Oligomorphic permutation groups

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Abstract

A permutation group G (acting on a set Ω, usually infinite) is said to be *oligomorphic* if G has only finitely many orbits on Ω^n (the set of n-tuples of elements of Ω). Such groups have traditionally been linked with model theory and combinatorial enumeration; more recently their group-theoretic properties have been studied, and links with graded algebras, Ramsey theory, topological dynamics, and other areas have emerged.

This paper is a short summary of the subject, concentrating on the enumerative and algebraic aspects but with an account of group-theoretic properties. The first section gives an introduction to permutation groups and to some of the more specific topics we require, and the second describes the links to model theory and enumeration. We give a spread of examples, describe results on the growth rate of the counting functions, discuss a graded algebra associated with an oligomorphic group, and finally discuss group-theoretic properties such as simplicity, the small index property, and "almost-freeness".

1 Introduction

Despite the history and importance of group theory, we have very little idea what an arbitrary group looks like. We have made important strides in understanding finite groups, by determining the finite simple groups; but we can only study general groups under some very strong condition, usually a

finiteness condition. We have theories of finitely generated groups, locally finite groups, residually finite groups, groups of finite cohomological dimension, linear groups, profinite groups, and so forth, but no theory of general groups.

Oligomorphic groups satisfy a rather different kind of finiteness condition; paradoxically, one which makes them "large" rather than "small". A permutation group G (a subgroup of the symmetric group on a set Ω) is said to be *oligomorphic* if G has only finitely many orbits on Ω^n for every natural number n. (An element of G acts componentwise on the set Ω^n of all n-tuples of points of Ω.)

Thus, by definition, an oligomorphic group G gives us a sequence of natural numbers, the numbers of orbits on n-tuples for $n = 0, 1, 2, \ldots$. Not surprisingly, the theory is connected with counting problems in various parts of mathematics (combinatorics, model theory, graded algebras). Curiously, there is rather less we can say about the groups themselves. If G is oligomorphic and H is a proper subgroup having the same orbits on n-tuples as G for all n, then counting cannot distinguish between G and H, even though they may be very different as groups (for example, G may be simple while H is a free group).

In the remainder of this section, we introduce some basics of permutation group theory and of the counting functions associated with oligomorphic groups. For further information about permutation groups, see [7, 14]. Note also that there are many connections between parts of the theory of oligomorphic groups and that of (combinatorial) species, as developed by Joyal [21].

1.1 Permutation groups

This section is a brief introduction to permutation groups. For more details, see [7].

The *symmetric group* $\mathrm{Sym}(\Omega)$ on a set Ω is the group of all permutations of Ω. If Ω is finite, say $\Omega = \{1, 2, \ldots, n\}$, we write the symmetric group as S_n. We write permutations on the right, so that the image of α under g is written αg.

An *orbit* of G is an equivalence class of the relation $\alpha \sim \beta$ if $\alpha g = \beta$ for some $g \in G$; in other words, a set of the form $\{\alpha g : g \in G\}$. We say that G is *transitive* if it has only one orbit. In a sense, any permutation group can be "resolved" into transitive groups.

There is a partial converse. If G_i is a transitive permutation group on Ω_i

for $i \in I$, where I is some index set, the *cartesian product* $\prod_{i \in I} G_i$ is the set of functions $f : I \to \bigcup_{i \in I} G_i$ satisfying $f(i) \in G_i$ for all $i \in I$. It has two natural actions:

- The *intransitive action* on the disjoint union of the sets Ω_i: if $\alpha \in \Omega_i$, then $\alpha f = \alpha f(i)$. If each group G_i is transitive, then the sets Ω_i are the orbits of the cartesian product.

- There is also a *product action*, componentwise on the cartesian product of the sets Ω_i.

If I is finite, we speak of the *direct product*, and write it as (for example) $G_1 \times \cdots \times G_k$, if $I = \{1, \ldots, k\}$.

Thus, if $G_1 = S_n$ and $G_2 = S_m$, then $G_1 \times G_2$ has an intransitive action on $m+n$ points, and a product action on mn points (which can be regarded as a rectangular grid with G_1 permuting the rows and G_2 the columns).

Now let G be transitive on Ω. A *congruence* is an equivalence relation on Ω which is G-invariant. There are two "trivial" congruences on Ω: the relation of equality, and the "universal" relation with a single equivalence class. We say that G is *primitive* if there are no other congruences. For example, the symmetric group is primitive.

Important examples of imprimitive groups are the *wreath products*, defined as follows. Let H be a permutation group on Γ, and K a permutation group on Δ. Let $\Omega = \Gamma \times \Delta$, regarded as a union of copies of Γ indexed by Δ: thus $\Gamma_\delta = \{(\gamma, \delta) : \gamma \in \Gamma\}$, for each $\delta \in \Delta$. Take a copy H_δ of H for each $\delta \in \Delta$, where H_δ acts on Γ_δ. Then the cartesian product $B = \prod_{\delta \in \Delta} H_\delta$ acts on Ω (in its intransitive action). Moreover, the group K acts on Ω by permuting the second components (i.e. by permuting the "fibres" Γ_δ. The *wreath product* $G = H \operatorname{Wr} K$ is the group generated by B and K; we call B the *base group* (it is a normal subgroup) and K the *top group* of the wreath product.

We note that there are different notations and conventions in other areas of mathematics. For example, in experimental design in statistics [2], direct product (product action) is called *crossing*, and wreath product is called *nesting*, but nesting is written with the top structure before the bottom one, e.g. Δ/Γ in our case.

If $|\Gamma| > 1$ and $|\Delta| > 1$, the wreath product is imprimitive: the relation $(\gamma, \delta) \equiv (\gamma', \delta')$ if $\delta = \delta'$ is a congruence. For this reason it is called the

imprimitive action. Any imprimitive permutation group can be embedded in a wreath product in a natural way.

There is another action of the wreath product, the *power action*, on the set Γ^Δ of functions from Δ to Γ. It bears a similar relation to the imprimitive action as the product action does to the intransitive action for the cartesian product. If we regard the domain of the imprimitive action as a fibred space, with fibres Γ_δ isomorphic to Γ indexed by Δ, then the domain for the power action is the set of global sections (subsets containing one point from each fibre).

