Transforms and minors for binary functions

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Abstract

In previous work, we introduced a continuum of generalised minor operations for the class of binary functions $f : 2^E \to \mathbb{R}$, with deletion and contraction being two distinct points in the continuum. The specialisation to graphs is obtained by letting f be the indicator function of the cutset space of a graph.

In this paper, we consider the consequences of using a different parameterisation of this family of operations, with parameters taken from the complex numbers. We give a family of transforms that extends duality, and which includes trinities and so on. Composition of these transforms corresponds to multiplication of their complex parameters. We establish how the transforms interact with our generalised minors, extending the classical matroid-theoretic relationship between duality and minors: $(M/e)^* = M^* \setminus e$. A new generalisation of the MacWilliams identity is given, using these transforms in place of ordinary duality. We also relate the weight enumerator of a linear code at a real argument < -1 to the transform, with parameter on the unit circle, of a close relative of the indicator function of the dual code. This result extends to arbitrary codes and arbitrary binary functions. The results on weight enumerators can also be recast in terms of the partition function of the Ising model from statistical mechanics.

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1 Introduction

The operations of *deletion* and *contraction* of an edge in a graph (or an element in a matroid) are of fundamental importance. Many of the most celebrated results in graph theory are characterisations of classes of structures in terms of the presence or absence of substructures (*minors*) that can be formed from some sequence of these operations. These range from Kuratowski's characterisation of planar graphs [10, 14] to the proof by Robertson and Seymour [13] of Wagner's conjecture that *any* minor-closed class of graphs can be characterised by a finite set of forbidden minors. Deletion and contraction also play a central role in enumeration problems, as may be seen in the Tutte-Whitney polynomials (see, e.g., [16]).

In previous work [3], the author extended these operations by giving a whole family of λ-*minor* operations. Deletion and contraction are obtained by taking $\lambda = 1$ and $\lambda = 0$, respectively. The theory of these operations is developed in [3, 4, 5]. It was shown that λ-*minor* operations commute and come in dual pairs, with duality described by the involution $\lambda^* = (1-\lambda)/(1+\lambda)$. A λ-*rank transform* was defined which, if given the indicator function of the cutset space of a graph (or cocircuit space of a binary matroid), produces the usual rank function when $\lambda = 1$ and its dual when $\lambda = 0$. It was shown to behave appropriately in terms of its relationship with duality and the λ-*minor* operations. The λ-rank transform was used to define a family of λ-*Whitney functions*, which generalise the usual Tutte-Whitney polynomials of graphs or matroids. These were shown to have a "deletion-contraction relation", analogous to that of the usual Tutte-Whitney polynomials, but using dual pairs of λ-minor operations. The λ-Whitney function was shown to contain (by means of appropriate partial evaluations) the weight enumerator of a code and the partition function of the symmetric Ashkin-Teller model from statistical mechanics. (This model was introduced in [1], and is one of the oldest graphical models in statistical mechanics.) This partition function cannot be found from the ordinary Tutte-Whitney polynomials of a graph.

The starting point of the present paper is to introduce a different parameterisation for the family of λ-minor operations. To do this, we use an appropriate fractional linear transformation $\mu = s(\lambda)$, under which duality corresponds simply to negation of the parameter μ. We also introduce a new family of transforms $L^{[\mu]}$, $\mu \in \mathbb{C}$, where $L^{[\mu]}$ specialises to the identity transform when $\mu = 1$ and to a scalar multiple of the Hadamard transform when $\mu = -1$. The transform $L^{[\mu]}$ generalises the Hadamard transform by providing trinity and other higher-order relationships that extend duality. As with the Hadamard transform, $L^{[\mu]}$ may be expressed in matrix form as a Kronecker power of an appropriate 2×2 matrix, and many of our proofs make use of this observation.

Our parameterisation, together with the transforms $L^{[\mu]}$, allows us to link the algebraic relationships among our various transforms (rank, inverse rank, and $L^{[\mu]}$) to the complex numbers under multiplication.

We begin by showing that composition of transforms $L^{[\mu]}$ (where $\mu \in \mathbb{C}$) corresponds exactly to multiplying their parameters μ, so that

$$L^{[\mu_1]}L^{[\mu_2]} = L^{[\mu_1\mu_2]}.$$

We then look at how the rank transform interacts with $L^{[\mu]}$. The rank transform (with μ_1) of $L^{[\mu_2]}$ is shown to be another rank transform, this time with parameter $\mu_1\mu_2$. The inverse rank transform (with μ_1) of a rank transform (with μ_2) is shown to be a multiple of $L^{[\mu_2/\mu_1]}$. So the new transforms $L^{[\mu]}$ allow us to describe composition of an inverse rank transform with a rank transform, even in cases where their parameters are different so that they are not mutual inverses.

Since the transforms $L^{[\mu]}$ generalise duality, it is natural to ask whether classical results about duality may be extended to them. We are able to show that the classical MacWilliams identity for binary linear codes, where the weight enumerator of a code is related to that of its dual, may be generalised to use $L^{[\mu]}$ for any $\mu \in \mathbb{C}$ instead of just duality ($\mu = -1$). We also prove analogues of Plancherel's and Parseval's Theorems.

With this new parameterisation, our generalised minor operations are shown to interact with the transform $L^{[\mu]}$ in a way that, once again, mimics the complex numbers under multiplication: the transform $L^{[\mu_1]}$ of a $[\mu_2]$-minor is essentially the $[\mu_1\mu_2]$-minor of the transform. This generalises the classical relationship between duality, deletion and contraction, and provides a pattern for how generalised minors may be expected to behave under trinity, quaternity and so on.

Finally, we link the weight enumerator of a binary linear code, evaluated at a real argument < -1, to the $L^{[\mu]}$ transform, with $\mu \in \mathbb{C}$ and $|\mu| = 1$, of a close relative of the indicator function of the dual code (for an appropriate μ). Our result extends to arbitrary codes and indeed to arbitrary binary functions.

Our results on weight enumerators can also be recast in terms of the partition function of the Ising model from statistical mechanics, via the close relationship between them.

All our results are established in the general setting of *binary functions* $f : 2^E \to \mathbb{C}$. The special case of graphs (respectively, binary matroids) is obtained by letting f be the indicator function of the cutset space of the graph (the cocircuit space of the binary matroid). The first investigation of minor operations at essentially this level of generality was by Kung [9], who showed (in different language) how to extend deletion and contraction to this level. This theme was taken up by the author in [2], where expressions for deletion and contraction in binary function language were given, and the Whitney rank generating function was generalised to arbitrary binary functions. That paper also linked the Hadamard transform to duality, even in cases where the objects being transformed do not come from graphs or binary matroids. The continuum of minor operations, interpolating between deletion and contraction, was introduced in [3], once again at the level of general binary functions. However, the algebraic connection between this work and the complex numbers under multiplication has not been investigated previously.

