseteq E(M)$, the rank of S is equal to the dimension of the subspace of V spanned by $f(S)$. Given a binary matroid M, we consider the set $2^{E(M)}$ of all subsets of $E(M)$ as a vector space over GF(2), where vector addition is given by symmetric difference $\oplus$. The *cycle space* and *cocycle space* of M are the subspaces of $2^{E(M)}$ spanned by $\mathcal{C}(M)$ and $\mathcal{K}(M)$, respectively. They have dimensions $|E(M)| - r(M)$ and $r(M)$, respectively. We refer to the elements of the cycle and cocycle spaces as *cycles* and *cocycles* of M. We need the following elementary lemma for binary matroids, see [4, Proposition 9.2.2].

Lemma 2.2 *Let M be a binary matroid and $S \subseteq E(M)$. Then S is a cycle (respectively cocycle) of M if and only if $|S \cap T|$ is even for all cocycles (respectively cycles) T of M.*

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[1] We show in [3] that if M_G is the cycle matroid of a graph G with vertex set V then $\Lambda(M_G) = \max\{\lambda(x, y) : x, y \in V\}$, where $\lambda(x, y)$ is the maximum number of pairwise edge-disjoint xy-paths in G.

[2] It is easy to see that if B is a basis for the cocycle space of a binary matroid M such that $\sum_{K\in B} |K|$ is as small as possible, then $\Lambda(M) = \max\{|K| : K \in B\}$.

We say that $(M, \mathbf{w})$ is a *weighted binary matroid* if M is a binary matroid and $\mathbf{w} = \{w_e\}_{e \in E(M)}$ is a set of nonnegative real weights for $E(M)$. Let

$$\Lambda(M, \mathbf{w}) \ = \ \min_{B \in \mathcal{B}(M)} \max_{K \in B} \sum_{e \in K} w_e \ . \tag{1}$$

cocycle space of M. The invariant $\Lambda(M)$ defined in the Introduction can be obtained from $\Lambda(M, \mathbf{w})$ by taking all weights equal to one. We consider the weighted version $\Lambda(M, \mathbf{w})$ for two reasons: the first is that our proofs for the weighted and unweighted versions are identical; the second is that we believe $\Lambda(M, \mathbf{w})$ can be used to bound the roots of generalisations of the characteristic polynomial of M, i.e. the Tutte polynomial of M and its multivariate extension, see [8].

Given a weighted binary matroid (M, w) and N a minor of M let $w(N) = \prod_{e \in E(N)} w(e)$. For $e \in E(M)$, let $\mathcal{D}_m(e, M)$ denote the set of all m-element 2-connected deletion minors of M which contain e, and put

$$d_m(e) = \sum_{N \in \mathcal{D}_m(e,M)} w(N).$$

The next two results will form the basis for an inductive proof of an upper bound on $d_m(e)$ in terms of m and $\Lambda(M)$.

Lemma 2.3 *Let M be a binary matroid, and let e be an element of M. Let B be a basis for the cocycle space of M, and choose $K_1 \in B$ with $e \in K_1$. Let $X = \{K \in B\colon\ e \in K\}$ and $Y = B \backslash X$.*

- *(a) If e is not a coloop of M, then $B_1 := \{K - \{e\}\colon\ K \in X\} \cup Y$ is a basis for the cocycle space of $M \backslash e$.*
- *(b) If e is not a loop of M, then $B_2 := \{K \oplus K_1\colon\ K \in X \backslash \{K_1\}\} \cup Y$ is a basis for the cocycle space of M/e.*

Proof. (a) Each element of B_1 is a cocycle of $M \backslash e$. Since e is not a coloop of M we have $r(M \backslash e) = r(M)$. Hence the cocycle spaces of M and $M \backslash e$ have the same dimension and it suffices to show that $B \backslash e$ is linearly independent. Suppose $[\bigoplus_{K \in X'} (K - \{e\})] \oplus [\bigoplus_{K \in Y'} K] = \varnothing$, for some $X' \subseteq X$ and $Y' \subseteq Y$ with $X' \cup Y' \neq \varnothing$. Then $\bigoplus_{K \in X' \cup Y'} K \subseteq \{e\}$. This is impossible: the left hand side of the above set inclusion cannot be empty since $\mathcal{B}$ is linearly independent, and cannot equal $\{e\}$ since it belongs to the cocycle space of M, and $\{e\}$ is not a cocycle of M (because it is not a coloop).

