

The number of degrees of freedom of two-dimensional turbulence

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Abstract

We derive upper bounds for the number of degrees of freedom of two-dimensional Navier-Stokes turbulence freely decaying from a smooth initial vorticity field $\omega(x, y, 0) = \omega_0$. This number, denoted by N , is defined as the minimum dimension such that for $n \geq N$, arbitrary n -dimensional balls in phase space centred on the solution trajectory $\omega(x, y, t)$, for $t > 0$, contract under the dynamics of the system linearized about $\omega(x, y, t)$. In other words, N is the minimum number of greatest Lyapunov exponents whose sum becomes negative. It is found that $N \leq C_1 R_e$ when the phase space is endowed with the energy norm, and $N \leq C_2 R_e (1 + \ln R_e)^{1/3}$ when the phase space is endowed with the enstrophy norm. Here C_1 and C_2 are constant and R_e is the Reynolds number defined in terms of ω_0 , the system length scale, and the viscosity ν . The linear (or nearly linear) dependence of N on R_e is consistent with the estimate for the number of active modes deduced from a recent mathematical bound for the viscous dissipation wave number. This result is in a sharp contrast to the forced case, for which well-known estimates for the Hausdorff dimension D_H of the global attractor scale highly superlinearly with ν^{-1} . We argue that the “extra” dependence of D_H on ν^{-1} is not an intrinsic property of the turbulent dynamics. Rather, it is a “removable artifact,” brought about by the employment of a time-independent forcing as a model for energy and enstrophy injection that drives the turbulence.

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INTRODUCTION

Chaotic dynamics are characterized by the stretching and folding of volume elements in phase space (solution space). In the presence of dissipation, these can be accompanied by volume contraction. For a finite-dimensional system, volume elements can eventually collapse onto complex sets of zero volume having fractal structures, whose generalized dimensions, such as the box-counting and Hausdorff dimensions, are significantly lower than the phase space dimension. For infinite-dimensional cases, volume contraction can occur for finite-dimensional volume elements. Furthermore, given a sufficiently large positive integer N (depending on physical parameters and initial conditions), this contraction can occur for arbitrarily oriented n -dimensional volume elements following a trajectory — solution “curve” in function phase space, provided that $n \geq N$. This is the case if the sum of the largest N Lyapunov exponents of the trajectory $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$, which can possibly be different for different trajectories, is negative. The smallest N (which will be denoted by N still) satisfying this condition thus defines the minimum dimension in phase space for which all n -dimensional ($n \geq N$) volume elements along a given trajectory contract during the course of evolution. This volume contraction means that the chaotic nature of the dynamics can be “captured” and “contained” within a linear subspace having dimension not higher than N . (Note that in principle this subspace may continually change along the trajectory, though its dimension does not exceed N .) For this reason, such an integer can be thought of as an effective dimension of the dynamical system in question, in the sense that its local dynamics can be adequately described by an N -dimensional model. When an attractor exists and N is common to trajectories having initial data containing the attractor, its box-counting and Hausdorff dimensions both are bounded from above by N [1]. More precisely, these are bounded from above by the Lyapunov dimension D_L , which satisfies $N - 1 \leq D_L < N$ and is defined by [2, 3]

$$D_L = N - 1 + \frac{1}{|\lambda_N|} \sum_{i=1}^{N-1} \lambda_i. \quad (1)$$

In this study we calculate upper bounds for N , which is conveniently defined as the number of degrees of freedom, for two-dimensional Navier–Stokes turbulence freely decaying from a smooth initial vorticity field ω_0 in a doubly periodic domain of length scale L . The bounds obtained are expressible in terms of physical parameters and found to scale linearly

or almost linearly (depending on the chosen norms for the phase space) with the Reynolds number R_e , which is defined in terms of ω_0 , L , and the viscosity ν . On the one hand, such scaling behaviors are in accord with heuristic arguments based on physical and mathematical estimates of the viscous dissipation wave number. On the other hand, these are in a sharp contrast to the forced case, for which well-known upper bounds for the Hausdorff dimension D_H of the global attractor have highly superlinear dependence on ν^{-1} . We discuss this discrepancy and argue that the superlinear dependence of D_H on ν^{-1} is not an intrinsic property of the turbulent dynamics. Rather, it appears to be a “removable artifact,” brought about by the particular form of the employed forcing as a model for energy and enstrophy input that drives the turbulence. Indeed, the “extra” dependence of D_H on ν^{-1} would be removed if the energy and enstrophy injection could be made viscosity independent (more precisely if the injection could be bounded independently of viscosity), provided that this forcing model does not jeopardize the existence of the global attractor.