A survey of work in this area to date is given in [4]. Connections with the work of Kung are discussed in [3, §2.2].

Other applications of the Hadamard transform and related Fourier transforms to graph theory are discussed in [6]. Other generalisations of Hadamard transforms, in a different direction to those of the present paper, are studied in [7].

For more information on matroid theory, see [12, 15].

In the next section we give background on the previous work upon which this paper is based. Notation and definitions are given in §3. The new parameterisation and transform family are described in §4. Our results follow in §§5–7. Results on composition of the various transforms are given in §5, along with our generalisation of the MacWilliams Identity. The main result on how the $L^{[\mu]}$ transforms interact with our minor operations is given in §6. Further results, including generalisations of Plancherel's and Parseval's Theorems and the other abovementioned result on weight enumerators, are in §7.

2 Background

Let $f : 2^E \to \{0, 1\}$ be the indicator function of the cutset space of a graph $G = (V, E)$. Expressions over $\mathbb{Q}$ for the indicator functions $f /\!\!/ e$ and $f \setminus\!\!\setminus e$ of the cutset spaces of the graphs G/e and $G \setminus e$, obtained by contracting and deleting $e \in E$ respectively, were given in [2]. These are:

$$f /\!\!/ e(X) = \frac{f(X)}{f(\emptyset)}, \qquad f \setminus\!\!\setminus e(X) = \frac{f(X) + f(X \cup \{e\})}{f(\emptyset) + f(\{e\})}. \tag{1}$$

The same paper also gave the following expression for the rank function Qf of the cycle matroid (E, Qf) of G:

$$Qf(X) = \log_2 \left(\frac{\sum_{Y \subseteq E} f(Y)}{\sum_{Y \subseteq E \setminus X} f(Y)} \right). \tag{2}$$

We call the operator Q the *rank transform*. It has an inverse $Q^\dagger$, defined by

$$(Q^\dagger \rho)(X) = (-1)^{|X|} \sum_{Y \subseteq X} (-1)^{|Y|} 2^{\rho(E) - \rho(E \setminus Y)}.$$

Note the similarity with the chromatic polynomial: in fact, when $X = E$, this has the form of the chromatic polynomial, evaluated at 2, treated as a function of ρ.

The paper [2] applies the expressions (1) and (2) to arbitrary functions on 2^E (provided $f(\emptyset) \neq 0$), and shows that the usual relationships between rank functions, minors and duality still hold, in some form, at that level of generality (although the "rank function" given by Qf is in general not integer-valued or submodular).

In [3], the author introduced a family of minor operations, extending contraction and deletion and interpolating between them. If $\lambda \in \mathbb{C}$, the λ-*minor* of f by e, denoted by $f \,\|_\lambda e$, is given by

$$f \,\|_\lambda e(X) = \frac{f(X) + \lambda f(X \cup \{e\})}{f(\emptyset) + \lambda f(\{e\})}.$$

The special cases $\lambda = 0$ and $\lambda = 1$ give contraction and deletion, respectively, as in (1). These minor operations come in dual pairs, where duality is described by the involution

$$\lambda^* = \frac{1-\lambda}{1+\lambda}.$$

The duality between the usual deletion and contraction operations is expressed by the fact that $0^* = 1$ and $1^* = 0$. Each value of the parameter λ has an associated *rank transform* $Q^{(\lambda)}$ defined by

$$Q^{(\lambda)} f(X) = \log_2 \left(\frac{(1+\lambda^*)^{|X|} \sum_{Y \subseteq E} \lambda^{|Y|} f(Y)}{\sum_{Y \subseteq E} \lambda^{|Y \setminus X|} (\lambda^*)^{|Y \cap X|} f(Y)} \right).$$

This definition is shown to be the right one to use with the λ-minor operations, in that once again the expected relations among minors, rank and duality can be shown to hold. If $\lambda \neq \lambda^*$ then $Q^{(\lambda)}$ has an inverse $Q^{(\lambda)}$, given by

$$\begin{aligned} Q^{\dagger(\lambda)} \rho(X) &= (\lambda - \lambda^*)^{-|E|} (-1)^{|X|} \sum_{Y \subseteq E} (-1)^{|Y|} (1+\lambda^*)^{-|Y|} (\lambda^*)^{|Y \setminus X|} \lambda^{|E \setminus (Y \cup X)|} \\ & \qquad \times 2^{\rho(E) - \rho(E \setminus Y)}. \end{aligned}$$

The two self-dual values of the parameter λ (i.e., where $\lambda^* = \lambda$) are $\alpha_1 := \sqrt{2} - 1$ and $\alpha_2 := -\sqrt{2} - 1$. These may be regarded as degenerate, in that the rank transform for such λ loses all information about values of f and so is not invertible [3, p. 250]:

$$Q^{(\alpha_1)} f(X) = |X|/2, \qquad Q^{(\alpha_2)} f(X) = |X| i\pi/2,$$

for all f and all $X \subseteq E$.

3 Definitions and notation

If $f : 2^E \to \mathbb{C}$, then its *Hadamard transform* $\hat{f} : 2^E \to \mathbb{C}$ is defined by

$$\hat{f}(X) = 2^{-|E|} \sum_{Y \subseteq E} (-1)^{|X \cap Y|} f(Y)$$

for all $X \subseteq E$.

If $z \in \mathbb{C}$ then the function $z^{|\bullet|} : 2^E \to \mathbb{C}$ maps $X \subseteq E$ to $z^{|X|}$.

Composition of transforms will be written by juxtaposition.

A function $f : 2^E \to \mathbb{C}$ may be regarded as a vector $\mathbf{f} \in \mathbb{C}^{2^m}$, with entries f_x indexed by $x = 0, \ldots, 2^m - 1$, where $m = |E|$. The integer x gives an m-bit vector $\mathbf{x}$, via its binary representation, which in turn is regarded as the characteristic vector of a set $X \subseteq E$. Under this correspondence, $f_x = f(\mathbf{x}) = f(X)$. The Hadamard transform may be viewed as a linear function on such vectors $\mathbf{f}$ with matrix $2^{-m} H_2^{\otimes m}$, using the m-th Kronecker power $H_2^{\otimes m}$ of the 2×2 Hadamard matrix

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

We write

$$A_f(\lambda) = \sum_{X \subseteq E} \lambda^{|X|} f(X).$$

When f is $\{0,1\}$-valued, A_f is the weight enumerator of the support of f.