(b) Each element of B_2 is a cocycle of M/e. Since e is not a loop of M we have $r(M/e) = r(M) - 1$. Hence the dimension of the cocycle space of M/e is one less than the dimension of the cocycle space of M and it suffices to show that B_2 is linearly independent. Suppose $[\bigoplus_{K \in X'} (K \oplus K_1)] \oplus [\bigoplus_{K \in Y'} K] = \varnothing$, for some $X' \subseteq X - \{K_1\}$

and $Y' \subseteq Y$. Then either $[\bigoplus_{K \in X' \cup Y'} K] = \varnothing$ or $[\bigoplus_{K \in X' \cup Y'} K] \oplus K_1 = \varnothing$. Both alternatives contradict the linear independence of $\mathcal{B}$. $\square$

Corollary 2.4 *Let $(M, \mathbf{w})$ be a weighted binary matroid and e be an element of M that is neither a loop nor a coloop. Then $\Lambda(M \backslash e, \mathbf{w}|_{E(M)-e}) \leq \Lambda(M, \mathbf{w})$ and $\Lambda(M/e, \mathbf{w}|_{E(M)-e}) \leq 2\Lambda(M, \mathbf{w})$.*

Proof. Immediate from Lemma 2.3. $\square$

Theorem 2.5 *Let $(M, \mathbf{w})$ be a weighted binary matroid and $e \in E(M)$. Then $d_m(e, M) \leq D(m) w_e \Lambda(M, \mathbf{w})^{m-1}$, where $D(1) = 1$ and $D(m) = \frac{1}{2}\prod_{i=0}^{m-2}(1 + 2^i)$ for $m \geq 2$.*

Proof. We use induction on m. Since $d_1(e, M) = w_e$, the theorem holds for $m = 1$. So suppose $m \geq 2$. If e is a loop or coloop of M, then $d_m(e, M) = 0$ for all $m \geq 2$. Hence we may suppose that e is not a loop or coloop of M. Let B be a basis for the cocycle space of M such that $\sum_{f \in K} w_f \leq \Lambda(M, \mathbf{w})$ for all $K \in B$. Choose $K_0 \in B$ with $e \in K_0$ and let $K_0 = \{e, e_1, \ldots, e_t\}$.

Suppose $m = 2$ and let $F = \{f \in E(M)\colon\ \{e, f\} \in \mathcal{C}(M)\}$. Then $d_2(e, M) = w_e \sum_{f \in F} w_f$. Since F is a subset of each cocycle of M which contains e, we have $\sum_{f \in F} w_f \leq \sum_{f \in K_0} w_f \leq \Lambda(M, \mathbf{w})$. Thus the theorem holds for $m = 2$ and we may assume that $m \geq 3$.

For each 2-connected deletion minor N of M with $e \in E(N)$, we have $|E(N) \cap K_0| \geq 2$ (since, if C is a circuit of N containing e, then C is a circuit of M and hence $|K_0 \cap C| \neq 1$ by Lemma 2.2). We shall classify such deletion minors N of M according to $p(N) := \min\{i\colon\ e_i \in E(N),\ 1 \leq i \leq t\}$. Let $\mathcal{D}^i = \{N \in \mathcal{D}_m(e, M)\colon\ p(N) = i\}$. Using Lemma 2.1, we may deduce that if $N \in \mathcal{D}^i$, then either $N - e_i \in \mathcal{D}_{m-1}(e)$ or $N/e_i \in \mathcal{D}_{m-1}(e)$ or both. Thus

$$d_m(e, M) \ \leq\ \sum_{i=1}^{t} w_{e_i}[d_{m-1}(e, M - e_i) + d_{m-1}(e, M/e_i)]\ . \tag{2}$$