PRELIMINARIES

In this section, we briefly recall a recently derived upper bound [4] for the enstrophy dissipation wave number k_d . We then deduce from this result an estimate for the number of active modes, by counting all modes having length scales larger than the dissipation length scale corresponding to k_d . A brief functional setting of the two-dimensional Navier-Stokes system in the vorticity and stream function formulation is described, and the problem of phase space volume evolution is formulated. We avoid much technical detail and use informal language, principally highlighting the necessary mathematical elements for the present purpose.

Number of active modes

The two-dimensional Navier-Stokes system written in terms of the stream function ψ and vorticity $\omega = \Delta\psi$ is

$$\omega_t + J(\psi, \omega) = \nu \Delta \omega, \quad (2)$$

where $J(\psi, \omega) = \psi_x \omega_y - \psi_y \omega_x$ and ν is the viscosity. We consider Eq. (2) in a doubly periodic domain of size $2\pi L$. The initial vorticity field ω_0 is assumed to be smooth and have zero aver-

age. Equation (2) preserves the zero-mean property. This, together with periodicity, allows ω (and ψ) to be expressible as Fourier series in terms of $\sin L^{-1}(\ell x + my)$ and $\cos L^{-1}(\ell x + my)$, where ℓ and m are integers not simultaneously zero. In other words, the phase space (solution function space) can be spanned by the infinite basis $\{\sin L^{-1}(\ell x + my), \cos L^{-1}(\ell x + my)\}$, hence infinite-dimensional function space. Here (ℓ, m) can be identified with lattices of unit spacing on the upper half plane with those on either half of the horizontal axis, including the origin, removed.

The advection term $J(\psi, \omega)$ possesses a wealth of conservation laws. In particular, the total kinetic energy $\|\nabla\psi\|^2/2 = \langle |\nabla\psi|^2 \rangle /2 = \int |\nabla\psi|^2 dx dy /2$, the total enstrophy $\|\omega\|^2/2$, and the peak vorticity $\|\omega\|_\infty$ are conserved. These are the most important conserved quantities and play prominent roles in the theory of turbulence. Under viscous effects, all these quantities decay, though in general at different rates. The enstrophy decays most rapidly, while the kinetic energy and the peak vorticity are far better conserved, with the latter probably best conserved [5].

For relatively small ν , the free decay of a general smooth vorticity field presumably becomes turbulent, featuring a wide range of dynamically interacting scales that extend to the viscous dissipation range. This range is characterized by the dissipation wave number k_ν , which, according to the phenomenological theory of turbulence [6], is given by $k_\nu = \chi^{1/6} / \nu^{1/2}$. Here $\chi = \nu \|\nabla\omega\|^2 / (4\pi^2 L^2)$ denotes the enstrophy dissipation rate per unit area. Recently, Tran [4] derived the upper bound

$$\nu \|\nabla\omega\|^2 \leq \|\omega\|_\infty \|\omega\|^2, \quad (3)$$

for the dissipation rate $\nu \|\nabla\omega\|^2$ at its peak. Since both vorticity norms on the right-hand side of Eq. (3) decay we have the bound

$$\|\nabla\omega\| \leq \frac{\|\omega_0\|_\infty^{1/2} \|\omega_0\|}{\nu^{1/2}}, \quad (4)$$

which is valid uniformly in time, and the bound

$$k_d = \frac{\|\nabla\omega\|}{\|\omega\|} \leq \frac{\|\omega_0\|_\infty^{1/2}}{\nu^{1/2}}, \quad (5)$$

which is valid at least up to (and probably beyond) the time of peak enstrophy dissipation. The bound for the newly defined enstrophy dissipation wave number k_d compares favorably to k_ν , as it could be significantly smaller than k_ν [4]. By the very definition (5), enstrophy