The partition function $Z_I(G;K)$ of the Ising model [8] on G is closely related to the weight enumerator of the cutset space:

$$Z_I(G;K) = e^{K|E|} 2^{c(G)} A_f(e^{-2K}),$$

where $c(G)$ is the number of connected components of G and f is the indicator function of its cutset space. This may be used to extend the Ising model partition function to any binary function f:

$$Z_I(f;K) := 2e^{K|E|} A_f(e^{-2K}). \tag{3}$$

The partition functions $Z_I(G;K)$ and $Z_I(f;K)$ are identical when G is connected and f is the indicator function of its cutset space. The definition (3) is similar in spirit to our extension of the Potts model partition function to binary functions in [3, §3].

4 The new parameterisation and associated transforms

Define $s : \mathbb{C} \setminus \{\alpha_2\} \to \mathbb{C} \setminus \{\alpha_2^2\}$ by

$$s(\lambda) = \frac{1 - \lambda/\alpha_1}{-1 + \lambda/\alpha_2} = -(3 + 2\sqrt{2}) \frac{\sqrt{2} - 1 - \lambda}{\sqrt{2} + 1 + \lambda}.$$

Then $s^{-1} : \mathbb{C} \setminus \{\alpha_2^2\} \to \mathbb{C} \setminus \{\alpha_2\}$ is defined by

$$s^{-1}(\mu) = \frac{1 + \mu}{1/\alpha_1 + \mu/\alpha_2} = \frac{1 + \mu}{\sqrt{2} + 1 - (\sqrt{2} - 1)\mu}.$$

Note that $s(\alpha_1) = 0$, and the maps s and s^{-1} are easily extended to all of $\mathbb{C} \cup \{\infty\}$ by putting $s(\alpha_2) = \infty$ and $s^{-1}(\infty) = \alpha_2$.

Observe that duality is described simply by negation of the parameter μ:

$$s(\lambda^*) = -s(\lambda), \qquad s^{-1}(-\mu) = (s^{-1}(\mu))^*. \tag{4}$$

This holds for all $\lambda \in \mathbb{C} \cup \{\infty\}$ provided we put $\infty^* = -1$, $(-1)^* = \infty$ and $-\infty = \infty$.

We use the mapping $\mu = s(\lambda)$ to give another way of parameterising the λ-minor operation. This new parameterisation turns out to have several advantages, and it will help to have succinct notation for it. To this end, we write $f \,\|_{[\mu]} e$ for the function $f \,\|_{s^{-1}(\mu)} e$, where $e \in E$. We call $f \,\|_{[\mu]} e$ the $[\mu]$-*minor* of f by e. In effect, we use $[\mu]$ as a shorthand for $s^{-1}(\mu)$, at times, in order to reduce notational clutter.

Let $\mu \in \mathbb{C}$. Put

$$\begin{aligned} a(\mu) &= \sqrt{2} - 1 + (\sqrt{2} + 1)\mu, \\ b(\mu) &= 1 - \mu, \\ c(\mu) &= \sqrt{2} + 1 + (\sqrt{2} - 1)\mu. \end{aligned}$$

Using these functions, we can write

$$s^{-1}(\mu) = \frac{b(-\mu)}{c(-\mu)}. \tag{5}$$

It will also be convenient to note that

$$1 + s^{-1}(-\mu) = \sqrt{2} \cdot \frac{c(-\mu)}{c(\mu)}. \tag{6}$$

The transform $L^{[\mu]}$ maps any $f : 2^E \to \mathbb{C}$ to the function $L^{[\mu]} f : 2^E \to \mathbb{C}$ defined for all $V \subseteq E$ by

$$(L^{[\mu]} f)(V) = (2\sqrt{2})^{-|E|} \sum_{X \subseteq E} a(\mu)^{|X \cap V|} b(\mu)^{|X \setminus V| + |V \setminus X|} c(\mu)^{|E \setminus (X \cup V)|} f(X).$$

In performing the exponentiations in the summand here, we replace any occurrence of 0^0 by 1. We frequently write $L^{[\mu]} f(V)$ for $(L^{[\mu]} f)(V)$. It is easy to verify that $L^{[1]}$ is the identity transform, while $L^{[-1]}$ is a multiple of the Hadamard transform:

$$L^{[-1]} f(V) = \sqrt{2}^{|E|} \cdot \hat{f}(V),$$

for all $V \subseteq E$. The prefactor on the right hand side may be regarded as a scaling factor to ensure that $L^{[-1]} f(\emptyset) = 1$.

Note that, for any f and any $\mu \in \mathbb{C} \setminus \{-\alpha_2^2\}$,

$$\begin{aligned} (L^{[\mu]} f)(\emptyset) &= (2\sqrt{2})^{-|E|} \sum_{X \subseteq E} b(\mu)^{|X|} c(\mu)^{|E \setminus X|} f(X) \\ &= \left(\frac{c(\mu)}{2\sqrt{2}} \right)^{|E|} \sum_{X \subseteq E} \left(\frac{b(\mu)}{c(\mu)} \right)^{|X|} f(X) \\ &= \left(\frac{c(\mu)}{2\sqrt{2}} \right)^{|E|} A_f(s^{-1}(-\mu)). \end{aligned}$$

Under the correspondence between functions and vectors described earlier, the transform $L^{[\mu]}$ is a linear map from $\mathbb{C}^{2^m}$ to itself, and its matrix is given by the m-th Kronecker power $M(\mu)^{\otimes m}$ of the 2×2 matrix

$$M(\mu) = \frac{1}{2\sqrt{2}} \begin{pmatrix} c(\mu) & b(\mu) \\ b(\mu) & a(\mu) \end{pmatrix}.$$

Computation of $L^{[\mu]}$ can be done in time $O(N \log N \log\log N)$, where N is input size, using the same kind of fast transform techniques that are used for the Hadamard transform. See also the discussion in [4, §3.8].

