MONOTONE UNITARY FAMILIES

DANIEL GRIESER

ABSTRACT. A unitary family is a family of unitary operators $U(x)$ acting on a finite dimensional hermitian vector space, depending analytically on a real parameter x. It is monotone if $\frac{1}{i}U'(x)U(x)^{-1}$ is a positive operator for each x. We prove a number of results generalizing standard theorems on the spectral theory of a single unitary operator U_0, which correspond to the 'commutative' case $U(x) = e^{ix}U_0$. Also, for a two-parameter unitary family – for which there is no analytic perturbation theory – we prove an implicit function type theorem for the spectral data under the assumption that the family is monotone in one argument.

1. INTRODUCTION

Let $U(x)$ be a family of unitary operators on a Hermitian vector space V of dimension $M < \infty$, depending real analytically on $x \in \mathbb{R}$ (or an interval in $\mathbb{R}$). Then

$$(1) \qquad D(x) := \frac{1}{i}U'(x)U(x)^{-1}$$

is symmetric. Here $U'(x)$ is the derivative with respect to x. We call U *monotone* if $D(x)$ is a positive operator for all x. Denote

$$W(x) = \operatorname{Ker}(I - U(x)) \quad \text{and } \mathcal{Z} = \{x : W(x) \neq \{0\}\}.$$

Thus $x \in \mathcal{Z}$ iff $U(x)$ has eigenvalue one.

A model case for this setup is $U(x) = e^{ix}U_0$ for a unitary U_0. Then $W(x)$ is the eigenspace of U_0 with eigenvalue e^{-ix}. Standard facts from the spectral theory of U_0 may be restated in terms of $\mathcal{Z}$ and $W(x)$, for example:

- $\mathcal{Z}$ is a 2π-periodic sequence, having exactly M terms in each half-open interval of length 2π (counting 'multiplicities').
- If I is an interval of length less than 2π then the spaces $W(x)$, $x \in I$, are linearly independent (even pairwise orthogonal).
- If $\|(I - U(x_0))\varphi\| \leq \varepsilon\|\varphi\|$ for some $\varphi \in V \setminus \{0\}$, $x_0 \in \mathbb{R}$ and $\varepsilon \geq 0$ then e^{-ix_0} lies within distance ε of an eigenvalue of U_0 (and so $\operatorname{dist}(x_0, \mathcal{Z}) \leq \pi\varepsilon/2$). Furthermore, if $\varepsilon' > \varepsilon$ and P denotes the orthogonal projection to $\bigoplus_{x:|e^{-ix}-e^{-ix_0}|<\varepsilon'} W(x)$ then

$$(2) \qquad \|\varphi - P\varphi\| \leq \frac{\varepsilon}{\varepsilon'}\|\varphi\|.$$

In this paper we prove generalizations of these facts to arbitrary monotone unitary families, see Theorem 1 in Section 2 and Theorems 2 and 3 in Section 3.

The estimates are expressed in terms of uniform bounds on the first and second derivatives of U: Assume $d_{\min}, d_{\max}, d_2 > 0$ are such that

$$(3) \qquad d_{\min} I \le D(x) \le d_{\max} I, \quad \|U''(x)\| \le d_2 \quad \text{for all } x.$$

Of course, such constants exist always locally. However, in applications of our results, see [2], it is essential that the estimates are uniform in terms of (3), and do not depend on additional data like separation of elements of $\mathcal{Z}$, see the explanation below.

An important difference between a general unitary family and the model case is that $U(x)$ and $U(x')$ do not in general commute for $x \ne x'$. A consequence of this is that, while one may take a logarithm of U, i.e. find an analytic family $A(x)$ of symmetric operators such that

$$(4) \qquad U(x) = e^{iA(x)} \quad \text{for all } x,$$

it is usually not true that $D(x)$ equals $A'(x)$, or even that positivity of $D(x)$ implies positivity of $A'(x)$. The opposite implication is true, however. See Section 4.