dissipation is strongest in the vicinity of k_d . The wave numbers greater than k_d are effectively suppressed by viscous forces and virtually inactive. The number of dynamically active modes N_c corresponding to $k \leq k_d$ are therefore given by

$$N_c \approx \frac{k_d^2}{k_0^2} \leq \frac{L^2 \|\omega_0\|_\infty}{\nu}, \quad (6)$$

where $k_0 = 1/L$ is the smallest wave number. The quantity $L \|\omega_0\|_\infty$ may be identified with the fluid velocity. Perhaps, $\|\omega_0\|$ is a better representative of the fluid velocity; nevertheless, when it comes to the definition of the Reynolds number R_e , we use $L \|\omega_0\|_\infty$ and $\|\omega_0\|$ interchangeably. With this identification, the term on the right-hand side of Eq. (6) may be defined as the Reynolds number R_e . Hence, Eq. (6) can be rewritten in a more compact form

$$N_c \leq R_e. \quad (7)$$

From our experience in numerical simulations of two-dimensional turbulence, the estimate (7) can be seen to be very sharp — in fact, spot on. For example, for the standard numerical domain $2\pi \times 2\pi$ and an initial vorticity maximum $\|\omega_0\|_\infty \approx 4\pi$, the simulations of Dritschel, Tran, and Scott [5] using $4\pi(8/3)^2/\nu$ grid points adequately resolve the dissipation scales. This resolution is obviously consistent with Eq. (7), within an order of magnitude. As will be seen in the next section, the estimate (7) for N_c fully agrees with the number of degrees of freedom discussed above.

Problem formulation

The problem of phase space volume element contraction (or expansion) is intimately related to the stability of solution with respect to disturbances. To investigate this problem, we consider the linear evolution of a deviation ϕ of the stream function ψ (corresponding to a deviation $\Delta\phi$ of the vorticity ω) governed by the linearised equation

$$\Delta\phi_t + J(\phi, \omega) + J(\psi, \Delta\phi) = \nu\Delta^2\phi, \quad (8)$$

where ω (and ψ) solves Eq. (2) with initial vorticity ω_0 (and initial stream function ψ_0). By taking the scalar product ($\langle \cdot \rangle$) of Eq. (8) with ϕ and $\Delta\phi$ we obtain the respective evolution equations for the energy norm $\|\nabla\phi\|$ and enstrophy norm $\|\Delta\phi\|$,

$$\|\nabla\phi\| \frac{d}{dt} \|\nabla\phi\| = \langle \phi J(\psi, \Delta\phi) \rangle - \nu \|\Delta\phi\|^2 \quad (9)$$

and

$$\|\Delta\phi\| \frac{d}{dt} \|\Delta\phi\| = -\langle \Delta\phi J(\phi, \omega) \rangle - \nu \|\nabla \Delta\phi\|^2. \quad (10)$$

The respective exponential growth (or decay) rates λ and Λ for $\|\nabla\phi\|$ and $\|\Delta\phi\|$ can be readily deduced and are given by

$$\lambda = \frac{d}{dt} \ln \|\nabla\phi\| = \frac{1}{\|\nabla\phi\|^2} (\langle \phi J(\psi, \Delta\phi) \rangle - \nu \|\Delta\phi\|^2) \quad (11)$$

and

$$\Lambda = \frac{d}{dt} \ln \|\Delta\phi\| = \frac{-1}{\|\Delta\phi\|^2} (\langle \Delta\phi J(\phi, \omega) \rangle + \nu \|\nabla \Delta\phi\|^2). \quad (12)$$

These rates provide a comprehensive picture of solution stability, quantitatively describing how solutions with nearby initial data disperse from one another.