It is a straightforward exercise to use the matrix view of the transform, together with the mixed product property for the Kronecker product, to describe the effect of the transform on the function $z^{|\bullet|}$ (for any $z \in \mathbb{C}$):

$$L^{[\mu]} z^{|\bullet|} = \left(\frac{c(\mu) + b(\mu) z}{2\sqrt{2}} \right)^{|E|} \cdot s^{-1}(\mu\, s(z))^{|\bullet|}.$$

Note that $L^{[\mu]}$ is invertible if and only if $\mu \neq 0$. If $\mu = 0$, then we find that

$$L^{[0]} f(V) = (2\sqrt{2})^{-1} A_f(\sqrt{2} - 1)(\sqrt{2} + 1)^{|E \setminus V|}.$$

So $L^{[0]}$ loses all information about f except for the size of E and the value of $A_f(\sqrt{2} - 1)$.

5 Composition of the transforms

It is easy to show that, for any $\mu_1, \mu_2 \in \mathbb{C}$,

$$\begin{aligned} a(\mu_1)a(\mu_2) + b(\mu_1)b(\mu_2) &= 2\sqrt{2} \cdot a(\mu_1\mu_2), & (7) \\ b(\mu_1)a(\mu_2) + c(\mu_1)b(\mu_2) &= 2\sqrt{2} \cdot b(\mu_1\mu_2), & (8) \\ a(\mu_1)b(\mu_2) + b(\mu_1)c(\mu_2) &= 2\sqrt{2} \cdot b(\mu_1\mu_2), & (9) \\ b(\mu_1)b(\mu_2) + c(\mu_1)c(\mu_2) &= 2\sqrt{2} \cdot c(\mu_1\mu_2). & (10) \end{aligned}$$

From this we obtain

Lemma 1 *For any $\mu_1, \mu_2 \in \mathbb{C}$,*

$$M(\mu_1)M(\mu_2) = M(\mu_1\mu_2).$$

□

We first show that composition of transforms $L^{[\mu]}$ corresponds to multiplication of the corresponding μ parameters.

Theorem 2 *For any $\mu_1, \mu_2 \in \mathbb{C}$,*

$$L^{[\mu_1]} L^{[\mu_2]} = L^{[\mu_1\mu_2]}.$$

Proof. Using the matrix representation of $L^{[\mu]}$:

$$\begin{aligned} \text{Matrix for } L^{[\mu_1]} L^{[\mu_2]} &= M(\mu_1)^{\otimes m} M(\mu_2)^{\otimes m} \\ &= (M(\mu_1)M(\mu_2))^{\otimes m} \\ &\quad \text{(by the mixed-product property for Kronecker product)} \\ &= M(\mu_1\mu_2)^{\otimes m} \qquad \text{(by Lemma 1)} \\ &= \text{Matrix for } L^{[\mu_1\mu_2]}. \end{aligned}$$

□

We now describe the effect of composing the $[\mu_1]$-rank transform with $L^{[\mu]}$.

Theorem 3 *If $\mu_1, \mu_2 \in \mathbb{C}$, $\mu_1 \neq \pm\alpha_2^2$ and $\mu_1\mu_2 \neq \pm\alpha_2^2$, then*

$$Q^{[\mu_1]}L^{[\mu_2]} = Q^{[\mu_1\mu_2]}.$$

Proof. Let $V \subseteq E$. We consider the main sum in the rank transform on the left hand side.

$$\begin{aligned} &\sum_{W\subseteq E} s^{-1}(\mu_1)^{|W\setminus V|}(s^{-1}(\mu_1)^*)^{|W\cap V|}L^{[\mu_2]}f(W) \\ &= V\text{-entry of: } \quad \begin{pmatrix} 1 & s^{-1}(\mu_1) \\ 1 & s^{-1}(-\mu_1) \end{pmatrix}^{\otimes m} M(\mu_2)^{\otimes m}\,\mathbf{f}. \end{aligned} \tag{11}$$

Now,

$$\begin{aligned} \begin{pmatrix} 1 & s^{-1}(\mu_1) \\ 1 & s^{-1}(-\mu_1) \end{pmatrix} &= \frac{1}{c(-\mu_1)}\begin{pmatrix} c(-\mu_1) & b(-\mu_1) \\ c(-\mu_1) & \frac{b(\mu_1)c(-\mu_1)}{c(\mu_1)} \end{pmatrix} \\ &= \frac{1}{c(-\mu_1)}\begin{pmatrix} 1 & 0 \\ 0 & \frac{c(-\mu_1)}{c(\mu_1)} \end{pmatrix}\begin{pmatrix} c(-\mu_1) & b(-\mu_1) \\ c(\mu_1) & b(\mu_1) \end{pmatrix} \end{aligned}$$

Hence

$$\begin{aligned} &c(-\mu_1)\begin{pmatrix} 1 & s^{-1}(\mu_1) \\ 1 & s^{-1}(-\mu_1) \end{pmatrix} M(\mu_2) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \frac{c(-\mu_1)}{c(\mu_1)} \end{pmatrix}\frac{1}{2\sqrt{2}}\begin{pmatrix} c(-\mu_1) & b(-\mu_1) \\ c(\mu_1) & b(\mu_1) \end{pmatrix}\begin{pmatrix} c(\mu_2) & b(\mu_2) \\ b(\mu_2) & a(\mu_2) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \frac{c(-\mu_1)}{c(\mu_1)} \end{pmatrix}\frac{1}{2\sqrt{2}}\begin{pmatrix} c(-\mu_1)c(\mu_2)+b(-\mu_1)b(\mu_2) & c(-\mu_1)b(\mu_2)+b(-\mu_1)a(\mu_2) \\ c(\mu_1)c(\mu_2)+b(\mu_1)b(\mu_2) & c(\mu_1)b(\mu_2)+b(\mu_1)a(\mu_2) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \frac{c(-\mu_1)}{c(\mu_1)} \end{pmatrix}\begin{pmatrix} c(-\mu_1\mu_2) & b(-\mu_1\mu_2) \\ c(\mu_1\mu_2) & b(\mu_1\mu_2) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \frac{c(-\mu_1)}{c(\mu_1)} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{c(\mu_1\mu_2)}{c(-\mu_1\mu_2)} \end{pmatrix}\begin{pmatrix} c(-\mu_1\mu_2) & b(-\mu_1\mu_2) \\ c(-\mu_1\mu_2) & \frac{b(\mu_1\mu_2)c(-\mu_1\mu_2)}{c(\mu_1\mu_2)} \end{pmatrix} \\ &= \begin{pmatrix} c(-\mu_1\mu_2) & 0 \\ 0 & \frac{c(-\mu_1)c(\mu_1\mu_2)}{c(\mu_1)} \end{pmatrix}\begin{pmatrix} 1 & \frac{b(-\mu_1\mu_2)}{c(-\mu_1\mu_2)} \\ 1 & \frac{b(\mu_1\mu_2)}{c(\mu_1\mu_2)} \end{pmatrix} \\ &= \begin{pmatrix} c(-\mu_1\mu_2) & 0 \\ 0 & \frac{c(-\mu_1)c(\mu_1\mu_2)}{c(\mu_1)} \end{pmatrix}\begin{pmatrix} 1 & s^{-1}(\mu_1\mu_2) \\ 1 & s^{-1}(-\mu_1\mu_2) \end{pmatrix}. \end{aligned}$$