The theorem now follows by applying Lemma 2.4 and induction, using the fact that $\sum_{e_i \in K_0} w_{e_i} \leq \Lambda(M, \mathbf{w})$. $\square$

If M is the cycle matroid of a graph, then it follows from a weighted version of Theorem 1.1, [3, Corollary 7.4], that the bound on $d_m(e, M)$ given in Theorem 2.5 can be reduced from $O(2^{m^2/2}\Lambda^m)$ to $O((2/\ln 2)^m \Lambda^m)$. It is an open problem to decide whether a similar strengthening holds for other families of binary matroids — e.g. cographic matroids, regular matroids, matroids for which the maxflow/mincut property holds — or even for all binary matroids. Some evidence in favour of this can be deduced from Theorem 2.8 below. We will need some further results on cocycle bases.

Lemma 2.6 *Let M be a binary matroid and e be an element of M. Let $B = \{K_1, K_2, \ldots, K_m\}$ be a basis for the cocycle space of $M\backslash e$.*

(a) If e is a coloop of M, then $B_1 = B \cup \{\{e\}\}$ is a basis for the cocycle space of M.

(b) If e is not a coloop of M, then there exists a basis $B_2 = \{K'_1, K'_2, \ldots, K'_m\}$ for the cocycle space of M such that $K_i \subseteq K'_i \subseteq K_i \cup \{e\}$ for all i, $1 \leq i \leq m$.

Proof. (a) Since e is a coloop of M, $\{e\}$ is a cocycle of M, and $r(M) = r(M\backslash e) + 1$. Hence the dimension of the cocycle space of M is $m+1$. It follows from the definition of $M\backslash e$ that either K_i or $K_i \cup \{e\}$ is a cocycle of M for all i, $1 \leq i \leq m$. However, if $K_i \cup \{e\}$ is a cocycle of M, then $(K_i \cup \{e\}) \oplus \{e\} = K_i$ is also a cocycle of M. Hence K_i is a cocycle of M for all i, $1 \leq i \leq m$. The linear independence of B_1 follows from the linear independence of B and the fact that $e \notin K_i$ for all i, $1 \leq i \leq m$.

(b) Since e is a not a coloop of M, $r(M) = r(M\backslash e)$, and hence the dimension of the cocycle space of M is m. It follows from the definition of $M\backslash e$ that either K_i or $K_i \cup \{e\}$ is a cocycle of M for all i, $1 \leq i \leq m$. Let K'_i be the cocycle of M with $K_i \subseteq K'_i \subseteq K_i \cup \{e\}$ and put $B_2 = \{K'_1, K'_2, \ldots, K'_m\}$. The linear independence of B_2 follows from the linear independence of B and the fact that $\{e\}$ is not a cocycle of M. □

Corollary 2.7 *Let M be a binary matroid and $S \subseteq M$. Let $B = \{K_1, K_2, \ldots, K_m\}$ be a basis for the cocycle space of $M\backslash S$. Then there exists a basis $B' = \{K'_1, K'_2, \ldots, K'_n\}$ for the cocycle space of M such that $K_i \subseteq K'_i \subseteq K_i \cup S$ for all i, $1 \leq i \leq m$, and $K'_i \subseteq S$ for all i, $m+1 \leq i \leq n$.*

Proof. This follows from Lemma 2.6 by induction on $|S|$. □

Let $(M, \mathbf{w})$ be a weighted binary matroid, m be a positive integer, and $S \subseteq E(M)$. Let $C_m(S) = \{C \in \mathcal{C}(M) \ : \ |C| = m, \ S \subseteq C\}$ and $c_m(S) = \sum_{C \in C_m(S)} \mathbf{w}(C)$.