Two natural norms for the present problem are the energy and enstrophy norms. We will refer to the phase space equipped with the energy (enstrophy) norm as the energy (enstrophy) space. In the course of evolution, consider a trajectory commencing from a given initial condition. At an arbitrary point on the trajectory (i.e., at an arbitrary instance in time $t > 0$), we calculate the greatest growth rate λ (Λ) and identify the corresponding most unstable “direction” by considering the problem of maximizing λ (Λ) with respect to all admissible ϕ . We denote by (λ_1, φ_1) $[(\Lambda_1, \vartheta_1)]$ the solution of this problem, where for convenience φ (ϑ) has been normalized, i.e., $\|\nabla\varphi_1\| = 1$ ($\|\Delta\vartheta_1\| = 1$). The second greatest rate λ_2 (Λ_2) and the corresponding second most unstable direction φ_2 (ϑ_2) orthogonal to φ_1 (ϑ_1) is obtained by the same maximization problem subject to the orthogonality constraint, i.e., $\langle \nabla\varphi_1, \nabla\varphi_2 \rangle = 0$ ($\langle \Delta\vartheta_1, \Delta\vartheta_2 \rangle = 0$). By repeating this procedure n times, we obtain the set $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ ($\{\vartheta_1, \vartheta_2, \dots, \vartheta_n\}$) of mutually orthonormal functions and the corresponding set of ordered rates $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ ($\Lambda_1 \geq \Lambda_2 \geq \dots \geq \Lambda_n$). These may be defined as the first n local Lyapunov exponents, and their existence is guaranteed since the maximization problems are expected to return unique solutions. Note that for the conventional Lyapunov exponents, existence can be a major issue, even for low-dimensional systems of a few degrees of freedom.

Now, in the linear subspace spanned by $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ ($\{\vartheta_1, \vartheta_2, \dots, \vartheta_n\}$), consider an n -dimensional ball $B(\cdot, r)$ of radius r centred at the point discussed above. The n -dimensional volumes v (in the energy subspace) and V (in the enstrophy subspace) of $B(\cdot, r)$ are given by

$v \propto r^n \|\nabla\varphi_1\| \|\nabla\varphi_2\| \cdots \|\nabla\varphi_n\| = r^n$ and $V \propto r^n \|\Delta\vartheta_1\| \|\Delta\vartheta_2\| \cdots \|\Delta\vartheta_n\| = r^n$, respectively. (See the book of Temam [7] for a formal definition of volume based on the related concept of exterior product.) The respective equations governing the evolution of v and V under the linearised dynamics described by Eq. (8) are

$$\frac{d}{dt} \ln v = \sum_{i=1}^n \lambda_i = \sum_{i=1}^n \left(\langle \varphi^i J(\psi, \Delta\varphi^i) \rangle - \nu \|\Delta\varphi^i\|^2 \right) \quad (13)$$

and

$$\frac{d}{dt} \ln V = \sum_{i=1}^n \Lambda_i = - \sum_{i=1}^n \left(\langle \Delta\vartheta^i J(\vartheta^i, \omega) \rangle + \nu \|\nabla \Delta\vartheta^i\|^2 \right). \quad (14)$$

In deriving Eqs. (13) and (14), we have used Eqs. (11) and (12), respectively. The sum $\sum_{i=1}^n \lambda_i$ ($\sum_{i=1}^n \Lambda_i$) represents the exponential growth or decay rate of v (V). When this sum is negative, the volume of the n -dimensional ball $B(\cdot, r)$ contracts exponentially. Note that by construction, $B(\cdot, r)$ is optimally “oriented” to be least contracting. This means that if $\sum_{i=1}^n \lambda_i$ ($\sum_{i=1}^n \Lambda_i$) is negative, then volume contraction becomes universal for all n - or higher-dimensional balls locally centred at the point in question. Furthermore, if this point is taken arbitrarily on the trajectory, which is the case in this study, then volume contraction becomes universal along the trajectory.

The determination of N then reduces to minimizing n in Eqs. (13) and (14) subject to the constraint that these expressions remain negative. We do this by deriving upper bounds for the right-hand sides of these equations using the mathematical techniques developed in the 1980s by Babin and Vishik [8] and Constantin, Foias, and Temam [9–11] for estimating the attractor dimension of forced two-dimensional Navier–Stokes turbulence. See also the paper of Doering and Gibbon [12] for the same treatment in the stream function and vorticity setting. As can be seen in the next section, the derivation of upper bounds for N is equivalent to that for the Hausdorff dimension of the global attractor in the forced case. The main difference is that although the present formulation is specifically designed to handle the decaying case, which apparently has a trivial attractor, its scope of application is broad. In general, the present notion of degrees of freedom makes sense for general dissipative dynamical systems, provided that bounded solutions exist. There are virtually no other technical requirements for the application of the method. In particular, no *a priori* knowledge of the existence of an attractor is required.