Our main sum (11) is therefore

$$\sum_{W\subseteq E} s^{-1}(\mu_1)^{|W\setminus V|}(s^{-1}(\mu_1)^*)^{|W\cap V|}L^{[\mu_2]}f(W)$$
$$= \left(\frac{c(-\mu_1)c(\mu_1\mu_2)}{c(\mu_1)}\right)^{|V|} c(-\mu_1\mu_2)^{|E\setminus V|}\sum_{X\subseteq E} f(X)\,(s^{-1}(\mu_1\mu_2)^*)^{|V\cap X|}s^{-1}(\mu_1\mu_2)^{|X\setminus V|} \tag{12}$$

Observe also that the special case $V=\emptyset$ of (12) gives

$$\sum_{W\subseteq E} s^{-1}(\mu_1)^{|W|}L^{[\mu_2]}f(W) = c(-\mu_1)^{-|E|}c(-\mu_1\mu_2)^{|E|}\sum_{X\subseteq E} f(X)s^{-1}(\mu_1\mu_2)^{|X|}. \tag{13}$$

We can now give the main line of the proof.

$$\begin{aligned}
2^{(Q^{[\mu_1]}L^{[\mu_2]}f)(V)} &= \frac{(1+s^{-1}(-\mu_1))^{|V|}\sum_{W\subseteq E} s^{-1}(\mu_1)^{|W|}L^{[\mu_2]}f(W)}{\sum_{W\subseteq E} s^{-1}(\mu_1)^{|W\setminus V|}s^{-1}(-\mu_1)^{|W\cap V|}L^{[\mu_2]}f(W)}\\
&= \frac{\sqrt{2}^{|V|}c(-\mu_1\mu_2)^{|V|}\sum_{X\subseteq E} f(X)s^{-1}(\mu_1\mu_2)^{|X|}}{c(\mu_1\mu_2)^{|V|}\sum_{X\subseteq E} f(X)(s^{-1}(\mu_1\mu_2)^*)^{|V\cap X|}s^{-1}(\mu_1\mu_2)^{|X\setminus V|}}\\
&\qquad\text{(by (6), (12) and (13))}\\
&= \frac{(1+s^{-1}(-\mu_1\mu_2))^{|V|}\sum_{X\subseteq E} f(X)s^{-1}(\mu_1\mu_2)^{|X|}}{\sum_{X\subseteq E} f(X)(s^{-1}(\mu_1\mu_2)^*)^{|V\cap X|}s^{-1}(\mu_1\mu_2)^{|X\setminus V|}}\\
&\qquad\text{(by (6))}\\
&= 2^{Q^{[\mu_1\mu_2]}f(V)}.
\end{aligned}$$

□

The next result describes how transforming a function f, using $L^{[\mu]}$, has the same effect on A_f as an appropriate transformation on its variable. This generalises the MacWilliams identity [11], which is obtained by taking f to be the indicator function of a linear code over $GF(2)$ and $\mu_2 = -\mu_1$.

Theorem 4 *If $\mu_1,\mu_2\in\mathbb{C}$, $\mu_1\notin\{0,\pm\alpha_2^2\}$ and $\mu_1\mu_2\neq\pm\alpha_2^2$, then*

$$A_{L^{[\mu_2/\mu_1]}f}(s^{-1}(\mu_1)^*) = \left(\frac{c(\mu_2)}{c(\mu_1)}\right)^m A_f(s^{-1}(\mu_2)^*).$$

Proof.

$$A_{L^{[\mu_2/\mu_1]}f}(s^{-1}(\mu_1)^*)$$
$$= \sum_{X\subseteq E}\left(\frac{b(\mu_1)}{c(\mu_1)}\right)^{|X|} L^{[\mu_2/\mu_1]}f(X)$$

$$= \begin{pmatrix} 1 & \frac{b(\mu_1)}{c(\mu_1)} \end{pmatrix}^{\otimes m} M(\mu_2/\mu_1)^{\otimes m} \mathbf{f}$$

(writing previous line in matrix form)

$$= \left(\begin{pmatrix} 1 & \frac{b(\mu_1)}{c(\mu_1)} \end{pmatrix} \cdot M(\mu_2/\mu_1) \right)^{\otimes m} \mathbf{f}$$

(by the mixed-product property)

$$= c(\mu_1)^{-m}(2\sqrt{2})^{-m} \times \begin{pmatrix} c(\mu_1)c(\mu_2/\mu_1) + b(\mu_1)b(\mu_2/\mu_1) & c(\mu_1)b(\mu_2/\mu_1) + b(\mu_1)a(\mu_2/\mu_1) \end{pmatrix}^{\otimes m} \mathbf{f}$$

$$= c(\mu_1)^{-m} \begin{pmatrix} c(\mu_2) & b(\mu_2) \end{pmatrix}^{\otimes m} \mathbf{f} \qquad \text{(by (10) and (8))}$$

$$= \left(\frac{c(\mu_2)}{c(\mu_1)}\right)^m \begin{pmatrix} 1 & \frac{b(\mu_2)}{c(\mu_2)} \end{pmatrix}^{\otimes m} \mathbf{f}$$

$$= \left(\frac{c(\mu_2)}{c(\mu_1)}\right)^m \begin{pmatrix} 1 & s^{-1}(\mu_2)^* \end{pmatrix}^{\otimes m} \mathbf{f} \qquad \text{(by (5) and (4))}$$

$$= \left(\frac{c(\mu_2)}{c(\mu_1)}\right)^m \sum_{X \subseteq E} (s^{-1}(\mu_2)^*)^{|X|} f(X)$$

$$= \left(\frac{c(\mu_2)}{c(\mu_1)}\right)^m A_f(s^{-1}(\mu_2)^*).$$

The result follows. □

We may rewrite this identity in terms of Ising model partition functions.