Finally, we prove a result on two-parameter perturbation theory. Recall the main result of one-parameter perturbation theory (see [4], [3]): For an analytic family $U(x)$ of unitary operators on V, there are real analytic functions $\mu_j(x)$, $\varphi_j(x)$ for $j = 1, \ldots, M = \dim V$ having values in $\mathbb{R}$ and V, respectively, such that for each x the eigenvalues of $U(x)$ are $e^{i\mu_j(x)}$, with corresponding orthonormal basis of eigenvectors $\varphi_j(x)$:

$$(5) \qquad U(x)\varphi_j(x) = e^{i\mu_j(x)}\varphi_j(x).$$

It is well-known that the analogous statement for two-parameter families of operators is false in general. However, we prove a related implicit function type theorem for the spectral data of a two-parameter family for which the dependence on one parameter is monotone. It may be regarded as the natural unitary family generalization of the one-parameter perturbation theory. See Theorem 7 in Section 5.

The analytic functions μ_j and φ_j play a central role in the proofs of our theorems. It is essential to control their derivatives. For μ_j this is easy from (3). However, φ_j may vary wildly whenever $e^{i\mu_j}$ is very close to another eigenvalue. To control this variation is the main technical problem in the proof of the generalization of (2), Theorem 3. Note that, unlike in the model case, one may not assume that the $e^{i\mu_j(x)}$ for fixed x but varying j are uniformly separated, or equivalently that the elements of $\mathcal{Z}$ are uniformly separated. This can already be seen in the simple example $U(x) = e^{ixL}$, where L is a diagonal matrix with positive diagonal entries that are independent over $\mathbb{Q}$: If $M \ge 2$ then for any $\varepsilon > 0$ there are $x, x' \in \mathcal{Z}$ satisfying $0 < |x - x'| < \varepsilon$.

The problems we study here arose in the context of a singular perturbation problem: In [2] we study the eigenvalues and eigenfunctions of the Laplacian on a space X^N which has a fixed compact part connected by cylindrical necks of length $N > 0$, and in particular their asymptotic behavior as $N \to \infty$. The unitary families arise from the scattering matrix of the limit problem (infinitely long 'necks').

2. Eigenvalue distribution

Theorem 1. *Let U be a monotone unitary family on $\mathbb{R}$. Then $\mathcal{Z} \subset \mathbb{R}$ is a discrete subset, and more precisely for all $A < B$*

$$(6) \qquad \left| \sum_{x:A<x<B} \dim W(x) - \frac{1}{2\pi} \int_A^B \operatorname{tr} D(x)\, dx \right| < M (:= \dim V)$$

In the special case $U(x) = e^{ixL} U_0$, $L > 0$, this implies the asymptotics

$$\sum_{x:0<x<B} \dim W(x) \sim \frac{\operatorname{tr} L}{2\pi} B, \quad B \to \infty.$$

This is the Weyl asympotitcs of a quantum graph, see for example [1]. We give a much simpler proof than [1].

First, we differentiate (5) and obtain monotonicity of the functions μ_j from monotonicity of U:

$$(7) \qquad \mu_j' = (\frac{1}{i} U' U^{-1} \varphi_j, \varphi_j) = (D\varphi_j, \varphi_j) > 0.$$

Also, for each x we have

$$(8) \qquad W(x) = \operatorname{span}\{\varphi_j(x) : \mu_j(x) \in 2\pi\mathbb{Z}\}$$

Proof of Theorem 1. We have

$$\sum_j (\mu_j(B) - \mu_j(A)) = \int_A^B \sum_j \langle D(x)\varphi_j(x), \varphi_j(x) \rangle\, dx$$

$$= \int_A^B \operatorname{tr} D(x)\, dx$$

Since the μ_j are strictly increasing, we get from (8)

$$\sum_{x:A<x<B} \dim W(x) = \sum_j \#\{k \in \mathbb{Z} : \mu_j(A) < 2\pi k < \mu_j(B)\} = \sum_j \left(\frac{\mu_j(B) - \mu_j(A)}{2\pi} + R_j \right)$$

with $|R_j| < 1$, and this gives (6). $\qquad \square$

3. Eigenspaces

In this section we consider monotone unitary families satisfying the estimates (3). By (7) we have

$$(9) \qquad d_{\min} \le \mu_j'(x) \le d_{\max} \quad \text{for all } x \text{ and } j = 1, \dots, M.$$

First, we have independence of eigenspaces.