1.2 Oligomorphic permutation groups

For a natural number n, a permutation group is *n-transitive* if it acts transitively on the set of n-tuples of distinct elements of Ω, and *n-set-transitive* if it acts transitively on the set of n-element subsets of Ω. We say a permutation group is *highly transitive* if it is n-transitive for all n, and *highly set-transitive* if it is n-set-transitive for all n.

Oligomorphic permutation groups generalise these classes. Thus, we let $F_n(G)$ denote the number of orbits of G on the set of n-tuples of distinct elements, and $f_n(G)$ the number of orbits on n-element subsets. So G is n-transitive (resp. n-set-transitive) if $F_n(G) = 1$ (resp. $f_n(G) = 1$). If the group G is clear, we drop it and write simply F_n, f_n.

The definition speaks of orbits on Ω^n, the set of all n-tuples. Let $F_n^*(G)$ denote the number of these orbits. It is clear that, for given n, one of these numbers is finite if and only if the others are; indeed, we have

- $F_n^* = \sum_{k=1}^{n} S(n,k)F_k$, where $S(n,k)$ is the Stirling number of the second kind (the number of partitions of an n-set into k parts);
- $f_n \le F_n \le n!\, f_n$.

As an example for the first point, if G is highly transitive, then

$$F_n^*(G) = \sum_{k=0}^{n} S(n,k) = B(n),$$

the nth *Bell number* (the number of partitions of an n-set).

In the second point, both bounds are attained:

- Let $G = \mathrm{Sym}(\Omega)$ (we will always denote this group by S). Then $f_n(S) = F_n(S) = 1$ for all n.

- Let G be the group of order-preserving permutations of the rational numbers (we will always denote this group by A). We can map any n rational numbers in increasing order to any other n such by an order-preserving permutation (by filling in the gaps to produce a piecewise-linear map); so $f_n(A) = 1$ and $F_n(A) = n!$.

Highly set-transitive groups must resemble the above types. All we can do is to modify the total order slightly. The following is proved in [3].

Theorem 1.1 *Let G be a highly set-transitive but not highly transitive permutation group. Then there is a linear or circular order preserved by G. In particular, G is not 4-transitive.*

1.3 Topology

We will only consider permutation groups of countable degree in this article. If we are interested in the sequences f_n and F_n, this is justified by the following result, which follows from the *Downward Löwenheim–Skolem Theorem* of model theory:

Proposition 1.2 *Let G be an oligomorphic permutation group on an infinite set. Then there is an oligomorphic permutation group G' on a countably infinite set such that $F_n(G') = F_n(G)$ and $f_n(G') = f_n(G)$ for all $n \in \mathbb{N}$.*

There is a natural topology on the symmetric group of countable degree. This is the topology of pointwise convergence, where a sequence (g_n) converges to g if $\alpha_i g_n = \alpha_i g$ for all sufficiently large n, where $(\alpha_1, \alpha_2, \ldots)$ is an enumeration of Ω. This topology can be derived from a complete metric. The topology is specified by the first part of the proposition below.

Proposition 1.3 *(a) G is open in $\mathrm{Sym}(\Omega)$ if and only if it contains the pointwise stabiliser of a finite set.*

(b) G is closed in Ω if and only if G is the automorphism group of a relational structure on Ω, that is, a family of relations (of various arities) on Ω.

The closure of a permutation group G consists of all permutations which preserve the G-orbits on Ω^n for all n. We remarked earlier that, as far as orbit-counting goes, we cannot distinguish between groups with the same orbits. The largest such group is necessarily closed in $\mathrm{Sym}(\Omega)$. So we may restrict our attention to closed groups of countable degree (that is, automorphism groups of countable relational structures) if we are interested in orbit-counting.

The fact that the topology on the symmetric group is derived from a complete metric means that the same is true for any closed subgroup. This permits use of the *Baire category theorem*. Recall that a subset of a complete metric space is *residual* if it contains the intersection of countably many open dense subsets. The Baire category theorem asserts that a residual set is non-empty. Indeed, residual sets are "large" (for example, they have non-empty intersection with any open set, and the intersection of countably many residual sets is residual). Often it is possible to give a non-constructive existence proof for some object by showing that objects of the required type form a residual set.

1.4 Cycle index

An important tool in combinatorial enumeration is the *cycle index* of a finite permutation group G, which is the polynomial in the indeterminates $s_1, s_2, \ldots, s_n$ (where n is the degree) given by

$$Z(G; s_1, s_2, \ldots, s_n) = \frac{1}{|G|} \sum_{g \in G} s_1^{c_1(g)} s_2^{c_2(g)} \cdots s_n^{c_n(g)},$$

where $c_i(g)$ is the number of i-cycles in the decomposition of the permutation g into disjoint cycles.

For its role in enumeration, see [18].

One cannot simply take this definition unchanged for infinite permutation groups, for several reasons: the number $c_i(g)$ may be infinite; there may be infinite cycles; and the denominator $|G|$ is infinite.

The trick to generalising it lies in the following result about finite groups, the *Shift Lemma*. Let $\mathcal{P}\Omega$ be the set of all subsets of the finite set Ω, and let $\mathcal{P}\Omega/G$ denote a set of representatives of the G-orbits on $\mathcal{P}\Omega$. Also, if X is any subset of Ω, we let $G[X]$ denote the permutation group on X induced by its setwise stabiliser in G.

Proposition 1.4 *For a finite permutation group G on Ω,*

$$\sum_{X \in \mathcal{P}\Omega/G} Z(G[X]; s_1, s_2, \ldots) = Z(G; s_1 + 1, s_2 + 1, \ldots).$$

Now let G be any oligomorphic permutation group on the (possibly infinite) set Ω. We define the *modified cycle index* of G to be the left-hand side of the Shift Lemma, with one small modification: we replace $\mathcal{P}\Omega$ by $\mathcal{P}_{\mathrm{fin}}\Omega$, the set of all finite subsets of Ω. Thus, the modified cycle index is

$$\tilde{Z}(G; s_1, s_2, \ldots) = \sum_{X \in \mathcal{P}_{\mathrm{fin}}\Omega/G} Z(G[X]; s_1, s_2, \ldots).$$

Each term in the sum is the cycle index of a finite permutation group. Also since G is oligomorphic, there are only a finite number of terms in the sum which contribute to the coefficient of a fixed monomial $s_1^{c_1} \cdots s_r^{c_r}$, namely those corresponding to the $f_n(G)$ orbits on sets of cardinality $n = c_1 + 2c_2 + \cdots + rc_r$. (We see here that the definition would fail if G were not oligomorphic.) So we have defined a formal power series in $s_1, s_2, \ldots$. We also see that if G happens to be a finite permutation group, then we have the ordinary cycle index with 1 added to each indeterminate.