Theorem 2.8 *Let $(M, \mathbf{w})$ be a weighted binary matroid, m be a positive integer, $S \subseteq E(M)$ and suppose $|S| = s \geq 1$. Let $\Lambda(M\backslash S, \mathbf{w}|_{E(M)-S}) = \Lambda$. Then*

$$\sum_{m=1}^{\infty} \Lambda^{-m+s} c_m(S) \ \leq \ \mathbf{w}(S). \tag{3}$$

Proof. We shall show that

$$\sum_{m=1}^{k} \Lambda^{-m+s} c_m(S) \ \leq \ \mathbf{w}(S) \tag{4}$$

for all $k \geq 1$. If $k < s$ then $c_m(S) = 0$ for all $1 \leq m \leq k$ and (4) holds trivially. Hence we may suppose that $k \geq s$. We sall proceed by induction on $k - s$. If $k = s$ then $c_m(S) = 0$ for all $1 \leq m < k$, $c_k(S) \leq \mathbf{w}(S)$ and again (4) holds. Hence suppose that $k > s$. Let B be a basis for the cocycle space of $M \backslash S$ such that $\sum_{e \in K} w_e \leq \Lambda$ for all $K \in B$. Let B' be a basis for the cocycle space of M obtained from B as in Corollary 2.7. Then

$$\sum_{e \in K-S} w_e \leq \Lambda \text{ for all } K \in B'. \tag{5}$$

Suppose $|S \cap K|$ is even for all $K \in B'$. Then $|S \cap K|$ is even for all cocycles K of M and hence S is a cycle of M. Thus $c_m(S) = 0$ if either $m \neq s$ or S is not a circuit of M, and $c_s(S) = \mathbf{w}(S)$ if S is a circuit of M. Thus (4) holds.

Hence we may assume that $|S \cap K_0|$ is odd for some $K_0 \in B'$. Let $K_0 = \{e_1, e_2, \ldots, e_n\}$. Choose $C \in C_m(S)$. Then $|C \cap K_0|$ is even. Since $|S \cap K_0|$ is odd, it follows that $|C \cap K_0| \not\subseteq S$. We shall classify the circuits $C \in C_m(S)$ according to $p(C) = \min\{i \;:\; e_i \in (C \cap K_0) - S\}$. Let $C^i = \{C \in C_m(S) \;:\; p(C) = i\}$. Note that $C^i \subseteq C_m(S \cup \{e_i\})$ for all $1 \leq i \leq n$. Using induction on $k - s$ we deduce that:

$$\begin{aligned}
\sum_{m=1}^{k} \Lambda^{-m+s} c_m(S) &\leq \sum_{m=1}^{k} \Lambda^{-m+s} \sum_{e_i \in K_0 - S} c_m(S + \cup\{e_i\}) \\
&= \Lambda^{-1} \sum_{e_i \in K_0 - S} \sum_{m=1}^{k} \Lambda^{-m+s+1} c_m(S + e_i) \\
&\leq \Lambda^{-1} \sum_{e_i \in K_0 - S} \mathbf{w}(S \cup \{e_i\}) \\
&= \mathbf{w}(S)\Lambda^{-1} \sum_{e_i \in K_0 - S} w_{e_i} \leq \mathbf{w}(S),
\end{aligned} \tag{6}$$

by (5) □

Theorem 2.8 has the following two corollaries for graphs. The special case when $|S| = 2$ of our first corollary is closely related to [3, Proposition 4.3].