RESULTS

This section presents the calculations described above, leading to upper bounds for N . The treatment is relatively self-contained. However, the reader, who is interested in further detail related to the analytic inequalities employed in various stages of the calculations, is referred to the cited papers and references therein.

Degrees of freedom in energy space

We begin by deriving an upper bound for N in the energy space. From Eq. (13) we have

$$\begin{aligned}
 \sum_{i=1}^n \lambda_i &= \sum_{i=1}^n \left(\langle \varphi^i J(\psi, \Delta \varphi^i) \rangle - \nu \|\Delta \varphi^i\|^2 \right) \\
 &= - \sum_{i=1}^n \left(\langle \Delta \varphi^i J(\psi, \varphi^i) \rangle + \nu \|\Delta \varphi^i\|^2 \right) \\
 &= - \sum_{i=1}^n \left(\langle \varphi^i J(\psi_x, \varphi_x^i) + \varphi^i J(\psi_y, \varphi_y^i) \rangle + \nu \|\Delta \varphi^i\|^2 \right) \\
 &= \sum_{i=1}^n \left(\langle \varphi_x^i J(\psi_x, \varphi^i) + \varphi_y^i J(\psi_y, \varphi^i) \rangle - \nu \|\Delta \varphi^i\|^2 \right) \\
 &\leq \sum_{i=1}^n \left(\langle |\nabla \varphi^i| (|\varphi_x^i| |\nabla \psi_x| + |\varphi_y^i| |\nabla \psi_y|) \rangle - \nu \|\Delta \varphi^i\|^2 \right) \\
 &\leq \sum_{i=1}^n \left(\langle |\nabla \varphi^i|^2 (|\nabla \psi_x|^2 + |\nabla \psi_y|^2)^{1/2} \rangle - \nu \|\Delta \varphi^i\|^2 \right) \\
 &\leq \|\omega\| \left\| \sum_{i=1}^n |\nabla \varphi^i|^2 \right\| - \nu \sum_{i=1}^n \|\Delta \varphi^i\|^2, \tag{15}
 \end{aligned}$$

where integration by parts and the Cauchy–Schwarz inequality have been used. For further estimates of the terms on the right-hand side of Eq. (15), we employ the following two analytic inequalities concerning the orthonormal set $\{\varphi^i\}_{i=1}^n$ with respect to the energy norm. First, we have the Lieb–Thirring inequality

$$\left\| \sum_{i=1}^n |\nabla \varphi^i|^2 \right\| \leq c_1 \left(\sum_{i=1}^n \|\Delta \varphi^i\|^2 \right)^{1/2}, \tag{16}$$

where c_1 is a non-dimensional constant independent of the set $\{\varphi^i\}_{i=1}^n$. Second, we know that for $n \gg 1$, there are approximately n basis functions (trigonometric functions mentioned earlier) within the wave number radius $\sqrt{n}/L$. Their (repeated) eigenvalues under $-\Delta$

are $(\ell^2 + m^2)/L^2$, where $\ell^2 + m^2 \leq n$. These constitute the first n eigenvalues (in non-decreasing order) of $-\Delta$ and sum up to approximately n^2/L^2 . It follows from the Rayleigh-Ritz principle that

$$\sum_{i=1}^n \|\Delta \varphi^i\|^2 \geq \frac{c_2^2}{L^2} n^2, \quad (17)$$

where c_2 is another non-dimensional constant independent of the set $\{\varphi^i\}_{i=1}^n$. By substituting Eqs. (16) and (17) into Eq. (15) we obtain

$$\sum_{i=1}^n \lambda_i \leq \left(\sum_{i=1}^n \|\Delta \varphi^i\|^2 \right)^{1/2} \left(c_1 \|\omega\| - \nu \frac{c_2}{L} n \right). \quad (18)$$

It follows that $\sum_{i=1}^n \lambda_i \leq 0$ when $n \geq c_1 L \|\omega\| / (c_2 \nu)$. Hence we deduce the bound

$$N \leq C_1 \frac{L \|\omega\|}{\nu} \leq C_1 \frac{L \|\omega_0\|}{\nu} = C_1 R_e, \quad (19)$$

where $C_1 = c_1/c_2$ and R_e has been redefined by replacing $L \|\omega_0\|_\infty$ with $\|\omega_0\|$. Note that the precise result should be that N is no greater than the least integral upper bound for $C_1 R_e$; however, in writing Eq. (19), we have opted to ignore this exceedingly minor detail. Equation (19) gives a clear linear dependence of N on R_e . Thus, we have essentially recovered the bound (7), up to the constant factor C_1 and a slight difference in the definition of R_e , which was obtained earlier by counting the active modes from the smallest wave number $k_0 = 1/L$ to the dissipation wave number $k_d = \|\nabla \omega\| / \|\omega\|$.