Corollary 5 *If $\mu_1, \mu_2 \in \mathbb{C}$, $\mu_1 \notin \{0, \pm\alpha_2^2\}$ and $\mu_1\mu_2 \neq \pm\alpha_2^2$, then*

$$c(\mu_1)^m Z_I(L^{[\mu_2/\mu_1]} f, K_1) = c(\mu_2)^m Z_I(f, K_2),$$

where, for $i = 1, 2$,

$$K_i = -\tfrac{1}{2} \ln s^{-1}(-\mu_i).$$

□

The transform $L^{[\mu]}$ allows us to describe the effect of composing parameterised versions of the rank transform and the inverse rank transform, in cases where the two transforms have different parameters.

Theorem 6 *If $\mu_1, \mu_2 \in \mathbb{C}$, $\mu_1 \notin \{0, \pm\alpha_2^2\}$ and $\mu_1\mu_2 \neq \pm\alpha_2^2$, then*

$$Q^{\dagger[\mu_1]} Q^{[\mu_2]} f = \left(\frac{c(\mu_1)}{c(\mu_2)}\right)^{|E|} \cdot A_f(s^{-1}(\mu_2)^*)^{-1} \cdot L^{[\mu_2/\mu_1]} f$$

Proof. Suppose $f : 2^E \to \mathbb{C}$ with $f(\emptyset) \neq 0$.

$$\begin{aligned} Q^{\dagger[\mu_1]} Q^{[\mu_2]} f &= Q^{\dagger[\mu_1]} Q^{[\mu_1 \cdot \mu_2/\mu_1]} f && \text{(since } \mu_1 \neq 0\text{)} \\ &= Q^{\dagger[\mu_1]} Q^{[\mu_1]} L^{[\mu_2/\mu_1]} f && \text{(by Theorem 3)} \\ &= A_{L^{[\mu_2/\mu_1]} f}(s^{-1}(\mu_1)^*)^{-1} \cdot L^{[\mu_2/\mu_1]} f && \text{(by [3, Theorem 6(a)])} \\ &= \left(\frac{c(\mu_1)}{c(\mu_2)}\right)^m A_f(s^{-1}(\mu_2)^*)^{-1} \cdot L^{[\mu_2/\mu_1]} f && \text{(by Theorem 4).} \end{aligned}$$

□

Theorem 7 *If $\mu_1, \mu_2, \mu_3 \in \mathbb{C} \setminus \{\pm\alpha_2^2\}$, $\mu_2 \neq 0$ and $\mu_1\mu_3/\mu_2 \neq \pm\alpha_2^2$, then*

$$Q^{[\mu_1]}Q^{\dagger[\mu_2]}Q^{[\mu_3]} = Q^{[\mu_1\mu_3/\mu_2]}.$$

Proof. Let $f : 2^E \to \mathbb{C}$. Then

$$\begin{aligned}
Q^{[\mu_1]}Q^{\dagger[\mu_2]}Q^{[\mu_3]}f &= Q^{[\mu_1]}(c \cdot L^{[\mu_3/\mu_2]}f) && \text{(for some } c\text{, by Theorem 6)}\\
&= Q^{[\mu_1]}L^{[\mu_3/\mu_2]}f && \text{(by [3, p. 250])}\\
&= Q^{[\mu_1\mu_3/\mu_2]}f && \text{(by Theorem 3).}
\end{aligned}$$

Note that the quantity c introduced in the first equation here depends on f, μ_1, μ_2 and μ_3. The elimination of this quantity in the next line is still valid. The only requirement of that elimination step is that c does not depend on the argument of the function being transformed, i.e., it is not itself a function on 2^E, but is, in effect, just a scalar. □

6 Minors

We now look at how the $L^{[\mu]}$ transform interacts with the $[\mu]$-minor operations. Our main result on this will use the next lemma.

In what follows, $f_e : 2^{E\setminus\{e\}} \to \mathbb{C}$ is defined for all $X \subseteq E \setminus \{e\}$ by $f_e(X) = f(X \cup \{e\})$. The function $f|_{E\setminus\{e\}}$ has the usual meaning of the restriction of f to $E \setminus \{e\}$.

Lemma 8 *If $\mu_1 \in \mathbb{C} \setminus \{0\}$, $\mu_2 \in \mathbb{C} \setminus \{\alpha_2^2\}$, $\mu_2/\mu_1 \neq \alpha_2^2$, $e \in E$ and $V \subseteq E \setminus \{e\}$,*

$$L^{[\mu_1]}_{E\setminus\{e\}}((f|_{E\setminus\{e\}})+s^{-1}(\mu_2)\cdot f_e)(V) = \frac{c(-\mu_2/\mu_1)}{c(-\mu_2)}((L^{[\mu_1]}_E f)(V)+s^{-1}(\mu_2/\mu_1)\cdot(L^{[\mu_1]}_E f)(V\cup\{e\}). \tag{14}$$

Proof. The left-hand side may be regarded as the V-entry of the vector obtained from the following matrix product:

$$M(\mu_1)^{\otimes(m-1)} \begin{pmatrix} I_{2^{m-1}} & s^{-1}(\mu_2)\cdot I_{2^{m-1}} \end{pmatrix} \mathbf{f} \;.$$

Now,

$$\begin{aligned}
&M(\mu_1)^{\otimes(m-1)} \begin{pmatrix} I_{2^{m-1}} & s^{-1}(\mu_2)\cdot I_{2^{m-1}} \end{pmatrix}\\
&= \begin{pmatrix} M(\mu_1)^{\otimes(m-1)} & \dfrac{b(-\mu_2)}{c(-\mu_2)}\cdot M(\mu_1)^{\otimes(m-1)} \end{pmatrix}\\
&= c(-\mu_2)^{-1} \begin{pmatrix} c(-\mu_2) & b(-\mu_2) \end{pmatrix} \otimes M(\mu_1)^{\otimes(m-1)}\\
&= c(-\mu_2)^{-1} \begin{pmatrix} c(-(\mu_2/\mu_1)\mu_1) & b(-(\mu_2/\mu_1)\mu_1) \end{pmatrix} \otimes M(\mu_1)^{\otimes(m-1)}
\end{aligned}$$