Theorem 2. *Let U be a monotone unitary family satisfying (3). Let I be an interval of length at most $\frac{2d_{\min}}{d_2 M}$. Then the spaces $W(x)$, $x \in I$, are independent, i.e.*

$$(10) \qquad \text{If } \varphi_x \in W(x) \text{ for each } x \in I \cap \mathcal{Z} \text{ and } \sum_x \varphi_x = 0 \text{ then } \varphi_x = 0 \ \forall x.$$

The following theorem gives a stable version of almost orthogonality.

Theorem 3. *Let U be a monotone unitary family satisfying (3). Assume $\varphi \in V \setminus 0$ satisfies*

(11) $$\|(I - U(x_0))\varphi\| \leq \varepsilon \|\varphi\|.$$

Then

(12) $$\mathrm{dist}(x_0, \mathcal{Z}) \leq \frac{\pi}{2} \frac{\varepsilon}{d_{min}}.$$

Furthermore, there is a constant C only depending on d_{min}, d_{max}, d_2, M such that the following holds: Suppose $0 < \varepsilon < \varepsilon' < C^{-1}$ and $\varepsilon/\varepsilon' < C^{-1}$. Denote by P_W the orthogonal projection to $W = \bigoplus_{|x-x_0| \leq \varepsilon'} W(x)$. Then

(13) $$\|\varphi - P_W \varphi\| \leq C \left(\frac{\varepsilon}{\varepsilon'}\right)^{\frac{1}{M+1}} \|\varphi\|.$$

In the proofs of Theorems 2 and 3 we will need the following estimate, which replaces orthogonality of the eigenspaces of a unitary operator.

Lemma 4. *If $\varphi \in W(x)$, $\psi \in W(y)$ and $x \neq y$ then*

(14) $$|\langle D(x)\varphi, \psi\rangle| \leq \frac{d_2}{2}|x - y| \cdot \|\varphi\| \cdot \|\psi\|.$$

Proof. By Taylor's formula,

$$U(y)U(x)^{-1} = I + (y - x)U'(x)U(x)^{-1} + (y - x)^2 R, \quad \|R\| \leq \frac{d_2}{2},$$

so if $U(x)\varphi = \varphi$, $U(y)\psi = \psi$ then

$$\langle \varphi, \psi\rangle = \langle U(x)^{-1}\varphi, U(y)^{-1}\psi\rangle = \langle U(y)U(x)^{-1}\varphi, \psi\rangle$$
$$= \langle \varphi, \psi\rangle + i(y - x)\langle D(x)\varphi, \psi\rangle + (y - x)^2\langle R\varphi, \psi\rangle,$$

and this gives (14). $\qquad\qquad\square$

Proof of Theorem 2. Let $\varphi_x \in W(x)$ for $x \in I \cap \mathcal{Z}$, and assume $\sum_x \varphi_x = 0$. Let φ_{x_0} have maximal norm among all φ_x. Then $0 = \langle D(x_0)\varphi_{x_0}, \sum_x \varphi_x\rangle$ gives with (3) and (14)

$$d_{\min}\|\varphi_{x_0}\|^2 \leq \langle D(x_0)\varphi_{x_0}, \varphi_{x_0}\rangle = \left|\sum_{x \neq x_0} \langle D(x_0), \varphi_{x_0}, \varphi_x\rangle\right|$$

$$\leq \sum_{x \neq x_0} \frac{d_2}{2}|I| \cdot \|\varphi_{x_0}\| \cdot \|\varphi_x\| \leq (M - 1)\frac{d_2}{2}|I| \cdot \|\varphi_{x_0}\|^2,$$

so if $d_{\min} > (M - 1)\frac{d_2}{2}|I|$ then $\varphi_{x_0} = 0$ and hence $\varphi_x = 0$ for all x. This implies the claim. $\qquad\qquad\square$