Corollary 2.9 *Let $(G, \mathbf{w})$ be a weighted graph, $S \subseteq E(G)$ and suppose $|S| = s \geq 1$. Let $\Lambda(G - S, \mathbf{w}|_{E(M)-S}) = \Lambda$. Let $C_m(S)$ be the set of all circuits of G which have length m and contain S, and $c_m(S) = \sum_{C \in C_m(S)} \mathbf{w}(C)$. Then*

$$\sum_{m=1}^{\infty} \Lambda^{-m+s} c_m(S) \;\leq\; \mathbf{w}(S). \tag{7}$$

□

Let $(G, \mathbf{w})$ be a weighted graph and $\mathcal{B}^*(G)$ be the cycle space of G. Put

$$\Lambda^*(G, \mathbf{w}) \;=\; \min_{B \in \mathcal{B}^*(G)} \max_{C \in B} \sum_{e \in C} w_e \;. \tag{8}$$

cycle space of M. A *cocircuit* of G is an element of the cocycle space of G which is minimal with respect to inclusion i.e. a cocycle K such that $G - K$ has one more components than G

Corollary 2.10 *Let $(G, \mathbf{w})$ be a weighted graph, $S \subseteq E(G)$ and suppose $|S| = s \geq 1$. Let $\Lambda^*(G/S, \mathbf{w}|_{E(M)-S}) = \Lambda^*$. Let $K_m(S)$ be the set of all cocircuits of G which have length m and contain S, and $k^*_m(S) = \sum_{K \in K_m(S)} \mathbf{w}(K)$. Then*

$$\sum_{m=1}^{\infty} (\Lambda^*)^{-m+s} k_m(S) \;\leq\; \mathbf{w}(S). \tag{9}$$

□

Problem 2.11 *Does there exist a universal constant $\alpha < \infty$ such that if $(M, \mathbf{w})$ is a weighted binary matroid and $e \in E(M)$, then $d_m(e, M) \leq \mathbf{w}(e)(\alpha\Lambda(M, \mathbf{w}))^m$?*

Problem 2.11 has an affirmative answer for graphic matroids, with $\alpha = 2/\ln 2$, by [3, Proposition 7.6]. (We also show in [3, Examples 7.4,7.5] that it has a negative answer for graphic matroids if we take $\alpha < 1/\ln 2$.) We have not been able to solve Problem 2.11 for any other family of binary matroids. In particular, it is still open for cographic matroids.

3 Roots of Characteristic Polynomials

The *characteristic polynomial* $P_M(q)$ of a matroid M with rank function r is the polynomial in q defined by

$$P_M(q) = \sum_{A \subseteq E(M)} (-1)^{|A|} t^{r(E)-r(A)}.$$

When M is is the cycle matroid of a graph G, $q^{-1}P_M(q)$ is the chromatic polynomial of G. Similarly when M is is the cocycle matroid of G, $P_M(q)$ is the flow polynomial of G.

As mentioned in the Introduction, our principal motivation for studying $\Lambda(M)$, for a binary matroid M, is the problem of deciding whether it can be used to bound the roots of the characteristic polynomial of M:

Conjecture 3.1 *[2, Conjecture 41] There exist universal constants* $C(\Lambda) < \infty$ *such that the roots (real or complex) of the characteristic polynomial of every loopless binary matroid* M *with* $\Lambda(M) = \Lambda$*, all lie in the disc* $|q| \leq C(\Lambda)$*.*

An analogous theorem for the chromatic polynomial of a graph G using the maximum degree of G rather than $\Lambda(G)$ was proven in [7]: the approach taken there is to decompose a spanning subgraph of G into its connected components and to treat these components as a "polymer gas". The desired bound on chromatic roots then follows from standard bounds on the zeros of a polymer-gas partition function, once one has an exponential bound in terms of maximum degree on the number of connected m-edge subgraphs of G containing a specified vertex. An affirmative answer to the unweighted version of Problem 2.11 would be a first step in adapting the approach of [7] to verify Conjecture 3.1. Similarly an affirmative answer to Problem 2.11 could lead to bounds on the roots of the multivariate Tutte polynomial of M, as was the case for graphs in [7].

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