Our previous counting functions can be extracted from the modified cycle index:

Proposition 1.5 *Let G be an oligomorphic permutation group.*

- *$F_G(x)$ is obtained from $\tilde{Z}(G)$ by substituting $s_1 = x$ and $s_i = 0$ for $i > 1$.*
- *$f_G(x)$ is obtained from $\tilde{Z}(G)$ by substituting $s_i = x^i$ for all i.*

Just as before, it is true that

- for any oligomorphic permutation group G, there is an oligomorphic group G' of countable degree satisfying $\tilde{Z}(G) = \tilde{Z}(G')$;
- an oligomorphic group of countable degree and its closure have the same modified cycle index.

So we may consider closed groups of countable degree.

We conclude this section by displaying the modified cycle index for the groups S (the infinite symmetric group) and A (the group of order-preserving permutations of $\mathbb{Q}$).

Proposition 1.6

$$\begin{aligned}\tilde{Z}(S) &= \exp\left(\sum_{i\geq 1}\frac{s_i}{i}\right);\\ \tilde{Z}(A) &= \frac{1}{1-s_1}.\end{aligned}$$

2 Connections

Oligomorphic permutation groups are closely connected with two other areas of mathematics: model theory, and combinatorial enumeration. In this section we discuss the connections.

2.1 Model theory

Model theory describes structures consisting of a set with a collection of constants, relations and functions. Much of mathematics can be fitted into this framework, often in different ways. For example, a group has a binary operation (composition), a unary operation (inversion), and a constant (the identity); but the second and third may be defined in terms of the first. A graph can be regarded as a set of vertices with a binary relation (adjacency), or as a set of vertices and edges with a unary relation (to distinguish the vertices) and a binary operation of incidence. The latter is appropriate for multigraphs.

Logic describes such structures by means of collections of formulae. The language includes symbols for the relations, functions, and constants, connectives and quantifiers, equality, brackets, and a supply of variables. We work in *first-order logic*: we are allowed to combine finitely many formulae with connectives, and to quantify over variables which range over the underlying set. I will assume that the language is countable. A *sentence* is a formula with no free variables (all variables are quantified). Thus for example the group axioms can be expressed as a single sentence (the conjunction of the associative, identity and inverse laws, each universally quantified over all variables).

A *theory* is a set of sentences; a structure is a *model* for a theory if every sentence in the theory is true in the model.

Sometimes we want a theory to have a wide range of models (this is the case with group theory). At other times, we have a particular model in mind, and want to capture as much of its essence as possible in a theory (this is the case with the Peano axioms for the natural numbers). It is known that, as long as a theory has infinite models, it cannot have a unique model; there will be models of arbitrarily large cardinality. The best we can do is ask that the theory is *α-categorical*, where α is an infinite cardinal, meaning that there is a unique model of cardinality α up to isomorphism. By a theorem of Vaught, there are only two types of categoricity, countable and uncountable: if a theory is α-categorical for some uncountable cardinal α, then it is α-categorical for all such.

Uncountable categoricity gives rise to a powerful structure theory, extending that for vector spaces over a fixed field or algebraically closed fields of fixed characteristic (where a single invariant, the rank, determines the model). By contrast, countable categoricity is related to symmetry, by the following remarkable theorem due independently to Engeler, Ryll-Nardzewski and Svenonius:

Theorem 2.1 *Let M be a countable model of a theory T over a countable language. Then T is $\aleph_0$-categorical if and only if the automorphism group of M is oligomorphic.*

In fact, if M is the unique countable model of T, and $G = \mathrm{Aut}(M)$, then $F_n^*(T)$ is the number of n-types over T (maximal consistent sets of formulae in n free variables). So our results are applicable to counting types in such theories.

Note in passing that the automorphism group of $\mathbb{N}$ is trivial; so the Peano axioms have countable "non-standard" models.

2.2 Combinatorial enumeration

The set-up is similar, but we restrict ourselves to relational structures (no function or constant symbols). We will see that the counting problems for orbits on sets and tuples of oligomorphic permutation groups are identical with those for unlabelled and labelled structures in so-called *oligomorphic Fraïssé classes* of relational structures. These include large numbers of combinatorially important classes of structures, so we have a rather general paradigm for interesting counting problems.

Take a fixed relational language $\mathbf{L}$ (this means, each set carries named relations of prescribed arities). For example, we could consider graphs, directed graphs, tournaments, partially ordered sets, k-edge-coloured graphs, graphs with a fixed bipartition, or much more recondite examples.

A *Fraïssé class* over $\mathbf{L}$ is a class $\mathcal{C}$ of finite relational structures over $\mathbf{L}$ satisfying the following four conditions:

(a) $\mathcal{C}$ is closed under isomorphism;

(b) $\mathcal{C}$ is closed under taking induced substructures (this means, take a subset of the domain, and all instances of all relations which are contained within this subset);

(c) $\mathcal{C}$ has only countably many members up to isomorphism;

(d) $\mathcal{C}$ has the *amalgamation property*: this means that, if $B_1, B_2 \in \mathcal{C}$ have isomorphic substructures, they can be "glued together" along these substructures (or possibly more) to form a structure in $\mathcal{C}$. Formally, if $A, B_1, B_2 \in \mathcal{C}$ and $f_i : A \to B_i$ is an embedding for $i = 1, 2$, then there is a structure $C \in \mathcal{C}$ and embeddings $h_i : B_i \to C$ for $i = 1, 2$ so that $h_1 f_1 = h_2 f_2$.