$$
\begin{aligned}
&= (2\sqrt{2})^{-1}c(-\mu_2)^{-1} \\
&\quad \cdot \begin{pmatrix} c(-\mu_2/\mu_1)c(\mu_1)+b(-\mu_2/\mu_1)b(\mu_1) & c(-\mu_2/\mu_1)b(\mu_1)+b(-\mu_2/\mu_1)a(\mu_1) \end{pmatrix} \\
&\quad \otimes M(\mu_1)^{\otimes(m-1)} \\
&= (2\sqrt{2})^{-1}\,\frac{c(-\mu_2/\mu_1)}{c(-\mu_2)} \\
&\quad \cdot \begin{pmatrix} c(\mu_1)+\dfrac{b(-\mu_2/\mu_1)}{c(-\mu_2/\mu_1)}\,b(\mu_1) & b(\mu_1)+\dfrac{b(-\mu_2/\mu_1)}{c(-\mu_2/\mu_1)}\,a(\mu_1) \end{pmatrix} \otimes M(\mu_1)^{\otimes(m-1)} \\
&= \frac{c(-\mu_2/\mu_1)}{c(-\mu_2)} \left(\begin{pmatrix} 1 & \dfrac{b(-\mu_2/\mu_1)}{c(-\mu_2/\mu_1)} \end{pmatrix} M(\mu_1) \right) \otimes \left(I_{2^{m-1}} M(\mu_1)^{\otimes(m-1)}\right) \\
&= \frac{c(-\mu_2/\mu_1)}{c(-\mu_2)} \left(\begin{pmatrix} 1 & \dfrac{b(-\mu_2/\mu_1)}{c(-\mu_2/\mu_1)} \end{pmatrix} \otimes I_{2^{m-1}} \right) M(\mu_1)^{\otimes m}.
\end{aligned}
$$

This expression, after postmultiplying by $\mathbf{f}$, gives a vector whose V-entry is the right hand side of (14), after converting back to transform notation. □

Our result on minors generalises [2, Theorem 4.6] and [3, Theorem 5], which in turn generalise the classical relationship between deletion, contraction and duality:

$$(M \setminus e)^* = M/e,$$

for any matroid M and any element e of the ground set of M.

Theorem 9 *For all $\mu_1 \in \mathbb{C} \setminus \{0\}$, $\mu_2 \in \mathbb{C}$ and $V \subseteq E \setminus \{e\}$,*

$$((L^{[\mu_1]} f) \,\|_{[\mu_2/\mu_1]} e)(V) = \frac{L^{[\mu_1]}(f \,\|_{[\mu_2]} e)(V)}{L^{[\mu_1]}(f \,\|_{[\mu_2]} e)(\emptyset)}.$$

Proof.

$$
\begin{aligned}
&(L^{[\mu_1]} f) \,\|_{[\mu_2/\mu_1]} e)(V) \\
&\quad = \frac{L_E^{[\mu_1]} f(V) + s^{-1}(\mu_2/\mu_1) L_E^{[\mu_1]} f(V \cup \{e\})}{L_E^{[\mu_1]} f(\emptyset) + s^{-1}(\mu_2/\mu_1) L_E^{[\mu_1]} f(\{e\})} \\
&\quad = \frac{(L_{E\setminus\{e\}}^{[\mu_1]}(f\,|_{E\setminus\{e\}} + s^{-1}(\mu_2)\cdot f_e))(V)}{(L_{E\setminus\{e\}}^{[\mu_1]}(f\,|_{E\setminus\{e\}} + s^{-1}(\mu_2)\cdot f_e))(\emptyset)} \\
&\qquad\qquad \text{(by Lemma 8 applied to numerator and (with } V=\emptyset\text{) denominator)} \\
&\quad = \frac{(L_{E\setminus\{e\}}^{[\mu_1]}((f(\emptyset)+s^{-1}(\mu_2)f(\{e\}))^{-1}(f\,|_{E\setminus\{e\}} + s^{-1}(\mu_2)\cdot f_e)))(V)}{(L_{E\setminus\{e\}}^{[\mu_1]}((f(\emptyset)+s^{-1}(\mu_2)f(\{e\}))^{-1}(f\,|_{E\setminus\{e\}} + s^{-1}(\mu_2)\cdot f_e)))(\emptyset)} \\
&\quad = \frac{L_{E\setminus\{e\}}^{[\mu_1]}(f \,\|_{[\mu_2]} e)(V)}{L_{E\setminus\{e\}}^{[\mu_1]}(f \,\|_{[\mu_2]} e)(\emptyset)}.
\end{aligned}
$$

The denominator may be regarded as a scaling factor to ensure that $(L^{[\mu_1]} f) \,\|_{[\mu_2/\mu_1]} e)(\emptyset) = 1$. □

7 Further results

We give a generalisation of Plancherel's Theorem.

Theorem 10 *For any* $\mu_1, \mu_2 \in \mathbb{C}$,

$$\sum_{X\subseteq E} L^{[\mu_1]}f(X) \cdot L^{[\mu_2]}g(X) = \sum_{X\subseteq E} L^{[\mu_1\mu_2]}f(X) \cdot g(X) = \sum_{X\subseteq E} f(X) \cdot L^{[\mu_1\mu_2]}g(X)\,.$$

Proof. The left hand side is the scalar product of two vectors whose X-entries are $L^{[\mu_1]}f(X)$ and $L^{[\mu_2]}g(X)$, respectively. Hence, using the matrix form of the transforms,

$$\begin{aligned}
\sum_{X\subseteq E} L^{[\mu_1]}f(X) \cdot L^{[\mu_2]}g(X) &= (M(\mu_1)^{\otimes m}\,\mathbf{f})^T (M(\mu_2)^{\otimes m}\,\mathbf{g}) \\
&= \mathbf{f}^T\,(M(\mu_1)^{\otimes m} M(\mu_2)^{\otimes m})\,\mathbf{g} \\
&= \mathbf{f}^T\,(M(\mu_1)M(\mu_2))^{\otimes m}\,\mathbf{g} \\
&= \mathbf{f}^T\,M(\mu_1\mu_2)^{\otimes m}\,\mathbf{g} \\
&= \sum_{X\subseteq E} f(X) \cdot (L^{[\mu_1\mu_2]}g)(X).
\end{aligned}$$

□

Corollary 11 *For any* $\mu \in \mathbb{C} \setminus \{0\}$,

$$\sum_{X\subseteq E} L^{[\mu]}f(X) \cdot L^{[\mu^{-1}]}g(X) = \sum_{X\subseteq E} f(X)g(X).$$

□

This gives a generalisation of Parseval's Theorem.

Corollary 12 *For any* $\mu \in \mathbb{C} \setminus \{0\}$,

$$\sum_{X\subseteq E} L^{[\mu]}f(X) \cdot L^{[\mu^{-1}]}f(X) = \sum_{X\subseteq E} f(X)^2.$$

□

We now work towards expressions for $L^{[\mu]}$ that are simpler in some respects than the original definition.