Proof of Theorem 3. The first estimate follows easily from the fact that, by the lower bound in (9), an eigenvalue close to one of $U(x_0)$ will turn into an eigenvalue equal to one of $U(x)$, for some x close to x_0: Let $B(x) = I - U(x)$ and let $\lambda_j(x) = 1 - e^{i\mu_j(x)}$ be the eigenvalues of $B(x)$. The assumption (11) implies that $|\lambda_j(x_0)| \leq \varepsilon$ for some j, and this implies $\mathrm{dist}(\mu_j(x_0), 2\pi\mathbb{Z}) \leq \frac{\pi}{2}\varepsilon$, and then $\mu'_j \geq d_{\min}$ shows that there is an x satisfying $|x - x_0| < \pi\varepsilon/2d_{\min}$ and $\mu_j(x) \in 2\pi\mathbb{Z}$, hence $x \in \mathcal{Z}$, so (12) follows.

For $\delta > 0$ let $P_\delta(x)$ denote the orthogonal projection to the sum of the eigenspaces of $B(x)$ with eigenvalues $|\lambda_j(x)| \leq \delta$. Then $\|B(x_0)\varphi\| \leq \varepsilon\|\varphi\|$ implies

$$(15) \qquad \|\varphi - P_\delta(x_0)\varphi\| \leq \frac{\varepsilon}{\delta}\|\varphi\|$$

(see (2), which also applies to normal operators). To make this a good estimate, we want to take $\delta >> \varepsilon$. Our goal is to replace $P_\delta(x_0)$ by P_W here. The idea is that eigenspaces of $B(x_0)$ with eigenvalue $|\lambda_j(x_0)| \leq \delta$ will turn into nullspaces of $B(x)$ for some x within $2\delta/d_{\min}$ of x_0, by the first part of this proof. However, the variation of eigenspaces is much less well behaved than the variation of eigenvalues: An eigenspace may change rapidly with x if the eigenvalue is very close to another eigenvalue. Therefore, we need to consider not single eigenspaces but rather clusters of eigenspaces.

The variation of eigenspaces is given as follows (see [3]): Fix x. If $B(x)$ has no eigenvalue on the circle $|\lambda| = \delta$ then, with a prime denoting derivative in x,

$$(16) \qquad P_\delta' = \sum_{j:|\lambda_j|<\delta}\sum_{k:|\lambda_k|>\delta} \frac{1}{\lambda_j - \lambda_k}(P_j B' P_k + P_k B' P_j).$$

Here, all quantities are evaluated at x, and P_j is the orthogonal projection to $\operatorname{span}\varphi_j$. Taking norms and using orthogonality of the P_j one obtains from this, using $\|B'\| \leq d_{\max}$,

$$(17) \qquad \|P_\delta'\| \leq d_{\max} M \left(\min_{|\lambda_j|<\delta,\,|\lambda_k|>\delta} |\lambda_j - \lambda_k|\right)^{-1}$$

We need to choose δ carefully to make the spectral gap not too small: Let $s = (\varepsilon'/\varepsilon)^{1/(M+1)}$ and consider the M disjoint subintervals

$$[\varepsilon s^k, \varepsilon s^{k+1}) \quad \text{for } k = 1, \ldots, M$$

of $(\varepsilon, \varepsilon')$. Since $B(x_0)$ has M eigenvalues and one of them has absolute value $\leq \varepsilon$, at least one of these intervals contains no $|\lambda_j(x_0)|$. Assume

$$(18) \qquad [\delta, \delta s), \ \delta = \varepsilon s^k, \ \text{contains no } |\lambda_j(x_0)|.$$

We then have:

A) The eigenvalues of $B(x_0)$ with $|\lambda_j(x_0)| \leq \delta$ are in 1-1 correspondence with those $x \in \mathcal{Z}$ (counted with multiplicity $\dim W(x)$) satisfying

$$|x - x_0| \leq \delta' := \frac{2\delta}{d_{\min}}.$$

   (Proof: Each such eigenvalue turns into a zero of $\lambda_j(x)$ for such an x, by the argument at the beginning of this proof. Conversely, if $\lambda_j(x) = 0$ then $|\lambda_j(x_0)| \leq |x-x_0|d_{\max}$ since $|\lambda_j'| = \mu_j' \leq d_{\max}$ and hence $|\lambda_j(x_0)| \leq 2\delta\frac{d_{\max}}{d_{\min}} < \delta s$ provided ε/ε' is sufficiently small (and therefore s big), and by (18) this implies further $|\lambda_j(x_0)| < \delta$.)