(The reader is warned that I allow the structure A to be empty here. Some authors exclude this, state the special case where $A = \emptyset$ separately, and call it the *joint embedding property.*)

For a simple example, finite graphs form a Fraïssé class: take the union of B_1 and B_2 identifying the isomorphic substructures, and put any or no edges between $B_1 \setminus A$ and $B_2 \setminus A$. It is just a little more difficult to show that finite total or partial orders form a Fraïssé class.

Let M be a structure over $\mathbf{L}$. We make two definitions:

- The *age* of M is the class of all finite $\mathbf{L}$-structures which are embeddable in M.
- M is *homogeneous* if any isomorphism between finite induced substructures of M can be extended to an automorphism of M.

For example, the totally ordered set $\mathbb{Q}$ is homogeneous, and its age consists of all finite totally ordered sets.

Fraïssé's Theorem asserts:

Theorem 2.2 *A class $\mathcal{C}$ of finite $\mathbf{L}$-structures is the age of a countable homogeneous $\mathbf{L}$-structure M if and only if it is a Fraïssé class. If these conditions hold, then M is unique up to isomorphism.*

We call the countable homogeneous structure M the *Fraïssé limit* of the class $\mathcal{C}$. Thus, $(\mathbb{Q}, <)$ is the Fraïssé limit of the class of finite totally ordered sets. The Fraïssé limit of the class of finite graphs is the celebrated *random graph*, or *Rado graph* R, which is extensively discussed in [4].

Now suppose that a Fraïssé class satisfies the following stronger version of condition (c):

(c′) For any $n \in \mathbb{N}$, the class $\mathcal{C}$ has only finitely many n-element members up to isomorphism.

We call such a class an *oligomorphic Fraïssé class*. All examples mentioned so far are oligomorphic; the condition certainly holds if $\mathbf{L}$ contains only finitely many relations. Let M be its Fraïssé limit, and let $G = \mathrm{Aut}(M)$ be the automorphism group of M. By homogeneity, it follows that

- $F_n(G)$ is the number of labelled n-element structures in $\mathcal{C}$ (that is, structures on the set $\{1, 2, \ldots, n\}$;
- $f_n(G)$ is the number of unlabelled n-element structures in $\mathcal{C}$ (that is, isomorphism classes of n-element structures).

Some Fraïssé classes satisfy a stronger version of the amalgamation property, called *strong amalgamation*. This is said to hold if the amalgam of any two structures can be produced without identifying any points not in the common substructure. More formally, if $h_1(b_1) = h_2(b_2)$, then $b_1 = f_1(a)$ and $b_2 = f_2(a)$ for some $a \in A$.

This holds in the above examples. An example where it fails is given by the class of graphs consisting of isolated vertices and edges. If B_1 and B_2 are edges and A a single vertex, then in the amalgam we are forced to identify the other ends of the two edges as well.

Proposition 2.3 *A Fraïssé class has the strong amalgamation property if and only if the automorphism group of its Fraïssé limit has the property that the stabiliser of any finite set of points fixes no additional points.*

Later we will see a still stronger version.

3 Constructions

There are two main sources for examples of (closed) oligomorphic groups: building new examples from old, or constructing Fraïssé classes (the group is then the automorphism group of the Fraïssé limit of the class).

3.1 Direct and wreath products

Suppose that G_1 and G_2 are oligomorphic permutation groups on Ω_1 and Ω_2 respectively. We can form their direct or wreath product, and each of these has two actions, which we now discuss.

Direct product, intransitive action The direct product $G = G_1 \times G_2$ acts on the disjoint union of Ω_1 and Ω_2, say Ω. An n-subset of Ω consists of k points of Ω_1 and $n-k$ points of Ω_2, for some k with $0 \le k \le n$; two n-sets are in the same G-orbit if and only if their intersections with Ω_i are in the same G_i-orbit for $i = 1, 2$. Thus $(f_n(G))$ is the convolution of $(f_n(G_1))$ and $(f_n(G_2))$:

$$f_n(G) = \sum_{k=0}^{n} f_k(G_1) f_{n-k}(G_2),$$

from which it follows that the generating functions multiply:

$$f_G(x) = f_{G_1}(x) f_{G_2}(x).$$

Similarly, for F_n, we have an exponential convolution, so that the exponential generating functions also multiply:

$$F_G(x) = F_{G_1}(x) F_{G_2}(x).$$

The modified cycle index is also multiplicative:

$$\tilde{Z}(G) = \tilde{Z}(G_1)\tilde{Z}(G_2).$$

Direct product, product action The direct product $G_1 \times G_2$ also has a product action on the cartesian product $\Omega_1 \times \Omega_2$. This is more difficult to analyse; the recent paper [9] describes the situation.

First, there is a multiplicative property: it is easy to see that

$$F_n^*(G) = F_n^*(G_1) F_n^*(G_2).$$

The modified cycle index of the product can be computed as follows. Define an operation on the indeterminates by

$$s_i \bullet s_j = \left(s_{\mathrm{lcm}(i,j)}\right)^{\gcd(i,j)}.$$

Extend this operation multiplicatively to monomials and then additively to sums of monomials. Then we have

$$\tilde{Z}(G) = \tilde{Z}(G_1) \bullet \tilde{Z}(G_2).$$

It should clearly be possible to deduce the first of these relations from the second; but this is surprisingly difficult (see the cited paper).

Here is an example. Consider first the group $G = A$, the order-preserving permutations of $\mathbb{Q}$. We have $F_n(G) = n!$, and hence

$$F_n^*(G) = \sum_{k=1}^{n} S(n,k)k! = P(n),$$

the number of *preorders* of an n-set (reflexive and transitive relations).

Now let $G = A \times A$ with the product action. We have $F_n^*(G) = P(n)^2$, and so

$$F_n(G) = \sum_{k=1}^{n} s(n,k)P(k)^2.$$

Moreover, since G (like A) has the property that the setwise stabiliser of an n-set fixes it pointwise, we have

$$f_n(G) = F_n(G)/n!.$$

Now the exponential generating function for $P(n)$ is $1/(2 - \mathrm{e}^x)$, with the nearest singularity to the origin at $\log 2$; so $P(n)$ is roughly $n!/(\log 2)^n$. So $F_n(G)$ is about $(n!)^2/(\log 2)^{2n}$. Since the Stirling numbers alternate in sign, it is not completely trivial to find the asymptotics of $f_n(G)$. This was achieved in [10], using three entirely different methods; it turns out that

$$f_n(G) \sim \frac{n!}{4}\mathrm{e}^{-\frac{1}{2}(\log 2)^2}\frac{1}{(\log 2)^{2n+2}}.$$

In broad-brush terms, "factorial times exponential".