Observe that, if $\mu \notin \{1, -\alpha_2^2\}$,

$$\begin{aligned}
&2\sqrt{2} \cdot M(\mu) \\
&= \begin{pmatrix} c(\mu) & b(\mu) \\ b(\mu) & a(\mu) \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= c(\mu)\begin{pmatrix} 1 & 0 \\ 0 & \frac{b(\mu)}{c(\mu)} \end{pmatrix}\begin{pmatrix} 1 & \frac{b(\mu)}{c(\mu)} \\ 1 & \frac{a(\mu)}{b(\mu)} \end{pmatrix} \\
&= c(\mu)\begin{pmatrix} 1 & 0 \\ 0 & \frac{b(\mu)}{c(\mu)} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & \frac{a(\mu)c(\mu)}{b(\mu)^2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{b(\mu)}{c(\mu)} \end{pmatrix} \qquad (15) \\
&= c(\mu)\begin{pmatrix} 1 & 0 \\ 0 & \frac{b(\mu)}{c(\mu)} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & \frac{a(\mu)c(\mu)}{b(\mu)^2} \end{pmatrix} H_2 \cdot \frac{1}{2} \cdot H_2 \begin{pmatrix} 1 & 0 \\ 0 & \frac{b(\mu)}{c(\mu)} \end{pmatrix} \\
&= 2c(\mu)\begin{pmatrix} 1 & 0 \\ 0 & \frac{b(\mu)}{c(\mu)} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{a(\mu)c(\mu)+b(\mu)^2}{2b(\mu)^2} & \frac{-a(\mu)c(\mu)+b(\mu)^2}{2b(\mu)^2} \end{pmatrix} \frac{1}{2} \cdot H_2 \begin{pmatrix} 1 & 0 \\ 0 & \frac{c(\mu)}{b(\mu)} \end{pmatrix} \\
&= 2c(\mu)\begin{pmatrix} 1 & 0 \\ 0 & \frac{b(\mu)}{c(\mu)} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{a(\mu)c(\mu)+b(\mu)^2}{2b(\mu)^2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & \frac{-a(\mu)c(\mu)+b(\mu)^2}{a(\mu)c(\mu)+b(\mu)^2} \end{pmatrix} \frac{1}{2} \cdot H_2 \begin{pmatrix} 1 & 0 \\ 0 & \frac{c(\mu)}{b(\mu)} \end{pmatrix} \\
&= 2c(\mu)\begin{pmatrix} 1 & 0 \\ 0 & \frac{a(\mu)c(\mu)+b(\mu)^2}{2b(\mu)c(\mu)} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & \frac{-a(\mu)c(\mu)+b(\mu)^2}{a(\mu)c(\mu)+b(\mu)^2} \end{pmatrix} \frac{1}{2} \cdot H_2 \begin{pmatrix} 1 & 0 \\ 0 & \frac{c(\mu)}{b(\mu)} \end{pmatrix}. \qquad (16)
\end{aligned}
$$

Note that the last two lines require $\mu \neq -1$.

It is routine to show that, if $\mu \neq 1$,

$$
\frac{a(\mu)c(\mu)}{b(\mu)^2} = 2\left(\frac{1+\mu}{1-\mu}\right)^2 - 1, \qquad (17)
$$

and if $\mu \neq -1$,

$$
\frac{-a(\mu)c(\mu)+b(\mu)^2}{a(\mu)c(\mu)+b(\mu)^2} = \left(\frac{1-\mu}{1+\mu}\right)^2 - 1. \qquad (18)
$$

Note that, if $|\mu| = 1$, then both these quantities are real and ≤ -1.

Applying the mixed-product property to (15), together with (17), gives another expression for the transform $L^{[\mu]}$.

Theorem 13 *If $\mu \in \mathbb{C} \setminus \{1, -\alpha_2^2\}$ and $V \subseteq E$,*

$$
L^{[\mu]} f(V) = \left(\frac{c(\mu)}{2\sqrt{2}}\right)^{|E|} \left(\frac{b(\mu)}{c(\mu)}\right)^{|V|} \sum_{X \subseteq E} \left(2\left(\frac{1+\mu}{1-\mu}\right)^2 - 1\right)^{|X \cap V|} \left(\frac{b(\mu)}{c(\mu)}\right)^{|X|} f(X).
$$

□

Similar use of (16) with (18) gives an expression for $L^{[\mu]}$ which may be simpler to use.

Theorem 14 *If $\mu \in \mathbb{C} \setminus \{\pm 1, -\alpha_2^2\}$ and $V \subseteq E$,*

$$
L^{[\mu]} f(V) = \left(\frac{c(\mu)}{\sqrt{2}}\right)^{|E|} \left(\frac{(1+\mu)^2}{b(\mu)c(\mu)}\right)^{|V|} \sum_{X \subseteq V} \left(\left(\frac{1-\mu}{1+\mu}\right)^2 - 1\right)^{|X|} \left(\widehat{\left(\frac{b(\mu)}{c(\mu)}\right)^{|\bullet|} \cdot f}\right)(X).
$$

□

Putting $V = E$ and renaming f gives

Corollary 15 *If* $\mu \in \mathbb{C} \setminus \{\pm 1, -\alpha_2^2\}$ *then*

$$L^{[\mu]}\left(2^{|E|} \cdot \left(\frac{b(\mu)}{c(\mu)}\right)^{|\bullet|} \cdot \hat{f}\right)(E) = \left(\frac{(1+\mu)^2}{\sqrt{2}\, b(\mu)}\right)^{|E|} A_f\left(\left(\frac{1-\mu}{1+\mu}\right)^2 - 1\right).$$

□

Consider the special case when f is the indicator function of a binary linear space. In that case, Corollary 15 relates the weight enumerator of a binary linear code at a real argument < -1 to the $L^{[\mu]}$ transform, for appropriate $\mu \in \mathbb{C}$ with $|\mu| = 1$, of a close relative of the indicator function of the dual code.

Recasting Corollary 15 in terms of the Ising model partition function, we obtain

$$L^{[\mu]}\left(2^{|E|} \cdot \left(\frac{b(\mu)}{c(\mu)}\right)^{|\bullet|} \cdot \hat{f}\right)(E) = \frac{1}{2}\left(\frac{(1+\mu)^2}{e^K \sqrt{2}\, b(\mu)}\right)^{|E|} Z_I(f; K), \tag{19}$$

where

$$K = -\frac{1}{2} \ln\left(\left(\frac{1-\mu}{1+\mu}\right)^2 - 1\right).$$

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