Degrees of freedom in enstrophy space

An upper bound for N in the enstrophy space is derived in a similar manner. From Eq. (14) we have

$$\begin{aligned} \sum_{i=1}^n \Lambda_i &= - \sum_{i=1}^n \left(\langle \Delta \vartheta^i J(\vartheta^i, \omega) \rangle + \nu \|\nabla \Delta \vartheta^i\|^2 \right) \\ &\leq \sum_{i=1}^n \left(\langle |\Delta \vartheta^i| |\nabla \vartheta^i| |\nabla \omega| \rangle - \nu \|\nabla \Delta \vartheta^i\|^2 \right) \\ &\leq \left\langle \left(\sum_{i=1}^n |\Delta \vartheta^i|^2 \sum_{i=1}^n |\nabla \vartheta^i|^2 \right)^{1/2} |\nabla \omega| \right\rangle - \nu \sum_{i=1}^n \|\nabla \Delta \vartheta^i\|^2, \\ &\leq \left\langle \left(\sum_{i=1}^n |\Delta \vartheta^i|^2 \sum_{i=1}^n |\nabla \vartheta^i|^2 \right)^2 \right\rangle^{1/4} \left\langle |\nabla \omega|^{4/3} \right\rangle^{3/4} - \nu \sum_{i=1}^n \|\nabla \Delta \vartheta^i\|^2, \end{aligned}$$

$$\leq \left\| \sum_{i=1}^n |\nabla \vartheta^i|^2 \right\|_\infty^{1/2} \left\| \sum_{i=1}^n |\Delta \vartheta^i|^2 \right\|_\infty^{1/2} (2\pi L)^{1/2} \|\nabla \omega\| - \nu \sum_{i=1}^n \|\nabla \Delta \vartheta^i\|^2, \quad (20)$$

where Hölder's inequalities with the pairs of conjugate exponents $4/3$ and 4 and $3/2$ and 3 have been used in the penultimate and final steps, respectively. For further estimates of the terms in this equation, we employ a few more analytic inequalities concerning the orthonormal set $\{\vartheta^i\}_{i=1}^n$ in the enstrophy space. First, we have [10, 11]

$$\left\| \sum_{i=1}^n |\nabla \vartheta^i|^2 \right\|_\infty \leq c_3^2 \left(1 + \ln \sum_{i=1}^n L^2 \|\nabla \Delta \vartheta^i\|^2 \right), \quad (21)$$

where c_3 is a non-dimensional constant independent of the set $\{\vartheta^i\}_{i=1}^n$. Second, a version of Eq. (16) for the present orthonormal set is

$$\left\| \sum_{i=1}^n |\Delta \vartheta^i|^2 \right\| \leq c_1 \left(\sum_{i=1}^n \|\nabla \Delta \vartheta^i\|^2 \right)^{1/2}. \quad (22)$$

Finally, a version of Eq. (17) for $\{\vartheta^i\}_{i=1}^n$ is

$$\sum_{i=1}^n \|\nabla \Delta \vartheta^i\|^2 \geq \frac{c_2^2}{L^2} n^2. \quad (23)$$