Wreath product, imprimitive action The wreath product $G_1 \mathrm{Wr} G_2$ has its imprimitive action on $\Omega_1 \times \Omega_2$; as we have seen, this should be thought of as a covering of Ω_2 with fibres isomorphic to Ω_1. The function $F_G(x)$ is found by substitution:

$$F_G(x) = F_{G_2}(F_{G_1}(x) - 1).$$

In particular, for any oligomorphic group G, we have

$$F_{S\mathrm{Wr}G}(x) = F_G(\mathrm{e}^x - 1) = F^*_G(x),$$

so that $F_n(S \,\mathrm{Wr}\, G) = F^*_n(G)$ for all n, as is easily seen directly.

For another example, we note that $F_{G\mathrm{Wr}S}(x) = \exp(F_G(x)-1)$. Thus, the substitution rule for the wreath product can be regarded as the prototype of a wide generalisation of the *exponential principle* in combinatorics [33, 16]. If G is the automorphism group of the Fraïssé limit of a Fraïssé class $\mathcal{C}$, then $G \,\mathrm{Wr}\, S$ is similarly associated with the class of disjoint unions of $\mathcal{C}$-structures; the exponential principle applies to this situation. In general, if H is associated with a Fraïssé class $\mathcal{D}$, then $G \,\mathrm{Wr}\, H$ is associated with the class of disjoint unions of $\mathcal{C}$-structures with a $\mathcal{D}$-structure on the set of parts. For example, if $H = A$, then we have ordered sequences of $\mathcal{C}$-structures; if H is the automorphism group of the random graph, we have a graph whose vertices are $\mathcal{C}$-structures; and so on.

The numbers $f_n(G)$ of orbits on unordered sets cannot be obtained from the sequences $(f_n(G_1))$ and $f_n(G_2))$ alone; we need the modified cycle index of the top group G_2. The generating function $f_G(x)$ is obtained from $\tilde{Z}(G_2)$ by substituting $f_{G_1}(x^i) - 1$ for the variable s_i, for $i = 1, 2, \ldots$.

Wreath product, power action The wreath product also has a power action on $\Omega_1^{\Omega_2}$, the set of functions from Ω_2 to Ω_1 (the set of transversals to the fibres in the imprimitive action). This action is not in general oligomorphic; it is so if G_2 is a finite group.

It is shown in [9] that $F^*_n(G)$ is obtained from the (ordinary) cycle index of the finite group G_2 by substituting $F^*_n(G_1)$ for all of the variables s_i.

3.2 Other examples

Further examples are most easily described as automorphism groups of relational structures. As we have seen, any closed oligomorphic group of countable degree is the automorphism group of a homogeneous relational structure,

which is the Fraïssé limit of a Fraïssé class of finite structures. Sometimes the easiest way to specify the group is to give the Fraïssé class.

The random graph and some of its relations. Let R be the countable random graph. This graph is the Fraïssé limit of the class of finite graphs. There are direct constructions for it, see [4]: for example, the vertices are the primes congruent to 1 mod 4, and p and q are joined if and only if p is a quadratic residue mod q. Alternatively, as Erdős and Rényi showed, if we form a countable random graph by choosing edges independently with probability 1/2 from the 2-subsets of a countable set, the resulting graph is isomorphic to R with probability 1.

The graph R is homogeneous, and contains all finite (and indeed all countable) graphs as induced subgraphs. So, if $G = \mathrm{Aut}(R)$, then $F_n(G)$ is the number of labelled graphs on n vertices (which is $2^{n(n-1)/2}$), while $f_n(G)$ is the number of unlabelled graphs on n vertices (which is asymptotically $F_n(G)/n!$). Note that these sequences grow so fast that the generating functions converge only at the origin.

There are several related Fraïssé classes. Here are a few examples.

- In place of graphs, we can take directed graphs, or oriented graphs, or tournaments, or k-uniform hypergraphs. Note that the automorphism group of the random k-uniform hypergraph is $(k-1)$-transitive but not k-transitive; so every degree of transitivity is possible for infinite permutation groups.

- Thomas [34] showed that there are only five closed supergroups of $\mathrm{Aut}(R)$, namely

  - $\mathrm{Aut}(R)$;
  - The group $D(R)$ of dualities (automorphisms and anti-automorphisms) of R, which is 2-transitive and contains $\mathrm{Aut}(R)$ as a normal subgroup of index 2;
  - The group $S(R)$ of switching automorphisms of R (see below), which is 2-transitive;
  - The group $B(R)$ of switching automorphisms and anti-automorphisms of R, which is 3-transitive and contains $S(R)$ as a normal subgroup of index 2;

– the symmetric group.

- Lachlan and Woodrow [24] determined all the countable homogeneous graphs. There are some trivial ones (disjoint unions of complete graphs of the same size, and their complements); some non-trivial ones, the Henson graphs and their complements; and the random graph. The *Henson graph* H_n is the Fraïssé limit of the class of all finite graphs which contain no complete graph of order n, for $n \geq 3$. They are not very well understood.

The operation σ_X of *switching* a graph with respect to a set X of vertices consists in replacing edges between X and its complement by non-edges, and non-edges by edges, while keeping things inside or outside X the same as before. A permutation g is a *switching automorphism* of Γ if $\Gamma^g = \sigma_X(\Gamma)$ for some set X. Switching anti-automorphisms are defined similarly.

One curious fact is that $F_n(S(R))$ and $f_n(S(R))$ are equal to the numbers of labelled and unlabelled *even graphs* on n vertices (graphs with all vertex degrees even), although the even graphs do not form a Fraïssé class.