B) The smaller interval $(\delta + 2\delta\frac{d_{\max}}{d_{\min}}, \delta s - 2\delta\frac{d_{\max}}{d_{\min}})$ contains no $|\lambda_j(x)|$ for any x with $|x - x_0| \leq \delta'$. (Follows directly from $|\lambda_j'| \leq d_{\max}$.)

The length of the interval in B) is $\delta(s - 1 - 4\frac{d_{\max}}{d_{\min}})$ which is $\geq \delta s/2 > 0$ if s is sufficiently big. Choose $\tilde{\delta}$ in this interval, then we get from (17) $\|P_{\tilde{\delta}}'(x)\| \leq$

$d_{\max}M\,(\delta s/2)^{-1}$ for $|x - x_0| \leq \delta'$. Integration gives

$$(19) \qquad \|P_{\tilde{\delta}}(x) - P_{\tilde{\delta}}(x_0)\| \leq \varepsilon_1 := \frac{4}{s}\frac{d_{\max}}{d_{\min}}M \quad \text{for } |x - x_0| \leq \delta'.$$

This implies

$$(20) \qquad \|\psi - P_{\delta}(x_0)\psi\| \leq \varepsilon_1\|\psi\| \quad \text{for } x \in \mathcal{Z}, \ |x - x_0| \leq \delta', \ \psi \in W(x)$$

since $P_{\tilde{\delta}}(x)\psi = \psi$ then and $P_{\tilde{\delta}}(x_0) = P_{\delta}(x_0)$.

Next we want to extend this estimate to $\psi \in W' := \bigoplus_{|x-x_0|\leq\delta'} W(x)$. For this it is essential that, by (14) and the positive definiteness of $D(x)$, the angles between different $W(x)$ are bounded away from zero. To carry this out, we first derive from (14) an estimate where all $D(x)$ are replaced by $D(x_0)$: From $D' = U''U^{-1} + D^2$ we have $\|D'\| \leq d_2 + d_{\max}{}^2$; integration yields $\|D(x_0)\| \leq \|D(x)\| + (d_2 + d_{\max}{}^2)\delta'$ for $|x - x_0| \leq \delta'$, and then (14) gives, with $D_0 := D(x_0)$,

$$(21) \qquad |\langle D_0\psi_x, \psi_y\rangle| \leq \delta''\|\psi_x\| \cdot \|\psi_y\|, \quad \delta'' := \delta'(d_2 + d_{\max}{}^2 + d_2)$$

for $\psi_x \in W(x)$, $\psi_y \in W(y)$, $x \neq y$ and $|x - x_0| \leq \delta'$, $|y - x_0| \leq \delta'$. Introduce the scalar product $(\varphi, \psi)_{D_0} := \langle D_0\varphi, \psi\rangle$ on V, with norm $\|\psi\|_{D_0} = \sqrt{\langle D_0\psi, \psi\rangle}$, then

$$(22) \qquad d_{\min}\|\psi\|^2 \leq \|\psi\|_{D_0}^2 \leq d_{\max}\|\psi\|^2$$

so (21) gives

$$(23) \qquad |(\psi_x, \psi_y)_{D_0}| \leq \frac{\delta''}{d_{\min}}\|\psi_x\|_{D_0} \cdot \|\psi_y\|_{D_0}$$

for the same ψ_x, ψ_y as there. By simple standard calculations this implies

$$(24) \qquad \|\sum_x \psi_x\|_{D_0}^2 \geq \left(1 - \frac{\delta''}{d_{\min}}(\tilde{M} - 1)\right)\sum_x \|\psi_x\|_{D_0}^2$$