Now by substituting Eqs. (4), (21), and (22) into Eq. (20) we obtain

$$\begin{aligned} \sum_{i=1}^n \Lambda_i &\leq C' \left(1 + \ln \sum_{i=1}^n L^2 \|\nabla \Delta \vartheta^i\|^2 \right)^{1/2} \left(\sum_{i=1}^n L^2 \|\nabla \Delta \vartheta^i\|^2 \right)^{1/4} \frac{\|\omega_0\|_\infty^{1/2} \|\omega_0\|}{\nu^{1/2}} - \nu \sum_{i=1}^n \|\nabla \Delta \vartheta^i\|^2 \\ &= \frac{\nu \xi^{1/4}}{L^2} \left(C' (1 + \ln \xi)^{1/2} \frac{L^2 \|\omega_0\|_\infty^{1/2} \|\omega_0\|}{\nu^{3/2}} - \xi^{3/4} \right) = \frac{\nu \xi^{1/4}}{L^2} \left(C' R_e^{3/2} (1 + \ln \xi)^{1/2} - \xi^{3/4} \right) \end{aligned} \quad (24)$$

where $C' = \sqrt{2\pi} c_1 c_3$, $\xi = \sum_{i=1}^n L^2 \|\nabla \Delta \vartheta^i\|^2$, and $R_e = (L^4 \|\omega_0\|_\infty \|\omega_0\|^2)^{1/3} / \nu$. Note that by Eq. (22) we have $\xi \geq c_2^2 n^2$. Hence without the logarithmic term, it would be straightforward to substitute this into Eq. (24) and deduce an upper bound for N similar to Eq. (19) with the newly defined R_e replacing its previously defined (and comparable) counterpart. Since we are interested in the case $\xi \gg 1$, the logarithmic term should introduce a small departure to the linear dependence of N on R_e only. In order to account for $\ln \xi$, we can “cover” it by a fraction of ξ , say $\xi/2$. By elementary calculus, we find that

$$C' R_e^{3/2} (1 + \ln \xi)^{1/2} - \frac{\xi^{3/4}}{2} \leq \sqrt{2} C' R_e^{3/2} (1 + \ln R_e)^{1/2}, \quad (25)$$

where we have dropped a negative term on the right-hand side. It follows that

$$\begin{aligned} C' R_e^{3/2} (1 + \ln \xi)^{1/2} - \xi^{3/4} &\leq \sqrt{2} C' R_e^{3/2} (1 + \ln R_e)^{1/2} - \frac{\xi^{3/4}}{2} \\ &\leq \sqrt{2} C' R_e^{3/2} (1 + \ln R_e)^{1/2} - \frac{(c_2 n)^{3/2}}{2}. \end{aligned} \quad (26)$$

The condition $\sum_{i=1}^n \Lambda_i \leq 0$ is satisfied when the right-hand side of Eq. (26) is non-positive. This requires a straightforward condition for n which in turn yields the result

$$N \leq C_2 R_e (1 + \ln R_e)^{1/3}, \quad (27)$$

where $C_2 = (8C'^2)^{1/3}/c_2$.

As expected, Eq. (27) gives an essentially linear scaling of N with R_e since the superlinear dependence on R_e represented by the logarithmic term is exceedingly slight for large R_e . Given the same linear scaling found earlier in the energy space, this is somewhat surprising. The reason is that the energy in two-dimensional turbulence is predominantly transferred to smaller wave numbers while the enstrophy is predominantly transferred to larger wave numbers. This undoubtedly implies that the enstrophy dynamics have relatively more degrees of freedom than the energy dynamics. Hence, it is somewhat counter-intuitive that Eqs. (19) and (27) do not differ by much. A possible explanation is that Eq. (19) may not be as optimal as Eq. (27). Some qualitative support for this possibility turns up in the next subsection.

Discussion

In the 1980s, estimates were derived for the Hausdorff dimension D_H of the global attractor of the two-dimensional Navier-Stokes system driven by a time-independent forcing $\mathbf{f}$ [8, 9, 11]. These estimates have been known to be sharp, allowing just minor improvements for the attractor dimension in the energy space only [13, 14]. In the present notations, the respective bounds for D_H in the energy and enstrophy spaces are given by

$$D_H \leq c' \frac{L \|\nabla^{-1} \mathbf{f}\|}{\nu^2} \leq c' \frac{L^2 \|\mathbf{f}\|}{\nu^2} = c' G \quad (28)$$

and

$$\begin{aligned} D_H &\leq c'' \left(\frac{L^2 \|\mathbf{f}\|}{\nu^2} \right)^{2/3} \left(1 + \ln \frac{L^2 \|\mathbf{f}\|}{\nu^2} \right)^{1/3} \\ &= c'' G^{2/3} (1 + \ln G)^{1/3}, \end{aligned} \quad (29)$$