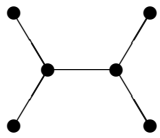
Ordered structures An old theorem of Skolem says that, if two subsets X and Y of $\mathbb{Q}$ are dense and have dense complements, then there is an order-preserving permutation of $\mathbb{Q}$ carrying X to Y. More generally, consider colourings of $\mathbb{Q}$ with m colours so that each colour class is dense. It is not hard to show that there is a unique such colouring up to order-preserving permutations. The automorpism group A_m of such a colouring (the group of permutations preserving the order and the colours) is oligomorphic, and $f_n(A_m) = m^n$. For, if $\{c_1, \ldots, c_m\}$ is the set of colours, then an n-set $\{q_1, \ldots, q_n\}$, with $q_1 < \cdots < q_m$, can be described by a word of length n in the alphabet $\{c_1, \ldots, c_m\}$, whose ith letter is the colour of q_i; two sets lie in the same orbit if and only if they are coded by the same word, and every word arises as the code of some subset.

We can modify this example in the same way we did for $\mathbb{Q}$ itself, allowing ourselves to preserve or reverse the order, or turning it into a circular order.

Treelike structures There are vast numbers of treelike structures; I cannot give even a brief overview, so I will concentrate on a couple of examples.

Consider the class of *boron trees*, that is, finite trees in which each vertex has degree 1 or 3. Consider any four leaves in such a tree. They are connected

by a graph of the following shape:



We see that one of the three partitions of the four leaves into two sets of two is distinguished by the fact that the paths joining vertices in the same pair do not intersect. Thus, there is a quaternary relation on the set of leaves. It is possible to show that this relation uniquely determines the boron tree, and that the class of finite structures carrying quaternary relations which arise in this way is a Fraïssé class. So there is a corresponding oligomorphic group G, which is 3-transitive but not 4-transitive, and satisfies $f_n(G) \sim An^{-5/2}c^n$, where $c = 2.483\ldots$.

This construction can be modified in many ways, of which a few are given below.

- We may consider trees with degrees $1, 3, 4, \ldots, m$, or with any possible degree except 2. (Divalent vertices are invisible in this construction.)
- We may consider internal vertices as well as leaves: there will be a ternary betweenness relation saying that one vertex is on the path joining the other two.
- By embedding the trees in the plane, we may impose a circular order on the set of leaves.

Many of these constructions give examples with exponential growth (roughly c^n for some $c > 1$).

Other examples The symmetric group S on a countable set Ω has an induced action on the set of k-element subsets of Ω, for any k. These groups are oligomorphic, but except in the case $k = 2$, not much is known about the asymptotics of the orbit-counting sequences (see [11] for the case $k = 2$).

Other examples are linear and affine groups on infinite-dimensional vector spaces over finite fields. See [12] for the asymptotics.

4 Growth rates

In this section we survey some known results about the rate of growth of the sequences $(f_n(G))$ and $(F_n(G))$ for an oligomorphic group G.

First, note that there is no upper bound for growth rates. Given any sequence (a_n) of natural numbers, let $\mathbf{L}$ be a relational language containing a_n n-ary relations $\mathcal{C}$ be the Fraïssé class consisting of all structures in which an n-ary relation holds only if all its arguments are distinct. If G is the automorphism group of the Fraïssé limit, then $f_n(G) \geq 2^{a_n}$ for all n. Indeed, it is slow growth which is most interesting!

For intransitive or imprimitive groups, we can have polynomial growth for (f_n). For a simple example, if G is the direct product of r symmetric groups (with the intransitive action), then $f_n(G) = \binom{n+r-1}{r-1}$, with generating function $f_G(x) = (1-x)^{-r}$.

For the same group, $F_n(G) = r^n$, since the orbit of an n-tuple is determined by the orbit containing each of its points. Recently it has been shown [1] that, if G is transitive, and the point stabiliser has m orbits on the remaining points, then $F_n(G) \geq m^{n-1}$. Equality is possible here for any m.

There is a gap above polynomial growth for (f_n). The next possible growth rate is fractional exponential, about $\exp(\sqrt{n})$. This is realised by the group $G = S \operatorname{Wr} S$, for which we have $f_n(G) = p(n)$, the number of partitions of the integer n, with growth asymptotically

$$\frac{1}{4n\sqrt{3}} \exp(\pi\sqrt{2n/3});$$

and $F_n(G) = B(n)$, the number of partitions of an n-set (the Bell number), whose growth is faster than exponential but slower than factorial.

However, if we insist that the group is primitive, there is a dramatic change [25, 29]:

Theorem 4.1 *There is a number $c > 1$ such that the following holds. Suppose that G is primitive but not highly set-transitive. Then*

- *$f_n(G) \geq c^n/q(n)$, for some polynomial q;*
- *$F_n(G) \geq n!\, c^n/q(n)$, for some polynomial q.*

In other words, for (f_n), there is a gap between constant and exponential growth! Merola's proof gives the result with $c = 1.324\ldots$; the best-known examples have $c = 2$ (these are circular versions of the rationals partitioned into two dense subsets, that we met earlier).

The main problems for exponential growth are:

- prove that the "exponential constant" $\lim_{n\to\infty}(f_n)^{1/n}$ always exists;
- find the possible values it can take;
- find a structural description of the examples where the growth is no faster than exponential.

What happens just above exponential growth? Here are some examples.

- Let $\mathcal{C}$ be the Fraïssé class each of whose members is a set with a pair of total orders, and G the automorphism group of its Fraïssé limit. Then $f_n(G) = n!$ and $F_n(G) = (n!)^2$.
- The group $S \operatorname{Wr} S_2$ with the power action is primitive. I do not know what the asymptotic behaviour of $(f_n(G))$ is. As explained earlier, we find that
$$F_n(G) \sim \tfrac{1}{2}B(n)(B(n)+1),$$
where $B(n)$ is the Bell number.
- For the permutation group induced by S on the set of 2-element subsets of the domain, we have
$$F_n(G) \sim B(2n)2^{-n}n^{-1/2}\exp\left(-[\tfrac{1}{2}\log(2n/\log n)]^2\right).$$

No clear evidence of a gap emerges from this limited data. Macpherson [26] has proved some theorems connecting growth just faster than exponential with model-theoretic properties such as stability and the strict order property.

For the automorphism group of the random graph, the growth rate is about $\exp(cn^2)$. For such growth, it doesn't make a lot of difference whether we consider F_n or f_n. We make one brief observation about this case.