where the sums are over all $x \in \mathcal{Z}$ with $|x - x_0| \leq \delta'$, $\psi_x \in W(x)$ are arbitrary and $\tilde{M}$ is the number of summands. Now by A) above $\tilde{M} \leq M$. Since $\delta < \varepsilon'$ the expression in parantheses is $\geq \frac{1}{2}$ for sufficiently small ε', so (24) gives $\sum_x \|\psi_x\|_{D_0}^2 \leq 2\|\sum_x \psi_x\|_{D_0}^2$, which with (22) gives $\sum_x \|\psi_x\|^2 \leq 2\frac{d_{\max}}{d_{\min}}\|\sum_x \psi_x\|^2$ and so

$$(25) \qquad \sum_x \|\psi_x\| \leq \sqrt{2M\frac{d_{\max}}{d_{\min}}}\|\sum_x \psi_x\|.$$

We return to (20). If $\psi \in W' = \bigoplus_{|x-x_0|\leq\delta'} W(x)$, $\psi = \sum_x \psi_x$ then we get from (25)

$$\|\psi - P_{\delta}(x_0)\psi\| \leq \sum_x \|\psi_x - P_{\delta}(x_0)\psi_x\| \leq \varepsilon_1\sum_x \|\psi_x\|$$

$$(26) \qquad \leq \varepsilon_2\|\psi\|, \quad \varepsilon_2 := \varepsilon_1\sqrt{2M\frac{d_{\max}}{d_{\min}}}$$

This means that W' is close to $W_0 := \operatorname{Ran} P_{\delta}(x_0)$. Now by A) above, W' and W_0 have the same dimension, so this implies by standard arguments that W_0 is also close to W', more precisely, with $P_{W'} : V \to W'$ the orthogonal projection,

$$(27) \qquad \|\psi - P_{W'}\psi\| \leq \varepsilon_2\|\psi\|, \quad \psi \in W_0.$$

Finally, assume that $\varphi \in V$ satisfies (11). Then (15) with δ as above together with (27) give, again by standard facts about distances of subspaces,

$$(28) \qquad \|\varphi - P_{W'}\varphi\| \le (\varepsilon_2 + \frac{\varepsilon}{\delta})\|\varphi\|.$$

Putting everything together, we have $\delta = \varepsilon s^k$ for some $k \in \{1, \dots, M\}$ with $s = (\varepsilon'/\varepsilon)^{1/(M+1)}$, which implies $\varepsilon/\delta \le s^{-1}$ and $\delta' \le \varepsilon'$. Also, $\varepsilon_1 = \frac{4}{s}\frac{d_{\max}}{d_{\min}}$ and $\varepsilon_2 = \varepsilon_1 \sqrt{2M\frac{d_{\max}}{d_{\min}}}$. Altogether, the right hand side of (28) is bounded by $Cs^{-1}\|\varphi\|$, and since the left hand side only decreases when replacing W' by the bigger space $\bigoplus_{|x-x_0|\le\varepsilon'} W(x)$, the Theorem is proven. $\qquad \square$

4. Monotonicity of U and of its logarithm

Denote by $\mathcal{S}(V)$ and $\mathcal{U}(V)$ the spaces of symmetric resp. unitary operators on V. The map $\mathcal{S}(V) \to \mathcal{U}(V)$, $A \mapsto e^{iA}$ has non-singular differential everywhere and is surjective, so it is a covering map. Hence any curve $U : x \mapsto U(x)$ in $\mathcal{U}(V)$ may be lifted to a curve $x \mapsto A(x)$ in $\mathcal{S}(V)$ (that is, $U(x) = e^{iA(x)}$ for all x) and the lift is unique if one prescribes it for one value of x. Furthermore, the lifted curve is analytic if U is.

Proposition 5. *Let $x \mapsto A(x)$ be a C^1 family of symmetric operators and $U(x) = e^{iA(x)}$. Then*

$$(29) \qquad \frac{1}{i}U'U^{-1} = \int_0^1 e^{i\tau A}A'e^{-i\tau A}\,d\tau.$$

Here, a prime denotes differentiation with respect to x, and U, U', A, A' are taken at a fixed x.