Proposition 4.2 *Let G be the automorphism group of a homogeneous structure over a finite relational language. Then $F_n(G)$ is bounded above by the exponential of a polynomial.*

The converse of this is not true. For example, the general linear group on a vector space of countable dimension over the field of two elements has $F_n(G)$ roughly $2^{n^2/16}$, but is not the automorphism group of a homogeneous structure over a finite relational language. (Take two n-tuples of vectors, the first linearly independent and the second satisfying the single linear relation that the sum of the vectors is zero. These two n-tuples cannot be distinguished by relations of arity less than n.)

An open problem here is to understand what the implications for G are of being the automorphism group of a homogeneous structure over a finite relational language.

5 Graded algebras

Another part of mathematics where sequences of positive integers occur is the theory of *graded algebras*. Such an algebra is a direct sum

$$A = \bigoplus_{n\in\mathbb{N}} A_n,$$

where the A_n are vector spaces over a field F, and, if $v \in A_m$ and $w \in A_m$, then the algebra product vw belongs to A_{m+n}. The subspaces A_n are the *homogeneous components*. If they are all finite-dimensional, then the sequence of their dimensions (or its ordinary generating function) is the *Hilbert series* of the algebra. If a graded algebra is finitely generated, then the dimensions grow no faster than a polynomial in n, and so the Hilbert series converges inside the unit circle.

We construct graded algebras as follows. First, let Ω be an infinite set, and F a field (for our purposes, always of characteristic zero). Let A_n be the vector space of functions from $\binom{\Omega}{n}$ (the set of n-element subsets of Ω) to F, and define a multiplication on the homogeneous components by the rule

$$fg(X) = \sum_{Y\in\binom{X}{n}} f(Y)g(X \setminus Y)$$

for $f \in A_n$, $g \in A_m$, and $X \in \binom{\Omega}{n+m}$. Extended linearly to $A = \bigoplus A_n$, this multiplication is commutative and associative, and makes A a graded algebra. The constant function 1 on $\binom{\Omega}{0} = \{\emptyset\}$ is the identity. (In fact A_0 is 1-dimensional, spanned by the identity.) This algebra is the *reduced*

incidence algebra of the poset of finite subsets of Ω, but we make no use of this fact.

The algebra A is much too large: its homogeneous components have infinite dimension (except for A_0 which has dimension 1), and there are many nilpotent elements.

But it has one important feature: if $e \in A_1$ is the constant function on Ω with value 1, then *e is not a zero-divisor*; that is, multiplication by e is a monomorphism from A_n to A_{n+1}. (This algebraic statement is really a fact about finite combinatorics: the incidence matrix of n-sets and $(n+1)$-sets of a set X of cardinality at least $2n+1$ has full rank.)

Now let G be a permutation group on Ω. There is a natural action of G on A_n, for all n; we let A_n^G be the set of functions invariant under G (that is, constant on the G-orbits), and $A^G = \bigoplus A_n^G$. Then A^G is a graded subalgebra of A. Moreover, if G is oligomorphic, then $\dim(A_n^G) = f_n(G)$, so that *the Hilbert series of A^G is $f_G(x)$.*

There are various interesting examples of groups for which A^G is finitely generated. For example, if $G = G_1 \times G_2$ in its intransitive action on $\Omega_1 \cup \Omega_2$, where G_i acts on Ω_i for $i = 1, 2$, we have

$$A^G \cong A^{G_1} \otimes_F A^{G_2},$$

for any field of characteristic zero. In particular, $A^{S \times S}$ is freely generated by (that is, a polynomial ring in) two generators of degree 1, and more generally, A^{S^r} freely generated by r generators of degree 1.

If H is a finite permutation group of degree n, and $G = S \operatorname{Wr} H$ (in its imprimitive action), then A^G is isomorphic to the ring of invariants of H (acting as a linear group via permutation matrices). In particular, if $H = S_n$, then A^G is the ring of symmetric polynomials in n variables, and is freely generated by the elementary symmetric polynomials $e_1, \ldots, e_n$ of degrees $1, 2, \ldots, n$, by Newton's Theorem.

However, it follows from Macpherson's Theorem that, if G is primitive but not highly homogeneous, then A^G cannot be finitely generated.

In the light of this, we are forced to look for other kinds of algebraic properties of A^G. An example is a remarkable recent theorem of Pouzet [30], confirming a 30-year-old conjecture:

Theorem 5.1 *Let G be a permutation group on an infinite set Ω. Then A^G is an integral domain if and only if G has no finite orbits on Ω.*

One direction is trivial. Suppose that Δ is a G-orbit of size n, and let $f \in A_n$ be the characteristic function of $\{\Delta\}$. Then $f \in A_n^G$ and $f^2 = 0$. The converse requires a new type of Ramsey-type theorem which is likely to have further applications. The result itself is applicable to growth rates, in view of the following result [6], whose proof uses some easy dimension theory from algebraic geometry:

Theorem 5.2 *If G is oligomorphic and A^G is an integral domain then $f_{m+n} \geq f_m + f_n - 1$.*

Another situation in which we can describe the structure of A^G is when G is associated with a Fraïssé class $\mathcal{C}$ (that is, G is the automorphism group of the Fraïssé limit of $\mathcal{C}$). In [5] it is shown that, if the class $\mathcal{C}$ has notions of connected components, disjoint union, and spanning substructures satisfying a few simple axioms, then A^G is a polynomial algebra whose generators are the characteristic functions of the connected structures in $\mathcal{C}$. (More precisely, for each isomorphism type of n-element connected structure in $\mathcal{C}$, let f be the function in A_n which takes the value 1 on n-sets whose induced substructure is of the given type and 0 elsewhere; the collection of all such functions is a set of free generators for A^G.) Here are a few, reasonably typical, examples.

- Let $\mathcal{C}$ be the class of finite graphs, so that G is the automorphism group of the random graph R. Then a basis for the nth homogeneous component of A^G consists the characteristic functions of all n-vertex graphs (so that $\dim(A_n^G) = f_n(G)$ is the number of unlabelled graphs on n vertices). Now the characteristic functions of connected graphs on n vertices are algebraically independent, and the set of all such elements forms a free generating set for A^G, which is a polynomial algebra (infinitely generated).