Proof. Let $W(t, x) = \frac{1}{i}\frac{\partial}{\partial x}e^{itA(x)}$. Then

$$\frac{\partial}{\partial t}W = \frac{1}{i}\frac{\partial}{\partial x}\frac{\partial}{\partial t}e^{itA} = \frac{\partial}{\partial x}Ae^{itA} = A'e^{itA} + iAW.$$

Now fix x, and let $B(t) = A'e^{itA}$. The solution of the ordinary differential equation $\frac{d}{dt}Y(t) = B(t) + iAY(t)$ with $Y(0) = 0$ is

$$Y(t) = \int_0^t e^{i(t-\tau)A}B(\tau)\,d\tau$$

as can be verified directly. (This is called Duhamel's principle.) Now $W(\cdot, x)$ satisfies the same first order ODE and initial condition as Y, so it follows that $W(t, x) = Y(t)$ for all t. Now rewrite $Y(t) = \int_0^t e^{i\tau A}B(t-\tau)\,d\tau$ and set $t = 1$ to obtain (29). $\qquad \square$

Corollary 6. *Let $x \mapsto A(x)$ be a C^1 family of symmetric operators. If $A'(x) > 0$ for each x then the unitary family $U(x) = e^{iA(x)}$ is monotone.*

Proof. Positivity of A' implies positivity of $e^{i\tau A}A'e^{-i\tau A}$ for each τ, so the claim follows from (29). $\qquad \square$

The converse is not true. As an example let $A_0 = \begin{pmatrix} 0 & -\pi \\ \pi & 0 \end{pmatrix}$, $B = \begin{pmatrix} -b & 0 \\ 0 & 1 \end{pmatrix}$ with $0 < b < 1$ and $A(x) = A_0 + xB$. Then $e^{i\tau A_0} = \begin{pmatrix} \cos \pi\tau & -\sin \pi\tau \\ \sin \pi\tau & \cos \pi\tau \end{pmatrix}$ is rotation

by $\pi\tau$, and a short calculation shows that $\int_0^1 e^{i\tau A_0} B e^{-i\tau A} \, d\tau = \frac{1-b}{2} I$ (this is also clear without calculation since the result must be rotation invariant with trace equal to $\operatorname{tr} B = 1 - b$; in essence, the negative direction of B gets averaged away against the positive direction). Therefore, $U(x) = e^{iA(x)}$ is monotone near $x = 0$ but $A'(0) = B$ is not positive.

5. TWO PARAMETER FAMILIES

Theorem 7. *Let $U(x, y)$ be a unitary operator in a finite-dimensional Hermitian vector space depending real analytically on $x, y \in \mathbb{R}$. Assume*

$$(30) \qquad\qquad \frac{1}{i}\frac{\partial U}{\partial x}U^{-1} > 0 \quad \text{at } (x_0, y_0).$$

Then the set $\{(x, y) : U(x, y) \text{ has eigenvalue one}\}$ is, in a neighborhood of (x_0, y_0), a union of real analytic curves $x = x_j(y)$. The corresponding projections $P_j(y)$ to the eigenspace of $U(x_j(y), y)$ with eigenvalue one are also analytic functions of $y \neq y_0$, extending analytically to $y = y_0$, and $\sum_j P_j(y_0)$ is the projection to $\ker(I - U(x_0, y_0))$.

Note that in general it is not true that the eigenvalues and eigenprojections of $U(x, y)$ may be arranged as real analytic functions of (x, y), see [3], II.6.1. While the example given there (in the analogous case of self-adjoint operators) does not satisfy the positivity assumption (30), it can be easily modified so it does, by adding a multiple of the identity. Explicitly, one may take $A(x, y) = \begin{pmatrix} 3x & y \\ y & x \end{pmatrix}$ and $U(x, y) = e^{iA(x,y)}$ and $(x_0, y_0) = (0, 0)$.

Note also that the statement of the theorem reduces to the well-known facts of one-parameter perturbation theory in case $U(x, y) = e^{ix}U(y)$, for an analytic one-parameter family of unitary operators $U(y)$.