- We saw earlier that, if G is associated with a Fraïssé class $\mathcal{C}$, then $G\mathrm{Wr}S$ is associated with the class of disjoint unions of $\mathcal{C}$-structures. There is an obvious notion of connectedness: structures with a single part are connected! Thus, $A^{G\mathrm{Wr}S}$ is a polynomial algebra, with $f_n(G)$ generators of degree n for all n. (Thus, the structure of $A^{G\mathrm{Wr}S}$ is independent of the structure of A^G except for numerical information about dimensions of components.)

- Consider the set $\mathbb{Q}$ with two kinds of structure: the order, and m subsets forming a partition of $\mathbb{Q}$, each one dense in $\mathbb{Q}$. (Think of a

colouring of $\mathbb{Q}$ with m colours $c_1, c_2, \ldots, c_m$, so that each colour class is dense.) Let G be the group of permutations preserving the order and the colours. An orbit of G on n-sets is described uniquely by a word of length n in the alphabet $\{c_1, c_2, \ldots, c_m\}$, recording the colours of the elements of the set in increasing order. Now the algebra A^G is the *shuffle algebra* over an alphabet of size m, which occurs in the theory of free Lie algebras [32]. One can develop an appropriate notion of connectedness, so that the connected words are the so-called *Lyndon words*. The fact that the shuffle algebra is a polynomial algebra generated by the Lyndon words was first proved by Radford [31], but emerges as a special case of the theory presented here.

6 Group structure

Relatively little is known about the structure of closed oligomorphic permutation groups, but the picture is rapidly changing. We look first at the normal structure in a couple of classical examples.

The countable symmetric group $S = \mathrm{Sym}(\Omega)$ has a normal subgroup $\mathrm{FSym}(\Omega)$, the *finitary symmetric group*, consisting of all permutations moving only finitely many points; this has a normal subgroup of index 2, the *alternating group* $\mathrm{Alt}(\Omega)$, consisting of finitary permutations which are even permutations of their supports. These are the only non-trivial proper normal subgroups; in particular, the quotients are simple.

The group A of order-preserving permutations of $\mathbb{Q}$ has two normal subgroups L and R; L consists of all permutations fixing all sufficiently large positive rationals, and R consists of permutations fixing all sufficiently large negative rationals. Their intersection consists of the order-preserving permutations of bounded support. These are the only non-trivial normal subgroups. In particular, $A/(L \cap R) \cong L/(L \cap R) \times R/(L \cap R)$, and the two factors are isomorphic.

The automorphism group of the countable random graph is simple [35]. Indeed, given any two elements g, h of this group with $g \neq 1$, it is possible to write h as the product of three copies of g or g^{-1}.

An important property which has had a lot of attention is the small index property. Let G be a permutation group of countable degree. A subgroup H has *small index* if $|G : H| < 2^{\aleph_0}$. (If the Continuum Hypothesis holds, this just says that H has finite or countable index.) The stabiliser of any

finite set has small index. We say that G has the *small index property* if any subgroup of G of small index contains the pointwise stabiliser of a finite set; it has the *strong small index property* if every subgroup of small index lies between the pointwise and setwise stabilisers of a finite set. If G is a closed oligomorphic group with the small index property, then the topology of G is determined by the group structure: a subgroup is open if and only if it has small index, so the subgroups of small index form a neighbourhood basis of the identity.

The symmetric group S, the group A of order-preserving permutations of $\mathbb{Q}$, and the automorphism group of the random graph all have the strong small index property [15, 19, 8]. A typical example of a permutation group which does not have the strong small index property is $S \operatorname{Wr} S$, in its imprimitive action; the stabiliser of a block of imprimitivity is not contained in the setwise stabiliser of any finite set. Examples which do not have the small index property can be constructed by producing automorphism groups of Fraïssé limits which have infinite elementary abelian 2-groups as quotients; they have "too many" subgroups of small index!

I mentioned earlier that the notion of Baire category is important for closed oligomorphic groups. Such a group G is said to have *generics* if there is a conjugacy class which is residual in G. In each of the three groups mentioned above, generics exist. Indeed, the nth direct power of the group has generics (in other words, the original group has generic n-tuples) for all n. This property is closely related to the small index property (see [19]).

We can also ask what group is generated by a "typical" n-tuple of elements. John Dixon proved that almost all pairs of elements of the finite symmetric group S_n generate S_n or the alternating group A_n. Later [13], he proved an analogue for the symmetric group of countable degree: almost all pairs of elements (in the sense of Baire category, that is, a residual set) generate a highly transitive free subgroup of S. (The existence of highly transitive free groups was first shown by McDonough [28].)

As noted, very recently these results have been extended to wider classes of groups.

Macpherson and Tent define the free amalgamation property, which is a strengthening of the strong amalgamation property, as follows. A Fraïssé class $\mathcal{C}$ has the *free amalgamation property* if, whenever B_1 and B_2 are structures in $\mathcal{C}$ with a common substructure A, there is an amalgam C of B_1 and B_2 such that

- the intersection of B_1 and B_2 in C is precisely A (this is the content of strong amalgamation, see Section 2);
- Every instance of a relation in C is contained in either B_1 or B_2.

This holds for graphs: we can choose to make the amalgam so that there are no edges between $B_1 \setminus A$ and $B_2 \setminus A$.

They prove the following theorem:

Theorem 6.1 *Let $\mathcal{C}$ be a nontrivial Fraïssé class (that is, there are some non-trivial relations) with the free amalgamation property, and G the automorphism group of its Fraïssé limit. Then G is simple.*

The trivial case must be excluded, since then G is the symmetric group S, and as we have seen this group is not simple.

Gartside and Knight [17] say that a Polish group (a topological group whose topology comes from a complete metric) is *almost free* if a residual subset of the n-tuples of elements of G freely generate a free group. They give a number of equivalent characterisations of such groups, and show:

Theorem 6.2 *Closed oligomorphic groups are almost free.*

Kechris and Rosendal [23] give conditions for the existence of generic conjugacy classes in a closed oligomorphic group.

In closing I mention a couple of important papers linking oligomorphic permutation groups, dynamical systems, and Ramsey theory: [20, 22].

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