Proof. Let w.l.o.g. $x_0 = y_0 = 0$.

We first consider the case $U(0, 0) = I$. Let $A = \frac{1}{i}\log U$ near $(x, y) = (0, 0)$. Then the operators $A(x, y)$ are self-adjoint, $A(0, 0) = 0$, and $\partial A/\partial x(0, 0) > 0$ since it equals $\frac{1}{i}\frac{\partial U}{\partial x}U^{-1}(0, 0)$ by (29), and we need to prove that the set $S = \{(x, y) : A(x, y) \text{ is not invertible }\}$ is a union of real analytic curves as claimed.[1]

If $A(x, y) = xA + yB$ is linear in x, y, then (since $A > 0$) A and B may be diagonalized simultaneously, hence may be assumed to be diagonal, and then it is obvious that S is a union of lines, $x_j(y) = yb_j/a_j$, where a_j, b_j are the diagonal entries of A, B, respectively. In general, write $A(x, y) = xA + yB + C(x, y)$ with $C(x, y) = O(|x, y|^2)$ and w.l.o.g. A, B diagonal. Then, if the dimension of the vector space is M,

$$\det A(x, y) = \prod_{j=1}^{M}(xa_j + yb_j) + O(|x, y|^{M+1}),$$

and a standard argument (using polar coordinates) shows that the zero set of this function is a union of real analytic lines $x = x_j(y)$, having tangents $xa_j + yb_j = 0$ at the origin.

If $U(0, 0)$ is arbitrary, let $W = U - I$ (where I denotes the identity) and $V_0 = \operatorname{Ker} W(0, 0)$ and V_1 its orthogonal complement. Let $W_{kl}(x, y)$, $k, l = 0, 1$, be the

---

[1] I am grateful to Y. Colin-de-Verdière for a fruitful discussion on this

'submatrices' of $W(x, y)$ corresponding to the decomposition $V_0 \oplus V_1$. Then W_{00}, W_{01} and W_{10} vanish at $(x, y) = (0, 0)$, and W_{11} is invertible at $(0, 0)$ and hence in a neighborhood. Then the equation $W(v_0 \oplus v_1) = 0$, where $v_0 \in V_0$, $v_1 \in V_1$, is equivalent to $W'v_0 = 0$, where $W' = W_{00} - W_{01}W_{11}^{-1}W_{10}$, and $v_1 = -W_{11}^{-1}W_{10}v_0$. Therefore, $U(x, y)$ has eigenvalue one iff the operator $U'(x, y) = W'(x, y) + I_{V_0}$ on V_0 has eigenvalue one. One easily checks that $U'(x, y)$ is unitary. Since $W'(0, 0) = 0$ the claim now follows from the case considered first.

Let $C_j(y) = U(x_j(y), y)$ and let $P_j(y)$ be the projection to $\operatorname{Ker} C_j(y)$. Since C_j is analytic in y, its eigenprojections are analytic for $y \neq 0$ (but near zero) and extend analytically to $y = 0$ (see [3]), so this is in particular true for P_j. □

References

1. S. Gnutzmann and U. Smilansky, *Quantum graphs: Applications to quantum chaos and universal spectral statistics*, Advances in Physics **55** (2006), no. 5-6, 527–625.
2. Daniel Grieser, *Spectra of graph neighborhoods and scattering*, Preprint arXiv:0710.3405, 2007.
3. Tosio Kato, *Perturbation theory for linear operators. 2nd ed.*, Grundlehren der mathematischen Wissenschaften. 132. Berlin-Heidelberg-New York: Springer-Verlag. , 1976.
4. Franz Rellich, *Störungstheorie der Spektralzerlegung. I. Analytische Störung der isolierten Punkteigenwerte eines beschränkten Operators.*, Math. Ann. **113** (1936), 600–619 (German).

INSTITUT FÜR MATHEMATIK, CARL VON OSSIETZKY UNIVERSITÄT OLDENBURG, D-26111 OLDENBURG
E-mail address: `grieser@mathematik.uni-oldenburg.de`